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Duality of chordal SLE, II

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Abstract. We improve the geometric properties of $\mathrm{SLE}(\kappa;\vec{\rho})$ processes derived in an earlier paper, which are then used to obtain more results about the duality of SLE. We find that for $\kappa\in(4,8)$, the boundary of a standard chordal $\mathrm{SLE}(\kappa)$ hull stopped on swallowing a fixed $x\in\mathbb{R}\setminus\{0\}$ is the image of some $\mathrm{SLE}(16/\kappa;\vec{\rho})$ trace started from a random point. Using this fact together with a similar proposition in the case that $\kappa\ge 8$, we obtain a description of the boundary of a standard chordal $\mathrm{SLE}(\kappa)$ hull for $\kappa>4$, at a finite stopping time. Finally, we prove that for $\kappa>4$, in many cases, a chordal or strip $\mathrm{SLE}(\kappa;\vec{\rho})$ trace a.s. ends at a single point.

Résumé. Nous améliorons des résultats précédemment obtenus concernant les propriétés géométriques des processus $\mathrm{SLE}(\kappa;\vec{\rho})$, que nous utilisons ensuite pour étudier la propriété dite de dualité des processus SLE.

Nous prouvons que pour $\kappa\in(4,8)$, la frontière de l'enveloppe d'un $\mathrm{SLE}(\kappa)$ chordal standard arrêté quand il disconnecte un point fixe $x\in\mathbb{R}\backslash\{0\}$ de l'infini est une courbe $\mathrm{SLE}(16/\kappa,\vec{\rho})$ issue d'un point aléatoire. Nous obtenons ainsi une description de la frontière de l'enveloppe d'un $\mathrm{SLE}(\kappa)$ pour $\kappa>4$. Finalement, nous démontrons que pour $\kappa>4$, dans de nombreux cas, la courbe de processus $\mathrm{SLE}(\kappa;\vec{\rho})$ généralisés (par exemple dans une bande) se termine presque sûrement en un point unique.

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Keywords: SLE; Duality; Coupling technique

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1. Introduction

This paper is a follow-up of the paper [9], in which we proved some versions of Duplantier's duality conjecture about Schramm's SLE process [8]. In the present paper, we will improve the technique used in [9], and derive more results about the duality conjecture.

Let us now briefly review some results in [9]. Let $\kappa_1<4<\kappa_2$ with $\kappa_1\kappa_2=16$. Let $x_1\ne x_2\in\mathbb{R}$. Let $N\in\mathbb{N}$ and $p_1,\dots,p_N\in\mathbb{R}\cup\{\infty\}\setminus\{x_1,x_2\}$ be distinct points. Let $C_1,\dots,C_N\in\mathbb{R}$ and $\rho_{j,m}=C_m(\kappa_j-4)$, $1\le m\le N$, $j=1,2$. Let $\vec{p}=(p_1,\dots,p_N)$ and $\vec{\rho}_j=(\rho_{j,1},\dots,\rho_{j,N})$, $j=1,2$. We used Itô's calculus and the Girsanov theorem to derive some commutation relation between two SLE-type processes in the upper half-plane $\mathbb{H}=\{z\in\mathbb{C}\colon \operatorname{Im} z>0\}$: one is a chordal $\mathrm{SLE}(\kappa_1;-\frac{\kappa_1}{2},\vec{\rho}_1)$ process, say $K_1(t)$, $0\le t<T_1$, started from $(x_1;x_2,\vec{p})$, the other a chordal $\mathrm{SLE}(\kappa_2;-\frac{\kappa_2}{2},\vec{\rho}_2)$ process $K_2(t)$, $0\le t<T_2$, started from $(x_2;x_1,\vec{p})$. Using the coupling technique obtained in [11], we obtained a coupling of the above two SLE processes such that for any $j\ne k\in\{1,2\}$, if $S_k<T_k$ is a stopping time for $(K_k(t))$, and if we conditioned on $K_k(t)$, $0\le t\le S_k$, then after a time-change, the part of $K_j(t)$ before hitting $K_k(S_k)$ is a chordal $\mathrm{SLE}(\kappa_j;-\frac{\kappa_j}{2},\vec{\rho}_j)$ process in $\mathbb{H}\setminus K_k(S_k)$ started from $(x_j;\beta_k(S_k),\vec{p})$, where $\beta_k(t)$ is the tract that corresponds to $(K_k(t))$. Moreover, some p_m could be degenerate, i.e., $p_m=x_j^+$ or x_j^-, if the corresponding force $\rho_{j,m}$ satisfies $\rho_{j,m}\ge\kappa_j/2-2$.

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This theorem was then applied to the case that $N = 3$; $x_1 < x_2$; $p_1 \in (-\infty, x_1)$ or $= x_1^-$; $p_2 \in (x_2, \infty)$, or $= \infty$, or $= x_2^+$; and $p_3 \in (x_1, x_2)$, or $= x_1^+$, or $= x_2^-$; $C_1 \le 1/2$, $C_2 = 1 - C_1$ and $C_3 = 1/2$. Using some study about the behavior of these SLE$(\kappa; \vec{\rho})$ traces at their end points, we concluded that $K_1(T_1^-) := \bigcup_{0\le t<T_1} K_1(t)$ is the outer boundary of $K_2(T_2^-) := \bigcup_{0\le t<T_2} K_2(t)$ in $\mathbb{H}$.

The following proposition, i.e., Theorem 5.2 in [9], is an application of the above result. It describes the boundary of a standard chordal SLE(κ) hull, where $\kappa \ge 8$, at the time when a fixed $x \in \mathbb{R} \setminus \{0\}$ is swallowed.

Proposition 1.1. *Suppose $\kappa \ge 8$, and $K(t)$, $0 \le t < \infty$ is a standard chordal* SLE(κ) *process. Let $x \in \mathbb{R} \setminus \{0\}$ and T_x be the first t such that $x \in \overline{K(t)}$. Then $\partial K(T_x) \cap \mathbb{H}$ has the same distribution as the image of a chordal* SLE$(\kappa'; -\frac{\kappa'}{2}, -\frac{\kappa'}{2}, \frac{\kappa'}{2} - 2)$ *trace started from $(x; 0, x^a, x^b)$, where $\kappa' = 16/\kappa$, $a = \mathrm{sign}(x)$ and $b = \mathrm{sign}(-x)$. So a.s. $\partial K(T_x) \cap \mathbb{H}$ is a crosscut in $\mathbb{H}$ connecting x with some $y \in \mathbb{R} \setminus \{0\}$ with* $\mathrm{sign}(y) = \mathrm{sign}(-x)$.

Here, a crosscut in $\mathbb{H}$ on $\mathbb{R}$ is a simple curve in $\mathbb{H}$ whose two ends approach to two different points on $\mathbb{R}$. Since $\kappa \ge 8$, the trace is space-filling, so a.s. x is visited by the trace at time T_x, and so x is an end point of $\overline{K(T_x)} \cap \mathbb{R}$. From this proposition, we see that the boundary of $K(T_x)$ in $\mathbb{H}$ is an SLE$(16/\kappa)$-type trace in $\mathbb{H}$ started from x.

The motivation of the present paper is to derive the counterpart of Proposition 1.1 in the case that $\kappa \in (4, 8)$. In this case, the trace, say γ, is not space-filling, so a.s. x is not visited by γ, at time T_x, and so x is an interior point of $\overline{K(T_x)} \cap \mathbb{R}$. Thus we can not expect that the boundary of $K(T_x)$ in $\mathbb{H}$ is a curve started from x.

This difficulty will be overcome by conditioning the process $K(t)$, $0 \le t < T_x$, on the value of $\gamma(T_x)$. In Section 3, we will prove that conditioned on $y = \gamma(T_x)$, $K(t)$, $0 \le t < T_x$, is a chordal SLE$(\kappa; -4, \kappa - 4)$ process started from $(0; y, x)$. This is the statement of Corollary 3.2.

In Section 4, we will improve the geometric results about SLE$(\kappa; \vec{\rho})$ processes that were derived in [9]. Using these geometric results, we will prove in Section 5 that Proposition 2.8 can be applied with $N = 4$ and suitable values of p_m and C_m for $1 \le m \le 4$, to obtain more results about duality. Especially, using Corollary 3.2, we will obtain the counterpart of Proposition 1.1 in the case that $\kappa \in (4, 8)$, which is Theorem 1.1 below.

Theorem 1.1. *Let $\kappa \in (4, 8)$, and $x \in \mathbb{R} \setminus \{0\}$. Let $K(t)$ and $\gamma(t)$, $0 \le t < \infty$, be standard chordal* SLE(κ) *process and trace, respectively. Let T_x be the first time that $x \in \overline{K(t)}$. Let $\bar{\mu}$ denote the distribution of $\partial K(T_x) \cap \mathbb{H}$. Let λ denote the distribution of $\gamma(T_x)$. Let $\kappa' = 16/\kappa$, $a = \mathrm{sign}(x)$ and $b = \mathrm{sign}(-x)$. Let $\bar{\nu}_y$ denote the distribution of the image of a chordal* SLE$(\kappa'; -\frac{\kappa'}{2}, \frac{3}{2}\kappa' - 4, -\frac{\kappa'}{2} + 2, \kappa' - 4)$ *trace started from $(y; 0, y^a, y^b, x)$. Then $\bar{\mu} = \int \bar{\nu}_y \, d\lambda(y)$. So a.s. $\partial K(T_x) \cap \mathbb{H}$ is a crosscut in $\mathbb{H}$ connecting some $y, z \in \mathbb{R} \setminus \{0\}$, where* $\mathrm{sign}(y) = \mathrm{sign}(x)$, $|y| > |x|$, *and* $\mathrm{sign}(z) = \mathrm{sign}(-x)$.

In Section 6, we will use Theorem 1.1 and Proposition 1.1 to study the boundary of a standard chordal SLE(κ) hull, say $K(t)$, at a finite positive stopping time T. Let $\gamma(t)$ be the corresponding SLE trace. We will find that if $\gamma(T) \in \mathbb{R}$, then $\partial K(T) \cap \mathbb{H}$ is a crosscut in $\mathbb{H}$ with $\gamma(T)$ as one end point; and if $\gamma(T) \in \mathbb{H}$, then $\partial K(T) \cap \mathbb{H}$ is the union of two semi-crosscuts in $\mathbb{H}$, which both have $\gamma(T)$ as one end point. Here a semi-crosscut in $\mathbb{H}$ is a simple curve in $\mathbb{H}$ whose one end lies in $\mathbb{H}$ and the other end approaches to a point on $\mathbb{R}$. Moreover, in the latter case, every intersection point of the two semi-crosscuts other than $\gamma(T)$ corresponds to a cut-point of $K(T)$. If $\kappa \ge 8$, then the two semi-crosscuts only meet at $\gamma(T)$, and so $\partial K(T) \cap \mathbb{H}$ is again a crosscut in $\mathbb{H}$ on $\mathbb{R}$.

In the last section of this paper, we will use the results in Section 6 to derive more geometric results about SLE$(\kappa; \vec{\rho})$ processes. We will prove that many propositions in [9] and Section 4 of this paper about the limit of an SLE$(\kappa; \vec{\rho})$ trace that hold for $\kappa \in (0, 4]$ are also true for $\kappa > 4$.

Julien Dubédat has a nice result ([4], Theorem 1) about the boundary arc of $K(t)$ straddling x, i.e., the boundary arc seen by x at time T_x^-, which says that the boundary arc is also an SLE$(16/\kappa)$ curve. His result is about the "inner" boundary of $K(T_x)$, while Theorem 1.1 in this paper is about the "outer" boundary. The author prefers to apply Theorem 1.1 to study the boundary of standard chordal SLE(κ) hulls at general stopping times, and to derive other related results.

2. Preliminary

In this section, we review some definitions and propositions in [9], which will be used in this paper.

If H is a bounded and relatively closed subset of $\mathbb{H} = \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$, and $\mathbb{H} \setminus H$ is simply connected, then we call H a hull in $\mathbb{H}$ w.r.t. ∞. For such H, there is φ_H that maps $\mathbb{H} \setminus H$ conformally onto $\mathbb{H}$, and satisfies $\varphi_H(z) = z + \frac{c}{z} + O(\frac{1}{z^2})$ as $z \to \infty$, where $c = \operatorname{hcap}(H) \geq 0$ is called the capacity of H in $\mathbb{H}$ w.r.t. ∞.

For a real interval I, we use $C(I)$ to denote the space of real continuous functions on I. For $T > 0$ and $\xi \in C([0, T))$, the chordal Loewner equation driven by ξ is

$$\partial_t \varphi(t, z) = \frac{2}{\varphi(t, z) - \xi(t)}, \qquad \varphi(0, z) = z.$$

For $0 \leq t < T$, let $K(t)$ be the set of $z \in \mathbb{H}$ such that the solution $\varphi(s, z)$ blows up before or at time t. We call $K(t)$ and $\varphi(t, \cdot)$, $0 \leq t < T$, chordal Loewner hulls and maps, respectively, driven by ξ. It turns out that $\varphi(t, \cdot) = \varphi_{K(t)}$ for each $t \in [0, T)$.

Let $B(t)$, $0 \leq t < \infty$, be a (standard linear) Brownian motion. Let $\kappa \geq 0$. Then $K(t)$ and $\varphi(t, \cdot)$, $0 \leq t < \infty$, driven by $\xi(t) = \sqrt{\kappa} B(t)$, $0 \leq t < \infty$, are called standard chordal SLE(κ) hulls and maps, respectively. It is known [5,7] that almost surely for any $t \in [0, \infty)$,

$$\gamma(t) := \lim_{\mathbb{H} \ni z \to \xi(t)} \varphi(t, \cdot)^{-1}(z) \tag{2.1}$$

exists, and $\gamma(t)$, $0 \leq t < \infty$, is a continuous curve in $\overline{\mathbb{H}}$. Moreover, if $\kappa \in (0, 4]$, then γ is a simple curve, which intersects $\mathbb{R}$ only at the initial point, and for any $t \geq 0$, $K(t) = \gamma((0, t])$; if $\kappa > 4$ then γ is not simple; if $\kappa \geq 8$ then γ is space-filling. Such γ is called a standard chordal SLE(κ) trace.

If $(\xi(t))$ is a semi-martingale, and $\mathrm{d}\langle \xi(t) \rangle = \kappa\, \mathrm{d}t$ for some $\kappa > 0$, then from Girsanov theorem (cf. [6]) and the existence of standard chordal SLE(κ) trace, almost surely for any $t \in [0, T)$, $\gamma(t)$ defined by (2.1) exists, and has the same property as a standard chordal SLE(κ) trace (depending on the value of κ) as described in the last paragraph.

Let $\kappa \geq 0$, $\rho_1, \ldots, \rho_N \in \mathbb{R}$, $x \in \mathbb{R}$, and $p_1, \ldots, p_N \in \widehat{\mathbb{R}} \setminus \{x\}$, where $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is a circle. Let $\xi(t)$ and $p_k(t)$, $1 \leq k \leq N$, be the solutions to the SDE:

$$\begin{cases} \mathrm{d}\xi(t) = \sqrt{\kappa}\, \mathrm{d}B(t) + \sum_{k=1}^N \frac{\rho_k}{\xi(t) - p_k(t)}\, \mathrm{d}t, \\ \mathrm{d}p_k(t) = \frac{2}{p_k(t) - \xi(t)}\, \mathrm{d}t, \quad 1 \leq k \leq N, \end{cases} \tag{2.2}$$

with initial values $\xi(0) = x$ and $p_k(0) = p_k$, $1 \leq k \leq N$. If $\varphi(t, \cdot)$ are chordal Loewner maps driven by $\xi(t)$, then $p_k(t) = \varphi(t, p_k)$. Suppose $[0, T)$ is the maximal interval of the solution. Let $K(t)$ and $\gamma(t)$, $0 \leq t < T$, be chordal Loewner hulls and trace driven by ξ. Let $\vec{\rho} = (\rho_1, \ldots, \rho_N)$ and $\vec{p} = (p_1, \ldots, p_N)$. Then $K(t)$ and $\gamma(t)$, $0 \leq t < T$, are called (full) chordal SLE($\kappa; \rho_1, \ldots, \rho_N$) or SLE($\kappa; \vec{\rho}$) process and trace, respectively, started from $(x; p_1, \ldots, p_N)$ or $(x; \vec{p})$. If $T_0 \in (0, T]$ is a stopping time, then $K(t)$ and $\gamma(t)$, $0 \leq t < T_0$, are called partial chordal SLE($\kappa; \vec{\rho}$) process and trace, respectively, started from $(x; \vec{p})$.

If we allow that one of the force points takes value x^+ or x^-, or two of the force points take values x^+ and x^-, respectively, then we obtain the definition of degenerate chordal SLE($\kappa; \vec{\rho}$) process. Let $\kappa \geq 0$; $\rho_1, \ldots, \rho_N \in \mathbb{R}$, and $\rho_1 \geq \kappa/2 - 2$; $p_1 = x^+$, $p_2, \ldots, p_N \in \widehat{\mathbb{R}} \setminus \{x\}$. Let $\xi(t)$ and $p_k(t)$, $1 \leq k \leq N$, $0 < t < T$, be the maximal solution to (2.2) with initial values $\xi(0) = p_1(0) = x$, and $p_k(0) = p_k$, $1 \leq k \leq N$. Moreover, we require that $p_1(t) > \xi(t)$ for any $0 < t < T$. Then the chordal Loewner hulls $K(t)$ and trace $\gamma(t)$, $0 \leq t < T$, driven by ξ, are called chordal SLE($\kappa; \rho_1, \ldots, \rho_N$) process and trace started from $(x; x^+, p_2, \ldots, p_N)$. If the condition $p_1(t) > \xi(t)$ is replaced by $p_1(t) < \xi(t)$, then we get chordal SLE($\kappa; \rho_1, \ldots, \rho_N$) process and trace started from $(x; x^-, p_2, \ldots, p_N)$. Now suppose $N \geq 2$, $\rho_1, \rho_2 \geq \kappa/2 - 2$, $p_1 = x^+$ and $p_2 = x^-$. Let $\xi(t)$ and $p_k(t)$, $1 \leq k \leq N$, $0 < t < T$, be the maximal solution to (2.2) with initial values $\xi(0) = p_1(0) = p_2(0) = x$, and $p_k(0) = p_k$, $1 \leq k \leq N$, such that $p_1(t) > \xi(t) > p_2(t)$ for all $0 < t < T$. Then we obtain chordal SLE($\kappa; \rho_1, \ldots, \rho_N$) process and trace started from $(x; x^+, x^-, p_3, \ldots, p_N)$. The force point x^+ or x^- is called a degenerate force point. Other force points are called generic force points. Let $\varphi(t, \cdot)$ be the chordal Loewner maps driven by ξ. Since for any generic force point p_j, we have $p_j(t) = \varphi(t, p_j)$, so we write $\varphi(t, p_j)$ for $p_j(t)$ in the case that p_j is a degenerate force point.

