The ODE method for some self-interacting diffusions on $\mathbb{R}^d$

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Received 23 January 2008; revised 20 May 2009; accepted 11 June 2009

Abstract. The aim of this paper is to study the long-term behavior of a class of self-interacting diffusion processes on $\mathbb{R}^d$. These are solutions to SDEs with a drift term depending on the actual position of the process and its normalized occupation measure $\mu_t$. These processes have so far been studied on compact spaces by Benaïm, Ledoux and Raimond, using stochastic approximation methods. We extend these methods to $\mathbb{R}^d$, assuming a confinement potential satisfying some conditions. These hypotheses on the confinement potential are required since in general the process can be transient, and is thus very difficult to analyze. Finally, we illustrate our study with an example on $\mathbb{R}^2$.

MSC: 60K35; 37C50

Keywords: Self-interaction diffusion; Reinforced processes; Stochastic approximation

1. Introduction

This paper addresses the long-term behavior of a class of “self-interacting diffusion” processes $(X_t, t \geq 0)$ living on $\mathbb{R}^d$. These processes are time-continuous and non-Markov. They are solutions to a kind of diffusion SDEs, whose drift term depends on the whole past of the path through the occupation measure of the process. Due to their non-Markovianity, they often exhibit an interesting ergodic behavior.

1.1. Previous results on self-interacting diffusions

Time-continuous self-interacting processes, also named “reinforced processes,” have already been studied in many contexts. Under the name of “Brownian polymers”, Durrett and Rogers [10] first introduced them as a possible mathematical model for the evolution of a growing polymer. They are solutions of SDEs of the form

$$dX_t = dB_t + dt \int_0^t ds \ f(X_t - X_s),$$

where $(B_t; t \geq 0)$ is a standard Brownian motion on $\mathbb{R}^d$ and $f$ is a given function. As the process $(X_t; t \geq 0)$ evolves in an environment changing with its past trajectory, this SDE defines a self-interacting diffusion, either self-repelling or self-attracting, depending on $f$. 

Another modelisation, with dependence on the normalized occupation measure \((\mu_t, t \geq 0)\), has been considered by Benaïm et al. [5]. They introduced a process living in a compact smooth connected Riemannian manifold \(M\) without boundary:

\[
dX_t = \sum_{i=1}^N F_i(X_t) \circ dB^i_t - \int_M \nabla_x W(X_t, y) \mu_t(dy) \, dt,
\]

where \(W\) is a (smooth) interaction potential, \((B^1, \ldots, B^N)\) is a standard Brownian motion on \(\mathbb{R}^N\) and the symbol \(\circ\) stands for the Stratonovich stochastic integration. The family of smooth vector fields \((F_i)_{1 \leq i \leq N}\) comes from the Hörmander “sum of squares” decomposition of the Laplace–Beltrami operator \(\Delta = \sum_{i=1}^N F_i^2\). The normalized occupation (or empirical) measure \(\mu_t\) is defined by

\[
\mu_t := \frac{r}{r+t} \mu + \frac{1}{r+t} \int_0^t \delta_{X_s} \, ds,
\]

where \(\mu\) is an initial probability measure and \(r\) is a positive weight. In the compact-space case, they showed that the asymptotic behavior of \(\mu_t\) can be related to the analysis of some deterministic dynamical flow defined on the space of the Borel probability measures. They went further in this study in [6] and gave sufficient conditions for the a.s. convergence of the empirical measure. When the interaction is symmetric, then \(\mu_t\) converges a.s. to a local minimum of a nonlinear free energy functional (each local minimum having a positive probability to be chosen). All these results are summarized in a recent survey of Pemantle [18].

The present paper follows the same lead and extends the results of Benaïm et al. [5] in the non-compact setting. We present all results in the Euclidean space \(\mathbb{R}^d\) for the sake of simplicity, but they can be extended to the case of a complete connected Riemannian manifold \(M\) without boundary with no further difficulty than the use of notations and a bit of geometry. One needs in particular to involve the Ricci curvature in the assumptions and work on the space \(M \setminus \text{cut}(o)\), where \(\text{cut}(o)\) is the cut locus of \(o\) (which has zero Lebesgue-measure).

### 1.2. Statement of the problem

Here we set the main definitions. Consider a confinement potential \(V : \mathbb{R}^d \to \mathbb{R}_+\) and an interaction potential \(W : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+\). For any bounded Borel measure \(\mu\), we consider the “convoled” function

\[
W \ast \mu : \mathbb{R}^d \to \mathbb{R}, \quad W \ast \mu(x) := \int_{\mathbb{R}^d} W(x, y) \mu(dy).
\]

Our main object of interest is the self-interacting diffusion solution to

\[
\begin{cases}
\, dX_t = dB_t - (\nabla V(X_t) + \nabla W \ast \mu_t(X_t)) \, dt, \\
\, d\mu_t = (\delta_{X_t} - \mu_t) \frac{dt}{r+t}, \\
\, X_0 = x, \quad \mu_0 = \mu,
\end{cases}
\]

where \((B_t)\) is a \(d\)-dimensional Brownian motion. Our goal is to study the long-term behavior of \((\mu_t, t \geq 0)\). Let us recall that the main difference with the work [5] is that the state space is \(\mathbb{R}^d\) and hence is not compact anymore. However, we are able to extend the results obtained in the compact case: the behavior of \(\mu_t\) is closely related to the behavior of a deterministic flow. We will also give some sufficient conditions on the interaction potential in order to prove ergodic results for \(X\).

Before stating the theorems proved in this paper, let us briefly describe the main results obtained so far on self-interacting diffusions in non-compact spaces. They concern the model of Durrett and Rogers, and can be classified in three categories. First, when \(f\) is real, non-negative and compactly supported, Cranston and Mountford [8] have solved a (partially proved) conjecture of Durrett and Rogers and shown that \(X_t/t\) converges a.s. Second, the attracting interaction on \(\mathbb{R}\) (i.e. \(xf(x) \leq 0\) for all \(x \in \mathbb{R}\)) has been studied in the constant case by Cranston and Le Jan [7] and its generalization by Raimond [19] for the case \(f(x) = -ax/\|x\|\), or by Herrmann and Roynette [11] for a local
interaction. Under some conditions, it is proved that $X_t$ converges a.s., whereas for a non-local interaction, it does not in general (but the paths are a.s. bounded for $f(x) = -\text{sign}(x)\mathbb{I}_{|x| \geq a}$). The third category concerns a non-integrable repulsive $f$ on $\mathbb{R}$ (i.e. $xf(x) \geq 0$ for all $x \in \mathbb{R}$) studied by Mountford and Tarrès [17] and solving a conjecture of Durrett and Rogers. They have proved that for $f(x) = x/(1 + |x|^{1+\beta})$, with $0 < \beta < 1$, there exists a positive $c$ such that with probability 1/2, the symmetric process $t^{-2/(1+\beta)}X_t$ converges to $c$.

These previous works have in common that the drift may overcome the noise, so that the randomness of the process is “controlled”. To illustrate that, let us mention, for the same model of Durrett and Rogers, the case of a compactly supported repulsive function, also conjectured in [10], which is still unsolved.

Conjecture 1 ([10]). Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an odd function, of compact support. Then $X_t/t$ converges a.s. to 0.

Coming back to our process of interest, the role of the confinement potential is to similarly “control” the drift term of the diffusion. Indeed, for the process (1.4) with $V = 0$, the interaction potential is in general not strong enough for the process to be recurrent, and the behavior is then very difficult to analyze. In particular, it is hard to predict the relative importance of the drift term in the evolution.

1.2.1. Technical assumptions on the potentials

First, we denote the Euclidian scalar product by $(\cdot, \cdot)$ and by (H) the following set of hypotheses:

(i) (regularity and positivity) $V \in C^2(\mathbb{R}^d)$, $W \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$ and $V \geq 1, W \geq 0$;

(ii) (growth) there exists $C > 0$ such that for all $x, y \in \mathbb{R}^d$

$$|\nabla V(x) - \nabla V(y)| \leq C(|x - y| \vee 1)(V(x) + V(y)) ;$$

(iii) (domination) there exists $\kappa \geq 1$ such that for all $x, y \in \mathbb{R}^d$,

$$W(x, y) \leq \kappa (V(x) + V(y)) \quad \text{and} \quad |\nabla^2 W(x, y)| + |\nabla_x W(x, y)| \leq \kappa (V(x) + W(x, y)),

$$

$$\lim_{|x| \to \infty} \sup_{y \in \mathbb{R}^d} \frac{|\nabla V(x)|^2 + 2(\nabla V(x), \nabla_x W(x, y))}{V(x) + W(x, y)} = +\infty ;

(iv) (curvature) there exist $\alpha, a > 0, \delta > 1$ and $M \in \mathbb{R}$ such that for all $x, y, \xi \in \mathbb{R}^d$,

$$(x, \nabla V(x) + \nabla_x W(x, y)) \geq a|x|^\delta - \alpha \quad \text{and} \quad ((V^2 V(x) + \nabla^2_{xx} W(x, y))\xi, \xi) \geq M|\xi|^2 .$$

Remark 1.1.

(1) The most important conditions are the domination and the curvature.

(2) The growth condition (1.5) on $V$ ensures that there exists $a > 0$ such that for all $x \in \mathbb{R}^d$,

$$\Delta V(x) \leq aV(x) .$$

(3) The positivity and domination conditions on the interaction potential are not so hard to be satisfied, since the self-interacting process will be invariant by the gauge transform $W(x, y) \mapsto W(x, y) + \phi(y)$ for any function $\phi$ that does not grow faster than $V$.

1.2.2. Results

We can now describe the behavior of $\mu_t$.

Theorem 1.2. Suppose (H). For any probability measure $\mu$ on $\mathbb{R}^d$, let $\Pi(\mu) := e^{-2(V + W*\mu)/Z(\mu)}$, where $Z(\mu)$ is the normalization constant:

(1) $P_{x, t, \mu}$-a.s., the $\omega$-limit set (i.e., the accumulation points) of $(\mu_t, t \geq 0)$ is weakly compact, invariant by the flow generated by $\dot{\mu} = \Pi(\mu) - \mu$ and admits no other (sub-)attractor than itself.
Some self-interacting diffusions on $\mathbb{R}^d$

(2) If $W$ is symmetric, then $\mathbb{P}_{x,r,\mu}$-a.s., the $\omega$-limit set of $(\mu_t, t \geq 0)$ is a connected subset of the set of fixed points of the application $\mu \mapsto \Pi(\mu)$.

Even if the model studied could at a first glance seem restrictive (because of $V$), the drift term competes with the Brownian motion. The evolution is non-trivial and strongly depends on the drift.

**Theorem 1.3.** Consider the diffusion (1.4) on $\mathbb{R}^2$, with $V(x) = V(|x|)$ and $W(x,y) = (x,Ry)$, where $R$ is the rotation matrix of angle $\theta$. For $\rho \geq 0$, define the probability measure $\gamma(d\rho) := e^{-2V(\rho)}\,d\rho/Z$. Then one of the following holds:

1. If $V$ is such that $\int_0^\infty \rho^2 \gamma(d\rho) \cos \theta > -1$, then a.s. $\mu_t \rightarrow \gamma$ (weakly);
2. Else, we get two different cases:
   1. if $\theta = \pi$, then there exists a random measure $\mu_\infty$ such that a.s. $\mu_t \rightarrow \mu_\infty$ (weakly),
   2. if $\theta \neq \pi$, then $\mu_t$ circles around.

1.3. Outline of contents

As mentioned earlier, the main difficulty here stems from the non-compactness of the state space. The way to get around it is first to introduce, in Section 2, the $V$-norm (also named “dual weighted norm”), compatible with non-bounded functions, controlled by $V$. The family of measures $(\mu_t, t \geq 0)$ will then prove being (uniformly) bounded (for $t$ large enough) for the dual $V$-norm in Section 5.1. Second, the dynamical system involved induces only a local semiflow and not necessarily a global one. The last important property is the following. Consider the Feller diffusion $X^\mu$, corresponding to the SDE (1.4) where $\mu_t$ is fixed to $\mu$. Its fundamental kernel (i.e. the inverse of the infinitesimal generator) is denoted by $Q_\mu$. In order to study the ergodicity (in the limit-quotient sense) of $X$, one has to find a (uniform in $\mu$) upper bound for the operator $Q_\mu$. More precisely, we will prove the ultracontractivity of the semigroup in Section 4.1.1.

The organization of this paper is as follows. In the next section, we introduce some notations and prove the existence and uniqueness of $X$. Section 3 is devoted to the presentation of the main results and is divided in three parts. First, we recall the former results and ideas of Benaïm et al. [5]. Then, we state the tightness of $(\mu_t)_t$ and introduce the uniform estimates on the Feller semigroup. We finally end by describing the behavior of $\mu_t$. Section 4 prepares the proofs of the main results by computing some useful estimates. First, we study in details the family of Markov semigroups, corresponding to $X^\mu$, for which we prove the uniform ultracontractivity property and the regularity of the operators $A_\mu$ and $Q_\mu$. After that, we analyze, in Section 4.2, the deterministic semiflow associated to the self-interacting diffusion and show its local existence. The proofs of the main results are given in Section 5, which heavily relies on the spectral analysis of Section 4.1. We first show the tightness of $(\mu_t)_t$. Then, Section 5.2 deals with the approximation of the normalized occupation measure $(\mu_t, t \geq 0)$ by a deterministic semiflow. In Section 5.3, we prove Theorem 1.2. Finally, Section 6 is devoted to the illustration in dimension $d = 2$ and the proof of Theorem 1.3.

2. Preliminaries and tools

2.1. Some useful spaces and results

In all the following, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ will be a filtered probability space satisfying the usual conditions.

2.1.1. Spaces and topology

We begin to introduce the weighted supremum norm (or $V$-norm)

$$ \|f\|_V := \sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{V(x)}, \quad (2.1) $$

and the space of continuous $V$-bounded functions

$$ \mathcal{C}^0(\mathbb{R}^d; V) := \{ f \in \mathcal{C}^0(\mathbb{R}^d) : \|f\|_V < \infty \}. \quad (2.2) $$
Similarly let $\mathcal{C}^k(\mathbb{R}^d; V) := \mathcal{C}^k(\mathbb{R}^d) \cap \mathcal{C}^0(\mathbb{R}^d; V)$ for all $k \geq 1$.

We denote by $\mathcal{M}(\mathbb{R}^d)$ the space of signed (bounded) Borel measures on $\mathbb{R}^d$ and by $\mathcal{P}(\mathbb{R}^d)$ its subspace of probability measures. We will need the following measure space:

$$\mathcal{M}(\mathbb{R}^d; V) := \left\{ \mu \in \mathcal{M}(\mathbb{R}^d); \int_{\mathbb{R}^d} V(y) |\mu|(dy) < \infty \right\},$$

(2.3)

where $|\mu|$ is the variation of $\mu$: $|\mu| := \mu^+ + \mu^-$ with $(\mu^+, \mu^-)$ the Hahn–Jordan decomposition of $\mu$. This space will enable us to always check the integrability of $\mu$ with respect to the measures to be considered. For example, it contains the measure

$$\gamma(dx) := e^{-2V(x)} dx.$$

(2.4)

We endow $\mathcal{M}(\mathbb{R}^d; V)$ with the following dual weighted supremum norm (or dual $V$-norm) defined by

$$\|\mu\|_V := \sup_{\varphi \in \mathcal{C}^0(\mathbb{R}^d; V); \|\varphi\|_V \leq 1} \left| \int_{\mathbb{R}^d} \varphi d\mu \right|, \quad \mu \in \mathcal{M}(\mathbb{R}^d; V).$$

(2.5)

This norm naturally arises in the approach of ergodic results for time-continuous Markov processes by Meyn and Tweedie [16]. It makes $\mathcal{M}(\mathbb{R}^d; V)$ a Banach space. Since we will mainly consider probability measures in the following, we set $\mathcal{P}(\mathbb{R}^d; V) := \mathcal{M}(\mathbb{R}^d; V) \cap \mathcal{P}(\mathbb{R}^d)$. The strong topology on $\mathcal{P}(\mathbb{R}^d; V)$ is the trace topology of the one defined on $\mathcal{M}(\mathbb{R}^d; V)$. It makes $\mathcal{P}(\mathbb{R}^d; V)$ a complete metric space.

In order to study the dynamical system in Section 4.2, we need to endow the space $\mathcal{P}(\mathbb{R}^d; V)$ with two different topologies. When nothing else is stated, we will consider that it is equipped with the strong topology defined by the dual weighted supremum norm $\| \cdot \|_V$. But, as the reader will notice, we will frequently need to switch from the strong topology to the weak topology of convergence of measures. We adopt here a non-standard definition compatible with possibly unbounded functions (yet dominated by $V$). For any sequence of probability measures $(\mu_n, n \geq 1)$ and any probability measure $\mu$ (all belonging to $\mathcal{P}(\mathbb{R}^d; V)$), we define the weak convergence as

$$\mu_n \rightharpoonup \mu \quad \text{if and only if} \quad \int_{\mathbb{R}^d} \varphi d\mu_n \longrightarrow \int_{\mathbb{R}^d} \varphi d\mu \quad \forall \varphi \in \mathcal{C}^0(\mathbb{R}^d; V).$$

(2.6)

We point out that our definition of the weak convergence is stronger than the usual one. We recall that $\mathcal{P}(\mathbb{R}^d; V)$, equipped with the weak topology, is a metrizable space. Since $\mathcal{C}^0(\mathbb{R}^d; V)$ is separable, we exhibit a sequence $(f_k)_k$ dense in $\{ f \in \mathcal{C}^0(\mathbb{R}^d; V); \| f \|_V \leq 1 \}$, and set for all $\mu, \nu \in \mathcal{P}(\mathbb{R}^d; V)$:

$$d(\mu, \nu) := \sum_{k=1}^{\infty} 2^{-k} |\mu(f_k) - \nu(f_k)|.$$

(2.7)

Then the weak topology is the metric topology generated by $d$.

Finally, for any $\beta > 1$, we introduce the subspace

$$\mathcal{P}_\beta(\mathbb{R}^d; V) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d; V); \int_{\mathbb{R}^d} V(y) \mu(dy) \leq \beta \right\}.$$

(2.8)

**Proposition 2.1.** Let $\beta > 1$. The set $\mathcal{P}_\beta(\mathbb{R}^d; V)$ is a weakly compact subset of $\mathcal{P}(\mathbb{R}^d; V)$.

**Proof.** Straightforward. \qed

2.1.2. **Preliminary results**

Through this paper, we will use (many times) some easy results. First, to illustrate the need of the space $\mathcal{M}(\mathbb{R}^d; V)$, we state
Lemma 2.2. For any $\mu \in \mathcal{M}(\mathbb{R}^d; V)$, the function $W * \mu$ belongs to $C^2(\mathbb{R}^d; V)$ and
$$|W * \mu(x)| \leq 2\kappa \|\mu\|_V V(x).$$

There exists $D > 0$ such that for all $\mu \in \mathcal{M}(\mathbb{R}^d; V)$,
$$\left|\Delta (V(x) + W * \mu(x))\right| \leq D(V(x) + W * \mu(x)). \quad (2.9)$$

Proof. It results from the growth (1.5) and domination (1.6) conditions. \hfill \Box

Corollary 2.3. Let $\beta > 1$. For $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$, we get
$$Z(\mu) = \int_{\mathbb{R}^d} e^{-2W*\mu(x)} \gamma(dx) \geq e^{-2\kappa \beta} \int_{\mathbb{R}^d} e^{-2\kappa V(x)} \gamma(dx) \geq \int_{\mathbb{R}^d} e^{-4\kappa \beta V(x)} \gamma(dx). \quad (2.10)$$

The following function will also prove being very useful, as a Lyapunov function
$$E_\mu(x) := V(x) + W * \mu(x). \quad (2.11)$$

Lemma 2.4. Let $\beta > 1$. For any $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$, we have the following upper bound:
$$E_\mu(x) \leq 3\kappa \beta V(x). \quad (2.12)$$

Proof. It follows from the domination (1.6) condition. \hfill \Box

For any probability measure $\mu \in \mathcal{P}(\mathbb{R}^d; V)$, let $(X^\mu_t, t \geq 0)$ be the Feller diffusion defined by the SDE
$$dX^\mu_t = dB_t - (\nabla V(X^\mu_t) + \nabla W * \mu(X^\mu_t)) dt, \quad X^\mu_0 = x. \quad (2.13)$$

Suppose that $X^\mu$ a.s. never explodes. We denote by $(P^\mu_t; t \geq 0)$ the associated Markov semigroup. Its infinitesimal generator is then the differential operator $A_\mu$ defined on $C^\infty(\mathbb{R}^d)$ by
$$A_\mu f := \frac{1}{2} \Delta f - (\nabla V + \nabla W * \mu, \nabla f). \quad (2.14)$$

We emphasize that $X^\mu$ is a positive-recurrent (reversible) diffusion. Denote by $\Pi(\mu) \in \mathcal{P}(\mathbb{R}^d; V)$ its unique invariant probability measure:
$$\Pi(\mu)(dx) := \frac{e^{-2W*\mu(x)}}{Z(\mu)} \gamma(dx), \quad (2.15)$$

where $Z(\mu) := \int_{\mathbb{R}^d} e^{-2W*\mu(x)} \gamma(dx) < +\infty$ is the normalization constant.

Proposition 2.5. The diffusion $X^\mu_t$ a.s. never explodes.

Proof. It is enough to check with Itô’s formula and (2.9) that $\mathcal{E}_\mu$, defined in (2.11), is a Lyapunov function: $A_\mu \mathcal{E}_\mu \leq D \mathcal{E}_\mu$. As a by-product we get the naive estimate
$$\mathbb{E} \mathcal{E}_\mu(X^\mu_t) \leq \mathcal{E}_\mu(x) e^{Dt}. \quad (2.16)$$

The classical ergodic (limit-quotient) theorem is true for $X^\mu$: a.s. we have, for all $f \in C^0(\mathbb{R}^d; V)$,
$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X^\mu_s) \, dx = \Pi(\mu) f =: \int_{\mathbb{R}^d} f \, d\Pi(\mu). \quad (2.17)$$
To end this part, for any $\mu \in \mathcal{P}(\mathbb{R}^d; V)$, let $L^2(\Pi(\mu))$ be the Lebesgue space of Borel square-integrable functions with respect to the measure $\Pi(\mu)$. We remark that this space depends on $\mu$, but we will consider mainly the subspace $C^0(\mathbb{R}^d; V) \subset L^2(\Pi(\mu))$. We denote the inner product on this space by

$$(f, g)_{\mu} := \int_{\mathbb{R}^d} f(x)g(x)\Pi(\mu)(dx)$$

and $\| \cdot \|_{2, \mu}$ is the associated norm. We introduce two operators: $Q_\mu$ (sometimes called the “fundamental kernel” as in Kontoyiannis and Meyn [12]) is the solution to Poisson’s equation, that is the “inverse” of $A_\mu$, defined for any function $f \in C^\infty(\mathbb{R}^d; V)$ by

$$Q_\mu f := \int_0^\infty \left( P_t^\mu f - \Pi(\mu) f \right) dt \quad (2.18)$$

and $K_\mu$ is the orthogonal projector defined by

$$K_\mu f := f - \Pi(\mu) f. \quad (2.19)$$

These operators are linked together by the Poisson equation:

$$\forall f \in C^\infty(\mathbb{R}^d; V), A_\mu \circ Q_\mu(f) = Q_\mu \circ A_\mu(f) = -K_\mu f.$$

**Remark 2.6.** The existence of $Q_\mu f$ comes from the uniform spectral gap obtained in Corollary 3.4.

### 2.2. The self-interacting diffusion

We recall the self-interacting diffusion considered here:

$$\begin{cases}
    dX_t = dB_t - \left( \nabla V(X_t) + \nabla W^*_{\mu_t}(X_t) \right) dt, & X_0 = x, \\
    d\mu_t = (\delta_{X_t} - \mu_t) \frac{dt}{r + \kappa}, & \mu_0 = \mu.
\end{cases}$$

**Proposition 2.7.** For any $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}(\mathbb{R}^d; V)$ and $r > 0$, there exists a unique global strong solution $(X_t, \mu_t, t \geq 0)$.

**Proof.** Let us introduce the increasing sequence of stopping times, $\tau_0 = 0,$ and

$$\tau_n := \inf \left\{ t \geq \tau_{n-1}; \mathcal{E}_{\mu_t}(X_t) + \int_0^t \| \nabla \mathcal{E}_{\mu_s}(X_s) \|^2 ds > n \right\}.$$

In order to show that the solution never explodes, we use again the Lyapunov functional $(x, \mu) \mapsto \mathcal{E}_\mu(x)$ defined in (2.11). As the process $(t, x) \mapsto \mathcal{E}_{\mu_t}(x)$ is of class $C^2$ (in the space variable) and is a $C^1$-semi-martingale (in the time variable), the generalized Itô formula (or Itô–Ventzell formula, see [13]), applied to $(t, x) \mapsto \mathcal{E}_{\mu_t \wedge \tau_n}(x)$ implies

$$\mathcal{E}_{\mu_t \wedge \tau_n}(X_{t \wedge \tau_n}) = \mathcal{E}_\mu(x) + \int_0^{t \wedge \tau_n} \left( \nabla \mathcal{E}_{\mu_s}(X_s), dB_s \right) - \int_0^{t \wedge \tau_n} \| \nabla \mathcal{E}_{\mu_s}(X_s) \|^2 ds$$

$$+ \frac{1}{2} \int_0^{t \wedge \tau_n} \Delta \mathcal{E}_{\mu_s}(X_s) \, ds + \int_0^{t \wedge \tau_n} \left( W(X_s, X_s) - W \ast \mu_s(X_s) \right) \frac{ds}{r + s}. \quad (2.20)$$

We note that $\int_0^{t \wedge \tau_n} (\nabla \mathcal{E}_{\mu_s}(X_s), dB_s)$ is a true martingale. Letting $k = a + 2\kappa/r + D$, we then get, similarly to (2.9),

$$\mathbb{E}\mathcal{E}_{\mu_t \wedge \tau_n}(X_{t \wedge \tau_n}) \leq \mathcal{E}_\mu(x) + k \log(1 + t) \int_0^t \mathbb{E}\mathcal{E}_{\mu_s \wedge \tau_n}(X_{s \wedge \tau_n}) \, ds.$$
So, Gronwall’s lemma leads to the same kind of estimate as for \( X^\mu \):

\[
\mathbb{E}V(X_{t \wedge \tau_n}) \leq \mathbb{E}\mathcal{E}_{\mu_X}^{t \wedge \tau_n}(X_{t \wedge \tau_n}) \leq \mathcal{E}_\mu(x)e^{kt\log(1+t)}.
\]

As \( \lim_{|x| \to \infty} V(x) = \infty \), the process \((X_t, t \geq 0)\) does not explode in a finite time and there exists a global strong solution. \( \square \)

3. Main results

3.1. Former tools and general idea

We remind how Benaïm et al. [5] handled the asymptotic behavior of \( \mu_t \) in a compact space. Indeed, we sketch here the general idea and explain why the tools introduced in Section 2 arise quite naturally.

First, suppose that the empirical measure appearing in the drift is “frozen” to some fixed measure \( \mu \). We obtain the Feller diffusion \( X^\mu \), for which there exists a spectral gap. The associated semigroup \((P^\mu_t; t \geq 0)\) is exponentially \( V \)-uniformly ergodic:

\[
\|P^\mu_t f - \Pi(\mu)f\|_V \leq c(\mu)\|f\|_V e^{-c(\mu)t}, \quad f \in C^0(\mathbb{R}^d; V).
\]

To get, as a by-product, the almost sure convergence of the empirical measure of \( X^\mu_t \)(as defined in (2.17)), a standard technique is to consider the operator \( Q_\mu \) defined by (2.18). Then, it is enough to apply Itô’s formula to \( Q_\mu f(X^\mu_t) \) and divide both members by \( t \) to get the desired result. Indeed, one has

\[
Q_\mu f(X^\mu_t) = Q_\mu f(x) + \int_0^t (\nabla Q_\mu f(X^\mu_s), dB_s) + \int_0^t A_\mu \circ Q_\mu f(X^\mu_s) \, ds.
\]

Thanks to some easy bounds on the semigroup \((P^\mu_t)\), one proves that the martingale term is negligible compared to \( t \) and then, one recognizes the last term since \( A_\mu \circ Q_\mu f = \Pi(\mu)f - f \).