For $h > 0$, let $\mathbb{S}_h = \{z \in \mathbb{C}\colon 0 < \operatorname{Im} z < h\}$ and $\mathbb{R}_h = \mathrm{i}h + \mathbb{R}$. If H is a bounded closed subset of $\mathbb{S}_\pi$, $\mathbb{S}_\pi \setminus H$ is simply connected, and has $\mathbb{R}_\pi$ as a boundary arc, then we call H a hull in $\mathbb{S}_\pi$ w.r.t. $\mathbb{R}_\pi$. For such H, there is a unique ψ_H that maps $\mathbb{S}_\pi \setminus H$ conformally onto $\mathbb{S}_\pi$, such that for some $c \geq 0$, $\psi_H(z) = z \pm c + o(1)$ as $z \to \pm\infty$ in $\mathbb{S}_\pi$. We call such c the capacity of H in $\mathbb{S}_\pi$ w.r.t. $\mathbb{R}_\pi$, and let it be denoted it by $\operatorname{scap}(H)$.

For $\xi \in C([0,T))$, the strip Loewner equation driven by ξ is

$$\partial_t \psi(t,z) = \coth\left(\frac{\psi(t,z)-\xi(t)}{2}\right), \qquad \psi(0,z)=z. \tag{2.3}$$

For $0 \le t < T$, let $L(t)$ be the set of $z \in \mathbb{S}_\pi$ such that the solution $\psi(s,z)$ blows up before or at time t. We call $L(t)$ and $\psi(t,\cdot)$, $0 \le t < T$, strip Loewner hulls and maps, respectively, driven by ξ. It turns out that $\psi(t,\cdot) = \psi_{L(t)}$ and $\operatorname{scap}(L(t)) = t$ for each $t \in [0,T)$. In this paper, we use $\coth_2(z)$, $\tanh_2(z)$, $\cosh_2(z)$ and $\sinh_2(z)$ to denote the functions $\coth(z/2)$, $\tanh(z/2)$, $\cosh(z/2)$ and $\sinh(z/2)$, respectively.

Let $\kappa \ge 0$, $\rho_1,\dots,\rho_N \in \mathbb{R}$, $x \in \mathbb{R}$, and $p_1,\dots,p_N \in \mathbb{R} \cup \mathbb{R}_\pi \cup \{+\infty,-\infty\} \setminus \{x\}$. Let $B(t)$ be a Brownian motion. Let $\xi(t)$ and $p_k(t)$, $1 \le k \le N$, be the solutions to the SDE:

$$\begin{cases} \mathrm{d}\xi(t) = \sqrt{\kappa}\,\mathrm{d}B(t) + \sum_{k=1}^N \frac{\rho_k}{2} \coth_2\big(\xi(t)-p_k(t)\big)\,\mathrm{d}t, \\ \mathrm{d}p_k(t) = \coth_2\big(p_k(t)-\xi(t)\big)\,\mathrm{d}t, \quad 1 \le k \le N, \end{cases} \tag{2.4}$$

with initial values $\xi(0)=x$ and $p_k(0)=p_k$, $1 \le k \le N$. Here, if some $p_k = \pm\infty$, then $p_k(t) = \pm\infty$ and $\coth_2(\xi(t)-p_k(t)) = \pm 1$ for all $t \ge 0$. Suppose $[0,T)$ is the maximal interval of the solution to (2.4). Let $L(t)$, $0 \le t < T$, be strip Loewner hulls driven by ξ. Let $\beta(t) = \lim_{\mathbb{S}_\pi \ni z \to \xi(t)} \psi(t,z)$, $0 \le t < T$. Then we call $L(t)$ and $\beta(t)$, $0 \le t < T$, (full) strip SLE$(\kappa;\vec{\rho})$ process and trace, respectively, started from $(x;\vec{p})$, where $\vec{\rho} = (\rho_1,\dots,\rho_N)$ and $\vec{p} = (p_1,\dots,p_N)$. If $T_0 \in (0,T]$ is a stopping time, then $L(t)$ and $\beta(t)$, $0 \le t < T_0$, are called partial strip SLE$(\kappa;\vec{\rho})$ process and trace, respectively, started from $(x;\vec{p})$.

The following two propositions are Lemmas 2.1 and 2.3 in [9]. They will be used frequently in this paper. Let S_1 and S_2 be two sets of boundary points or prime ends of a domain D. We say that K does not separate S_1 from S_2 in D if there are neighborhoods U_1 and U_2 of S_1 and S_2, respectively, in D such that U_1 and U_2 lie in the same pathwise connected component of $D \setminus K$.

Proposition 2.1. *Suppose $\kappa \ge 0$ and $\vec{\rho} = (\rho_1,\dots,\rho_N)$ with $\sum_{m=1}^N \rho_m = \kappa - 6$. For $j=1,2$, let $K_j(t)$, $0 \le t < T_j$, be a generic or degenerate chordal* SLE$(\kappa;\vec{\rho})$ *process started from $(x_j;\vec{p}_j)$, where $\vec{p}_j = (p_{j,1},\dots,p_{j,N})$, $j=1,2$. Suppose W is a conformal or conjugate conformal map from $\mathbb{H}$ onto $\mathbb{H}$ such that $W(x_1)=x_2$ and $W(p_{1,m}) = p_{2,m}$, $1 \le m \le N$. Let $p_{1,\infty} = W^{-1}(\infty)$ and $p_{2,\infty} = W(\infty)$. For $j=1,2$, let $S_j \in (0,T_j]$ be the largest number such that for $0 \le t < S_j$, $K_j(t)$ does not separate $p_{j,\infty}$ from ∞ in $\mathbb{H}$. Then $(W(K_1(t)), 0 \le t < S_1)$ has the same law as $(K_2(t), 0 \le t < S_2)$ up to a time-change. A similar result holds for the traces.*

Proposition 2.2. *Suppose $\kappa \ge 0$ and $\vec{\rho} = (\rho_1,\dots,\rho_N)$ with $\sum_{m=1}^N \rho_m = \kappa - 6$. Let $K(t)$, $0 \le t < T$, be a chordal* SLE$(\kappa;\vec{\rho})$ *process started from $(x;\vec{p})$, where $\vec{p} = (p_1,\dots,p_N)$. Let $L(t)$, $0 \le t < S$, be a strip* SLE$(\kappa;\vec{\rho})$ *process started from $(y;\vec{q})$, where $\vec{q} = (q_1,\dots,q_N)$. Suppose W is a conformal or conjugate conformal map from $\mathbb{H}$ onto $\mathbb{S}_\pi$ such that $W(x)=y$ and $W(p_k)=q_k$, $1 \le k \le N$. Let $I = W^{-1}(\mathbb{R}_\pi)$ and $q_\infty = W(\infty)$. Let $T' \in (0,T]$ be the largest number such that for $0 \le t < T'$, $K(t)$ does not separate I from ∞ in $\mathbb{H}$. Let $S' \in (0,S]$ be the largest number such that for $0 \le t < S'$, $L(t)$ does not separate q_∞ from $\mathbb{R}_\pi$. Then $(W(K(t)), 0 \le t < T')$ has the same law as $(L(t), 0 \le t < S')$ up to a time-change. A similar result holds for the traces.*

Now we recall some geometric results of SLE$(\kappa;\vec{\rho})$ traces derived in [9].

Let $\kappa > 0$, and $\rho_+,\rho_- \in \mathbb{R}$ be such that $\rho_+ + \rho_- = \kappa - 6$. Suppose $\beta(t)$, $0 \le t < \infty$, is a strip SLE$(\kappa;\rho_+,\rho_-)$ trace started from $(0;+\infty,-\infty)$. In the following propositions, Proposition 2.3 is a combination of Lemma 3.1, Lemma 3.2 and the argument before Lemma 3.2 in [9]; Propositions 2.4 and 2.5 are Theorem 3.3, and Theorem 3.4, respectively, in [9].

Proposition 2.3. *If $|\rho_+ - \rho_-| < 2$, then a.s. $\beta([0,\infty))$ is bounded, and $\overline{\beta([0,\infty))}$ intersects $\mathbb{R}_\pi$ at a single point $J + \pi\mathrm{i}$. And the distribution of J has a probability density function w.r.t. the Lebesgue measure, which is proportional to* $\exp(\frac{1}{\kappa}(\rho_- - \rho_+)x)(\cosh_2 x)^{-4/\kappa}$.

Proposition 2.4. *If $\kappa \in (0,4]$ and $|\rho_+ - \rho_-| < 2$, then a.s. $\lim_{t\to\infty} \beta(t) \in \mathbb{R}_\pi$.*

Proposition 2.5. *If $\kappa\in(0,4]$ and $\pm(\rho_+-\rho_-)\ge 2$, then a.s.* $\lim_{t\to\infty}\beta(t)=\mp\infty$.

The following two propositions are Theorems 3.1 and 3.2 in [9].

Proposition 2.6. *Let $\kappa>0$, $N_+,N_-\in\mathbb{N}$, $\vec\rho_\pm=(\rho_{\pm1},\dots,\rho_{\pm N_\pm})\in\mathbb{R}^{N_\pm}$ with $\sum_{j=1}^k\rho_{\pm j}\ge\kappa/2-2$ for $1\le k\le N_\pm$, $\vec p_\pm=(p_{\pm1},\dots,p_{\pm N_\pm})$ with $0<p_1<\cdots<p_{N_+}$ and $0>p_{-1}>\cdots>p_{-N_-}$. Let $\gamma(t)$, $0\le t<T$, be a chordal* SLE$(\kappa;\vec\rho_+,\vec\rho_-)$ *trace started from $(0;\vec p_+,\vec p_-)$. Then a.s. $T=\infty$ and ∞ is a subsequential limit of $\gamma(t)$ as $t\to\infty$.*

Proposition 2.7. *Let $\kappa\in(0,4]$, $\rho_+,\rho_-\ge\kappa/2-2$. Suppose $\gamma(t)$, $0\le t<\infty$, is a chordal* SLE$(\kappa;\rho_+,\rho_-)$ *trace started from $(0;p_+,p_-)$. If $p_+=0^+$ and $p_-=0^-$, or $p^+\in(0,\infty)$ and $p^-\in(-\infty,0)$, then a.s.* $\lim_{t\to\infty}\gamma(t)=\infty$.

The following proposition is Theorem 4.1 in [9] in the case that $\kappa_1<4<\kappa_2$.

Proposition 2.8. *Let $0<\kappa_1<4<\kappa_2$ be such that $\kappa_1\kappa_2=16$. Let $x_1\ne x_2\in\mathbb{R}$. Let $N\in\mathbb{N}$. Let $p_1,\dots,p_N\in\mathbb{R}\cup\{\infty\}\setminus\{x_1,x_2\}$ be distinct points. For $1\le m\le N$, let $C_m\in\mathbb{R}$ and $\rho_{j,m}=C_m(\kappa_j-4)$, $j=1,2$. There is a coupling of $K_1(t)$, $0\le t<T_1$, and $K_2(t)$, $0\le t<T_2$, such that* (i) *for $j=1,2$, $K_j(t)$, $0\le t<T_j$, is a chordal* SLE$(\kappa_j;-\frac{\kappa_j}{2},\vec\rho_j)$ *process started from $(x_j;x_{3-j},\vec p)$; and* (ii) *for $j\ne k\in\{1,2\}$, if $\bar t_k$ is an $(\mathcal{F}^k_t)$-stopping time with $\bar t_k<T_k$, then conditioned on $\mathcal{F}^k_{\bar t_k}$, $\varphi_k(\bar t_k,K_j(t))$, $0\le t\le T_j(\bar t_k)$, has the same distribution as a time-change of a partial chordal* SLE$(\kappa_j;-\frac{\kappa_j}{2},\vec\rho_j)$ *process started from $(\varphi_k(\bar t_k,x_j);\xi_k(\bar t_k),\varphi_k(\bar t_k,\vec p))$, where $\varphi_k(t,\vec p)=(\varphi_k(t,p_1),\dots,\varphi_k(t,p_N))$; $\varphi_k(t,\cdot)$, $0\le t<T_k$, are chordal Loewner maps for the hulls $K_k(t)$, $0\le t<T_k$; $T_j(\bar t_k)\in(0,T_j]$ is the largest number such that $\overline{K_j(t)}\cap\overline{K_k(\bar t_k)}=\varnothing$ for $0\le t<T_j(\bar t_k)$; and $(\mathcal{F}^j_t)$ is the filtration generated by $(K_j(t))$, $j=1,2$. This still holds if some p_m take(s) value $x_1^\pm$ or $x_2^\pm$.*

3. Integration of SLE measures

Let $\kappa>0$, $\rho_+,\rho_-\in\mathbb{R}$, $\rho_++\rho_-=\kappa-6$, and $|\rho_+-\rho_-|<2$. Suppose $\xi(t)$, $0\le t<\infty$, is the driving function of a strip SLE$(\kappa;\rho_+,\rho_-)$ process started from $(0;+\infty,-\infty)$. Let $\sigma=(\rho_--\rho_+)/2$. Then there is a Brownian motion $B(t)$ such that $\xi(t)=\sqrt{\kappa}B(t)+\sigma t$, $0\le t<\infty$.

Let μ denote the distribution of ξ. We consider μ as a probability measure on $C([0,\infty))$. Let $(\mathcal{F}_t)$ be the filtration on $C([0,\infty))$ generated by coordinate maps. Then the total σ-algebra is $\mathcal{F}_\infty=\bigvee_{t\ge0}\mathcal{F}_t$. For each $x\in\mathbb{R}$, let ν_x denote the distribution of the driving function of a strip SLE$(\kappa;-4,\rho_-+2,\rho_++2)$ process started from $(0;x+\pi i,+\infty,-\infty)$, which is also a probability measure on $C([0,\infty))$. Then we have the following lemma.

Lemma 3.1. *We have*

$$\mu=\frac{1}{Z}\int_{\mathbb{R}}\nu_x\exp(2\sigma x/\kappa)(\cosh_2 x)^{-4/\kappa}\,\mathrm{d}x,$$

where dx *is Lebesgue measure, $Z=\int_{\mathbb{R}}\exp(2\sigma x/\kappa)(\cosh_2 x)^{-4/\kappa}\,\mathrm{d}x$, which is finite because $|\sigma|<1$, and the integral means that for any $A\in\mathcal{F}_\infty$,*

$$\mu(A)=\frac{1}{Z}\int_{\mathbb{R}}\nu_x(A)\exp(2\sigma x/\kappa)(\cosh_2 x)^{-4/\kappa}\,\mathrm{d}x. \tag{3.1}$$

Proof. Let $f(x)=\frac{1}{Z}\exp(2\sigma x/\kappa)(\cosh_2 x)^{-4/\kappa}$, $x\in\mathbb{R}$. Then $\int_{\mathbb{R}}f(x)\,\mathrm{d}x=1$, and

$$\frac{f'(x)}{f(x)}=\frac{2}{\kappa}(\sigma-\tanh_2 x),\quad x\in\mathbb{R}, \tag{3.2}$$

$$\frac{\kappa}{2}f''(x)+f'(x)(-\sigma+\tanh_2 x)+\frac{f(x)}{2}(\cosh_2 x)^{-2}=0,\quad x\in\mathbb{R}. \tag{3.3}$$

Note that the collection of A's that satisfy (3.1) is a monotone class, and $\bigcup_{t\ge 0}\mathcal{F}_t$ is an algebra. From Monotone Class theorem, we suffice to show that (3.1) holds for any $A\in\mathcal{F}_t$, $t\in[0,\infty)$. This will be proved by showing that $\nu_x|_{\mathcal{F}_t}\ll\mu|_{\mathcal{F}_t}$ for all $x\in\mathbb{R}$ and $t\in[0,\infty)$, and if $R_t(x)$ is the Radon–Nikodym derivative, then $\int_{\mathbb{R}} R_t(x)f(x)\,\mathrm{d}x=1$.