Now when \( \mu_t \) changes in time, we still can write a convenient extended form of Itô’s formula, which let appear the time derivative of \( Q_\mu f \), but we need to improve the remainder of the argument. Intuitively, as for stochastic approximation processes, one expects the trajectories of \( \mu_t \) to approximate the trajectories of a deterministic semiflow. This very last remark conveyed to Benaïm et al. [5] the idea to compare the asymptotic evolution of \((\mu_t; t \geq 0)\) with a semiflow.

Definition 3.1. A continuous function \( \xi : \mathbb{R}_+ \to \mathcal{P}(\mathbb{R}^d; V) \) is an asymptotic pseudotrajectory (or asymptotic pseudo-orbit) for the semiflow \( \Phi \) if for all \( T > 0 \),

\[
\lim_{t \to +\infty} \sup_{0 \leq s \leq T} d(\xi(t+s), \Phi_s(\xi(t))) = 0.
\]

The notion of asymptotic pseudotrajectory has been introduced by Benaïm and Hirsch [4]. It is particularly useful to analyze the long-term behavior of stochastic processes, considered as approximations of solutions of ordinary differential equation (the “ODE method”). In Section 5, we prove that the empirical measure is an asymptotic pseudotrajectory for the semiflow \( \Phi \) induced by \( \Pi(\mu) - \mu \).

3.2. New tools: Tightness and ultracontractivity

The paper of Benaïm et al. [5] crucially relies on the compactness of the manifold where the diffusion lives. It readily implies that the measure \( \mu_t \) is close to \( \Pi(\mu_t) \). On the contrary, if the state space is \( \mathbb{R}^d \) and \( V \equiv 0 \), then \( X \) will escape from any compact set. Indeed, the confinement potential \( V \) forces the process \((\mu_t, t \geq 0)\) to remain in a (weakly) compact space of measures, for \( t \) large, and \( X \) is then positive-recurrent.

Theorem 3.2. \( \mathbb{P}_{x,r,\mu} - a.s., \beta := \sup\{\int V \, d\mu_t; t \geq 0\} < +\infty. \)
The proof is postponed to Section 5 and we emphasize that $\beta$ is a random variable.

We also need some precise bounds on the family of semigroups $(P^\mu_t, t \geq 0)$ where $\mu \in \mathcal{P}(\mathbb{R}^d; V)$. A priori, it is not obvious that the semigroup $(P^\mu_t)$ admits a (uniform) spectral gap. Indeed, we will prove a stronger result: $(P^\mu_t)$ is uniformly bounded as an operator from $L^2(\Pi(\mu))$ to $L^\infty$. Section 4.1 will be devoted to those uniform properties. In the following, define $\|P^\mu_t f\|_\infty := \text{ess sup}_{x \in \mathbb{R}^d} |P^\mu_tf(x)|$.

**Proposition 3.3.** The family of semigroups $(P^\mu_t, t \geq 0, \mu \in \mathcal{P}(\mathbb{R}^d; V))$ is uniformly ultracontractive: there exists $c > 0$ independent of $\mu$ such that for all $1 \geq t > 0$ and $\mu \in \mathcal{P}(\mathbb{R}^d; V)$, we have

$$\sup_{f \in C^\infty(\mathbb{R}^d; V) \setminus \{0\}} \frac{\|P^\mu_t f\|_\infty}{\|f\|_{L^2(\Pi(\mu))}} \leq \exp\left(ct^{-b/(b-1)}\right).$$

The proof is postponed to Section 4.1.

**Corollary 3.4.** The family of measures $(\Pi(\mu), \mu \in \mathcal{P}(\mathbb{R}^d; V))$ satisfies a uniform (in $\mu$) logarithmic Sobolev inequality and admits a uniform spectral gap. So, there exists $C > 0$, independent of $\mu$, such that for all $f \in C^\infty(\mathbb{R}^d; V)$, for all $t \geq 0$:

$$\|P^\mu_t(K_\mu f)\|_{L^2(\Pi(\mu))} \leq e^{-t/C} \|K_\mu f\|_{L^2(\Pi(\mu))}.$$  

**Proof.** When a semigroup is ultracontractive, then it is hypercontractive. As being hypercontractive is equivalent to satisfy a logarithmic Sobolev inequality, we conclude (see, for instance, Bakry [1]). The given inequality is a consequence of the logarithmic Sobolev one. □

### 3.3. The $\omega$-limit set

First, let us define an $\omega$-limit set:

**Definition 3.5.** For every continuous function $\xi : \mathbb{R}_+ \to \mathcal{P}(\mathbb{R}^d; V)$, the $\omega$-limit set of $\xi$, denoted by $\omega(\xi_t, t \geq 0)$, is the set of limits of weak convergent sequences $\xi(t_k), t_k \uparrow \infty$, that is

$$\omega(\xi_t, t \geq 0) := \bigcap_{t \geq 0} \overline{\xi([t, \infty))},$$

where $\overline{\xi([t, \infty))}$ stands for the closure of $\xi([t, \infty))$ according to the weak topology.

Let $\Phi : \mathbb{R}_+ \times \mathcal{P}(\mathbb{R}^d; V) \to \mathcal{P}(\mathbb{R}^d; V)$ be the semiflow generated by

$$\Phi_t(\mu) = e^{-t} \mu + e^{-t} \int_0^t e^s \Pi(\Phi_s(\mu)) \, ds, \quad \Phi_0(\mu) = \mu.$$  

We will prove the local existence of the semiflow in Section 4.2, and for $W$ symmetric or bounded, we will show it never explodes. In other cases, we will assume the global existence of the semiflow. Section 5 is devoted to the study of $\mu_t$. Indeed, the time-changed process $\mu_{h(t)}$ (and not $\mu_t$) is an asymptotic pseudotrajectory for $\Phi$, where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$h(t) := r(e^t - 1).$$

This deterministic time-change $h$ comes from the normalization of the occupation measure $\mu_t$. The factor $(r + t)^{-1}$ disappears while considering

$$\frac{d}{dt} \mu_{h(t)} = \delta_{X_{h(t)}} - \mu_{h(t)}.$$
Theorem 3.6. Under $\mathbb{P}_{x,r,\mu}$, the function $t \mapsto \mu_{h(t)}$ is almost surely an asymptotic pseudotrajectory for the semiflow $\Phi$.

The proof is given in Section 5. This result enables us to describe the $\omega$-limit set of $(\mu_t, t \geq 0)$:

Corollary 3.7. $\mathbb{P}_{x,r,\mu}$-a.s., $\omega(\mu_t, t \geq 0)$ is weakly compact, invariant by $\Phi$ and the flow restricted to $\omega(\mu_t, t \geq 0)$ contains no attractor (other than itself). The convex hull of the image of $\Pi$ contains $\omega(\mu_t, t \geq 0)$.

In some cases, we state and prove a more precise description of $\omega(\mu_t, t \geq 0)$ in Section 5.

Theorem 3.8. Assume that $W$ is symmetric. Then, $\mathbb{P}_{x,r,\mu}$-a.s., $\omega(\mu_t, t \geq 0)$ is a connected subset of the fixed points of $\Pi$.

Corollary 3.9. Suppose that $W$ is symmetric. If $\Pi$ admits only finitely many fixed points, then $\mathbb{P}_{x,r,\mu}$-a.s., $(\mu_t; t \geq 0)$ converges to one of them.

4. Estimates on the semigroups and dynamical system

4.1. The family of semigroups

In this part, we exhibit the ultracontractivity (implying the existence of a spectral gap) for the family of semigroups $(P_t^\mu, \mu \in \mathcal{P}(\mathbb{R}^d; V))$. Since we consider these semigroups altogether for all the measures $\mu \in \mathcal{P}(\mathbb{R}^d; V)$, we will prove that the constants involved are uniform in $\mu$. The need for ultracontractivity will impose some kind of boundedness on the convolution term in the SDE that cannot be easily removed. Finally, we compute several estimates preparing Section 5.

4.1.1. Uniform ultracontractivity

The notion of ultracontractivity and its relation to the analysis of Markov semigroups were first studied by Davies and Simon [9] and recently by Röckner and Wang [20] for more general diffusions. To prove that the family of semigroups $(P_t^\mu, t \geq 0, \mu \in \mathcal{P}(\mathbb{R}^d; V))$ is uniformly ultracontractive, we will rely on the following result of Röckner and Wang:

Lemma 4.1 ([20], Corollary 2.5). Let $(P_t, t \geq 0)$ be a Markov semigroup, with infinitesimal generator $A := \frac{1}{2} \Delta - (\nabla U, \nabla)$, and $\nabla^2 U \geq -K$ with $K > 0$. Assume that there exists a continuous increasing map $\chi : \mathbb{R}_+ \mapsto \mathbb{R}_+ \setminus \{0\}$ such that:

1. $\lim_{r \to \infty} \frac{\chi(r)}{r} = \infty$,
2. the mapping $g_\chi(r) := r \chi(m \log r)$ is convex on $[1, \infty)$ for any $m > 0$,
3. $A|x|^2 \leq b - \chi(|x|^2)$ for some $b > 0$.

Then $P_t$ admits a unique invariant probability measure. If $\int_2^\infty \frac{\mathrm{d}r}{r \chi(m \log r)} < \infty, m > 0$, then $P_t$ is ultracontractive.

If, moreover, $\chi(r) = r^\delta$, with $\chi > 0, \delta > 1$, then there exists $c = c(b, \chi) > 0$ such that for all $t \in (0, 1],$

$$\sup_{f\in C_b^\infty(\mathbb{R}^d) \setminus \{0\}} \frac{\|P_t f\|_\infty}{\|f\|_2} \leq \exp \left( ct^{\frac{1}{\delta} / (\delta - 1)} \right).$$

Proof of Proposition 3.3. First, we prove that there exist $c_1, c_2$ independent of $\mu$ such that $\|P_t^\mu f(x)\| \leq e^{c_1 + c_2|x|^2}/t$ for all $t \in (0, 1)$. Let $M$ be the constant involved in the curvature condition (1.8) and denote $m_t := \frac{M}{1 - e^{-2tm_t}}$. By Wang [24], Lemma 2.1, it appears that for all $x, y \in \mathbb{R}^d$,

$$\left| P_t^\mu f(x) \right|^2 \leq P_t^\mu f^2(y) \exp \{ m_t |x - y|^2 \}.$$
As \( \Pi_t^{(\mu)} := \frac{e^{-(V+W^*\mu)}}{Z_{1}} \), where \( Z_{1} := \int_{|y| \leq 1} e^{-2(V+W^*\mu)(y)} \, dy \), is an invariant measure for the process \( X^{\mu} \), we have that \( \int_{|y| \leq 1} e^{-m_t|y|^2} \Pi_t^{(\mu)}(dy) \geq e^{-m_t}. \) So, we get
\[
\int_{\mathbb{R}^d} e^{-m_t|x-y|^2} \Pi_t^{(\mu)}(dy) \geq e^{-2m_t(|x|^2+1)}.
\]

It remains to choose \( f \in C^\infty(\mathbb{R}^d; V) \) such that \( \Pi_t^{(\mu)} f = 1 \) to conclude that
\[
|P_t^{(\mu)} f(x)|^2 e^{-2m_t(|x|^2+1)} \leq |P_t^{(\mu)} f(x)|^2 \int_{\mathbb{R}^d} e^{-m_t|x-y|^2} \Pi_t^{(\mu)}(dy) \leq 1.
\]

Now, we apply Lemma 4.1 with \( U := V + W^*\mu \) to show that each \( (P_t^{(\mu)})_{t \geq 0} \) is ultracontractive. Indeed, the curvature condition (1.8) implies that there exist \( a, b > 0 \) such that for any \( \mu \in \mathcal{P}(\mathbb{R}^d; V) \),
\[
A_\mu|x|^2 = d - 2(x, \nabla V(x) + \nabla W^*\mu(x)) \leq b - a|x|^{2\delta}.
\]

As \( \chi(r) = r^\delta \) with \( \delta > 1 \), the constant \( c \) is uniform in \( \mu \).

We are now able to derive some useful bounds on the operator \( Q_\mu \). As we need these bounds being uniform (in \( \mu \)), and depending on \( x \) only through \( V(x) \), the ultracontractivity is essential.

**Proposition 4.2.** For all \( \varepsilon > 0 \), there exists a positive constant \( K(\varepsilon) \) such that for all \( \mu \in \mathcal{P}_2(\mathbb{R}^d; V) \), \( x \in \mathbb{R}^d \), \( f \in C^0(\mathbb{R}^d; V) \):
\[
|Q_\mu f(x)| \leq (\varepsilon V(x) + K(\varepsilon))\|f\|_V.
\]

**Proof.** Let \( t_0 \in (0, 1] \) (we will choose it precisely later). We have
\[
|Q_\mu f(x)| \leq \int_0^{\infty} |P_t^{(\mu)} (K_\mu f)(x)| \, dt = \int_0^{t_0} |P_t^{(\mu)} (K_\mu f)(x)| \, dt + \int_{t_0}^{\infty} |P_t^{(\mu)} (K_\mu f)(x)| \, dt.
\]

We begin to work with the second right-hand term. Using the composition property of the semigroup, the uniform ultracontractivity and uniform spectral gap, we have
\[
\int_{t_0}^{\infty} |P_t^{(\mu)} (K_\mu f)(x)| \, dt \leq \exp\left(c t_0^{-\delta/2}\right) \int_{t_0}^{\infty} e^{-t/C} \, dt \|K_\mu f\|_{2,\mu}.
\]

As \( K_\mu \) is an orthogonal projector, \( \|K_\mu f\|_{2,\mu} \leq \|f\|_{2,\mu} \leq \left( \int V^2 \, d\Pi(\mu) \right)^{1/2} \|f\|_V \), and we get
\[
\int_{t_0}^{\infty} |P_t^{(\mu)} (K_\mu f)(x)| \, dt \leq C \|f\|_V \exp\left(c t_0^{-\delta/2}\right) \left( \int V^2 \, d\Pi(\mu) \right)^{1/2}.
\]

We now have to work with the first right-hand term. We get
\[
|P_t^{(\mu)} f(x)| \leq \|f\|_V P_t^{(\mu)} V(x) \leq \|f\|_V \mathbb{E}\mathcal{E}_{(\mu)}(X_t^{(\mu)}) \leq \mathcal{E}_{(\mu)}(x) e^{D t} \|f\|_V.
\]

By Proposition 5.1, we get that \( \int_0^t \mathbb{E}\mathcal{E}_{(\mu)}(X_t^{(\mu)}) \, ds = O(t) \) and so, we choose \( t_0 \) small enough such that \( \int_0^{t_0} \mathbb{E}\mathcal{E}_{(\mu)}(X_t^{(\mu)}) \, ds \leq \varepsilon \) to conclude.

**Proposition 4.3.** For all \( \varepsilon > 0 \), there exists \( K_1(\varepsilon) > 0 \) such that for all \( \mu \in \mathcal{P}_2(\mathbb{R}^d; V) \), \( x \in \mathbb{R}^d \), \( f \in C^\infty(\mathbb{R}^d; V) \), we have \( Q_\mu f \in C^1(\mathbb{R}^d) \) and
\[
|\nabla Q_\mu f(x)| \leq (\varepsilon V(x) + K_1(\varepsilon))\|f\|_V.
\]

\[
\square
\]
The same definition applies to functions with values in a Banach space or even in Section 4.1.2.

Regularity with respect to the measure $\mu$

Consider measures $\mu \in \mathcal{P}(\mathbb{R}^d; V)$ used only for differentiating functions defined on $\mathcal{P}(\mathbb{R}^d; V)$, which is also needed in the study of the semiflow in Section 4.2.

For any $\mu \in \mathcal{P}(\mathbb{R}^d; V)$ we consider the set $C^k(\mu)$ ($k \geq 1$) of (germs of) curves defined on some neighborhood of zero $(-\varepsilon, \varepsilon)$ with values in $\mathcal{P}(\mathbb{R}^d; V)$, passing through $\mu$ at time zero and that are of class $C^k$ when they are considered as functions with values in the Banach space $\mathcal{M}(\mathbb{R}^d; V)$. Now we say that a function $\phi : \mathcal{P}(\mathbb{R}^d; V) \to \mathbb{R}$ is of class $C^k$ if for any $\mu \in \mathcal{P}(\mathbb{R}^d; V)$ and any curve $f \in C^k(\mu)$ the real function $\phi \circ f$ is of class $C^k$. This enables to define the differential of such a function $\phi$. For any $\mu$ the tangent space at $\mu$ to $\mathcal{P}(\mathbb{R}^d; V)$ can be identified with the space $\mathcal{M}_0(\mathbb{R}^d; V)$ of zero-mass measures in $\mathcal{M}(\mathbb{R}^d; V)$, that is $\nu(\mathbb{R}^d) = 0$. The differential is then the linear operator

$$D\phi(\mu) \cdot \nu = \frac{d}{dt} \phi(\mu + t\nu) \bigg|_{t=0}, \quad \nu \in \mathcal{M}_0(\mathbb{R}^d; V).$$

The same definition applies to functions with values in a Banach space or even in $\mathcal{P}(\mathbb{R}^d; V)$. As an example, the maps $\mu \mapsto W \ast \mu(x)$ (for any point $x$) and $\Pi$ (applying Lebesgue’s theorem) are $C^\infty$.

First, consider the Banach space $B$ of bounded linear operators from $C^\infty(\mathbb{R}^d; V) \subset L^2(\gamma)$, endowed with the norm $\| f \|_{2, \mu, 1} := \| f \|_{2, \mu} + \| A_\mu f \|_{2, \mu}$, to the same space equipped with the standard quadratic norm. We endow $B$ with the operator norm. Then, $A_\mu$ obviously belongs to the closed subset of $B$ consisting in operators $A$ such that $A 1 = 0$.

Proposition 4.4. The mappings $\mu \mapsto A_\mu$ and $\mu \mapsto K_\mu$ are $C^\infty$. For any function $f \in C^\infty(\mathbb{R}^d; V)$, the application $\mu \mapsto Q_\mu f$ is $C^\infty$ for the strong topology and the differentials are (for any $\mu \in \mathcal{P}(\mathbb{R}^d; V)$, $\nu \in \mathcal{M}_0(\mathbb{R}^d; V)$):

$$D(A_\mu f) \cdot \nu = -(\nabla W \ast \nu, \nabla f), \quad D(K_\mu f) \cdot \nu = -(D\Pi(\mu) \cdot \nu)(f), \quad D(Q_\mu f) \cdot \nu = (D\Pi(\mu) \cdot \nu)(Q_\mu f) + Q_\mu(\nabla W \ast \nu, \nabla Q_\mu f).$$

Proof. Consider measures $\mu \in \mathcal{P}(\mathbb{R}^d; V)$. As $\mu \mapsto W \ast \mu$ and $\Pi$ are $C^\infty$, there is nothing to prove in case of $A_\mu$ or $K_\mu$. To look at $Q_\mu$, we need to consider the resolvent operator of $P^\mu_t$:

$$R^\mu_\lambda := \int_0^\infty e^{-\lambda t} P^\mu_t \, dt = (\lambda - A_\mu)^{-1}, \quad \forall \lambda > 0.$$  

(4.5)

For $\lambda > 0$, we define the approximation of $Q_\mu$,

$$Q_\mu(\lambda) := \int_0^\infty e^{-\lambda t} P^\mu_t K_\mu \, dt = K_\mu(\lambda - A_\mu)^{-1}.$$  

(4.6)

As $\mu \mapsto K_\mu$ and $\mu \mapsto A_\mu$ are $C^\infty$, the map $\mu \mapsto Q_\mu(\lambda) f$ is $C^\infty$ by composition.
The uniform spectral gap shows the existence of $C, C_1 > 0$ such that
\[
\| Q_\mu f - Q_\mu(\lambda)f \|_V \leq \int_0^\infty (1 - e^{-\lambda t}) \| P_t^\mu K_\mu f \|_V \, dt \leq \lambda C \| f \|_V \int_0^\infty te^{-tC_1} \, dt.
\]
Hence the convergence of $Q_\mu(\lambda)$ towards $Q_\mu$ is uniform with respect to $\mu$. As a by-product, $\mu \mapsto Q_\mu f$ is continuous.

The differential of $Q_\mu(\lambda)$ is
\[
D Q_\mu(\lambda) \cdot v = (D K_\mu \cdot v)(\lambda - A_\mu)^{-1} + K_\mu(\lambda - A_\mu)^{-1}(D A_\mu \cdot v)(\lambda - A_\mu)^{-1}.
\]
Replacing $D K_\mu$ and $D A_\mu$ by their expressions, we will prove that each right-hand side term of the equality converges uniformly. For the first one, as $(D \Pi(\mu) \cdot v)((\lambda - A_\mu)^{-1} f) = (D \Pi(\mu) \cdot v)(K_\mu(\lambda - A_\mu)^{-1} f)$, we have uniformly
\[
\lim_{\lambda \to 0} (D \Pi(\mu) \cdot v)((\lambda - A_\mu)^{-1} f) = (D \Pi(\mu) \cdot v)(Q_\mu f).
\]
To prove the convergence of the second term, remark that
\[
K_\mu(\lambda - A_\mu)^{-1} (\nabla W \ast v, \nabla)((\lambda - A_\mu)^{-1} f) = Q_\mu(\lambda)(\nabla W \ast v, \nabla Q_\mu(\lambda)f).
\]
It remains now to show that $\nabla Q_\mu(\lambda)f$ converges (uniformly in $\mu$) to $\nabla Q_\mu f$. By definition of $Q_\mu(\lambda)$, we find
\[
|\nabla Q_\mu f - \nabla Q_\mu(\lambda)f| \leq \int_0^\infty |\nabla (P_t^\mu K_\mu f)| (1 - e^{-\lambda t}) \, dt.
\]
We use the inequality (4.3) to prove that this family of differentials converges uniformly with respect to $\mu$; so $\mu \mapsto Q_\mu f$ is actually $C^1$ with the stated differential.

Remark 4.5. Looking at the differential $D(Q_\mu f)$, we see that it is itself a $C^1$ function of $\mu$, so by induction one proves that $\mu \mapsto Q_\mu f$ is $C^\infty$ and also that $\mu \mapsto P_t^\mu f$ is $C^\infty$.

Corollary 4.6. For every $f \in C^\infty(\mathbb{R}^d; V)$, we have the uniform inequality
\[
| (DQ_\mu \cdot v)(f)(x) | \leq (\varepsilon V^2(x) + K_2(\varepsilon)) \| f \|_V \| v \|_V.
\]

Proof. We easily get the inequality
\[
| (DQ_\mu \cdot v)(f)(x) | \leq | (D\Pi(\mu) \cdot v)(Q_\mu f)(x) | + | Q_\mu(\nabla W \ast v(x), \nabla Q_\mu f(x)) |.
\]

If we consider the second right-hand term, we find (using Cauchy’s inequality)
\[
| Q_\mu(\nabla W \ast v(x), \nabla Q_\mu f(x)) | \leq (\varepsilon V^2(x) + K(\varepsilon)) \| (\nabla W \ast v, \nabla Q_\mu f) \|_V \| f \|_V \| v \|_V.
\]

We work now with the other member of the inequality:
\[
| (D\Pi(\mu) \cdot v)(Q_\mu f) | \leq 2 \int | Q_\mu f(x) \| W \ast v(x) - \int W \ast v d\Pi(\mu) \Pi(\mu)(dx) \| f \|_V \| v \|_V \int V^2(x) \Pi(\mu)(dx) = C \| f \|_V \| v \|_V.
\]

4.2. The dynamical system

Define the semiflow $\Phi : \mathbb{R}_+ \times \mathcal{P}(\mathbb{R}^d; V) \to \mathcal{P}(\mathbb{R}^d; V)$ by
\[
\Phi_t(\mu) = e^{-t} \mu + e^{-t} \int_0^t e^{s} \Pi(\Phi_s(\mu)) \, ds, \quad \Phi_0(\mu) = \mu.
\]

(4.7)
4.2.1. Existence of the semiflow
We first prove the local existence of the semiflow and then give sufficient conditions on the potentials for non-explosion. To show the local existence of a solution, since \( \mathcal{P}(\mathbb{R}^d; V) \) is not a vector space, we will proceed directly by approximation. The following lemma is helpful in order to find a good security cylinder.

**Lemma 4.7.** For any \( \beta > 1 \), the application \( \Pi \) restricted to \( \mathcal{P}_\beta(\mathbb{R}^d; V) \) is bounded and Lipschitz.

**Proof.** By equation (2.10), we have the following bound for \( \Pi(\mu) \):

\[
\| \Pi(\mu) \|_V \leq \left( \int_{\mathbb{R}^d} e^{-4\kappa_\beta V(x)} \gamma(\mathrm{d}x) \right)^{-1} \int_{\mathbb{R}^d} V(x) \gamma(\mathrm{d}x) =: C_\beta. \tag{4.8}
\]

Remind, that \( \Pi \) is \( C^\infty \) on \( \mathcal{P}(\mathbb{R}^d; V) \) equipped with the strong topology. Its differential (at \( \mu \)) is the continuous linear operator \( D\Pi(\mu) : \mathcal{M}_0(\mathbb{R}^d; V) \to \mathcal{M}_0(\mathbb{R}^d; V) \) defined by

\[
D\Pi(\mu) \cdot v(\mathrm{d}x) := -2 \left( W \ast v(x) - \int_{\mathbb{R}^d} W \ast v(y) \Pi(\mu)(\mathrm{d}y) \right) \Pi(\mu)(\mathrm{d}x). \tag{4.9}
\]

Fix \( v \in \mathcal{M}_0(\mathbb{R}^d; V) \). Lemma 2.2 implies that

\[
\| D\Pi(\mu) \cdot v \|_V \leq 4\kappa (1 + C_\beta) \| v \|_V \int_{\mathbb{R}^d} V^2(x) \Pi(\mu)(\mathrm{d}x).
\]

For \( \mu \in \mathcal{P}_\beta(\mathbb{R}^d; V) \), the computation used for the bound of \( \Pi(\mu) \) enables to control the last integral, hence we get a bound (call it \( C'_\beta \)) on the differential and \( \Pi \) is Lipschitz as stated. \( \square \)

**Proposition 4.8.** For all \( \mu \in \mathcal{P}(\mathbb{R}^d; V) \), the Eq. (4.7) admits a local solution. This defines a \( C^\infty \) semiflow \( \Phi \) for the strong topology.