Let $\psi(t,\cdot)$, $0\le t<\infty$, be the strip Loewner maps driven by ξ. For $x\in\mathbb{R}$ and $t\ge 0$, let $X(t,x)=\mathrm{Re}(\psi(t,x+\pi\mathrm{i})-\xi(t))$. Note that $\psi(t,x+\pi\mathrm{i})\in\mathbb{R}_\pi$ for any $t\ge 0$. From (2.3), for any fixed $x\in\mathbb{R}$, $X(t,x)$ satisfies the SDE

$$\partial_t X(t,x)=-\sqrt{\kappa}\,\partial B(t)-\sigma\,\partial t+\tanh_2\big(X(t,x)\big)\,\partial t. \tag{3.4}$$

If t is fixed, then $\partial_x X(t,x)=\partial_x\psi(t,x+\pi\mathrm{i})$. From (2.3), we have

$$\begin{aligned}\partial_t\,\partial_x X(t,x)=\partial_t\,\partial_x\psi(t,x+\pi\mathrm{i})&=-\frac{1}{2}\sinh_2^{-2}\big(\psi(t,x+\pi\mathrm{i})-\xi(t)\big)\,\partial_x\psi(t,x+\pi\mathrm{i})\\&=\frac{1}{2}\cosh_2^{-2}\big(X(t,x)\big)\,\partial_x X(t,x).\end{aligned} \tag{3.5}$$

For $x\in\mathbb{R}$ and $t\ge 0$, define $M(t,x)=f(X(t,x))\,\partial_x X(t,x)$. From (3.2)–(3.5) and Itô's formula (cf. [6]), we find that for any fixed x, $(M(t,x))$ is a local martingale, and satisfies the SDE:

$$\frac{\partial_t M(t,x)}{M(t,x)}=-\frac{f'(X(t,x))}{f(X(t,x))}\sqrt{\kappa}\,\partial B(t)=-\frac{2}{\sqrt{\kappa}}\big(\sigma-\tanh_2\big(X(t,x)\big)\big)\,\partial B(t).$$

From the definition, f is bounded on $\mathbb{R}$. From (3.5) and that $\partial_x X(0,x)=1$, it follows that $|\partial_x X(t,x)|\le\exp(t/2)$. Thus, for any fixed $t_0>0$, $M(t,x)$ is bounded on $[0,t_0]\times\mathbb{R}$. So $(M(t,x)\colon 0\le t\le t_0)$ is a bounded martingale. Then we have $\mathbf{E}[M(t_0,x)]=M(0,x)=f(x)$ for any $x\in\mathbb{R}$. Now define the probability measure $\nu_{t_0,x}$ such that $\mathrm{d}\nu_{t_0,x}/\mathrm{d}\mu=M(t_0,x)/f(x)$, and let

$$\widetilde{B}(t)=B(t)+\int_0^t\frac{2}{\sqrt{\kappa}}\big(\sigma-\tanh_2\big(X(s,x)\big)\big)\,\mathrm{d}s,\quad 0\le t\le t_0.$$

From the Girsanov theorem, under the probability measure $\nu_{t_0,x}$, $\widetilde{B}(t)$, $0\le t\le t_0$, is a partial Brownian motion. Now $\xi(t)$, $0\le t\le t_0$, satisfies the SDE:

$$\begin{aligned}\mathrm{d}\xi(t)&=\sqrt{\kappa}\,\mathrm{d}\widetilde{B}(t)+\sigma\,\mathrm{d}t-2\big(\sigma-\tanh_2\big(X(t,x)\big)\big)\,\mathrm{d}t\\&=\sqrt{\kappa}\,\mathrm{d}\widetilde{B}(t)-\sigma\,\mathrm{d}t-\frac{-4}{2}\coth_2\big(\psi(t,x+\pi\mathrm{i})-\xi(t)\big)\,\mathrm{d}t.\end{aligned}$$

Since $\xi(0)=0$, so under $\nu_{t_0,x}$, $(\xi(t),0\le t\le t_0)$ has the distribution of the driving function of a strip SLE$(\kappa;-4,\rho_-+2,\rho_++2)$ process started from $(0;x+\pi\mathrm{i},+\infty,-\infty)$. So we conclude that $\nu_{t_0,x}|_{\mathcal{F}_{t_0}}=\nu_x|_{\mathcal{F}_{t_0}}$. Thus, $\nu_x|_{\mathcal{F}_{t_0}}\ll\mu|_{\mathcal{F}_{t_0}}$, and the Radon–Nikodym derivative is $R_{t_0}(x)=M(t_0,x)/f(x)$. Thus,

$$\int_{\mathbb{R}}R_{t_0}(x)f(x)\,\mathrm{d}x=\int_{\mathbb{R}}M(t_0,x)\,\mathrm{d}x=\int_{\mathbb{R}}f(X(t_0,x))\,\partial_x X(t_0,x)\,\mathrm{d}x=\int_{\mathbb{R}}f(y)\,\mathrm{d}y=1. \qquad\square$$

Theorem 3.1. *Let $\kappa>0$, and $\rho_+,\rho_-\in\mathbb{R}$ satisfy $\rho_++\rho_-=\kappa-6$ and $|\rho_+-\rho_-|<2$. Let $\bar\mu$ denote the distribution of a strip* SLE$(\kappa;\rho_+,\rho_-)$ *trace $\beta(t)$, $0\le t<\infty$, started from $(0;+\infty,-\infty)$. Let λ denote the distribution of the intersection point of $\overline{\beta([0,\infty))}$ with $\mathbb{R}_\pi$. For each $p\in\mathbb{R}_\pi$, let $\bar\nu_p$ denote the distribution of a strip* SLE$(\kappa;-4,\rho_-+2,\rho_++2)$ *trace started from $(0;p,+\infty,-\infty)$. Then $\bar\mu=\int_{\mathbb{R}_\pi}\bar\nu_p\,\mathrm{d}\lambda(p)$.*

Proof. This follows from Proposition 2.3 and the above lemma. $\square$

Remark. *A special case of the above theorem is that $\kappa=2$ and $\rho_+=\rho_-=-2$, so $\rho_++2=\rho_-+2=0$. From* [10], *a strip* SLE$(2;-2,-2)$ *trace started from $(0;+\infty,-\infty)$ is a continuous LERW in $\mathbb{S}_\pi$ from 0 to $\mathbb{R}_\pi$; a strip* SLE$(2;-4,0,0)$ *trace started from $(0;p,+\infty,-\infty)$ is a continuous LERW in $\mathbb{S}_\pi$ from 0 to p; and the above theorem in this special case follows from the convergence of discrete LERW to continuous LERW.*

Corollary 3.1. *Let $\kappa > 0$, $\rho \in (\kappa/2-4, \kappa/2-2)$, and $x \neq 0$. Let $\bar{\mu}$ denote the distribution of a chordal* SLE$(\kappa;\rho)$ *trace $\gamma(t)$, $0 \le t < T$, started from $(0;x)$. Let λ denote the distribution of the subsequential limit of $\gamma(t)$ on $\mathbb{R}$ as $t \to T$, which is a.s. unique. For each $y \in \mathbb{R}$, let $\bar{\nu}_y$ denote the distribution of a chordal* SLE$(\kappa; -4, \kappa-4-\rho)$ *trace started from $(0; y, x)$. Then $\bar{\mu} = \int_{\mathbb{R}} \bar{\nu}_y \,\mathrm{d}\lambda(y)$.*

Proof. This follows from the above theorem and Proposition 2.2. □

Corollary 3.2. *Let $\kappa \in (4,8)$ and $x \neq 0$. Let $\gamma(t)$, $0 \le t < \infty$, be a standard chordal* SLE(κ) *trace. Let T_x be the first t that $\gamma([0,t])$ disconnects x from ∞ in $\mathbb{H}$. Let $\bar{\mu}$ denote the distribution of $(\gamma(t), 0 \le t < T_x)$. Let λ denote the distribution of $\gamma(T_x)$. For each $y \in \mathbb{R}$, let $\bar{\nu}_y$ denote the distribution of a chordal* SLE$(\kappa; -4, \kappa-4)$ *trace started from $(0; y, x)$. Then $\bar{\mu} = \int_{\mathbb{R}} \bar{\nu}_y \,\mathrm{d}\lambda(y)$.*

Proof. This is a special case of the above corollary because $\gamma(t)$, $0 \le t < T_x$, is a chordal SLE$(\kappa; 0)$ trace started from $(0;x)$, and $0 \in (\kappa/2-4, \kappa/2-2)$. □

4. Geometric properties

In this section, we will improve some results derived in Section 3 of [9]. We first derive a simple lemma.

Lemma 4.1. *Suppose $\psi(t,\cdot)$, $0 \le t < T$, are strip Loewner maps driven by ξ. Suppose $\xi(0) < x_1 < x_2$ or $\xi(0) > x_1 > x_2$, and $\psi(t,x_1)$ and $\psi(t,x_2)$ are defined for $0 \le t < T$. Then for any $0 \le t < T$,*

$$\left|\int_0^t \coth_2\bigl(\psi(s,x_1)-\xi(s)\bigr)\,\mathrm{d}s - \int_0^t \coth_2\bigl(\psi(s,x_2)-\xi(s)\bigr)\,\mathrm{d}s\right| < |x_1-x_2|.$$

Proof. By symmetry, we only need to consider the case that $\xi(0) < x_1 < x_2$. For any $0 \le t < T$, we have $\xi(t) < \psi(t,x_1) < \psi(t,x_2)$, which implies that $\coth_2(\psi(t,x_1)-\xi(t)) > \coth_2(\psi(t,x_2)-\xi(t)) > 0$. Also, note that $\partial_t \psi(t,x_j) = \coth_2(\psi(t,x_j)-\xi(t))$, $j=1,2$, so for $0 \le t < T$,

$$\begin{aligned}
0 &\le \int_0^t \coth_2\bigl(\psi(s,x_1)-\xi(s)\bigr)\,\mathrm{d}s - \int_0^t \coth_2\bigl(\psi(s,x_2)-\xi(s)\bigr)\,\mathrm{d}s \\
&= \bigl(\psi(t,x_1)-\psi(0,x_1)\bigr) - \bigl(\psi(t,x_2)-\psi(0,x_2)\bigr) \\
&= \psi(t,x_1)-\psi(t,x_2)+x_2-x_1 < x_2-x_1 = |x_1-x_2|.
\end{aligned}$$

□

From now on, in this section, we let $\kappa > 0$, $N_+, N_- \in \mathbb{N} \cup \{0\}$, $\vec{\rho}_\pm = (\rho_{\pm 1}, \ldots, \rho_{\pm N_\pm}) \in \mathbb{R}^{N_\pm}$, and $\chi_\pm = \sum_{m=1}^{N_\pm} \rho_{\pm m}$. Let $\tau_+, \tau_- \in \mathbb{R}$ be such that $\chi_+ + \tau_+ + \chi_- + \tau_- = \kappa - 6$. Let $\vec{p}_\pm = (p_{\pm 1}, \ldots, p_{\pm N_\pm})$ be such that $p_{-N_-} < \cdots < p_{-1} < 0 < p_1 < \cdots < p_{N_+}$. Suppose $\beta(t)$, $0 \le t < T$, is a strip SLE$(\kappa; \vec{\rho}_+, \vec{\rho}_-, \tau_+, \tau_-)$ trace started from $(0; \vec{p}_+, \vec{p}_-, +\infty, -\infty)$. Let $\xi(t)$ and $\psi(t,\cdot)$, $0 \le t < T$, be the driving function and strip Loewner maps for β. Then there is a Brownian motion $B(t)$ such that for $0 \le t < T$, $\xi(t)$ satisfies the SDE

$$\begin{aligned}
\mathrm{d}\xi(t) &= \sqrt{\kappa}\,\mathrm{d}B(t) - \sum_{m=1}^{N_+} \frac{\rho_m}{2} \coth_2\bigl(\psi(t,p_m)-\xi(t)\bigr)\,\mathrm{d}t \\
&\quad - \sum_{m=1}^{N_-} \frac{\rho_{-m}}{2} \coth_2\bigl(\psi(t,p_{-m})-\xi(t)\bigr)\,\mathrm{d}t - \frac{\tau_+-\tau_-}{2}\,\mathrm{d}t.
\end{aligned} \tag{4.1}$$

For $0 \le t < T$, we have

$$\psi(t,p_{-N_-}) < \cdots < \psi(t,p_{-1}) < \xi(t) < \psi(t,p_1) < \cdots < \psi(t,p_{N_+}). \tag{4.2}$$

Since $\partial_t\psi(t,x)=\coth_2(\psi(t,x)-\xi(t))$, so $\partial_t\psi(t,p_m)>1$ for $1\le m\le N_+$, and $\partial_t\psi(t,p_{-m})<-1$ for $1\le m\le N_-$. Thus, for $0\le t<T$, $\psi(t,p_m)$ increases in t, and $\psi(t,p_m)>t$ for $1\le m\le N_+$; $\psi(t,p_{-m})$ decreases in t, and $\psi(t,p_{-m})<-t$ for $1\le m\le N_-$. We say that some force point p_s is swallowed by β if $T<\infty$ and $\psi(t,p_s)-\xi(t)\to 0$ as $t\to T$. In fact, if $T<\infty$ then some force point on $\mathbb{R}$ must be swallowed by β, and from (4.2) we see that either p_1 or p_{-1} is swallowed.

Lemma 4.2. (i) *If $\sum_{j=1}^k\rho_j\ge\kappa/2-2$ for $1\le k\le N_+$, then a.s. p_1 is not swallowed by β.* (ii) *If $\sum_{j=1}^k\rho_{-j}\ge\kappa/2-2$ for $1\le k\le N_-$, then a.s. p_{-1} is not swallowed by β.*

Proof. From symmetry, we only need to prove (i). Suppose $\sum_{j=1}^k\rho_j\ge\kappa/2-2$ for $1\le k\le N_+$. Let $\mathcal{E}$ denote the event that p_1 is swallowed by β. Let $\mathbf{P}$ be the probability measure we are working on. We want to show that $\mathbf{P}(\mathcal{E})=0$. Assume that $\mathbf{P}(\mathcal{E})>0$. Assume that $\mathcal{E}$ occurs. Then $\lim_{t\to T}\xi(t)=\lim_{t\to T}\psi(t,p_1)\ge T$. For $1\le m\le N_-$, since $\psi(t,p_{-m})<-t$, $0\le t<T$, so $\psi(t,p_{-m})-\xi(t)$ on $[0,T)$ is uniformly bounded above by a negative number. Thus $\coth_2(\psi(t,p_{-m})-\xi(t))$ on $[0,T)$ is uniformly bounded for $1\le m\le N_-$. For $0\le t<T$, let $\widetilde{B}(t)=B(t)+\int_0^t a(s)\,\mathrm{d}s$, where

$$a(t)=-\frac{\kappa/2-2}{2\sqrt{\kappa}}+\frac{\kappa/2-4-\chi_+}{2\sqrt{\kappa}}\coth_2\bigl(\psi(t,\pi\mathrm{i})-\xi(t)\bigr)$$
$$-\sum_{m=1}^{N_-}\frac{\rho_{-m}}{2\sqrt{\kappa}}\coth_2\bigl(\psi(t,p_{-m})-\xi(t)\bigr)-\frac{\tau_+-\tau_-}{2\sqrt{\kappa}}.$$

For $0\le t<T$, since $\psi(t,\pi\mathrm{i})-\xi(t)\in\mathbb{R}_\pi$, so $|\coth_2(\psi(t,\pi\mathrm{i})-\xi(t))|\le 1$. From the previous discussion, we see that if $\mathcal{E}$ occurs, then $T<\infty$ and $a(t)$ is uniformly bounded on $[0,T)$, and so $\int_0^T a(t)^2\,\mathrm{d}t<\infty$. For $0\le t<T$, define

$$M(t)=\exp\biggl(-\int_0^t a(s)\,\mathrm{d}B(s)-\frac12\int_0^t a(s)^2\,\mathrm{d}s\biggr).\tag{4.3}$$

Then $(M(t),0\le t<T)$ is a local martingale and satisfies $\mathrm{d}M(t)/M(t)=-a(t)\,\mathrm{d}B(t)$. In the event $\mathcal{E}$, since $\int_0^T a(t)^2\,\mathrm{d}t<\infty$, so a.s. $\lim_{t\to T}M(t)\in(0,\infty)$. For $N\in\mathbb{N}$, let $T_N\in[0,T]$ be the largest number such that $M(t)\in(1/(2N),2N)$ on $[0,T_N)$. Let $\mathcal{E}_N=\mathcal{E}\cap\{T_N=T\}$. Then $\mathcal{E}=\bigcup_{N=1}^\infty\mathcal{E}_N$ a.s., and $\mathbf{E}[M(T_N)]=M(0)=1$, where $M(T):=\lim_{t\to T}M(t)$. Since $\mathbf{P}(\mathcal{E})>0$, so there is $N\in\mathbb{N}$ such that $\mathbf{P}(\mathcal{E}_N)>0$. Define another probability measure $\mathbf{Q}$ such that $\mathrm{d}\mathbf{Q}/\mathrm{d}\mathbf{P}=M(T_N)$. Then $\mathbf{P}\ll\mathbf{Q}$, and so $\mathbf{Q}(\mathcal{E}_N)>0$. By the Girsanov theorem, under the probability measure $\mathbf{Q}$, $\widetilde{B}(t)$, $0\le t<T_N$, is a partial Brownian motion. From (4.1), $\xi(t)$, $0\le t<T$, satisfies the SDE:

$$\mathrm{d}\xi(t)=\sqrt{\kappa}\,\mathrm{d}\widetilde{B}(t)-\sum_{m=1}^{N_+}\frac{\rho_m}{2}\coth_2\bigl(\psi(t,p_m)-\xi(t)\bigr)\,\mathrm{d}t$$
$$+\frac{\kappa/2-2}{2}\,\mathrm{d}t-\frac{\kappa/2-4-\chi_+}{2}\coth_2\bigl(\psi(t,\pi\mathrm{i})-\xi(t)\bigr)\,\mathrm{d}t,$$

so under $\mathbf{Q}$, $\beta(t)$, $0\le t<T_N$, is a partial strip SLE$(\kappa;\vec{\rho}_+,\frac{\kappa}{2}-2,\frac{\kappa}{2}-4-\chi_+)$ trace started from $(0;\vec{p}_+,-\infty,\pi\mathrm{i})$. In the event $\mathcal{E}_N$, since $\psi(t,p_1)-\xi(t)\to 0$ as $t\to T_N=T$, so $\beta(t)$, $0\le t<T_N$, is a full trace under $\mathbf{Q}$. Note that

$$\sum_{m=1}^{N_+}\rho_m+\Bigl(\frac{\kappa}{2}-2\Bigr)+\Bigl(\frac{\kappa}{2}-4-\chi_+\Bigr)=\kappa-6.$$

From Proposition 2.2, Proposition 2.6, and that $\sum_{j=1}^k\rho_j\ge\kappa/2-2$ for $1\le k\le N_+$, we see that on $\mathcal{E}_N$, $\mathbf{Q}$-a.s. $\pi\mathrm{i}$ is a subsequential limit of $\beta(t)$ as $t\to T_N$, which implies that the height of $\beta((0,t])$ tends to π as $t\to T_N$, and so $T_N=\infty$. This contradicts that $T_N=T<\infty$ on $\mathcal{E}_N$ and $\mathbf{Q}(\mathcal{E}_N)>0$. Thus $\mathbf{P}(\mathcal{E})=0$. $\square$

Lemma 4.3. (i) *If* $\chi_+ \ge -2$ *and* $\chi_+ + \tau_+ > \kappa/2 - 4$, *then* $T = \infty$ *a.s. implies that* $\liminf_{t\to\infty}(\psi(t, p_m) - \xi(t))/t > 0$ *for* $1 \le m \le N_+$. (ii) *If* $\chi_- \ge -2$ *and* $\chi_- + \tau_- > \kappa/2 - 4$, *then a.s.* $T = \infty$ *implies that* $\limsup_{t\to\infty}(\psi(t, p_{-m}) - \xi(t))/t < 0$ *for* $1 \le m \le N_-$.