**Proof.** Let \( \mu \) belong to \( \mathcal{P}(\mathbb{R}^d; V) \) and choose \( \beta > 2\| \mu \|_V \) (so that \( \mu \in \mathcal{P}_\beta(\mathbb{R}^d; V) \)). We introduce the classic Picard approximation scheme

\[
\begin{cases}
\mu_t^{(0)} := \mu, \\
\mu_t^{(n)} := e^{-t} \mu + \int_0^t e^{s - t} \Pi(\mu_s^{(n-1)}) \, \mathrm{d}s.
\end{cases}
\]

We set \( \varepsilon \) small enough such that \( \| \mu \|_V (1 - e^{-\varepsilon}) C_\beta \leq \beta \) and \( \varepsilon C'_\beta < 1 \) where both constants were defined in Lemma 4.7. Then, for all \( n \), \( \mu_t^{(n)} \) is defined and belongs to \( \mathcal{P}_\beta(\mathbb{R}^d; V) \), which makes \( [0, \varepsilon) \times \mathcal{P}_\beta(\mathbb{R}^d; V) \) a good security cylinder. We have, for \( t < \varepsilon \),

\[
\left\| \mu_t^{(n+1)} - \mu_t^{(n)} \right\|_V \leq (1 - e^{-\varepsilon}) C'_\beta \sup_{t < \varepsilon} \left\| \mu_t^{(n)} - \mu_t^{(n-1)} \right\|_V.
\]

Now the series with general term sup \( t < \varepsilon \) \( \left\| \mu_t^{(n+1)} - \mu_t^{(n)} \right\|_V \) converges and thus the sequence of functions \( \mu^{(n)} \) is Cauchy for the topology of uniform convergence. Since \( \mathcal{P}(\mathbb{R}^d; V) \) is complete, we have successfully built a solution on \( [0, \varepsilon) \). As the map \( \Pi \) is \( C^\infty \) for the strong topology, every Picard approximation \( \mu \mapsto \mu_t^{(n)} \) is \( C^\infty \) by induction, and it is enough to take the limit uniformly in \( \mu \) on \( \mathcal{P}_\beta(\mathbb{R}^d; V) \) to conclude that the semiflow is smooth. \( \square \)

**Definition 4.9.** A subset \( A \) of \( \mathcal{P}(\mathbb{R}^d; V) \) is positively invariant (negatively invariant, invariant) for \( \Phi \) provided \( \Phi_t(A) \subset A \) (\( A \subset \Phi_t(A) \), \( \Phi_t(A) = A \)) for all \( t \geq 0 \).

For a symmetric \( W \), we introduce the free energy (up to a multiplicative constant) for any \( \mu \in \mathcal{P}(\mathbb{R}^d; V) \) absolutely continuous with respect to Lebesgue’s measure:

\[
\mathcal{F}(\mu) := \int_{\mathbb{R}^d} \log \left( \frac{\mathrm{d}\mu}{\mathrm{d}\gamma} \right) \, \mathrm{d}\mu + \int_{\mathbb{R}^d \times \mathbb{R}^d} W(x, y) \mu(\mathrm{d}x) \mu(\mathrm{d}y). \tag{4.10}
\]
and $\mathcal{F}(\mu) = +\infty$ if $\mu$ has a singular part with respect to Lebesgue. This functional is the sum of an entropy and an interacting energy term. Under some convexity, the competition between them implies the existence of a unique minimizer for $\mathcal{F}$ (see [23]).

**Proposition 4.10.** Whenever that $W$ is either symmetric or uniformly bounded in the second variable ($W(x, y) \leq \kappa V(x)$), the semiflow $\Phi$ does not explode.

**Proof.** Suppose that $W(x, y)$ is bounded in $y$: $W(x, y) \leq \kappa V(x)$. Mimicking the proof of Lemma 4.7, we show that $\Pi$ is globally bounded (call $C$ the upper bound). This means that $\Phi_t(\mu)$ remains in the space $\mathcal{P}_C(\mathbb{R}^d; V)$, so it cannot explode.

Let us now assume that $W$ is symmetric. We point out that the free energy $\mathcal{F}$ is not a Lyapunov function for the semiflow $\Phi$ because, in general, the measure $\Phi_t(\mu)$ is not absolutely continuous with respect to Lebesgue’s measure and so, $\mathcal{F}(\Phi_t(\mu)) = \infty$. Consider the Lyapunov function $\mathcal{I}(\mu) := \mathcal{F}(\Pi(\mu))$, which can be viewed as $\mathcal{F}$ restricted to absolutely continuous probability measures, is a $C_\infty$ function for the strong topology. We compute (thanks to the symmetry of $W$) for $v \in \mathcal{M}_0(\mathbb{R}^d; V)$

$$D\mathcal{F}(\mu) \cdot v = \int_{\mathbb{R}^d} \left[ \log \left( \frac{d\mu}{d\gamma}(x) \right) + 2W \ast \mu(x) \right] dv(x). \quad (4.11)$$

But $\Pi$ is $C_\infty$ and its differential is given by (4.9). Computing the differential of $\mathcal{I}(\mu)$ by composition, we obtain

$$D\mathcal{I}(\mu) \cdot v = -4 \int_{\mathbb{R}^d} \left( W \ast \Pi(\mu) - W \ast \mu \right) \left( W \ast v - \int_{\mathbb{R}^d} W \ast v \, d\Pi(\mu) \right) \, d\Pi(\mu).$$

We choose $v = \Pi(\mu) - \mu$ and get

$$\frac{1}{4} \frac{d}{dr} \mathcal{I}(\Phi_t(\mu)) = - \int_{\mathbb{R}^d} (W \ast v)^2 \, d\Pi(\mu) + \left( \int_{\mathbb{R}^d} W \ast v \, d\Pi(\mu) \right)^2 \leq 0.$$

So, for all $c > 0$, the sets $\{ \mu; \mathcal{I}(\mu) \leq c \}$ are positively invariant. As they are (weakly) compact, the semiflow cannot explode.

We have defined the smooth dynamical system $\Phi$, with respect to the strong topology. But, in order to study the asymptotic behavior of $(\mu_t, t \geq 0)$, it is technically easier to work with the weak topology. Therefore, we will also consider the semiflow $\Phi$ with the weak topology:

**Proposition 4.11.** $\Phi$ induces a continuous semiflow with respect to the weak topology.

**Proof.** Since $\mu \mapsto W \ast \mu(x)$ is readily weakly continuous (by the domination condition again), $\Pi$ is weakly continuous. Now, going back to the Picard approximation scheme, it results that $\mu \mapsto \mu_{t_n}$ is weakly continuous for every $n$ and $t$. Passing to the limit, we conclude.  

4.2.2. The free energy

We show how the free energy functional $\mathcal{F}$ helps to find the fixed points of $\Pi$. From now on, we restrict ourselves to the set of absolutely continuous measures.

**Lemma 4.12.** Suppose that $W$ is symmetric. Then the fixed points of $\Pi$ are the minima of $\mathcal{F}$.

**Proof.** Equation (4.11) readily implies that $D\mathcal{F}(\mu) \cdot v = 0$ for all $v \in \mathcal{M}_0(\mathbb{R}^d; V)$ if and only if $\mu = \Pi(\mu)$. So, the fixed points of $\Pi$ are the critical points of $\mathcal{F}$. Indeed, $\mathcal{F}$ is a $C_\infty$ functional, with second differential $D^2\mathcal{F}(\mu)$. Let $v_1, v_2 \in \mathcal{M}_0(\mathbb{R}^d; V)$. We have:

$$D^2\mathcal{F}(\mu) \cdot (v_1, v_2) = \int_{\mathbb{R}^d} v_1(x)v_2(x)\mu(x)^{-1}v(x) \, dx + 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(x, y)v_1(dx)v_2(dy) \geq 0.$$
Suppose that $F$ is symmetric and that for all $\mu \in \mathcal{P}(\mathbb{R}^d)$, the function $x \mapsto V(x) + W(x, y)$ is strictly convex. Then, $F$ admits a unique minimizer $\mu_\infty$ and this probability measure $\mu_\infty$ satisfies $\lim_{t \to \infty} \Phi_t(\mu) = \mu_\infty$. \hfill $\square$

**Proof.** Under strict convexity, McCann [15] has proved that $F$ admits a unique critical point $\mu_\infty$, which is a unique global minimum and also the unique fixed point of $\Pi$. \hfill $\square$

**Corollary 4.13.** Suppose that $W$ is symmetric and for all $y \in \mathbb{R}^d$, the function $x \mapsto V(x) + W(x, y)$ is strictly convex. Then, $F$ admits a unique minimizer $\mu_\infty$ and this probability measure $\mu_\infty$ satisfies $\lim_{t \to \infty} \Phi_t(\mu) = \mu_\infty$. \hfill $\square$

**Proposition 4.14.** Whenever that $W$ is either symmetric or bounded in the second variable, then the set $\{ \mu \in \mathcal{P}(\mathbb{R}^d); V; \Pi(\mu) = \mu \}$ is a nonempty compact (for the weak topology) subset of $\mathcal{P}(\mathbb{R}^d); V$.

**Proof.** Suppose first that $W$ is bounded in $y$: $W(x, y) \leq \kappa V(x)$ and let $\beta := \frac{\int_0^t V(x)(y) dx}{\int_0^t e^{-\beta V(x)}(y) dx}$. By Lemma 4.7, $\Pi$ maps (weakly) continuously the compact convex space $\mathcal{P}_\beta(\mathbb{R}^d); V)$ into itself. The Leray–Schauder fixed point theorem then ensures that the set $\{ \mu \in \mathcal{P}_\beta(\mathbb{R}^d); V); \Pi(\mu) = \mu \}$ is nonempty.

Suppose now that $W$ is symmetric. We use again the free energy $\mathcal{I} = F \circ \Pi$. Let $m := \inf\{ \mathcal{I}(\mu); \mu \in \mathcal{P}(\mathbb{R}^d); V)\}$. There exists a sequence of probability measures $(\mu_n)$ absolutely continuous with respect to Lebesgue’s measure such that $m \leq \mathcal{I}(\mu_n) \leq m + 1/n$. As for any $c > 0$, the set $\{ \mu; \mathcal{I}(\mu) \leq c \}$ is compact, we extract a subsequence $(\mu_{n_k})$ converging (weakly) to $\mu_\infty$. As $\mu \mapsto W * \mu$ and $\mu \mapsto \Pi(\mu)$ are continuous, $\mu \mapsto \mathcal{I}(\mu)$ is also weakly continuous and so $\mathcal{I}(\mu_\infty) = m$. Lemma 4.12 permits to conclude. \hfill $\square$

## 5. Behavior of the occupation measure

### 5.1. Tightness of $(\mu_t, t \geq 0)$

Thanks to the potential $V$, we manage to obtain a weak form of compactness for the empirical measure, the tightness.

**Proof of Theorem 3.2.** Set $\phi(t) := \int_0^t V(X_s) ds$. All we need to prove is that $\phi(t) = O(t)$ a.s. We use again the Lyapunov functional $\mathcal{E}_\mu(x) = V(x) + W \ast \mu(x)$ and remark Itô’s formula (2.20) for $\mathcal{E}_\mu(X_t)$. Moreover, Eq. (1.7) implies that for all $\epsilon > 0$, there exists $\eta > 0$ such that for any $|x| \geq \eta$, we have that $V(x) + W(x, y) \leq \epsilon (|V(x)|^2 + 2(V(x), \nabla_x W(x, y)))$. So, for all $\epsilon > 0$, there exists $k_\epsilon$ such that $\mathcal{E}_{\mu_t}(x) \leq k_\epsilon + \epsilon |\nabla \mathcal{E}_{\mu_t}(x)|^2$. On one hand, if $\int_0^t |\nabla \mathcal{E}_{\mu_t}(x)|^2 ds < \infty$ a.s., then the strong LLN for martingales asserts that $\int_0^\infty (\nabla \mathcal{E}_{\mu_t}(X_s), dB_s)$ converges a.s. to $M_\infty$ and the proof is then similar to the following. (Indeed the ergodic theorem implies that this case does not happen). On the other, if $\int_0^\infty |\nabla \mathcal{E}_{\mu_t}(X_s)|^2 ds = \infty$ a.s., then a.s. there exists $T(\omega)$ such that for all $t \geq T$, we have $\int_0^t |\nabla \mathcal{E}_{\mu_t}(X_s)|^2 ds \leq \frac{1}{2} \int_0^t |\nabla \mathcal{E}_{\mu_t}(X_s)|^2 ds$. So, we get the a.s. inequality for $t$ (random) large enough:

$$\int_0^t |\nabla \mathcal{E}_{\mu_t}(X_s)|^2 ds \leq 2 \mathcal{E}_\mu(x) + \int_0^t \Delta \mathcal{E}_{\mu_t}(X_s) ds + \frac{2}{r} \int_0^t W(X_s, X_s) ds.$$

The domination condition (1.6) leads to $W(X_s, X_s) \leq 2 \kappa V(X_s) \leq 2 \kappa (k_\epsilon + \epsilon |\nabla \mathcal{E}_{\mu_t}(X_s)|^2)$. Moreover, it also implies:

$$\Delta \mathcal{E}_{\mu_t} \leq D \mathcal{E}_{\mu_t} \leq D(k_\epsilon + \epsilon |\nabla \mathcal{E}_{\mu_t}(X_s)|^2).$$

So, putting all the pieces together, we get

$$(1 - (D + 4 \kappa / r) \epsilon) \int_0^t |\nabla \mathcal{E}_{\mu_t}(X_s)|^2 ds \leq 2 \mathcal{E}_\mu(x) + (D + 4 \kappa / r) k_\epsilon t.$$

It remains to choose $\epsilon = (D + 4 \kappa / r)^{-1}/2$ and then we obtain the desired inequality: for some $C > 0$,

$$\phi(t) \leq \int_0^t \mathcal{E}_{\mu_t}(X_s) ds \leq C(1 + t).$$
We finally conclude that \( \beta(\omega) := \sup\{\mu_t(V); t \geq 0\} < +\infty \) a.s. \(\square\)

**Proposition 5.1.** For all \( n \in \mathbb{N} \), we have that \( \int_0^t \mathbb{E}_{x,r,\mu}(V^n(X_s)) \, ds = O(t) \).

**Proof.** We drop the subscripts \( x, r, \mu \) in the following. We prove the result for the Lyapunov function \( \mathcal{E}_\mu(x) \) instead of \( V \). For \( n = 1 \), it suffices to adapt the previous proof to show that for all \( t > 0 \)

\[
\int_0^t \mathbb{E} |\nabla \mathcal{E}_\mu(X_s)|^2 \, ds \leq \mathcal{E}_\mu(x) + \int_0^t \mathbb{E} \Delta \mathcal{E}_\mu(X_s) \, ds + \frac{2}{r} \int_0^t \mathbb{E} W(X_s, X_s) \, ds.
\]

The result follows.