Proof. We will only prove (i) since (ii) follows from symmetry. Suppose $\chi_+ \ge -2$ and $\chi_+ + \tau_+ > \kappa/2 - 4$. Then $\Delta := 1 + \frac{\chi_+}{2} + \frac{\tau_+}{2} - \frac{\chi_-}{2} - \frac{\tau_-}{2} > 0$. Let $X(t) = \psi(t, p_1) - \xi(t)$, $t \ge 0$. From (4.2) we suffice to show that $T = \infty$ a.s. implies that $\liminf_{t\to\infty} X(t)/t > 0$. Now assume that $T = \infty$. From (2.3) and (4.1), for any $0 \le t_1 \le t_2$,

$$X(t_2) - X(t_1) = -\sqrt{\kappa}B(t_2) + \sqrt{\kappa}B(t_1) + \frac{\tau_+ - \tau_-}{2}(t_2 - t_1) + \int_{t_1}^{t_2} \coth_2\big(X(t)\big)\,dt$$
$$+ \sum_{m=1}^{N_+} \frac{\rho_m}{2} \int_{t_1}^{t_2} \coth_2\big(\psi(t, p_m) - \xi(t)\big)\,dt + \sum_{m=1}^{N_-} \frac{\rho_{-m}}{2} \int_{t_1}^{t_2} \coth_2\big(\psi(t, p_{-m}) - \xi(t)\big)\,dt. \quad (4.4)$$

Let $M_+ = \sum_{m=1}^{N_+} |\rho_m||p_m - p_1|$. From Lemma 4.1, for any $0 \le t_1 \le t_2$,

$$\sum_{m=1}^{N_+} \frac{\rho_m}{2} \int_{t_1}^{t_2} \coth_2\big(\psi(t, p_m) - \xi(t)\big)\,dt \ge \frac{\chi_+}{2} \int_{t_1}^{t_2} \coth_2\big(X(t)\big)\,dt - M_+. \quad (4.5)$$

Let $\varepsilon_1 = \min\{\Delta, 1\}/6 > 0$. There is a random number $A_0 = A_0(\omega) > 0$ such that a.s.

$$\big|\sqrt{\kappa}B(t)\big| \le A_0 + \varepsilon_1 t \quad \text{for any } t \ge 0. \quad (4.6)$$

Let $\chi_-^* = \sum_{m=1}^{N_-} |\rho_{-m}|$, and $\varepsilon_2 = \frac{\Delta}{\chi_-^*+1} > 0$. Choose $R > 0$ such that if $x < -R$ then $|\coth_2(x) - (-1)| < \varepsilon_2$. Suppose $X(t) \le t$ on $[t_1, t_2]$, where $t_2 \ge t_1 \ge R$. Then for $1 \le m \le N_-$ and $t \in [t_1, t_2]$, from $\psi(t, p_{-m}) < -t$ and $\psi(t, p_1) > t$, we have

$$\psi(t, p_{-m}) - \xi(t) = \psi(t, p_{-m}) - \psi(t, p_1) + X(t) < -t - t + t = -t \le -R,$$

and so $|\coth_2(\psi(t, p_{-m}) - \xi(t)) - (-1)| < \varepsilon_2$. Then

$$\sum_{m=1}^{N_-} \frac{\rho_{-m}}{2} \int_{t_1}^{t_2} \coth_2\big(\psi(t, p_{-m}) - \xi(t)\big)\,dt \ge \left(-\frac{\chi_-}{2} - \frac{\chi_-^*}{2}\varepsilon_2\right)(t_2 - t_1). \quad (4.7)$$

Suppose $X(t_0) \ge t_0$ for some $t_0 \ge \max\{R, 2M_+ + 4A_0 + 2\}$. We claim that a.s. for any $t \ge t_0$, we have $X(t) \ge \varepsilon_1 t$. If this is not true, then there are $t_2 > t_1 \ge t_0$ such that $X(t_1) = t_1$, $X(t_2) = \varepsilon_1 t_2$ and $X(t) \le t$ for $t \in [t_1, t_2]$. From (4.4)–(4.7), we have a.s.

$$X(t_2) - X(t_1) \ge -2A_0 - \varepsilon_1 t_1 - \varepsilon_1 t_2 + \frac{\tau_+ - \tau_-}{2}(t_2 - t_1)$$
$$+ \left(1 + \frac{\chi_+}{2}\right) \int_{t_1}^{t_2} \coth_2\big(X(t)\big)\,dt - M_+ - \frac{\chi_- + \chi_-^*\varepsilon_2}{2}(t_2 - t_1)$$
$$\ge -M_+ - 2A_0 - 2\varepsilon_1 t_2 + \left(\Delta - \frac{\chi_-^*\varepsilon_2}{2}\right)(t_2 - t_1), \quad (4.8)$$

where in the last inequality we use the facts that $\coth_2(X(t)) > 1$ and $1 + \frac{\chi_+}{2} \ge 0$. Since $X(t_1) = t_1$ and $X(t_2) = \varepsilon_1 t_2$, so we have

$$M_+ + 2A_0 \ge \big(\Delta - \chi_-^*\varepsilon_2/2 - 3\varepsilon_1\big)(t_2 - t_1) + (1 - 3\varepsilon_1)t_1.$$

Since $\Delta - \chi_-^* \varepsilon_2/2 - 3\varepsilon_1 \geq \Delta - \Delta/2 - \Delta/2 \geq 0$ and $1 - 3\varepsilon_1 \geq 1/2$, so

$$M_+ + 2A_0 \geq t_1/2 \geq t_0/2 \geq (2M_+ + 4A_0 + 2)/2 = M_+ + 2A_0 + 1,$$

which is a contradiction. Thus, if $X(t_0) \geq t_0$ for some $t_0 \geq \max\{R, 2M_+ + 4A_0 + 2\}$, then a.s. $X(t) \geq \varepsilon_1 t$ for any $t \geq t_0$, and so $\liminf_{t\to\infty} X(t)/t \geq \varepsilon_1 > 0$. The other possibility is that $X(t_0) < t_0$ for all $t_0 \geq \max\{R, 2M_+ + 4A_0 + 2\}$. Let $t_1 = \max\{R, 2M_+ + 4A_0 + 2\}$ and $t_2 \geq t_1$. Then (4.4)–(4.7) still hold, so we have (4.8) again. Let both sides of (4.8) be divided by t_2 and let $t_2 = t \to \infty$. Then we have a.s.

$$\liminf_{t\to\infty} X(t)/t \geq \Delta - \chi_-^* \varepsilon_2/2 - 2\varepsilon_1 \geq \Delta/6 > 0. \qquad \square$$

The following theorem improves Theorem 3.6 in [9].

Theorem 4.1. *If $\kappa \in (0,4]$, $\sum_{j=1}^k \rho_{\pm j} \geq \kappa/2 - 2$, $1 \leq k \leq N_\pm$, and $|\chi_+ + \tau_+ - \chi_- - \tau_-| < 2$, then a.s. $T = \infty$ and $\lim_{t\to\infty} \beta(t) \in \mathbb{R}_\pi$.*

Proof. From Lemma 4.2, a.s. neither p_1 nor p_{-1} is swallowed by β, so $T = \infty$. Since $|\chi_+ + \tau_+ - \chi_- - \tau_-| < 2$ and $\chi_+ + \tau_+ + \chi_- + \tau_- = \kappa - 6$, so $\chi_\pm + \tau_\pm > \kappa/2 - 4$. If $N_+ \geq 1$, then $\chi_+ = \sum_{m=1}^{N_+} \rho_\pm \geq \kappa/2 - 2 \geq -2$, so from Lemma 4.3, a.s. $\liminf_{t\to\infty}(\psi(t, p_m) - \xi(t))/t > 0$ for $1 \leq m \leq N_+$. If $N_+ = 0$, this is also true since there is nothing to check. Similarly, $\limsup_{t\to\infty}(\psi(t, p_{-m}) - \xi(t))/t < 0$ for $1 \leq m \leq N_-$. For $0 \leq t < \infty$, let $\widetilde{B}(t) = B(t) + \int_0^t a(s)\,\mathrm{d}s$, where

$$a(t) = \sum_{m=1}^{N_+} \frac{\rho_m}{2\sqrt{\kappa}}\big(1 - \coth_2\big(\psi(t, p_m) - \xi(t)\big)\big) - \sum_{m=1}^{N_-} \frac{\rho_{-m}}{2\sqrt{\kappa}}\big(1 + \coth_2\big(\psi(t, p_{-m}) - \xi(t)\big)\big).$$

Then $\int_0^\infty a(t)^2\,\mathrm{d}t < \infty$, and $\xi(t)$, $0 \leq t < \infty$, satisfies the SDE:

$$\mathrm{d}\xi(t) = \sqrt{\kappa}\,\mathrm{d}\widetilde{B}(t) - \frac{\tau_+ + \chi_+ - \tau_- - \chi_-}{2}\,\mathrm{d}t. \tag{4.9}$$

For $0 \leq t < \infty$, define $M(t)$ by (4.3). Then $(M(t))$ is a local martingale, satisfies the SDE: $\mathrm{d}M(t)/M(t) = -a(t)\,\mathrm{d}B(t)$, and a.s. $M(\infty) := \lim_{t\to\infty} M(t) \in (0,\infty)$. For $N \in \mathbb{N}$, let $T_N \in [0,\infty]$ be the largest number such that $M(t) \in (1/(2N), 2N)$ on $[0, T_N)$. Then $\mathbf{E}[M(T_N)] = M(0) = 1$. Let $\mathcal{E}_N = \{T_N = \infty\}$. Let $\mathbf{P}$ be the probability measure we are working on. Fix $\varepsilon > 0$. There is $N \in \mathbb{N}$ such that $\mathbf{P}[\mathcal{E}_N] > 1 - \varepsilon$. Define another probability measure $\mathbf{Q}$ such that $\mathrm{d}\mathbf{Q}/\mathrm{d}\mathbf{P} = M(T_N)$. By the Girsanov theorem, under $\mathbf{Q}$, $\widetilde{B}(t)$, $0 \leq t < T_N$, is a partial Brownian motion, which together with (4.9) implies that $\beta(t)$, $0 \leq t < T_N$, is a partial strip SLE$(\kappa; \rho_+, \rho_-)$ trace started from $(0; +\infty, -\infty)$, where $\rho_\pm = \chi_\pm + \tau_\pm$. Since $\rho_+ + \rho_- = \kappa - 6$ and $|\rho_+ - \rho_-| < 2$, so from Proposition 2.4, $\mathbf{Q}$-a.s. $\lim_{t\to T_N} \beta(t) \in \mathbb{R}_\pi$ on $\{T_N = \infty\} = \mathcal{E}_N$. Since $\mathbf{P} \ll \mathbf{Q}$, so ($\mathbf{P}$-)a.s. $\lim_{t\to T_N} \beta(t) \in \mathbb{R}_\pi$ on $\mathcal{E}_N$. Since $\mathbf{P}[\mathcal{E}_N] > 1 - \varepsilon$, so the probability that $\lim_{t\to\infty} \beta(t) \in \mathbb{R}_\pi$ is greater than $1 - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, so ($\mathbf{P}$-)a.s. $\lim_{t\to\infty} \beta(t) \in \mathbb{R}_\pi$. $\square$

The following theorem improves Theorem 3.1 in [9] when $\kappa \in (0,4]$.

Theorem 4.2. *Suppose $\kappa \in (0,4]$; $N_+, N_- \in \mathbb{N} \cup \{0\}$; $\vec{\rho}_\pm = (\rho_{\pm 1}, \ldots, \rho_{\pm N_\pm}) \in \mathbb{R}^{N_\pm}$; $\sum_{j=1}^k \rho_{\pm j} \geq \kappa/2 - 2$, $1 \leq k \leq N_\pm$; $\vec{p}_\pm = (p_{\pm 1}, \ldots, p_{\pm N_\pm}) \in \mathbb{R}^{N_\pm}$; $p_{-N_-} < \cdots < p_{-1} < 0 < p_1 < \cdots < p_{N_+}$. Let $\gamma(t)$, $0 \leq t < T$, be a chordal SLE$(\kappa; \vec{\rho}_+, \vec{\rho}_-)$ trace started from $(0; \vec{p}_+, \vec{p}_-)$. Then a.s. $\lim_{t\to T} \gamma(t) = \infty$.*

Proof. If $N_+ = N_- = 0$, then γ is a standard chordal SLE(κ) trace, so the conclusion follows from Theorem 7.1 in [7]. If $N_+ = 0$ and $N_- = 1$, or $N_+ = 1$ and $N_- = 0$, the conclusion follows from Propositions 2.2 and 2.5. If $N_+ = N_- = 1$, this follows from Proposition 2.7. For other cases, we will prove the theorem by reducing the number of force points.

Now consider the case that $N_- = 0$ and $N_+ \geq 2$. Choose W that maps $\mathbb{H}$ conformally onto $\mathbb{S}_\pi$ such that $W(0) = 0$, $W(\infty) = -\infty$, and $W(p_{N_+}) = +\infty$. Let $N_+' = N_+ - 1$; $\vec{q} = (q_1, \ldots, q_{N_+'})$, where $q_m = W(p_m)$, $1 \leq m \leq N_+'$. Then

$0 < q_1 < \cdots < q_{N'_+}$. Let $\vec{\rho} = (\rho_1, \ldots, \rho_{N'_+}) \in \mathbb{R}^{N'_+}$. Then $\sum_{j=1}^{k} \rho_j \ge \kappa/2 - 2$ for $1 \le k \le N'_+$. Let $\chi_+ = \sum_{m=1}^{N'_+} \rho_m$. Then $\chi_+ \ge \kappa/2 - 2 \ge -2$. Let $\tau_+ = \rho_{N_+}$ and $\tau_- = \kappa - 6 - \chi_+ - \tau_+$. Then $\chi_+ + \tau_+ + \tau_- = \kappa - 6$ and $\chi_+ + \tau_+ = \sum_{m=1}^{N_+} \rho_m \ge \kappa/2 - 2 > \kappa/2 - 4$. From Proposition 2.2, a time-change of $W \circ \gamma(t)$, $0 \le t < T$, say $\beta(t)$, $0 \le t < S$, is a strip SLE$(\kappa; \tau_-, \tau_+, \vec{\rho})$ trace started from $(0; -\infty, +\infty, \vec{q})$. Let $\xi(t)$ and $\psi(t, \cdot)$, $0 \le t < S$, be the driving function and strip Loewner maps for β. Then there is a Brownian motion $B(t)$ such that for $0 \le t < S$, $\xi(t)$ satisfies the SDE

$$\mathrm{d}\xi(t) = \sqrt{\kappa}\,\mathrm{d}B(t) - \frac{\tau_+ - \tau_-}{2}\,\mathrm{d}t - \sum_{m=1}^{N'_+} \frac{\rho_m}{2} \coth_2\bigl(\psi(t, q_m) - \xi(t)\bigr)\,\mathrm{d}t.$$

From Lemmas 4.2 and 4.3, a.s. $S = \infty$ and $\liminf_{t\to\infty}(\psi(t, q_m) - \xi(t))/t > 0$ for $1 \le m \le N'_+$. Let $\widetilde{B}(t) = B(t) + \int_0^t a(s)\,\mathrm{d}s$, where

$$a(t) = \sum_{m=1}^{N'_+} \frac{\rho_m}{2\sqrt{\kappa}} \bigl(1 - \coth_2\bigl(\psi(t, p_m) - \xi(t)\bigr)\bigr).$$

Then $\int_0^\infty a(t)^2\,\mathrm{d}t < \infty$. Now $\xi(t)$ satisfies the SDE

$$\mathrm{d}\xi(t) = \sqrt{\kappa}\,\mathrm{d}\widetilde{B}(t) - \frac{\chi_+ + \tau_+ - \tau_-}{2}\,\mathrm{d}t.$$

Note that $(\chi_+ + \tau_+) - \tau_- \ge 2$. We observe that if $\widetilde{B}(t)$ is a Brownian motion, then β is a strip SLE$(\kappa; \chi_+ + \tau_+, \tau_-)$ trace started from $(0; +\infty, -\infty)$, and so from Proposition 2.5, we have $\lim_{t\to\infty} \beta(t) = -\infty$. Using the argument at the end of the proof of Theorem 4.1, we conclude that a.s. $\lim_{t\to\infty} \beta(t) = -\infty$, and so $\lim_{t\to T} \gamma(t) = W^{-1}(-\infty) = \infty$.