We conclude the general case \( n \geq 1 \) by induction. Indeed, we have for all \( \varepsilon > 0 \):

\[
\mathcal{E}_\mu^n(x) \leq k \mathcal{E}_\mu^{n-1}(x) + \varepsilon \mathcal{E}_\mu^{n-1}(x) |\nabla \mathcal{E}_\mu(x)|^2.
\]

Moreover, by Itô's formula, we also find for all \( s < t \) that

\[
\int_s^t \mathbb{E} \mathcal{E}_\mu^{n-1}(X_u) |\nabla \mathcal{E}_\mu(X_u)|^2 \, du
\]

\[
\leq \int_s^t \frac{2k}{r + u} \mathbb{E} \mathcal{E}_\mu^n(X_u) \, du + (n - 1) \int_s^t \mathbb{E} \mathcal{E}_\mu^{n-2}(X_u) |\nabla \mathcal{E}_\mu(X_u)|^2 \, du
\]

\[
+ k \int_s^t \mathbb{E} \mathcal{E}_\mu^n(X_u) \, du + \int_s^t \frac{\kappa}{r + u} \mathbb{E} \left( \mathcal{E}_\mu^{n-1}(X_u) \int_0^u \mathcal{E}_\mu(X_v) \, dv \right) \, du.
\]

Young’s inequality: \( x^{n-1} y \leq \frac{n-1}{n} x^n + \frac{1}{n} y^n \), with \( x = \mathcal{E}_\mu^{n-1}(X_u) \) and \( y = \mathcal{E}_\mu(X_u) \), yields to the existence of \( \alpha, A > 0 \) such that

\[
\int_s^t \mathbb{E} \mathcal{E}_\mu^{n-1}(X_u) |\nabla \mathcal{E}_\mu(X_u)|^2 \, du \leq \alpha \int_s^t \mathbb{E} \mathcal{E}_\mu^n(X_u) \, du + A \int_s^t \frac{du}{r + u} \int_0^u \mathbb{E} \mathcal{E}_\mu^n(X_v) \, dv.
\]

We thus obtain:

\[
\int_s^t \mathbb{E} \mathcal{E}_\mu^n(X_u) \, du
\]

\[
\leq k \int_s^t \mathbb{E} \mathcal{E}_\mu^{n-1}(X_u) \, du + \varepsilon \left( \alpha \int_s^t \mathbb{E} \mathcal{E}_\mu^n(X_u) \, du + A \int_s^t \frac{du}{r + u} \int_0^u \mathbb{E} \mathcal{E}_\mu^n(X_v) \, dv \right)
\]

\[
\leq k(t - s) + \int_s^t \frac{du}{r + u} \int_0^u \mathbb{E} \mathcal{E}_\mu^n(X_v) \, dv.
\]

Let \( x(t) := \int_0^t \mathbb{E} \mathcal{E}_\mu^n(X_s) \, ds \). Solving the preceding inequality boil down to solve \( \dot{x} \leq M + x/(r + t) \). The solution satisfies \( x(t) = O(t) \) and we finally conclude. \(\square\)

**Corollary 5.2.** For all \( n \in \mathbb{N} \), we have that \( \mathbb{E}_{x,r,\mu}(V^n(X_t)) = O(t) \).

5.2. Asymptotic behavior

Define the family of measures \( \{\varepsilon_{t,t+s}; t \geq 0, s \geq 0\} \) by

\[
\varepsilon_{t,t+s} := \int_t^{t+s} (\delta_{X_{h(u)}} - \Pi(\mu_{h(u)})) \, du.
\]

(5.1)

This family will be essential for proving that \( t \mapsto \mu_{h(t)} \) is an asymptotic pseudotrajectory for \( \Phi \).
Proposition 5.3. For all $T > 0$ and all $f \in C^\infty(\mathbb{R}^d; V)$, we have $\mathbb{P}_{x,\mu} - a.s.$,

$$\lim_{t \to \infty} \sup_{0 \leq s \leq T} |\varepsilon_{t,t+s} f| = 0.$$

Proof. First, we need the uniform estimates on the family of semigroups $(P_t^\mu)$. Let $f \in C^\infty(\mathbb{R}^d; V)$. We begin to rewrite

$$\varepsilon_{t,t+s} f = \int_{h(t)}^{h(t+s)} A_{\mu_u} \circ Q_{\mu_u} f \, du.$$

We consider the $C^2$-valued process $(t, x) \mapsto Q_{\mu_{h(t)}} f(x)$, which is of class $C^2$ and a $C^1$-semimartingale. Indeed, it is easy to see that $t \mapsto \mu_{h(t)}$ is a.s. a bounded variation process with values in $\mathcal{M}(\mathbb{R}^d; V)$ (since Proposition 4.3 shows that $\mu \mapsto Q_{\mu} f$ is also $C^1$, the claim follows by composition). So, we apply the generalized Itô formula to $(t, x) \mapsto h(t)^{-1} Q_{\mu_{h(t)}} f(x)$ and decompose $\varepsilon_{t,t+s}$ in four parts:

$$\varepsilon_{t,t+s} f = \varepsilon_{t,t+s}^{(1)} f + \varepsilon_{t,t+s}^{(2)} f + \varepsilon_{t,t+s}^{(3)} f + \varepsilon_{t,t+s}^{(4)} f$$

with

$$\varepsilon_{t,t+s}^{(1)} f = - \frac{1}{h(t+s)} Q_{\mu_{h(t+s)}} f(X_{h(t+s)}) + \frac{1}{h(t)} Q_{\mu_{h(t)}} f(X_{h(t)}),$$

$$\varepsilon_{t,t+s}^{(2)} f = - \int_{h(t)}^{h(t+s)} Q_{\mu_u} f(X_u) \, du \frac{du}{(r+u)^2},$$

$$\varepsilon_{t,t+s}^{(3)} f = \int_{h(t)}^{h(t+s)} \frac{\partial}{\partial u} Q_{\mu_u} f(X_u) \, du \frac{du}{r+u},$$

$$\varepsilon_{t,t+s}^{(4)} f = M_{h(t+s)}^f - M_{h(t)}^f,$$

where $M_{h(t)}^f$ is the local martingale $M_{h(t)}^f := \int_0^{h(t)} \nabla Q_{\mu_u} f(X_u) \, dB_u$.

Before controlling each term separately, we remind the estimates of Propositions 4.2 and 4.3: for all $f \in C^\infty(\mathbb{R}^d; V)$,

$$|Q_{\mu_{h(t)}} f(X_{h(t)})| \leq \|f\|_V (\varepsilon V(X_{h(t)}) + K(\varepsilon)),$$

$$|\nabla Q_{\mu_{h(t)}} f(X_{h(t)})| \leq \|f\|_V (\varepsilon V(X_{h(t)}) + K_1(\varepsilon)).$$

We also remark that $\int_0^t V(X_s) \, ds = O(t)$ a.s. and $\int_0^t \mathbb{E} V(X_s) \, ds = O(t)$. Now, we are able to find for all $\varepsilon > 0$:

$$|\varepsilon_{t,t+s}^{(1)} f| \leq h(t) \leq (\varepsilon V(X_{h(t)}) + \varepsilon V(X_{h(t)})) + K(\varepsilon),$$

and so $\lim_{t \to \infty} \sup_{0 \leq s \leq T} |\varepsilon_{t,t+s}^{(1)} f| \leq \varepsilon \|f\|_V$ a.s. As the latter is true for all $\varepsilon > 0$, we deduce that a.s.

$$\lim_{t \to \infty} \sup_{0 \leq s \leq T} |\varepsilon_{t,t+s}^{(1)} f|$$

vanishes. Similarly, with $\varepsilon = 1$, there exists $C_2$ such that

$$|\varepsilon_{t,t+s}^{(2)} f| \leq \int_{h(t)}^{h(t+s)} (V(X_u) + K) \, du \frac{du}{(r+u)^2} \|f\|_V \leq \frac{C_2 \|f\|_V}{h(t)^2} \int_{h(t)}^{h(t+s)} V(X_u) \, du,$$

and so $\sup_{0 \leq s \leq T} |\varepsilon_{t,t+s}^{(2)} f| \leq C_2 h(t)^{-1} \|f\|_V$ a.s.
By Markov’s inequality and using the bound on the differential of $Q_u$ given in Corollary 4.6, we get:

$$\mathbb{P}\left(\sup_{0 \leq s \leq T} |\epsilon_{t,t+s}^{(3)} f| \geq \delta\right) \leq \delta^{-2} \int_{h(t)}^{h(t+T)} \mathbb{E}\left|DQ_{\mu_u} \cdot \dot{\mu_u}(f)(X_u)\right|^2 \frac{du}{r+u} \leq \frac{C}{\delta^2} \|f\|^2 \int_{h(t)}^{h(t+T)} \mathbb{E}(V^6(X_u)) \frac{du}{(r+u)^3}.$$  

As for all $\varepsilon > 0$ and $n \in \mathbb{N}$ we have $\int_0^T \mathbb{E}^N(X_s) \, ds = O(t)$, there exists $C_3 > 0$ such that

$$\mathbb{P}\left(\sup_{0 \leq s \leq T} |\epsilon_{t,t+s}^{(3)} f| \geq \delta\right) \leq \frac{C_3}{\delta^2} h(t)^{-1} \|f\|^2_V.$$  

Since the quadratic variation of $M_{h(t+s)}^f - M_{h(t)}^f$ is bounded by $\|f\|^2 \int_{h(t)}^{h(t+T)} (\varepsilon V(X_u) + K_1(\varepsilon))^2 \frac{du}{(r+u)^2}$, Burkholder–Davis–Gundy’s inequality (BDG) implies

$$\mathbb{P}_{x,r,\mu}\left(\sup_{s \in [0,T]} |\epsilon_{t,t+s}^{(4)} f| \geq \delta\right) \leq \frac{C_4}{\delta^2} h(t)^{-1} \|f\|^2_V. \quad (5.2)$$  

It only remains to prove that a.s.

$$\lim_{t \to \infty} \sup_{0 \leq s \leq T} |\epsilon_{t,t+s}^{(4)} f| = \lim_{t \to \infty} \sup_{0 \leq s \leq T} |\epsilon_{t,t+s}^{(3)} f| = 0.$$  

First, for all $\varepsilon > 0$, we have by Doob’s inequality together with BDG’s inequality that

$$\mathbb{P}_{x,r,\mu}\left(\sup_{n \leq t < n+1} \sup_{s \in [0,T]} |\epsilon_{t,t+s}^{(4)} f| \geq \delta\right) \leq \frac{C}{\delta^2} \|f\|^2_V \sup_{n \leq t < n+1} h(t)^{-1} = \frac{C}{\delta^2} \|f\|^2_V h(n)^{-1}.$$  

Since the series $\sum_n h(n)^{-1}$ converges, we conclude by Borel–Cantelli’s lemma that a.s.

$$\lim_{n \to \infty} \sup_{n \leq t < n+1} \sup_{0 \leq s \leq T} |\epsilon_{t,t+s}^{(4)} f| = 0.$$  

The same argument for $|\epsilon_{t,t+s}^{(3)} f|$ permits to conclude.  

\[\square\]

**Lemma 5.4.** If for all $T > 0$, all $f \in C^\infty(\mathbb{R}^d; V)$, we have

$$\lim_{t \to \infty} \sup_{0 \leq s \leq T} |\epsilon_{t,t+s} f| = 0 \quad \text{a.s.,}$$

then the time-changed process, given by $\mathbb{R}_+ \to \mathcal{P}(\mathbb{R}^d; V), t \mapsto \mu_{h(t)}$ is a.s. an asymptotic pseudotrajectory for $\Phi$ (for the weak topology of measures).

**Proof.** The family $(\mu_t, t \geq 0)$ is a.s. tight and by Prokhorov’s theorem (because $\mathcal{P}(\mathbb{R}^d; V)$ is a Polish space), it is equivalent to the relative compactness of $(\mu_t, t \geq 0)$. Benaïm [3], Theorem 3.2, asserts that a continuous map $v : \mathbb{R}_+ \to \mathcal{P}(\mathbb{R}^d; V)$ is an asymptotic pseudotrajectory for the semiflow $\Phi$ if and only if $v$ is (weakly) uniformly continuous and every limit point of $\{v(t + \cdot); t \geq 0\}$ is an orbit of $\Phi$. We first show that $\mu_{h(t)}$ is uniformly continuous. By definition of $\mu_t$, we have

$$|\mu_{h(t+s)} f - \mu_{h(t)} f| \leq \int_{t}^{t+s} \left( |\mu_{h(u)} f| + |f(X_{h(u)})| \right) \, du.$$  

As asymptotically $\mu_t \in \mathcal{P}_b(\mathbb{R}^d; V)$ a.s., we get for all $t$ large enough

$$|\mu_{h(t+s)} f - \mu_{h(t)} f| \leq 2 \beta s \|f\|_V. \quad (5.3)$$
We put these estimates in (2.7) and the uniform continuity follows. As a.s. \( \mu_{h(t)} \) belongs to a compact set (for \( t \) large enough), Ascoli’s theorem implies that there exist an increasing sequence \( (t_n)_n \) and \( \tilde{\mu} \in \mathcal{P}(\mathbb{R}^d; V) \) such that \( (\mu_{h(t_n+s)})_s \geq 0 \) converges weakly to \( (\tilde{\mu}_s)_s \geq 0 \). Then, we have \( \mu_{h(t_n+s)} = \mu_{h(t_n)} + \varepsilon_{t_n, t_n+s} + \int_{t_n}^{t_n+s} (\Pi(\mu_{h(u)}) - \mu_{h(u)}) \, du \). As \( \mu_{h(t_n+s)} \) converges weakly to \( \tilde{\mu} \) and \( \varepsilon_{t_n, t_n+s} \) goes to 0, the limit \( \tilde{\mu} \) satisfies (4.7).

Suppose that \( \mu_{h(t)} \) is not an asymptotic pseudotrajectory for \( \Phi \). It means that:

\[
\exists \tau > 0, \exists \varepsilon_0 > 0, \exists n \uparrow \infty, \exists s_n \in [0, T] \text{ such that } d(\mu_{h(t_n+s_n)}, \Phi_{s_n}(\mu_{h(t_n)})) \geq \varepsilon_0.
\]

It implies, denoting by \( s \) the limit of \( s_n \) and \( \tilde{\mu} \) the limit of \( \mu_{h(t_n)} \), that \( d(\tilde{\mu}_s, \Phi_s(\tilde{\mu})) \geq \varepsilon_0 \), which contradicts that \( \tilde{\mu} \) is an orbit of \( \Phi \).