For the case $N_- = 1$ and $N_+ \ge 2$, we define W and β as in the above case, and conclude that $\lim_{t\to\infty} \beta(t) = -\infty$ using the same argument as above except that now we use Proposition 2.2 and the conclusion of this theorem in the case $N_+ = N_- = 1$ to prove that a.s. $\lim_{t\to\infty} \beta(t) = -\infty$. So again we conclude that a.s. $\lim_{t\to T} \gamma(t) = \infty$. The cases that $N_+ \in \{0, 1\}$ and $N_- \ge 2$ are symmetric to the above two cases. For the case that $N_+, N_- \ge 2$, we define W and β as in the case that $N_- = 0$ and $N_+ \ge 2$, and conclude that a.s. $\lim_{t\to\infty} \beta(t) = -\infty$ using the same argument as in that case except that now we use Proposition 2.2 and the conclusion of this theorem in the case $N_- \ge 2$ and $N_+ = 1$. So we also have a.s. $\lim_{t\to T} \gamma(t) = \infty$. □

5. Duality

Let γ be a simple curve in a simply connected domain Ω. We call γ a crosscut in Ω if its two ends approach to two different boundary points or prime ends of Ω. We call γ a degenerate crosscut in Ω if its two ends approach to the same boundary point or prime end of Ω. We call γ a semi-crosscut in Ω if its one end approaches to some boundary point or prime end of Ω, and the other end stays inside Ω. In the above definitions, if $\Omega = \mathbb{H}$, and no end of γ is ∞, then γ is called a crosscut, or degenerate crosscut, or semi-crosscut, respectively, in $\mathbb{H}$ on $\mathbb{R}$. For example, $\mathrm{e}^{\mathrm{i}\theta}$, $0 < \theta < \pi$, is a crosscut in $\mathbb{H}$ on $\mathbb{R}$; $\mathrm{e}^{\mathrm{i}\theta}$, $0 < \theta \le \pi/2$, is a semi-crosscut in $\mathbb{H}$ on $\mathbb{R}$; $\mathrm{i} + \mathrm{e}^{\mathrm{i}\theta}$, $-\pi/2 < \theta < 3\pi/2$, is a degenerate crosscut in $\mathbb{H}$ on $\mathbb{R}$. If γ is a crosscut in $\mathbb{H}$ on $\mathbb{R}$, $\mathbb{H} \setminus \gamma$ has two connected components. We use $D_{\mathbb{H}}(\gamma)$ to denote the bounded component.

In Proposition 2.8, let $N = 4$; choose $p_1 < x_1 < p_3 < p_4 < x_2 < p_2$; choose $C_2, C_4 \ge 1/2$, let $C_1 = 1 - C_2$, $C_3 = 1/2 - C_4$ and $\rho_{j,m} = C_m(\kappa_j - 4)$, $1 \le m \le 4$, $j = 1, 2$. Let $K_j(t)$, $0 \le t < T_j$, $j = 1, 2$, be given by Proposition 2.8. Let $\varphi_j(t, \cdot)$ and $\gamma_j(t)$, $0 \le t < T_j$, $j = 1, 2$, be the corresponding chordal Loewner maps and traces.

Since $\kappa_1 \in (0, 4)$, so $\gamma_1(t)$, $0 \le t < T_j$, is a simple curve, and $\gamma_1(t) \in \mathbb{H}$ for $0 < t < T_j$. From Theorem 4.1 and Proposition 2.2, a.s. $\gamma_1(T_1) := \lim_{t\to T_1} \gamma_1(t) \in (x_2, p_2)$. Thus, γ_1 is a crosscut in $\mathbb{H}$ on $\mathbb{R}$. Note that γ_1 disconnects x_2 from ∞ in $\mathbb{H}$. If $\bar{t}_2 \in [0, T_2)$ is an $(\mathcal{F}^2_t)$-stopping time, then conditioned on $\mathcal{F}^2_{\bar{t}_2}$, after a time-change, $\varphi_2(\bar{t}_2, \gamma_1(t))$, $0 \le t < T_1(\bar{t}_2)$, has the same distribution as a chordal SLE$(\kappa_1; -\frac{\kappa_1}{2}, \vec{\rho}_1)$ trace started from $(\varphi_2(\bar{t}_2, x_1); \xi_2(\bar{t}_2), \varphi_2(\bar{t}_2, \vec{p}))$. Then we find that a.s. $\lim_{t\to T_1(\bar{t}_2)} \varphi_2(\bar{t}_2, \gamma_1(t)) \in (\xi_2(\bar{t}_2), \varphi_2(\bar{t}_2, p_2))$. Thus, $\varphi_2(\bar{t}_2, \gamma_1(t))$, $0 \le t < T_1(\bar{t}_2)$, disconnects

$\xi_2(\bar{t}_2)$ from ∞ in $\mathbb{H}$, and so γ_1 disconnects $\gamma_2(\bar{t}_2)$ from ∞ in $\mathbb{H}\setminus L_2(\bar{t}_2)$. By choosing a sequence of $(\mathcal{F}_t^2)$-stopping times that are dense in $[0,T_2)$, we conclude that a.s. $\overline{K_2(T_2^-)}\subset\overline{D_{\mathbb{H}}(\gamma_1)}$, where $K_2(T_2^-)=\bigcup_{0\le t<T_2}K_2(t)$. From Propositions 2.6 and 2.1, a.s. x_1 is a subsequential limit of $\gamma_2(t)$ as $t\to T_2$. Similarly, for every $(\mathcal{F}_t^1)$-stopping time $\bar{t}_1\in(0,\bar{T}_1)$, $\gamma_1(\bar{t}_1)$ is a subsequential limit of $\gamma_2(t)$ as $t\to T_2(\bar{t}_1)$. By choosing a sequence of $(\mathcal{F}_t^1)$-stopping times that are dense in $[0,T_1)$, we conclude that a.s. $\gamma_1(t)\in\overline{K_2(T_2^-)}$ for $0\le t<T_1$. So we have the following lemma and theorem. Here, $\partial_{\mathbb{H}}^{\text{out}}S$ is defined for bounded $S\subset\mathbb{H}$, which is the intersection of $\mathbb{H}$ with the boundary of the unbounded component of $\mathbb{H}\setminus S$. For detailed proof of the lemma, please see Lemma 5.1 in [9].

Lemma 5.1. *Almost surely $\partial_{\mathbb{H}}^{\text{out}}K_2(T_2^-)$ is the image of $\gamma_1(t)$, $0<t<T_1$.*

Theorem 5.1. *Suppose $\kappa>4$; $p_1<x_1<p_3<p_4<x_2<p_2$; $C_2,C_4\ge 1/2$, $C_1=1-C_2$, and $C_3=1/2-C_4$. Let $K(t)$, $0\le t<T$, be chordal* SLE$(\kappa;-\frac{\kappa}{2},C_1(\kappa-4),C_2(\kappa-4),C_3(\kappa-4),C_4(\kappa-4))$ *process started from $(x_2;x_1,p_1,p_2,p_3,p_4)$. Let $K(T^-)=\bigcup_{0\le t<T}K(t)$. Then a.s. $K(T^-)$ is bounded, and $\partial_{\mathbb{H}}^{\text{out}}K(T^-)$ has the distribution of the image of a chordal* SLE$(\kappa';-\frac{\kappa'}{2},C_1(\kappa'-4),C_2(\kappa'-4),C_3(\kappa'-4),C_4(\kappa'-4))$ *trace started from $(x_1;x_2,p_1,p_2,p_3,p_4)$, where $\kappa'=16/\kappa$.*

The above lemma and theorem still hold if we let $p_1\in(-\infty,x_1)$, or $=x_1^-$; let $p_2\in(x_2,\infty)$, or $=\infty$, or $=x_2^+$; let $p_3\in(x_1,x_2)$, or $=x_1^+$; let $p_4\in(x_1,x_2)$ or $=x_2^-$. Here if $p_2=x_2^+$, we use Theorem 4.2 instead of Theorem 4.1 to prove that the image of γ_1 in Lemma 5.1 is a crosscut in $\mathbb{H}$ on $\mathbb{R}$.

Proof of Theorem 1.1. First suppose $x<0$. Then λ is supported by $(-\infty,x)$, and $\bar{\mu}=\int\bar{\nu}_y\,d\lambda(y)$ follows from Corollary 3.2 and Theorem 5.1 with $x_1=y$, $x_2=0$, $p_1=y^-$, $p_2=\infty$, $p_3=y^+$, $p_4=x$, $C_1=\frac{\kappa-6}{\kappa-4}$, $C_2=\frac{2}{\kappa-4}$, $C_3=-1/2$ and $C_4=1$. From Theorem 4.1 and Proposition 2.2, for each $y\in(-\infty,x)$, $\bar{\nu}_y$ is supported by the space of crosscuts in $\mathbb{H}$ from y to some point on $(0,\infty)$. Thus a.s. $\partial_{\mathbb{H}}^{\text{out}}K(T_x)$ is a crosscut in $\mathbb{H}$ on $\mathbb{R}$ connecting some $y\in(-\infty,x)$ with some $z\in(0,\infty)$. The case that $x>0$ is symmetric. □ □

Let $S\subset\mathbb{H}$. Suppose $\overline{S}\cap(a,\infty)=\varnothing$ for some $a\in\mathbb{R}$. Then there is a unique component of $\mathbb{H}\setminus\overline{S}$, which has (a,∞) as part of its boundary. Let D_+ denote this component. Then $\partial D_+\cap\mathbb{H}$ is called the right boundary of S in $\mathbb{H}$. Let it be denoted by $\partial_{\mathbb{H}}^+S$. Similarly, if $\overline{S}\cap(-\infty,a)=\varnothing$ for some $a\in\mathbb{R}$. Then there is a unique component of $\mathbb{H}\setminus\overline{S}$, which has $(-\infty,a)$ as part of its boundary. Let D_- denote this component. Then $\partial D_-\cap\mathbb{H}$ is called the left boundary of S in $\mathbb{H}$. Let it be denoted by $\partial_{\mathbb{H}}^-S$. The following theorem improves Theorem 5.3 in [9].

Theorem 5.2. *Let $\kappa>4$ and $C_r,C_l\ge 1/2$. Let $K(t)$, $0\le t<\infty$, be a chordal* SLE$(\kappa;C_r(\kappa-4),C_l(\kappa-4))$ *process started from $(0;0^+,0^-)$. Let $K(\infty)=\bigcup_{t\ge 0}K(t)$. Let $\kappa'=16/\kappa$ and $W(z)=1/\bar{z}$. Then* (i) *$W(\partial_{\mathbb{H}}^+K(\infty))$ has the same distribution as the image of a chordal* SLE$(\kappa';(1-C_r)(\kappa'-4),(1/2-C_l)(\kappa'-4))$ *trace started from* $(0;0^+,0^-)$; (ii) *$W(\partial_{\mathbb{H}}^-K(\infty))$ has the same distribution as the image of a chordal* SLE$(\kappa';(1/2-C_r)(\kappa'-4),(1-C_l)(\kappa'-4))$ *trace started from* $(0;0^+,0^-)$; *and* (iii) *a.s.* $\overline{K(\infty)}\cap\mathbb{R}=\{0\}$.

Proof. Let $W_0(z)=1/(1-z)$. Then W_0 maps $\mathbb{H}$ conformally onto $\mathbb{H}$, and $W_0(0)=1$, $W_0(\infty)=0$, $W_0(0^\pm)=1^\pm$. From Proposition 2.1, after a time-change, $(W_0(K(t)))$ has the same distribution as a chordal SLE$(\kappa;(\frac{3}{2}-C_r-C_l)(\kappa-4)-\frac{\kappa}{2},C_r(\kappa-4),C_l(\kappa-4))$ process started from $(1;0,1^+,1^-)$. Applying Theorem 5.1 with $x_1=0$, $x_2=1$, $p_1=0^-$, $p_2=1^+$, $p_3=0^+$, $p_4=1^-$, $C_1=1-C_r$, $C_2=C_r$, $C_3=1/2-C_l$, and $C_4=C_l$, we find that $\partial_{\mathbb{H}}^{\text{out}}W_0(K(\infty))$ has the same distribution as the image of a chordal SLE$(\kappa';(C_2+C_4)(\kappa'-4)-\frac{\kappa'}{2},C_1(\kappa'-4),C_3(\kappa'-4))$ trace started from $(0;1,0^-,0^+)$. Let γ denote this trace. From Proposition 2.1 and Theorem 4.2, γ is a crosscut in $\mathbb{H}$ from 0 to 1. Thus $\partial_{\mathbb{H}}^+K(\infty)=W_0^{-1}(\gamma)$, and so $W(\partial_{\mathbb{H}}^+K(\infty))=W\circ W_0^{-1}(\gamma)$. Let $W_1=W\circ W_0^{-1}$. Then $W_1(z)=\bar{z}/(\bar{z}-1)$. So $W_1(0)=0$, $W_1(1)=\infty$, $W_1(0^\pm)=0^\mp$. From Proposition 2.1, after a time-change, $W_1(\gamma)$ has the same distribution as a chordal SLE$(\kappa';C_1(\kappa'-4),C_3(\kappa'-4))$ trace started from $(0;0^+,0^-)$. Since $C_1=1-C_r$ and $C_3=1/2-C_l$, so we have (i). Now (ii) follows from symmetry. Finally, from (i), (ii) and Proposition 2.7, $\partial_{\mathbb{H}}^+K(\infty)$ and $\partial_{\mathbb{H}}^-K(\infty)$ are two crosscuts in $\mathbb{H}$ that connect ∞ with 0, so we have (iii). □

In the proof of the above theorem, if we choose p_2 and p_4 to be generic force points, then we may obtain the following theorem using a similar argument.

Theorem 5.3. *Let $\kappa > 4$, $C_r, C_l \ge 1/2$, and $p_r > 0 > p_l$. Suppose $K(t)$, $0 \le t < \infty$, is a chordal* SLE$(\kappa; C_r(\kappa - 4), C_l(\kappa - 4))$ *process started from $(0; p_r, p_l)$. Let $K(\infty) = \bigcup_{t\ge 0} K(t)$ and $\kappa' = 16/\kappa$. Then $\partial^+_{\mathbb{H}} K(\infty)$ is a crosscut in $\mathbb{H}$ from ∞ to some point on $(0, p_r)$; $\partial^-_{\mathbb{H}} K(\infty)$ is a crosscut in $\mathbb{H}$ from ∞ to some point on $(p_l, 0)$; and $K(\infty)$ is bounded away from $(-\infty, p_l]$ and $[p_r, +\infty)$.*

6. Boundary of chordal SLE

In this section, we use Theorem 1.1 and Proposition 1.1 to study the boundary of standard chordal SLE(κ) hulls for $\kappa > 4$.

Let $\kappa > 4$. Let $K(t)$, $0 \le t < \infty$, be a standard chordal SLE(κ) process. Let $\xi(t)$, $\varphi(t, \cdot)$ and $\gamma(t)$, $0 \le t < \infty$, be the corresponding driving function, chordal Loewner maps, and trace. Then there is a Brownian motion $B(t)$ such that $\xi(t) = \sqrt{\kappa} B(t)$, $t \ge 0$. For each $t > 0$, let $a(t) = \inf(\overline{K(t)} \cap \mathbb{R})$ and $b(t) = \sup(\overline{K(t)} \cap \mathbb{R})$, then $a(t) < 0 < b(t)$, and $\varphi(t, \cdot)$ maps $(-\infty, a(t))$ and $(b(t), +\infty)$ onto $(-\infty, c(t))$ and $(d(t), +\infty)$ for some $c(t) < 0 < d(t)$. And we have $c(t) \le \xi(t) \le d(t)$, $t > 0$. For each $t > 0$, $f_t := \varphi(t, \cdot)^{-1}$ extends continuously to $\overline{\mathbb{H}}$ with $f_t(c(t)) = a(t)$, $f_t(d(t)) = b(t)$, $f_t(\xi(t)) = \gamma(t)$, and $K(t)$ is bounded by $f_t([c(t), d(t)])$ and $\mathbb{R}$. We have the following theorem.

Theorem 6.1. *Let $T \in (0, \infty)$ be a stopping time w.r.t. the filtration generated by $(\xi(t))$. Then $\gamma(T) \in \mathbb{R}$ a.s. implies that $\xi(T) = c(T)$ or $= d(T)$, and the curve $f_t(x)$, $c(T) < x < d(T)$, is a crosscut in $\mathbb{H}$ on $\mathbb{R}$ with dimension $1 + 2/\kappa$ everywhere; and $\gamma(T) \in \mathbb{H}$ a.s. implies that $c(T) < \xi(T) < d(T)$, and the two curves $f_t(x)$, $c(T) < x \le \xi(T)$, and $f_t(x)$, $\xi(T) \le x < d(T)$, are both semi-crosscuts in $\mathbb{H}$ on $\mathbb{R}$ with dimension $1 + 2/\kappa$ everywhere. Moreover, $\overline{K(T)}$ is connected, and has no cut-point on $\mathbb{R}$.*

Here, a curve α is said to have dimension d everywhere if any nondegenerate subcurve of α has Hausdorff dimension d. From the main theorem in [3], every standard chordal SLE(κ) trace has dimension $(1 + \kappa/8) \wedge 2$ everywhere. From the Girsanov theorem and Proposition 2.2, this is also true for any chordal or strip SLE$(\kappa; \vec{\rho})$ trace. For a connected set $K \subset \mathbb{C}$, $z_0 \in K$ is called a cut-point of K, if $K \setminus \{z_0\}$ is not connected. Such cut-point must lie on the boundary of K.

We need a lemma to prove this theorem. For each $p \in \mathbb{R} \setminus \{0\}$, let T_p denote the first time that p is swallowed by $K(t)$. Then $T_p > 0$ is a finite stopping time because $\kappa > 4$.