**Remark 5.5.** Combine Proposition 5.3 with Lemma 5.4 to deduce Theorem 3.6: \( \mathbb{P}_{x,v} \)-a.s., the function \( t \mapsto \mu_{h(t)} \) is an asymptotic pseudotrajectory for \( \Phi \).

### 5.3. Back to the dynamical system: A global attractor for the semiflow

As explained in Section 4.2, we will consider from now on the semiflow \( \Phi \) with the weak topology. A good candidate to be an attractor of the semiflow is the \( \omega \)-limit set of \( (\mu_t) \),

\[
\omega(\mu_t, t \geq 0) := \bigcap_{t \geq 0} [\mu_t; s \geq t]
\]

(5.4)

which is (a.s.) weakly compact, since it is contained in \( \mathcal{P}_\beta(\mathbb{R}^d; V) \) a.s.

We introduce here a crucial set to analyse the dynamical system \( \Phi \). Let

\[
\text{Im}(\Pi) := \{ \Pi(\mu); \mu \in \mathcal{P}(\mathbb{R}^d; V) \},
\]

(5.5)

and denote its convex hull by \( \text{Im}(\Pi) \).

**Proposition 5.6.** \( \text{Im}(\Pi) \) is a positively invariant set for the semiflow \( \Phi \) and contains every negatively invariant bounded subset of \( \mathcal{P}(\mathbb{R}^d; V) \).

**Proof.** By Jensen’s inequality applied to the convex combination \( \Phi_t(\mu) = e^{-t} \mu + e^{-t} \int_0^t e^s \Pi(\Phi_s(\mu)) \, ds \) and to the convex map \( \mu \mapsto d_V(\mu, \text{Im}(\Pi)) \), we show, for every \( \mu \in \mathcal{P}(\mathbb{R}^d; V) \) and every \( t \geq 0 \), that

\[
d_V(\Phi_t(\mu), \text{Im}(\Pi)) \leq e^{-t} d_V(\mu, \text{Im}(\Pi)),
\]

(5.6)

where \( d_V(\mu, X) := \inf\{\|\mu - v\|_V; v \in X\} \). So, for any negatively invariant bounded subset \( A \) of \( \mathcal{P}(\mathbb{R}^d; V) \), we get for all \( t \geq 0 \): \( d_V(A, \text{Im}(\Pi)) \leq d_V(\Phi_t(A), \text{Im}(\Pi)) \leq e^{-t} d_V(A, \text{Im}(\Pi)) \).

Now, we need to recall a short list of important definitions coming from the theory of Dynamical Systems.

**Definition 5.7.** (a) A subset \( A \) of \( \mathcal{P}(\mathbb{R}^d; V) \) is an attracting set (respectively attractor) for \( \Phi \) provided:

1. \( A \) is nonempty, weakly compact and positively invariant (respectively invariant) and
2. \( A \) has a neighborhood \( \mathcal{N} \subset \mathcal{P}(\mathbb{R}^d; V) \) such that \( d(\Phi_t(\mu), A) \rightarrow 0 \) as \( t \rightarrow +\infty \) uniformly in \( \mu \in \mathcal{N} \).

(b) The basin of attraction of an attractor \( K \subset A \) for \( \Phi|A = (\Phi_t|A) \) is the positively invariant open set (in \( A \)) comprising all points whose orbits are asymptotically in \( K \):

\[
B(K, \Phi|A) := \left\{ \mu \in A; \lim_{t \rightarrow \infty} d(\Phi_t(\mu), K) = 0 \right\}.
\]

(c) A global attracting set (respectively global attractor) is an attracting set (respectively attractor) whose basin is the whole space \( \mathcal{P}(\mathbb{R}^d; V) \).
An attractor-free set is a nonempty compact invariant set \( A \) such that \( \Phi|A \) has no attractor except \( A \) itself.

Our aim is now to describe the limit set of \( \mu_t \) and find a global attracting set for \( \Phi \). The natural candidate is the set \( \omega(\mu_t, t \geq 0) \). First, we describe it dynamically.

**Theorem 5.8.** The \( \omega \)-limit set of \( \{\mu_t, t \geq 0\} \) is \( \mathbb{P}_{x,r,\mu} \)-almost surely an attractor-free set of \( \Phi \).

**Proof.** It results from Theorem 3.6 and [4]. □

**Corollary 5.9.** \( \mathbb{P}_{x,r,\mu}(\lim_{t \to +\infty} |X_t| = +\infty) = 1 \).

**Proof.** Let \( A \) be a open subset of \( \mathbb{R}^d \) such that \( \gamma(A) > 0 \). Since the measure \( \gamma \) is diffusive, for all \( v \in \text{Im}(1) \cap \omega(\mu_t, t \geq 0) \), there exist \( m, M > 0 \) such that \( m \gamma \leq v \leq M \gamma \). If we consider a sequence \( (v_{t_n}, n \geq 0) \) in \( \mathcal{P}(\mathcal{P}(\mathbb{R}^d; V)) \), the limits of its convergent subsequences will belong to \( \text{Im}(1) \cap \omega(\mu_t, t \geq 0) \), because \( \omega(\mu_t, t \geq 0) \) is a.s. an attractor-free set of \( \Phi \). Thus, there exists a subsequence \( (v_{t_{n_k}}) \) such that a.s. \( v_{t_{n_k}} \) converges (weakly) to \( v \); for any smooth function \( \varphi \) of compact support, we have that \( v_{t_{n_k}}(\varphi) \) converges to \( v(\varphi) \). If we consider a function \( \varphi \) such that \( \varphi(x) = 1 \) for \( x \in A \) and \( \varphi(x) = 0 \) for \( x \notin B \), \( A \subset B \), we find that \( v(\varphi) \geq v(A) > 0 \). Thus

\[
v(B) \geq \limsup v_t(\varphi) \geq \liminf v_t(\varphi) \geq v(A) \geq m \gamma(A).
\]

So, \( \int_0^t \gamma(X_s(A)) ds = \infty \) a.s. Then, for all \( K > 0 \), \( \int_0^t \gamma(X_s(\mathbb{R}^d \setminus B_K)) ds = \infty \) a.s., where \( B_K \) is the closed ball of radius \( K \). Finally,

\[
\mathbb{P}_{x,r,\mu}(\bigcap_{n \geq 0} \left\{ \int_0^t \mathbb{1}_{|X_s| \geq K} ds = \infty \right\}) = 1.
\]

Second, we consider the (nonempty) set \( \text{Im}(1) \cap \omega(\mu_t, t \geq 0) \).

**Theorem 5.10.** The set \( \text{Im}(1) \cap \omega(\mu_t, t \geq 0) \) is a.s. a global attracting set for \( \Phi \).

**Proof.** We begin to notice that \( \text{Im}(1) \cap \omega(\mu_t, t \geq 0) \) is weakly compact a.s. and by definition, it is also positively invariant. Let \( \mu \in \omega(\mu_t, t \geq 0) \). Since \( \omega(\mu_t, t \geq 0) \) is an attractor-free set for \( \Phi \), for all \( s \geq 0 \), we have \( \Phi_s(\mu) \in \omega(\mu_t, t \geq 0) \). By Proposition 5.6, we know that \( \lim_{t \to +\infty} d(\Phi_s(\mu), \text{Im}(1)) = 0 \) (uniformly in \( \mu \)). So, \( d(\Phi_s(\mu), \omega(\mu_t, t \geq 0) \cap \text{Im}(1)) \) converges to 0 uniformly in \( \mu \). Using again Jensen’s inequality, we show that the basin of attraction of \( \Phi \) is the whole space. □

**Corollary 5.11.** \( \omega(\mu_t, t \geq 0) \) is a.s. a subset of \( \text{Im}(1) \).

**Proof.** As \( \omega(\mu_t, t \geq 0) \) is attractor-free, Theorem 5.10 implies that \( \omega(\mu_t, t \geq 0) \) is the only attractor of \( \Phi \) restricted to this set. So, \( \text{Im}(1) \cap \omega(\mu_t, t \geq 0) = \omega(\mu_t, t \geq 0) \).

When \( W \) is symmetric, we can give a better description of \( \omega(\mu_t, t \geq 0) \). Let begin with the following:

**Theorem 5.12 (Tromba [22]).** Let \( B \) be a \( C^\infty \) Banach manifold, \( F \) a \( C^\infty \) vector field on \( B \) and \( \mathcal{I}: B \to \mathbb{R} \) a \( C^\infty \) function. Assume that:

1. \( D\mathcal{I}(\mu) = 0 \) if and only if \( F(\mu) = 0 \);
2. \( F^{-1}(0) \) is compact;
3. for each \( \mu \in F^{-1}(0) \), \( D\mathcal{I}(\mu) \) is a Fredholm operator.

Then \( \mathcal{I}(F^{-1}(0)) \) has an empty interior.
Proposition 5.13 ([3], Proposition 6.4). Let $\Lambda$ be a compact invariant set for a semiflow $\Phi$ on a metric space $E$. Assume that there exists a continuous function $V: E \to \mathbb{R}$ such that:

1. $V(\Phi_t(x)) < V(x)$ for $x \in E \setminus \Lambda$ and $t > 0$;
2. $V(\Phi_t(x)) = V(x)$ for $x \in \Lambda$ and $t > 0$.

If $V$ has an empty interior, then every attractor-free set $A$ for $\Phi$ is contained in $\Lambda$. Furthermore, $V$ restricted to $A$ is constant.

Proof of Theorem 3.8. The fixed points of $\Pi$ form a nonempty compact subset of $\mathcal{P}(\mathbb{R}^d; V)$ thanks to Proposition 4.14. Let $F(\mu) := \Pi(\mu) - \mu$. We already know that $F^{-1}(0)$ is compact for the weak topology. If we show that $\mathcal{I}(F^{-1}(0))$ has an empty interior, then the result is a consequence of Proposition 5.13 with the Lyapunov function $\bar{\Pi}(\mu)$. So, the set $\{DF(\mu) \cdot v; \|v\|_V \leq 1\}$ is a.s. bounded. For $x, y \in \mathbb{R}^d$, we get

\[
|DF(\mu) \cdot v(x) - DF(\mu) \cdot v(y)| \leq 2|W \ast v(x)\Pi(\mu)(x) - W \ast v(y)\Pi(\mu)(y)|
\]

\[
+ 2 \int W \ast v d\Pi(\nu)(\Pi(\mu)(x) - \Pi(\mu)(y))
\]

\[
\leq M \left[ |V(x) - V(y)| + \|\mu(x) - \mu(y)\| + \|W(y, \cdot) - W(x, \cdot)\|_V \right].
\]

So, the map $DF(\mu) \cdot v(\|v\|_V \leq 1)$ is equicontinuous and by Ascoli’s theorem, we conclude that the preceding set is relatively compact in $C^0(\mathbb{R}^d; V)$ and thus the operator $DF(\mu)$ is compact. Moreover, it is self-adjoint. It follows from the spectral theory of compact self-adjoint operators that $DF$ has at most countably many real eigenvalues and the set of nonzero eigenvalues is either finite or can be ordered as $|\lambda_1| > |\lambda_2| > \cdots > 0$ with $\lim_{n \to \infty} \lambda_n = 0$. So, by Tromba, $\mathcal{I}(F^{-1}(0))$ has an empty interior.

\[\square\]

6. Illustration in dimension $d = 2$

When $W$ is not symmetric, it can happen that no Lyapunov function exists and that the $\omega$-limit set is a non-trivial orbit. Suppose for instance that $(d = 2)$ $W(x, y) = (x, Ry)$ where $R$ is a rotation matrix and $V(x) = V(|x|) \geq a|x|^4 + b|x|^2 + 1$ (with $a, b \geq 1$). Note, that the measure $\gamma(dx) = e^{-2V(x)} dx / Z$ is invariant by rotation. Then, one expects, depending on $R$ and $V$, that either the unique invariant set for the semiflow is $\{\gamma\}$ and so a.s. $\mu_t$ converges to $\gamma$; or a.s. $\mu_t$ converges to a random measure, related to the critical points of the free energy; or $\omega(\mu_t, t \geq 0)$ is a periodic orbit related to $\gamma$. Remark that, equivalently considering $W(x, y) + \frac{1}{2}(b|x|^2 + |y|^2/b)$ or $W$, the set of conditions (H) is satisfied. Denote $p := (\frac{1}{2})$.