Lemma 6.1. *For $p_- < 0 < p_+$, the events $\{T_{p_-} < T_{p_+}\}$ and $\{T_{p_+} < T_{p_-}\}$ both have positive probabilities.*

Proof. Let $T = T_{p_-} \wedge T_{p_+}$. Let $X_\pm(t) = \varphi(t, p_\pm) - \xi(t)$, $0 \le t < T$. Then $X_\pm(t)$ satisfies the SDE: $\mathrm{d}X_\pm(t) = -\sqrt{\kappa}\,\mathrm{d}B(t) + \frac{2}{X_\pm(t)}\,\mathrm{d}t$. Let $Y_\pm(t) = \ln(|X_\pm(t)|)$, $0 \le t < T$. From Itô's formula, $Y_\pm(t)$ satisfies the SDE:

$$\mathrm{d}Y_\pm(t) = -\frac{\sqrt{\kappa}}{X_\pm(t)}\,\mathrm{d}B(t) + \left(2 - \frac{\kappa}{2}\right)\frac{\mathrm{d}t}{X_\pm(t)^2}.$$

Let $Y(t) = Y_+(t) - Y_-(t)$, $0 \le t < T$. Then $Y(t)$ satisfies the SDE:

$$\mathrm{d}Y(t) = -\sqrt{\kappa}\left[\frac{1}{X_+(t)} - \frac{1}{X_-(t)}\right]\mathrm{d}B(t) + \left(2 - \frac{\kappa}{2}\right)\left[\frac{1}{X_+(t)^2} - \frac{1}{X_-(t)^2}\right]\mathrm{d}t.$$

Let $u(t) = \int_0^t (1/X_+(s) - 1/X_-(s))^2\,\mathrm{d}s$, $0 \le t < T$. Let $Z(t) = Y(u^{-1}(t))$, $0 \le t < u(T)$. Then there is a Brownian motion $\widetilde{B}(t)$ such that $Z(t)$ satisfies the SDE:

$$\begin{aligned}\mathrm{d}Z(t) &= -\sqrt{\kappa}\,\mathrm{d}\widetilde{B}(t) + \left(2 - \frac{\kappa}{2}\right)\frac{X_-(u^{-1}(t)) + X_+(u^{-1}(t))}{X_-(u^{-1}(t)) - X_+(u^{-1}(t))}\,\mathrm{d}t \\ &= -\sqrt{\kappa}\,\mathrm{d}\widetilde{B}(t) + \left(\frac{\kappa}{2} - 2\right)\tanh_2(Z(t))\,\mathrm{d}t.\end{aligned}$$

From the chordal Loewner equation, $X_+(t) - X_-(t) = \varphi(t, p_+) - \varphi(t, p_-)$ increases in t. If $T = T_{p_-}$, as $t \to T^-$, $X_-(t) = \varphi(t, p_-) - \xi(t) \to 0$, so $|X_+(t)|/|X_-(t)| \to \infty$, which implies that $Z(t) \to +\infty$ as $t \to u(T)$. Similarly, if $T = T_{p_+}$, then $Z(t) \to -\infty$ as $t \to u(T)$. Thus as $t \to T$, either $Z(t) \to +\infty$ or $Z(t) \to -\infty$. For $x \in \mathbb{R}$, let $h(x) = \int_0^x (\cosh_2 s)^{2/\kappa - 2}\, ds$. Since $2/\kappa - 2 < 0$, so h maps $\mathbb{R}$ onto a finite interval, say $(-L, L)$. And we have $\frac{\kappa}{2} h''(x) + (\frac{\kappa}{2} - 2) h'(x) \tanh_2(x) = 0$ for any $x \in \mathbb{R}$. Let $W(t) = h(Z(t))$, $0 \le t < u(T)$. Then as $t \to u(T)$, either $W(t) \to L$ or $W(t) \to -L$. From Itô's formula, $(W(t))$ is a bounded martingale. Thus, the probability that $\lim_{t \to u(T)} W(t) = L$ is $(W(0) - (-L))/(2L) > 0$. So the probability that $T_{p_-} < T_{p_+}$, i.e., $T = T_{p_-}$, is positive. Similarly, the probability that $T_{p_+} < T_{p_-}$ is also positive. $\square$

Proof of Theorem 6.1. Let $\kappa' = 16/\kappa \in (0, 4)$. If $T = T_p$ for some $p \in \mathbb{R} \setminus \{0\}$, then $\gamma(T) \in \mathbb{R}$, and $\xi(T) = c(T)$ or $d(T)$, depending on whether $p < 0$ or $p > 0$. From Theorem 1.1 and Proposition 1.1, $\partial K(T) \cap \mathbb{H} = \{f_T(x)\colon c(T) < x < d(T)\}$ is the image of a chordal SLE$(\kappa', \vec{\rho})$ trace, and so it has dimension $1 + \kappa'/8 = 1 + 2/\kappa$ everywhere. We also see that this curve is a crosscut in $\mathbb{H}$ on $\mathbb{R}$, so $K(T)$ is the hull bounded by this crosscut. Thus, $\overline{K(T)}$ is connected, and has no cut-point.

Now consider the general case. We first prove (i): $\xi(T) = c(T)$ a.s. implies that $f_t(x)$, $c(T) < x < d(T)$, is a crosscut in $\mathbb{H}$ on $\mathbb{R}$ with dimension $1 + 2/\kappa$ everywhere. Let $\mathcal{E}$ denote the event that $\xi(T) = c(T)$, but $f_t(x)$, $c(T) < x < d(T)$, is not a crosscut in $\mathbb{H}$ on $\mathbb{R}$, or does not have dimension $1 + 2/\kappa$ anywhere. Assume that $\mathbf{P}(\mathcal{E}) > 0$. For each $n \in \mathbb{N}$, let

$$\begin{aligned}\mathcal{E}_n := &\bigl\{\xi(T) = c(T)\bigr\} \cap \bigl\{-n < a(T)\bigr\} \cap \bigl\{d(T) - c(T) > 1/n\bigr\} \\ &\cap \bigl\{f_t(x), c(T) + 1/n \le x < d(T), \text{ is not a semi-crosscut in } \mathbb{H} \text{ on } \mathbb{R}, \\ &\text{or does not have dimension } 1 + 2/\kappa \text{ everywhere}\bigr\}.\end{aligned}$$

Since $f_T(c(T)) = a(T) \in \mathbb{R}$, and $a(T) < b(T) = f_T(d(T))$, so $\mathcal{E} = \bigcup_{n=1}^\infty \mathcal{E}_n$. Then there is $n_0 \in \mathbb{N}$ such that $\mathbf{P}(\mathcal{E}_{n_0}) > 0$.

Let $(\widetilde{K}(t), 0 \le t < \infty)$ be a standard chordal SLE(κ) process that is independent of $(K(t))$. Let $\widetilde{\mathcal{E}}_{n_0}$ denote the event that $\widetilde{K}(t)$ swallows $\varphi(T, -n_0) - \xi(T)$ before swallowing $1/n_0$, and let $\widetilde{T}$ denote the first time that $\widetilde{K}(t)$ swallows $\varphi(T, -n_0) - \xi(T)$. From Lemma 6.1, the probability of $\widetilde{\mathcal{E}}_{n_0}$ is positive. Let $\widehat{\mathcal{E}}_{n_0} = \mathcal{E}_{n_0} \cap \widetilde{\mathcal{E}}_{n_0}$. Then $\widehat{\mathcal{E}}_{n_0}$ also has positive probability.

Define $\widehat{K}(t) = K(t)$ for $0 \le t \le T$; and $\widehat{K}(t) = K(T) \cup f_T(\widetilde{K}(t - T) + \xi(T))$ for $t > T$. Then $(\widehat{K}(t))$ has the same distribution as $(K(t))$. Let $\widehat{T}_{-n_0}$ denote the first time that $\widehat{K}(t)$ swallows $-n_0$. Then $\partial \widehat{K}(\widehat{T}_{-n_0}) \cap \mathbb{H}$ is a.s. a crosscut in $\mathbb{H}$ on $\mathbb{R}$ with dimension $1 + 2/\kappa$ everywhere. Since on $\widehat{\mathcal{E}}_{n_0}$, $\widehat{T}_{-n_0} = T + \widetilde{T}$, and $\overline{\widetilde{K}(\widetilde{T})} \cap \mathbb{R}$ is bounded above by $1/n_0$, so $\{f_T(x), c(T) + 1/n_0 \le x < d(T)\}$ is a subset of the boundary of $\widehat{K}(\widehat{T}_{-n_0}) = K(T) \cup f_T(\widetilde{K}(\widetilde{T}) + \xi(T))$ in $\mathbb{H}$, which implies that a.s. $f_T(x)$, $c(T) + 1/n_0 \le x < d(T)$, is a semi-crosscut with dimension $1 + 2/\kappa$ everywhere. This contradicts that $\widehat{\mathcal{E}}_{n_0}$ has positive probability. So we have (i). Symmetrically, we have (ii): $\xi(T) = d(T)$ a.s. implies that $f_t(x)$, $c(T) < x < d(T)$, is a crosscut in $\mathbb{H}$ on $\mathbb{R}$ with dimension $1 + 2/\kappa$ everywhere.

If $\gamma(T) = f_T(\xi(T)) \in \mathbb{H}$, then $\gamma(T) \notin \{c(T), d(T)\}$, so $c(T) < \xi(T) < d(T)$. Using the same argument as in (i), we can prove (iii): $\gamma(T) \in \mathbb{H}$ a.s. implies that $f_t(x)$, $\xi(T) \le x < d(T)$, is a semi-crosscut in $\mathbb{H}$ on $\mathbb{R}$ with dimension $1 + 2/\kappa$ everywhere. Symmetrically, we have (iv): $\gamma(T) \in \mathbb{H}$ a.s. implies that $f_t(x)$, $c(T) < x \le \xi(T)$, is a semi-crosscut in $\mathbb{H}$ on $\mathbb{R}$ with dimension $1 + 2/\kappa$ everywhere. From (iii) and (iv), we see that $\gamma(T) \in \mathbb{H}$ a.s. implies that $\overline{K(T)}$ is connected, and has no cut-point on $\mathbb{R}$. Similarly, we have (v): $c(T) < \xi(T) < d(T)$ and $\gamma(T) \in \mathbb{R}$ a.s. implies that $f_T(x)$, $\xi(T) < x < d(T)$, and $f_T(x)$, $c(T) < x < \xi(T)$, are both crosscuts or degenerate crosscuts in $\mathbb{H}$ on $\mathbb{R}$. Moreover, these two curves intersect at only one point: $\gamma(T)$, since the curve $\alpha(y) := \overline{f_T(\xi(T) + iy)}$, $y > 0$, connects $\gamma(T)$ with ∞, and does not intersect the above two curves. So $\gamma(T)$ is a cut-point of $\overline{K(T)}$ on $\mathbb{R}$.

To finish the proof, it remains to prove (vi): $\gamma(T) \in \mathbb{R}$ a.s. implies that $\xi(T) = c(T)$ or $= d(T)$. Let $\mathcal{E}$ denote the event that $\gamma(T) \in \mathbb{R}$ and $c(T) < \xi(T) < d(T)$. We suffice to show that $\mathbf{P}(\mathcal{E}) = 0$. Assume that $\mathbf{P}(\mathcal{E}) > 0$. Assume that $\mathcal{E}$ occurs. From (v), we know that $\overline{K(T)} = K_1 \cup K_2$, where K_1 and K_2 are hulls bounded by crosscut or degenerate crosscut in $\mathbb{H}$ on $\mathbb{R}$, and $\overline{K_1} \cap \overline{K_2} = \{\gamma(T)\}$. Since $\kappa > 4$, so a.s. $K(T)$ contains a neighborhood of 0 in $\mathbb{H}$. We may label K_1 and K_2 such that K_1 contains a neighborhood of 0 in $\mathbb{H}$. Then $\gamma(T) \neq 0$. Let $\mathcal{S} = \{\overline{\mathbf{B}(x + iy; r)}\colon x, y, r \in \mathbb{Q}, y, r > 0, r < y/2\}$, where $\mathbf{B}(z_0; r) := \{z \in \mathbb{C}\colon |z - z_0| < r\}$. Then $\mathcal{S}$ is countable, and every $A \in \mathcal{S}$ is contained in $\mathbb{H}$. For $A \in \mathcal{S}$, let $\mathcal{E}_A$ denote the intersection of $\mathcal{E}$ with the event that $A \cap \partial K_2 \neq \varnothing$ and $A \cap K_1 = \varnothing$. Then $\mathcal{E} = \bigcup_{A \in \mathcal{S}} \mathcal{E}_A$.

So there is $A_0 \in \mathcal{S}$ such that $\mathbf{P}(\mathcal{E}_{A_0}) > 0$. Let T_0 be the first time that $\gamma(t)$ hits A_0. Let $T_1 = T \wedge T_0$. Then T_1 is a finite stopping time. Assume $\mathcal{E}_{A_0}$ occurs. Since $\gamma(t)$, $0 \le t \le T$, visits every point on $\partial K_2 \cap \mathbb{H} \subset \partial K(T) \cap \mathbb{H}$, so $T_0 \le T$, and so $T_1 = T_0$. We have $\gamma(T_1) = \gamma(T_0) \in A_0$. Since $A_0 \cap \mathbb{R} = \emptyset$, so $\gamma(T_1) \in \mathbb{H}$. Since $\gamma(0) = 0 \in \overline{K_1}$, and $\gamma(T_1) \in \overline{K_2}$, which are both different from $\gamma(T)$, so $\gamma(T) \in \overline{K_1} \cap \overline{K_2}$ is a cut-point of $\overline{K(T_1)}$. However, since T_1 is a positive finite stopping time, and $\gamma(T_1) \in \mathbb{H}$ on $\mathcal{E}_{A_0}$, so from (iii) and (iv) in the above proof, a.s. $\overline{K(T_1)}$ has no cut-point on $\mathbb{R}$ in the event $\mathcal{E}_{A_0}$. This contradicts that $\mathbf{P}(\mathcal{E}_{A_0}) > 0$. So $\mathbf{P}(\mathcal{E}) = 0$. □

Corollary 6.1. *For any stopping time $T \in (0, \infty)$, a.s. $f_T(x) \notin \mathbb{R}$ for $x \in (c(T), d(T))$; $\partial K(T) \cap \mathbb{H}$ has Hausdorff dimension $1 + 2/\kappa$; $\overline{K(T)}$ is connected, and has no cut-point on $\mathbb{R}$; and for every $x \in (a(T), b(T))$, $K(T)$ contains a neighborhood of x in $\mathbb{H}$.*

In the above theorem, when $\gamma(T) \in \mathbb{H}$, $\partial K(T) \cap \mathbb{H}$ is composed of two semi-crosscuts in $\mathbb{H}$ on $\mathbb{R}$, which are $f_T(x)$, $c(T) < x \le \xi(T)$ and $f_T(x)$, $\xi(T) \le x < d(T)$. If the two semi-crosscuts intersect only at $\gamma(T) = f_T(\xi(T))$, then we get a crosscut $f_T(x)$, $c(T) < x < d(T)$. If the two semi-crosscuts intersect at any point z_0 other than $\gamma(T)$, then z_0 is a cut-point of $K(T)$. To see this, suppose $f_T(x_1) = f_T(x_2) = z_0$, where $c(T) < x_1 < \xi(T) < x_2 < d(T)$. Then $f_T(x)$, $c(T) < x \le x_1$ and $f_T(x)$, $x_2 \le x < d(T)$ are two semi-crosscuts in $\mathbb{H}$ on $\mathbb{R}$, which together bound a hull in $\mathbb{H}$ on $\mathbb{R}$. Let it be denoted by K_1. The simple curves $f_T(x)$, $x_1 \le x \le \xi(T)$, and $f_T(x)$, $\xi(T) \le x \le x_2$, together bound a closed bounded set in $\mathbb{H}$. Let it be denoted by K_2. Then $K(T) = K_1 \cup K_2$ and $K_1 \cap K_2 = \{z_0\}$. On the other hand, every cut-point of $K(T)$ corresponds to an intersection point between $f_T(x)$, $c(T) < x < \xi(T)$, and $f_T(x)$, $\xi(T) < x < d(T)$, and so such cut-point disconnects $\gamma(T)$ from $\xi(0) = 0$ in $K(T)$. From Theorem 5 in [2], if $\kappa > 8$ and $T > 0$ is a constant, then a.s. $K(T)$ has no cut-point, so $f_T(x)$, $c(T) < x < d(T)$, is a crosscut in $\mathbb{H}$ on $\mathbb{R}$. We now make some improvement over this result.

Theorem 6.2. *If $\kappa \ge 8$ and $T \in (0, \infty)$ is a stopping time, then a.s. $K(T)$ has no cut-point, and so $f_T(x)$, $c(T) < x < d(T)$, is a crosscut in $\mathbb{H}$ on $\mathbb{R}$.*

Proof. First suppose $\kappa > 8$. Let $\mathcal{E}$ denote the event that $K(T)$ has a cut-point. We suffice to show that $\mathbf{P}(\mathcal{E}) = 0$. Assume that $\mathbf{P}(\mathcal{E}) > 0$. For each $n \in \mathbb{N}$, let $\mathcal{E}_n$ denote the event that $c(T) + 1/n < \xi(T) < d(T) - 1/n$, and the two curves $f_T(x)$, $c(T) < x \le \xi(T) - 1/n$, and $f_T(x)$, $\xi(T) + 1/n \le x < d(T)$, are not disjoint. Then $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$. So there is $n_0 \in \mathbb{N}$ such that $\mathbf{P}(\mathcal{E}_{n_0}) > 0$.

Let $(\widetilde{K}(t))$ be a standard chordal SLE(κ) process that is independent of $(K(t))$. There is a small $h > 0$ such that the probability that $\overline{\widetilde{K}(h)} \cap \mathbb{R} \subset (-1/n_0, 1/n_0)$ is positive. There is $t_0 \in [0, \infty)$ such that $\mathbf{P}(\mathcal{E}_{n_0} \cap \{t_0 - h \le T \le t_0\}) > 0$. Let $\widehat{\mathcal{E}}$ denote the intersection of $\mathcal{E}_{n_0} \cap \{t_0 - h \le T \le t_0\}$ with $\{\overline{\widetilde{K}(h)} \cap \mathbb{R} \subset (-1/n_0, 1/n_0)\}$. Then $\widehat{\mathcal{E}}$ also has positive probability. Define $\widehat{K}(t) = K(t)$ for $0 \le t \le T$; and $\widehat{K}(t) = K(T) \cup f_T(\widetilde{K}(t - T) + \xi(T))$ for $t > T$. Then $(\widehat{K}(t))$ has the same distribution as $(K(t))$. From Theorem 5 in [2], a.s. $\widehat{K}(t_0)$ has no cut-point. Since $T \le t_0 \le T + h$, so $K(T) \subset \widehat{K}(t_0) \subset K(T) \cup f_T(\widetilde{K}(h) + \xi(T))$. In the event $\widehat{\mathcal{E}}$, since $\overline{\widetilde{K}(h)} \cap \mathbb{R} \subset (-1/n_0, 1/n_0)$, so $f_T(x)$, $c(T) < x \le \xi(T) - 1/n_0$, and $f_T(x)$, $\xi(T) + 1/n_0 \le x < d(T)$, are subarcs of $\partial \widehat{K}(t_0) \cap \mathbb{H}$. However, in the event $\widehat{\mathcal{E}}$, the above two curves are not disjoint, so $\widehat{K}(t_0)$ has a cut-point, which contradicts that $\widehat{\mathcal{E}}$ has positive probability. Thus $\mathbf{P}(\mathcal{E}) = 0$.

Now suppose $\kappa = 8$. Let $\gamma^R(t) = \gamma(1/t)$, $0 < t < \infty$. Since chordal SLE(8) trace is reversible (cf. [5]), so after a time-change, γ^R has the distribution of a chordal SLE(8) trace in $\mathbb{H}$ from ∞ to 0. Thus, a.s. there is a crosscut α in $\mathbb{H} \setminus \gamma^R((0, 1/T]) = \mathbb{H} \setminus \gamma([T, \infty))$ connecting $\gamma^R(1/T) = \gamma(T)$ with 0. Then $\alpha \subset K(T)$ and does not intersect $\partial K(T)$. If $K(T)$ has any cut-point, the cut-point must disconnect $\gamma(T)$ from 0 in $K(T)$, so such α does not exist. Thus a.s. $K(T)$ has no cut-point. □

If $\kappa \in (4, 8)$, this theorem does not hold since from Theorem 5 in [2], the probability that $K(1)$ has cut-point is positive.

7. More geometric results

The description of the boundary of SLE(κ) hulls for $\kappa > 4$ enables us to obtain some results about the limit of SLE($\kappa; \vec{\rho}$) traces when $\kappa > 4$. We will prove that the limits of the traces exist when certain conditions are satisfied.

Let $\kappa > 4$. In this section, $L(t)$, $0 \le t < T_e$, is a strip SLE$(\kappa; \vec{\rho})$ process started from $(0; \vec{p})$, where no force point is degenerate. Let $\xi(t)$, $\psi(t, \cdot)$, and $\beta(t)$, $0 \le t < T_e$, be the corresponding driving function, strip Loewner maps, and trace. For $t \in (0, T_e)$, let $a(t) = \inf(\overline{L(t)} \cap \mathbb{R}) < 0$ and $b(t) = \sup(\overline{L(t)} \cap \mathbb{R}) > 0$. Then $\psi(t, \cdot)$ maps $(-\infty, a(t))$ and $(b(t), +\infty)$ onto $(-\infty, c(t))$ and $(d(t), +\infty)$ for some $c(t) < 0 < d(t)$, and we have $c(t) \le \xi(t) \le d(t)$. For each $t > 0$, $f_t := \psi(t, \cdot)^{-1}$ extends continuously to $\overline{\mathbb{S}_\pi}$ such that $f_t(c(t)) = a(t)$, $f_t(d(t)) = b(t)$ and $f_t(\xi(t)) = \beta(t)$. From Theorem 6.1, Proposition 2.2 and the Girsanov theorem, we have the following lemma.

Lemma 7.1. *If $T \in (0, T_e)$ is a stopping time, then a.s. $f_T(x) \in \mathbb{S}_\pi$ for $c(T) < x < d(T)$, and for every $x \in (a(T), b(T))$, $L(T)$ contains a neighborhood of x in $\mathbb{S}_\pi$.*

Lemma 7.2. *Let $T \in [0, T_e)$ be a stopping time. Define $\beta_T(t) = \psi(T, \beta(T + t)) - \xi(T)$, $0 \le t < T_e - T$. Suppose $\vec{p} = (p_1, \dots, p_N)$. If $\psi(T, p_m) - \xi(T) = p_m$ for $1 \le m \le N$, then β_T has the same distribution as β. In the general case, conditioned on $\beta(t)$, $0 \le t \le T$, β_T is a strip* SLE$(\kappa; \vec{\rho})$ *trace started from $(0; \vec{q})$, where $\vec{q} = (q_1, \dots, q_N)$ and $q_m = \psi(T, p_m) - \xi(T)$, $1 \le m \le N$.*

Proof. This follows from the definition of strip SLE$(\kappa; \vec{\rho})$ process and the property that Brownian motion has i.i.d. increment. □

Lemma 7.3. *Let $\kappa > 4$, $\rho_+, \rho_- \in \mathbb{R}$, $\rho_+ + \rho_- = \kappa - 6$, and $\rho_- - \rho_+ \ge 2$. Suppose $\beta(t)$, $0 \le t < \infty$, is a strip* SLE$(\kappa; \rho_+, \rho_-)$ *trace started from $(0; +\infty, -\infty)$. Then a.s. any subsequential limit of $\beta(t)$ as $t \to \infty$ does not lie on $\mathbb{R} \cup \mathbb{R}_\pi \cup \{-\infty\}$.*

Proof. Let Q denote the set of subsequential limits of $\beta(t)$ as $t \to \infty$. Let $\sigma = (\rho_- - \rho_+)/2 \ge 1$. Then there is a Brownian motion $B(t)$ such that $\xi(t) = \sqrt{\kappa} B(t) + \sigma t$, $0 \le t < \infty$. Thus, a.s. there is a random number $A_0 < 0$ such that $\xi(t) \ge A_0$ for $0 \le t < \infty$. From (2.3), for any $z \in \mathbb{S}_\pi$ with $\operatorname{Re} z < A_0$, $\psi(t, z)$ never blows up for $0 \le t < \infty$. Thus, a.s. $\beta([0, \infty)) \subset \{z \in \overline{\mathbb{S}_\pi}\colon \operatorname{Re} z \ge A_0\}$. So a.s. $-\infty \notin Q$. Moreover, for any $\varepsilon > 0$, there is $R_\varepsilon > 0$ such that the probability that $\operatorname{Re}\beta(t) \ge -R_\varepsilon$ for $0 \le t < \infty$ is at least $1 - \varepsilon$.

Fix $x_0 \in \mathbb{R}$. Let $X(t) = \operatorname{Re}\psi(t, x_0 + \pi \mathrm{i}) - \xi(t)$, $0 \le t < \infty$. Then $X(t)$ satisfies the SDE: $\mathrm{d}X(t) = -\sqrt{\kappa}\,\mathrm{d}B(t) + \tanh_2(X(t))\,\mathrm{d}t - \sigma\,\mathrm{d}t$. Define h on $\mathbb{R}$ such that

$$h'(x) = \exp(2\sigma x/\kappa)(\cosh_2 x)^{-4/\kappa}, \quad x \in \mathbb{R}.$$

Since $\sigma \ge 1$, so h maps $\mathbb{R}$ onto (L, ∞) for some $L \in \mathbb{R}$. Let $Y(t) = h(X(t))$, $0 \le t < \infty$. From Itô's formula, $Y(t)$ satisfies the SDE: $\mathrm{d}Y(t) = -h'(X(t))\sqrt{\kappa}\,\mathrm{d}B(t)$. Define $u(t) = \int_0^t \kappa h'(X(s))^2\,\mathrm{d}s$, $0 \le t < \infty$, and $u(\infty) = \sup u([0, \infty))$. Then $Y(u^{-1}(t))$, $0 \le t < u(\infty)$, has the distribution of a partial Brownian motion. Since $Y(u^{-1}(t)) \in (L, \infty)$ for $0 \le t < u(\infty)$, so a.s. $u(\infty) < \infty$ and $\lim_{t\to\infty} Y(t) = \lim_{t\to u(\infty)} Y(u^{-1}(t)) \in [L, \infty)$. Note that $\lim_{t\to\infty} Y(t) \in (L, \infty)$ implies that $\lim_{t\to\infty} X(t) \in \mathbb{R}$ and so $X(t)$, $0 \le t < \infty$, is bounded. If X is bounded on $[0, \infty)$, from the definition of u, $u'(t)$ is uniformly bounded below by a positive constant, which implies that $u(\infty) = \infty$. Since a.s. $u(\infty) < \infty$, so $\lim_{t\to\infty} Y(t) \notin (L, \infty)$. Thus, a.s. $\lim_{t\to\infty} Y(t) = L$, and so $\lim_{t\to\infty} X(t) = -\infty$.

Fix $\varepsilon > 0$. Let T be the first time such that $X(t) \le -R_\varepsilon - 1$. Then T is a finite stopping time. Let β_T be defined as in Lemma 7.2. Then β_T has the same distribution as β. So the probability that $\operatorname{Re}\beta_T(t) \ge -R_\varepsilon$ for any $0 \le t < \infty$ is at least $1 - \varepsilon$. Let Q_T denote the set of subsequential limits of $\beta_T(t)$ as $t \to \infty$. Then the probability that $Q_T \cap (\pi\mathrm{i} + (-\infty, -R_\varepsilon - 1]) = \varnothing$ is at least $1 - \varepsilon$. If for any $x \le x_0$, $x + \pi\mathrm{i} \in Q$, then $\psi(T, x + \pi\mathrm{i}) - \xi(T) \in Q_T$. Since $x \le x_0$, so $\operatorname{Re}\psi(T, x + \pi\mathrm{i}) - \xi(T) \le X(T) \le -R_\varepsilon - 1$, and so $\psi(T, x + \pi\mathrm{i}) - \xi(T) \in Q_T \cap (\pi\mathrm{i} + (-\infty, -R_\varepsilon - 1])$. Thus, the probability that $Q \cap (\pi\mathrm{i} + (-\infty, x_0]) = \varnothing$ is at least $1 - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, so a.s. $Q \cap (\pi\mathrm{i} + (-\infty, x_0]) = \varnothing$. Since this holds for any $x_0 \in \mathbb{N}$, so a.s. $Q \cap \mathbb{R}_\pi = \varnothing$.

Fix $\varepsilon > 0$ and $x_0 \ge R_\varepsilon + 1$. Let $X_0(t) = \psi(t, x_0) - \xi(t)$, $0 \le t < T_0$, where $[0, T_0)$ is the largest interval on which $\psi(t, x_0)$ is defined. Then $X_0(t)$ satisfies the SDE: $\mathrm{d}X_0(t) = -\sqrt{\kappa}\,\mathrm{d}B(t) + \coth_2(X_0(t))\,\mathrm{d}t - \sigma\,\mathrm{d}t$. Define h_0 on $(0, \infty)$ such that

$$h_0'(x) = \exp(2\sigma x/\kappa)(\sinh_2 x)^{-4/\kappa}, \quad 0 < x < \infty.$$

Since $\kappa > 4$ and $\sigma \ge 1$, so h_0 maps $(0,\infty)$ onto (L,∞) for some $L \in \mathbb{R}$. From Itô's formula, $Y_0(t) := h_0(X_0(t))$, $0 \le t < T_0$, satisfies the SDE: $\mathrm{d}Y_0(t) = -h_0'(X_0(t))\sqrt{\kappa}\,\mathrm{d}B(t)$. Using a similar argument as before, we conclude that a.s. $T_0 < \infty$ and $\lim_{t\to T_0} X_0(t) = 0$. So T_0 is a finite stopping time. Let β_{T_0} be the β_T in Lemma 7.2 with $T = T_0$. Then β_{T_0} has the same distribution as β. Let Q_{T_0} denote the set of subsequential limits of $\beta_{T_0}(t)$ as $t \to \infty$. Then $Q_{T_0} = \psi(T_0, Q) - \xi(T_0)$.

Since x_0 is swallowed at time T_0, so $\xi(T_0) = d(T_0)$ and $b(T_0) \ge x_0$. Since the extremal distance (cf. [1]) between $(-\infty, a(T_0))$ and $(b(T_0), \infty)$ in $\mathbb{S}_\pi \setminus L(T_0)$ is not less than the extremal distance between them in $\mathbb{S}_\pi$, so from the properties of f_{T_0}, we have $d(T_0) - c(T_0) \ge b(T_0) - a(T_0)$. Thus,

$$c(T_0) - \xi(T_0) = c(T_0) - d(T_0) \le a(T_0) - b(T_0) \le -b(T_0) \le -x_0 \le -R_\varepsilon - 1.$$

If $Q \cap (-\infty, a(T_0)] \ne \varnothing$, then since $Q_{T_0} = \psi(T_0, Q) - \xi(T_0)$, so $Q_{T_0} \cap (-\infty, c(T_0) - \xi(T_0)] \ne \varnothing$, which happens with probability less than ε since β_{T_0} has the same distribution as β, and $c(T_0) - \xi(T_0) \le -R_\varepsilon - 1$. From Lemma 7.1, for every $x \in (a(T_0), b(T_0))$, $L(T_0)$ contains a neighborhood of x in $\mathbb{S}_\pi$. Since β does not cross its past, so $Q \cap (a(T_0), b(T_0)) = \varnothing$. Thus, the probability that $Q \cap (-\infty, b(T_0)) \ne \varnothing$ is less than ε. Since $b(T_0) \ge x_0$, and $x_0 \ge R_\varepsilon + 1$ is arbitrary, so the probability that $Q \cap \mathbb{R} \ne \varnothing$ is less than ε. Since $\varepsilon > 0$ is arbitrary, so a.s. $Q \cap \mathbb{R} = \varnothing$. □

Corollary 7.1. *Let $\kappa > 4$ and $\rho \ge \kappa/2 - 2$. Suppose $\gamma_*(t)$, $0 \le t < \infty$, is a chordal* SLE$(\kappa;\rho)$ *trace started from* $(0; 1)$. *Then a.s. γ_* has no subsequential limit on $\mathbb{R}$.*

Proof. This follows from the above lemma and Proposition 2.2. □

Theorem 7.1. *Let $\kappa > 4$ and $\rho \ge \kappa/2 - 2$. Suppose $\gamma(t)$, $0 \le t < \infty$, is a chordal* SLE$(\kappa;\rho)$ *trace started from* $(0; 0^+)$ *or* $(0; 0^-)$. *Then a.s.* $\lim_{t\to\infty}\gamma(t) = \infty$.

Proof. By symmetry, we only need to consider the case that the trace is started from $(0, 0^+)$. Let Q be the set of subsequential limits of γ. From Proposition 2.1, for any $a > 0$, $(a\gamma(t))$ has the same distribution as $(\gamma(a^2 t))$. Thus, aQ has the same distribution as Q for any $a > 0$. To prove that a.s. $Q = \{\infty\}$, we suffice to show that a.s. $0 \notin Q$.

Let $\zeta(t)$ and $\varphi(t,\cdot)$, $0 \le t < \infty$, be the driving function and chordal Loewner maps for γ. Let $X(0) = 0$ and $X(t) = \varphi(t, 0^-) - \zeta(t)$ for $t > 0$. Then $(X(t)/\sqrt{\kappa})$ is a Bessel process with dimension $\frac{2}{\kappa}(2+\rho) + 1 \ge 2$. So a.s. $\limsup_{t\to\infty} X(t) = \infty$. Let T be the first time that $X(t) = 1$. Then T is a finite stopping time. Let $\gamma_*(t) = \varphi(T, \gamma(T+t)) - \zeta(T)$, $t \ge 0$. Then γ_* is a chordal SLE$(\kappa;\rho)$ trace started from $(0; 1)$. From the last corollary, γ_* has no subsequential limit on $\mathbb{R}$. Let $g_T = \varphi(T,\cdot)^{-1}$. Then g_T extends continuously to $\overline{\mathbb{H}}$, and $\gamma(T+t) = g_T(\gamma_*(t) + \zeta(T))$. From the property of $\varphi(T,\cdot)$, we have $g_T(z) = z + o(1)$ as $z \to \infty$, so $g_T^{-1}(0) - \zeta(T) \subset \mathbb{R}$ is bounded. If $0 \in Q$, then γ_* has a subsequential limit on $g_T^{-1}(0) - \zeta(T) \subset \mathbb{R}$, which a.s. does not happen. Thus, a.s. $0 \notin Q$. □

Corollary 7.2. *Let γ_* be as in Corollary 7.1. Then a.s.* $\lim_{t\to\infty}\gamma_*(t) = \infty$.

Proof. Let γ be a chordal SLE$(\kappa;\rho)$ trace started from $(0; 0^+)$. Let $\zeta(t)$ and $\varphi(t,\cdot)$, $0 \le t < \infty$, be the driving function and chordal Loewner maps for γ. Let $X(0) = 0$ and $X(t) = \varphi(t, 0^-) - \zeta(t)$ for $t > 0$. Let T be the first time that $X(t) = 1$. Then T is a finite stopping time. Let $\gamma_1(t) = \varphi(T, \gamma(T+t)) - \zeta(T)$, $t \ge 0$. Then γ_1 has the same distribution as γ_*. Since a.s. $\lim_{t\to\infty}\gamma(t) = \infty$, so a.s. $\lim_{t\to\infty}\gamma_1(t) = \infty$. Since γ_1 has the same distribution as γ_*, so a.s. $\lim_{t\to\infty}\gamma_*(t) = \infty$. □

Theorem 7.2. *Proposition 2.5 also holds for $\kappa > 4$.*

Proof. This follows from the above corollary and Proposition 2.2. □

Let $\kappa > 4$, $p_0 = x_0 + \pi\mathrm{i} \in \mathbb{R}_\pi$, $\rho_+, \rho_-, \rho_0 \in \mathbb{R}$, and $\rho_+ + \rho_- + \rho_0 = \kappa - 6$. Let $\beta(t)$, $0 \le t < \infty$, be a strip SLE$(\kappa; \rho_+, \rho_-, \rho_0)$ trace started from $(0; +\infty, -\infty, p_0)$. Let $\xi(t)$, $\psi(T,\cdot)$ and $L(t)$, $0 \le t < \infty$, be the corresponding driving function, strip Loewner maps and hulls. Then there is some Brownian motion $B(t)$ such that $\xi(t)$ satisfies the SDE:

$$\mathrm{d}\xi(t) = \sqrt{\kappa}\,\mathrm{d}B(t) - \frac{\rho_+ - \rho_-}{2}\,\mathrm{d}t - \frac{\rho_0}{2}\coth_2\bigl(\psi(t, p_0) - \xi(t)\bigr)\,\mathrm{d}t.$$

Let

$$X(t) = \operatorname{Re}\psi(t, p_0) - \xi(t), \quad 0 \le t < \infty. \tag{7.1}$$

Then $X(t)$ satisfies the SDE:

$$\mathrm{d}X(t) = -\sqrt{\kappa}\,\mathrm{d}B(t) + \frac{\rho_+ - \rho_-}{2}\,\mathrm{d}t + \left(\frac{\kappa}{2} - 2 - \frac{\rho_+ + \rho_-}{2}\right)\tanh_2\bigl(X(t)\bigr)\,\mathrm{d}t.$$

Define h on $\mathbb{R}$ such that

$$h'(x) = \exp\left(\frac{1}{\kappa}(\rho_- - \rho_+)x\right)(\cosh_2 x)^{-(4/\kappa)\cdot(\kappa/2-2-(\rho_++\rho_-)/2)}, \quad x \in \mathbb{R}.$$

Let $Y(t) = h(X(t))$, $0 \le t < \infty$. From Itô's formula, $Y(t)$ satisfies the SDE: $\mathrm{d}Y(t) = -h'(X(t))\sqrt{\kappa}\,\mathrm{d}B(t)$. For $0 \le t < \infty$, let $u(t) = \int_0^t \kappa h'(X(s))^2\,\mathrm{d}s$. Then $Y(u^{-1}(t))$, $0 \le t < u(\infty) := \sup u([0,\infty))$, is a partial Brownian motion. The behavior of $X(t)$ as $t \to \infty$ depends on the values of ρ_+ and ρ_-. Now we suppose that $\rho_+, \rho_- \ge \kappa/2 - 2$. Then h maps $\mathbb{R}$ onto $\mathbb{R}$. If $u(\infty) < \infty$, then a.s. $Y(u^{-1}(t))$ is bounded on $[0, u(\infty))$, so $X(t)$ is bounded on $[0,\infty)$. This then implies that $u'(t)$ is uniformly bounded below by a positive constant, and so $u(\infty) = \infty$, which is a contradiction. Thus, a.s. $u(\infty) = \infty$, and so $\limsup_{t\to u(\infty)} Y(u^{-1}(t)) = \infty$ and $\liminf_{t\to u(\infty)} Y(u^{-1}(t)) = -\infty$, which implies that $\limsup_{t\to\infty} X(t) = \infty$ and $\liminf_{t\to\infty} X(t) = -\infty$.

Lemma 7.4. *Let β be as above. If $\rho_+, \rho_- \ge \kappa/2 - 2$, then a.s. β has no subsequential limit on $\mathbb{R} \cup \{+\infty, -\infty\} \cup \mathbb{R}_\pi \setminus \{p_0\}$.*

Proof. Let Q denote the set of subsequential limits of $\beta(t)$ as $t \to \infty$. Let $L(\infty) = \bigcup_{t\ge 0} L(t)$. From Theorem 5.3 and Proposition 2.2, a.s. $p_0 \in \overline{L(\infty)}$, and $L(\infty)$ is bounded by two crosscuts in $\mathbb{S}_\pi$ that connect p_0 with a point on $(-\infty, 0)$ and a point on $(0,\infty)$, respectively. Thus, a.s. $Q \cap (\mathbb{R}_\pi \cup \{+\infty, -\infty\} \setminus \{p_0\}) = \varnothing$. Moreover, for any $\varepsilon > 0$, there is $R_\varepsilon > 0$ such that the probability that $\overline{L(\infty)} \cap \mathbb{R} \subset [-R_\varepsilon, R_\varepsilon]$ is at least $1 - \varepsilon$.

For $r \in (0,1)$, let $A_r = \{z\colon r < |z - p_0| < \pi\}$. If $\operatorname{dist}(p_0, L(t)) \le r$, then any curve in $\mathbb{S}_\pi \setminus L(t)$ that connects the arc $[p_0, +\infty) \subset \mathbb{R}_\pi$ with $(-\infty, a(t))$ must connect the two boundary components of A_r. Thus, the extremal distance between $[p_0, +\infty)$ and $(-\infty, a(t))$ in $\mathbb{S}_\pi \setminus L(t)$ is at least $(\ln(\pi) - \ln(r))/\pi$. So the extremal distance between $[\psi(t, p_0), +\infty)$ and $(-\infty, c(t))$ in $\mathbb{S}_\pi$ is at least $(\ln(\pi) - \ln(r))/\pi$, which tends to ∞ as $r \to 0$. This implies that $\operatorname{Re}\psi(t, p_0) - c(t) \to \infty$ as $\operatorname{dist}(p_0, L(t)) \to 0$. Similarly, $d(t) - \operatorname{Re}\psi(t, p_0) \to \infty$ as $\operatorname{dist}(p_0, L(t)) \to 0$. Fix $\varepsilon > 0$. There is $r \in (0,1)$ such that if $\operatorname{dist}(p_0, L(t)) \le r$, then $\operatorname{Re}\psi(t, p_0) - c(t), d(t) - \operatorname{Re}\psi(t, p_0) \ge R_\varepsilon + |x_0| + 1$. Let T_0 be the first t such that $\operatorname{dist}(p_0, \beta(t)) = r$. Since a.s. $p_0 \in \overline{L(\infty)}$, so T_0 is a finite stopping time.

Let $X(t)$ be defined as in (7.1). Let T be the first $t \ge T_0$ such that $X(t) = x_0 = \operatorname{Re} p_0$. Since $\limsup_{t\to\infty} X(t) = +\infty$ and $\liminf_{t\to\infty} X(t) = -\infty$, so T is also a finite stopping time. Let β_T be defined as in Lemma 7.2, then β_T has the same distribution as β. So the probability that $\overline{\beta_T([0,\infty))} \cap \mathbb{R} \subset [-R_\varepsilon, R_\varepsilon]$ is at least $1 - \varepsilon$. Since $\operatorname{dist}(p_0, L(T)) \le \operatorname{dist}(p_0, L(T_0)) = r$, so $\operatorname{Re}\psi(T, p_0) - c(T), d(T) - \operatorname{Re}\psi(T, p_0) \ge R_\varepsilon + |x_0| + 1$. Since $X(T) = \operatorname{Re}\psi(T, p_0) - \xi(T) = x_0$, so $\xi(T) - c(T), d(T) - \xi(T) \ge R_\varepsilon + 1$, and so $[-R_\varepsilon, R_\varepsilon] \subset [c(T) - \xi(T), d(T) - \xi(T)]$. Thus, the probability that $\overline{\beta_T([0,\infty))} \cap \mathbb{R} \subset [c(T) - \xi(T), d(T) - \xi(T)]$ is at least $1 - \varepsilon$. Since for every $x \in (a(T), b(T))$, $L(T)$ contains a neighborhood of x in $\mathbb{S}_\pi$, and β does not cross its past, so $Q \cap (a(T), b(T)) = \varnothing$. If $Q \cap (-\infty, a(T)] \cup [b(T), \infty) \ne \varnothing$, then β_T has a subsequential limit on $(-\infty, c(T) - \xi(T)] \cup [d(T) - \xi(T), \infty)$, which happens with probability at most ε. Thus, the probability that $Q \cap \mathbb{R} \ne \varnothing$ is at most ε. Since $\varepsilon > 0$ is arbitrary, so a.s. $Q \cap \mathbb{R} = \varnothing$. $\square$

Corollary 7.3. *Let $\kappa > 4$, $\rho_+, \rho_- \ge \kappa/2 - 2$, and $p_- < 0 < p_+$. Let $\gamma_1(t)$, $0 \le t < \infty$, be a chordal* SLE$(\kappa; \rho_+, \rho_-)$ *trace started from $(0; p_+, p_-)$. Then a.s. γ_1 has no subsequential limit on $\mathbb{R}$.*

Proof. This follows from the above lemma and Proposition 2.2. $\square$

Theorem 7.3. *Let $\kappa > 4$ and $\rho_+, \rho_- \ge \kappa/2 - 2$. Let $\gamma(t)$, $0 \le t < \infty$, be a chordal* SLE$(\kappa; \rho_+, \rho_-)$ *trace started from* $(0; 0^+, 0^-)$. *Then a.s.* $\lim_{t\to\infty} \gamma(t) = \infty$.

Proof. Let Q be the set of subsequential limits of γ. From Proposition 2.1, for any $a > 0$, $(a\gamma(t))$ has the same distribution as $(\gamma(a^2 t))$. Thus aQ has the same distribution as Q for any $a > 0$. So we suffice to show that a.s. $0 \notin Q$.

Let $\varphi(t,\cdot)$ and $\zeta(t)$ be the chordal Loewner maps and driving function for the trace γ. Then for $t > 0$, $\varphi(t, 0^-) < \zeta(t) < \varphi(t, 0^+)$. Let $p_\pm = \varphi(1, 0^\pm) - \zeta(1)$. Let $\gamma_1(t) = \varphi(1, \gamma(1+t)) - \zeta(1)$. Then conditioned on $\gamma(t)$, $0 \le t \le 1$, γ_1 is a chordal SLE$(\kappa; \rho_+, \rho_-)$ trace started from $(0; p_+, p_-)$. From the argument in the proof of Theorem 7.1, we see that if $0 \in Q$, then γ_1 has a subsequential limit on $\mathbb{R}$. From Corollary 7.3, this a.s. does not happen. Thus, a.s. $0 \notin Q$. $\square$

Theorem 7.4. *Let β be as in Lemma 7.4. Then a.s.* $\lim_{t\to\infty} \beta(t) = p_0$.

Proof. Let $\gamma(t)$, $0 \le t < \infty$, be a chordal SLE$(\kappa; \rho_+, \rho_-)$ trace started from $(0; 0^+, 0^-)$. Let $\varphi(t,\cdot)$ and $\zeta(t)$ be the chordal Loewner maps and driving function for the trace γ. Let $\gamma_1(t) = \varphi(1, \gamma(1+t)) - \zeta(1)$. Let $p_\pm = \varphi(1, 0^\pm) - \zeta(1)$. Then conditioned on $\gamma(t)$, $0 \le t \le 1$, γ_1 is a chordal SLE$(\kappa; \rho_+, \rho_-)$ trace started from $(0; p_+, p_-)$. Choose W that maps $\mathbb{H}$ conformally onto $\mathbb{S}_\pi$ such that $W(0) = 0$ and $W(p_\pm) = \pm\infty$. Let $p_* = W(\infty) \in \mathbb{R}_\pi$, and $\rho_0 = \kappa - 6 - \rho_+ - \rho_-$. From Proposition 2.2, there is a time-change function $u(t)$ such that $\beta_*(t) := W(\gamma_1(u^{-1}(t)))$, $0 \le t < \infty$, is a strip SLE$(\kappa; \rho_+, \rho_-, \rho_0)$ trace started from $(0; +\infty, -\infty, p_*)$. Let $\xi_*(t)$ and $\psi_*(t,\cdot)$, $0 \le t < \infty$, denote the driving function and strip Loewner maps for the trace β_*. Let $X_*(t) = \operatorname{Re}\psi_*(t, p_*) - \xi_*(t)$, $0 \le t < \infty$. Let T be the first time such that $X_*(t) = x_0 = \operatorname{Re} p_0$. Since $\rho_+, \rho_- \ge \kappa/2 - 2$, so $\limsup_{t\to\infty} X_*(t) = \infty$ and $\liminf_{t\to\infty} X_*(t) = -\infty$. Thus T is a finite stopping time. Let $\beta_T(t) = \psi_*(T, \beta_*(T+t)) - \xi_*(T)$, $t \ge 0$. Then β_T is a strip SLE$(\kappa; \rho_+, \rho_-)$ trace started from $(0; +\infty, -\infty, p_0)$. From Theorem 7.3, we have a.s. $\lim_{t\to\infty} \gamma(t) = \infty$, which implies that $\lim_{t\to\infty} \gamma_1(t) = \infty$, and so $\lim_{t\to\infty} \beta_*(t) = p_*$. Thus, a.s. $\lim_{t\to\infty} \beta_T(t) = \psi(T, p_*) - \xi_*(T) = X_*(T) + \pi\mathrm{i} = p_0$. Since $(\beta_T(t))$ has the same distribution as $(\beta(t))$, so a.s. $\lim_{t\to\infty} \beta(t) = p_0$. $\square$

Corollary 7.4. *Let γ_1 be as in Corollary 7.3. Then a.s.* $\lim_{t\to\infty} \gamma_1(t) = \infty$.

Theorem 7.5. *Proposition 2.4 also holds for $\kappa > 4$.*

Proof. This follows from Theorems 3.1 and 7.4. $\square$

Theorem 7.6. *Theorem 4.1 also holds for $\kappa > 4$.*

Proof. The proof of Theorem 4.1 still works here except that Theorem 7.5 should be used instead of Proposition 2.4. $\square$

Theorem 7.7. *Theorem 4.2 also holds for $\kappa > 4$.*

Proof. The proof of Theorem 4.2 still works here except that Theorems 7.2 and 7.4 should be used instead of Propositions 2.5 and 2.7. $\square$

Let γ be as in Theorem 7.7. Let $K(t)$, $0 \le t < \infty$, be the chordal Loewner hulls generated by γ. Let $K(\infty) = \bigcup_{t\ge 0} K(t)$. Let $\kappa' = 16/\kappa$, $\rho'_{\pm m} = C_{\pm m}(\kappa' - 4)$, $1 \le m \le N_\pm$, $\vec{\rho}'_\pm = (\rho'_{\pm 1}, \dots, \rho'_{\pm N_\pm})$, $C_\pm = \sum_{m=1}^{N_\pm} C_{\pm m}$, $W(z) = 1/\bar{z}$, $p'_{\pm m} = W(p_{\pm m})$, $1 \le m \le N_\pm$, and $\vec{p}'_\pm = (p'_{\pm 1}, \dots, p'_{\pm N_\pm})$. In Lemma 5.1, if we take $N_\mp + 1$ force points, one of which is x_1^+, on (x_1, x_2), and take $N_\pm + 1$ force points, one of which is x_1^-, outside $[x_1, x_2]$, then we have the following theorem.

Theorem 7.8. (i) *If* $N_+ \geq 1$, *then* $W(\partial_{\mathbb{H}}^+ K(\infty))$ *has the same distribution as a chordal* $\mathrm{SLE}(\kappa'; (1 - C_+)(\kappa' - 4), (1/2 - C_-)(\kappa' - 4), \vec{\rho}'_+, \vec{\rho}'_-)$ *trace started from* $(0; 0^+, 0^-, \vec{p}'_+, \vec{p}'_-)$. *And* $\partial_{\mathbb{H}}^+ K(\infty)$ *is a crosscut in* $\mathbb{H}$ *that connects* ∞ *with some point that lies on* $(0, p_1)$.

(ii) *If* $N_- \geq 1$, *then* $W(\partial_{\mathbb{H}}^- K(\infty))$ *has the same distribution as a chordal* $\mathrm{SLE}(\kappa'; (1/2 - C_+)(\kappa' - 4), (1 - C_-)(\kappa' - 4), \vec{\rho}'_+, \vec{\rho}'_-)$ *trace started from* $(0; 0^+, 0^-, \vec{p}'_+, \vec{p}'_-)$. *And* $\partial_{\mathbb{H}}^- K(\infty)$ *is a crosscut in* $\mathbb{H}$ *that connects* ∞ *with some point that lies on* $(p_{-1}, 0)$.

Let $\beta(t)$, $X(t)$ and $h(x)$ be defined as before Lemma 7.4. Then $(h(X(t)))$ is a local martingale. Let $I_1 = [\kappa/2 - 2, \infty)$, $I_2 = (\kappa/2 - 4, \kappa/2 - 2)$ and $I_3 = (-\infty, \kappa/2 - 4]$. Let case (jk) denote the case that $\rho_+ \in I_j$ and $\rho_- \in I_k$. We have studied case (11). In cases (12) and (13), h maps $\mathbb{R}$ onto $(-\infty, L)$ for some $L \in \mathbb{R}$, and we conclude that a.s. $\lim_{t\to\infty} X(t) = \infty$. Symmetrically, in cases (21) and (31), a.s. $\lim_{t\to\infty} X(t) = \infty$. In cases (22), (23), (32) and (33), h maps $\mathbb{R}$ onto (L_1, L_2) for some $L_1 < L_2 \in \mathbb{R}$, and we conclude that for some $p \in (0, 1)$, with probability p, $\lim_{t\to\infty} X(t) = \infty$; and with probability $1 - p$, $\lim_{t\to\infty} X(t) = -\infty$. Now we are able to prove the counterpart of Theorem 3.5 in [9] when $\kappa > 4$.

Theorem 7.9. *In case* (11), *a.s.* $\lim_{t\to\infty} \beta(t) = p_0$. *In case* (12), *a.s.* $\lim_{t\to\infty} \beta(t) \in (-\infty, p_0)$. *In case* (21), *a.s.* $\lim_{t\to\infty} \beta(t) \in (p_0, +\infty)$. *In case* (13), *a.s.* $\lim_{t\to\infty} \beta(t) = -\infty$. *In case* (31), *a.s.* $\lim_{t\to\infty} \beta(t) = +\infty$. *In case* (22), *a.s.* $\lim_{t\to\infty} \beta(t) \in (-\infty, p_0)$ *or* $\in (p_0, +\infty)$. *In case* (23), *a.s.* $\lim_{t\to\infty} \beta(t) = -\infty$ *or* $\in (p_0, +\infty)$. *In case* (32), *a.s.* $\lim_{t\to\infty} \beta(t) \in (-\infty, p_0)$ *or* $= +\infty$. *In case* (33), *a.s.* $\lim_{t\to\infty} \beta(t) = -\infty$ *or* $= +\infty$. *And in each of the last four cases*, *both events happen with some positive probability.*

Proof. This follows from the same argument as in the proof of Theorem 3.5 in [9] except that here we use Theorems 7.2, 7.4 and 7.5. □

We believe that for any chordal or strip $\mathrm{SLE}(\kappa; \vec{\rho})$ trace $\beta(t)$, $0 \leq t < T$, it is always true that a.s. $\lim_{t\to T} \beta(t)$ exists.

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