Lemma 6.1 ([5], Lemma 4.6). For all continuous $\varphi: \mathbb{R} \to \mathbb{R}$, for all $y \in \mathbb{S}^1$ we have

\[
\int_{\mathbb{R}^2} [\varphi((x, y)) - \varphi((x, p))] \gamma(dx) = \int_{\mathbb{R}^2} \varphi((x, y))(x - (x, y)y) \gamma(dx) = 0.
\]

Proof. For all $y \in \mathbb{S}^1$, there exists $g \in O(2)$ such that $y = gp$. We show the first equality by a change of variable in the integral (because $V(x) = V(|x|)$). Define $\phi(y) := \int_{\mathbb{R}^2} \varphi((x, y))(x - (x, y)y) \gamma(dx)$. We have $\phi(y, y) = 0$ and the rotation-invariance of $\gamma$ implies for the antisymmetry matrix $j$, that $\phi(p) = j\phi(p)$. So, $\phi(p) = 0$ and thus $\phi(y) = 0$. \[\square\]

For any $\mu \in \mathcal{P}(\mathbb{R}^2; V)$, define its mean by $\bar{\mu} := \int_{\mathbb{R}^2} x \mu(dx)$. Let the probability measure

\[
\tilde{\Pi}(\bar{\mu})(dx) := \frac{e^{-2(x, R\bar{\mu})}}{Z(\bar{\mu})} \gamma(dx).
\] (6.1)
Here, $\Pi(\bar{\mu}) = \Pi(\mu)$. If we let $\Bar{\Pi}(\mu) := \int_{\mathbb{R}^2} x \Pi(\mu)(dx)$, then $\Phi_t(\mu)$ is readily the semiflow
\[
\Phi_t(\mu) = e^{-t} \bar{\mu} + e^{-t} \int_0^t e^{s} \overline{\Pi}(\Phi_s(\mu)) \, ds, \quad \Phi_0(\mu) = \bar{\mu}.
\] (6.2)

**Lemma 6.2.** Let $m = \rho v$ with $\rho \geq 0$ and $v \in \mathbb{S}^1$. Then, we get
\[
\Pi(m) = \int_{\mathbb{R}^2} x \Pi(m)(dx) = -\frac{1}{2} \frac{d}{d\rho} \log \left( \int_{\mathbb{R}^2} e^{-2\rho(x,v)} \gamma(dx) \right) R_v.
\]

**Proof.** It follows from differentiating the function $\alpha \mapsto \log(\int_{\mathbb{R}^2} e^{-2\alpha(x,v)} \gamma(dx))$ and Lemma 6.1. \qed

Let $m = \rho v$ be the solution to $\dot{m} = \Pi(m) - m$, with $\rho = |m|$ and $v \in \mathbb{S}^1$. Then, Lemma 6.2 implies that $\dot{v} = 0$. If we let $\alpha = 2\rho$, then $\alpha$ satisfies the one-dimensional ODE
\[
\dot{\alpha} = J(\alpha) = -\alpha + 2 \bar{\alpha}_d \log \left( \int_{\mathbb{R}^2} e^{-\alpha(x,R_p)} \gamma(dx) \right).
\] (6.3)

The problem expressed in polar coordinates becomes $J(\alpha) = -\alpha (1 - 2 \frac{\tilde{\alpha}(\alpha)}{H(\alpha)})$, where
\[
H(\alpha) := \int_0^\infty d\rho \gamma(\rho) \int_0^{2\pi} dv e^{-\alpha \rho \cos v},
\]
\[
\tilde{H}(\alpha) := \int_0^\infty d\rho \gamma(\rho) \rho^2 \int_0^{2\pi} dv \sin^2 v e^{-\alpha \rho \cos v}.
\]

**Remark 6.3.** The function $t \mapsto \int_0^{2\pi} e^{-t \cos v} \, dv$ is the Bessel function $I_0(t)$.

6.1. The case $R = -Id$

Here, $W$ is a symmetric function.

**Proposition 6.4.** If $\int_0^\infty \rho^2 \gamma(\rho) \, d\rho \leq 1$, then $0$ is the unique equilibrium of (6.3) and $0$ is stable. Its basin of attraction is $\mathbb{R}_+^+$. If $\int_0^\infty \rho^2 \gamma(\rho) \, d\rho > 1$, then $0$ is linearly unstable and there is another stable equilibrium $\alpha_1$, whose basin of attraction is $\mathbb{R}_+^+$.

**Proof.** Remark, that $J$ is $C^\infty$. A computation yields to
\[
J^{(3)}(\alpha) = 2 \frac{H^{(4)}(\alpha)}{H(\alpha)} - 8 \frac{H^{(3)}(\alpha) H'(\alpha)}{H^2(\alpha)} + 24 \frac{H''(\alpha)}{H(\alpha)} \left( \frac{H'(\alpha)}{H(\alpha)} \right)^2 - 12 \left( \frac{H'(\alpha)}{H(\alpha)} \right)^4.
\]

The point is to determine the sign of $J^{(3)}$. This function corresponds to (twice) the kurtosis of the projection on the first coordinate of a random variable $X$ (expressed in polar coordinates) having the law $\gamma$. As the graph of the symmetric part of the density function cuts exactly twice the graph of the corresponding Gaussian variable (with same mean and variance), the kurtosis of $X$ is negative: $J^{(3)}(\alpha) < 0$ for $\alpha > 0$ and $J^{(3)}(0) = 0$. So, for all $\alpha \geq 0$, we have $J''(\alpha) \leq J''(0) = 0$. Similarly, we find
\[
J'(\alpha) \leq J'(0) = -1 + \int_0^\infty \rho^2 \gamma(\rho) \, d\rho.
\]

So, if $J'(0) \leq 0$, then $J$ decreases and, as $J(0) = 0$, the first result follows. Else $J'(0) > 0$. As $J'$ is monotonic and $\lim_{\alpha \to \infty} J'(\alpha) = -1$, by continuity of $J'$, there exists $\alpha_0 > 0$ such that $J'(\alpha_0) = 0$. Moreover, we have
lim_{α→∞} J(α) = −∞. Finally, there exists a positive solution to J(α) = 0 if and only if \( \int_0^∞ ρ^2 γ(ρ) dρ > 1 \). In that case, 0 is unstable and there exists a stable equilibrium. □

The next result shows that we can reduce the problem in studying the semiflow generated by (6.2) and then deduce the same (asymptotic) statements for \( μ \).

**Lemma 6.5 ([5], Proposition 3.9–Corollary 3.10).**

(1) Let \( L \subset \mathcal{P}_β(\mathbb{R}^d; V) \) be an attractor-free set for \( Φ \) and \( A \subset \mathcal{P}_β(\mathbb{R}^d; V) \) an attractor for \( Φ \), with basin of attraction \( B(A) \). If \( L \cap B(A) \neq \emptyset \), then \( L \subset A \).

(2) Let \( (E, d) \) be a metric space, \( \Phi : E × \mathbb{R} → E \) a semiflow on \( E \) and \( G : \mathcal{P}_β(\mathbb{R}^d; V) → E \) a continuous function. Assume that \( G \circ Φ_t = Φ_t \circ G \). Then, almost surely \( G(ω(μ_t, t ≥ 0)) \) is attractor-free for \( Φ \).

We can now state and prove the following:

**Theorem 6.6.** Consider the self-interacting diffusion on \( \mathbb{R}^2 \), where \( W(x, y) = −(x, y) \). Then, we have two different cases:

1. If \( \int_0^∞ dρ \ y(ρ) ρ^2 \leq 1 \), then a.s. \( μ_t \xrightarrow{w} γ \);
2. If \( \int_0^∞ dρ \ y(ρ) ρ^2 > 1 \), then there exists a random variable \( v ∈ \mathbb{S}^1 \) such that a.s. \( μ_t \xrightarrow{w} μ_∞^v \) with

\[
μ_∞^v(dx) = \frac{e^{α_1(x,v)}}{Z_1} γ(dx),
\]

where \( α_1 \) is the unique positive solution to

\[
J(α) = −α + 2 H(α) \frac{H(α)}{H(α)} = 0.
\]

**Proof.** Let \( G : \mathcal{P}_β(\mathbb{R}^d; V) → \mathbb{R}^2 \) be the mapping defined by \( G(μ) = \bar{μ} \). By Lemma 6.5, the limit set of \( \bar{μ}_t \) is a.s. attractor-free for \( Φ \). If \( \int_0^∞ dρ \ y(ρ) ρ^2 \leq 1 \), then 0 is a global attractor for the semiflow generated by \( Φ \). So, each attractor-free set of \( Φ \) reduces to 0, and a.s. \( \bar{μ}_t → 0 \) and \( ω(μ_t, t ≥ 0) \subset G(0) \). The definitions of \( Π(μ) \) and \( J(α) \) imply that \( G(0) \) is invariant for \( Φ \) and, as \( Π(Φ_0) = γ \), we have

\[
Φ|_{G(0)}(μ) = e^{−T} (μ_0 + γ).
\]

So, \( γ \) is a global attractor for \( Φ|_{G(0)} \) and each attractor-free set reduces to \( γ \). By Theorem 5.8, we conclude that \( ω(μ_t, t ≥ 0) = \{γ\} \).

Suppose now that 0 is unstable for \( Π - Id \). For all \( f ∈ C^∞(\mathbb{R}^2; V) \), it holds

\[
\frac{d}{dt} μ_h(t) f = −μ_h(t) f + Π(μ_h(t)) f + \frac{d}{ds} Π_{t+s} |_{s=0} f.
\]

If we consider the projection map \( P_t(x) = x_t \), then \( δ_t \bar{μ}_h(t) = Π_t(μ_h(t)) − \bar{μ}_h(t) + η_t \), where \( η_t \) is the random vector \( η_t = \frac{d}{ds} Π_{t+s} |_{s=0} P_t, P_s^T \). As 0 is an unstable linear equilibrium for \( Π - Id \), by Tarres [21] we get that \( \mathbb{P}(lim_{t→∞} Π_0 = 0) = 0 \). Using Theorem 3.6, we obtain that \( lim_{t→∞} sup_{0 ≤ s ≤ t} |Π_h(t+s) − Φ_s(μ_h(t))| = 0 \). Denote by \( α_1 \) the unique positive solution to

\[
−α + 2 H(α) \frac{H(α)}{H(α)} = 0
\]

and consider the \( Φ \)-invariant set \( A := \{m = ρ v; ρ = α_1, v ∈ \mathbb{S}^1\} \). By Lemma 6.5, the limit set of \( Π_h(t) \) is attractor-free, so \( ω(μ_h(t), t ≥ 0) \) either reduces to \( \{0\} \), or is included in \( A \). But, as \( \mathbb{P}(lim_{t→∞} Π_h(t) = 0) = 0 \), it is a.s. a subset of \( A \). Finally, as \( \dot{v} = 0 \), we have \( Φ_t A = Id A \) and so, \( Π_h(t) \) is a Cauchy sequence in \( A \). Then, there exists \( v ∈ \mathbb{S}^1 \) such that

\[
lim_{t→∞} |Π_h(t) − α_1 v| = 0.
\]

To conclude, on one side, \( ω(μ_t, t ≥ 0) \) is an attractor-free set for \( Φ|_{G(0)} \) and on the other side, the semiflow \( Φ|_{G(0)} \) admits \( μ_∞^v \) as a global attractor. This leads to \( ω(μ_t, t ≥ 0) = μ_∞^v \). □
6.2. When \( R \) is a rotation

We assume that \( R \) is the rotation matrix \( R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \), with \( 0 \leq \theta < 2\pi \). We emphasize that (unless \( \theta = 0, \pi \)) \( W \) is not a symmetric function. We state and prove a more precise version of Theorem 1.3.

Theorem 6.7. Consider the self-interacting diffusion on \( \mathbb{R}^2 \) associated with \( W(x, y) = (x, Ry) \). Then one of the following holds:

1. If \( V \) is such that \( \int_0^\infty \rho^2 \gamma(\rho) \, d\rho \, (\cos\theta) > -1 \), then a.s. \( \mu_t \xrightarrow{w} \gamma \);
2. If \( V \) is such that \( \int_0^\infty \rho^2 \gamma(\rho) \, d\rho \, (\cos\theta) \leq -1 \), then:
   
   a. if \( \theta = \pi \), then there exists a random variable \( v \in S^1 \) such that a.s. \( \mu_t \xrightarrow{w} \mu^v_{\infty} \) with \( \mu^v_{\infty}(dx) = \frac{e^{\int_0^1 0}(x,v)}{Z_1} \gamma(dx) \),

where \( \alpha_1 \) is the unique positive solution to \( -\alpha + 2 \frac{H'(\alpha)}{H(\alpha)} = 0 \),

b. if \( \theta \neq \pi \), then \( \omega(\mu_t, t \geq 0) = \{v(\delta), 0 \leq \delta < 2\pi\} \) a.s., where \( v(\delta) = \frac{1}{e^{\theta_\alpha - 1}} \int_0^{T_\theta} e^{\int_0^s v,\theta} \, ds \), with \( T_\theta \) = \( 2\pi (\tan\theta)^{-1} \) and \( \mu^v_{\infty} \) is the unique positive solution to \( -\alpha + 2 \cos\theta \frac{H'(\alpha)}{H(\alpha)} = 0 \).

Proof. Let \( v = gp \) with \( g \in O(2) \) and \( m = \alpha v/2 \). We remind the equations

\[
\dot{\alpha} = -\alpha - 2 \frac{H'(\alpha)}{H(\alpha)} (Rv, v); \quad \dot{v} = -\frac{2}{\alpha} \frac{H'(\alpha)}{H(\alpha)} ((Rv, v)v - Rv).
\]

By definition of \( R \) and \( v = (-\sin\theta, \cos\theta) \), a simple computation yields to

\[
\dot{\alpha} = -\alpha - 2 \frac{H'(\alpha)}{H(\alpha)} \cos\theta; \quad \dot{\sigma} = -\frac{2}{\alpha} \frac{H'(\alpha)}{H(\alpha)} \sin\theta.
\]

We recall that \( \frac{H'(\alpha)}{H(\alpha)} > 0 \) for \( \alpha > 0 \). By Proposition 6.4, we have a bifurcation: if \( \cos\theta \int_0^\infty \gamma(\rho) \rho^2 \geq 1 \), then the set \( \{0, \alpha; \dot{\alpha} = 0\} \) is a global attracting set for the semiflow generated by (6.4) and so a.s. \( \mu_t \xrightarrow{w} \gamma \). Let \( \alpha_0 \) be such that \( \dot{\alpha}_0 = 0 \). If \( \cos\theta \int_0^\infty \gamma(\rho) \rho^2 < 1 \), then \( \tilde{A} := \{(\sigma, \alpha); \alpha = \alpha_0\} \) is a global attracting set. On \( \tilde{A} \), the dynamics is given by

\[
\dot{\sigma} = -\frac{2}{\alpha_0} \frac{H'(\alpha_0)}{H(\alpha_0)} \sin\theta = \tan\theta.
\]

By Theorem 6.6, there exists a random variable \( \sigma_0 \) such that a.s.

\[
\lim_{t \to \infty} \left| \bar{\mu}_{h(t)} - \frac{\alpha_0}{2} v(t \tan\theta + \sigma_0) \right| = 0.
\]

At that point, we know the dynamics on the set \( \tilde{A} \). But, we need to study the system defined on \( M(\mathbb{R}^2; V) \times \mathbb{R}^2 \) by

\[
\dot{m} = -m + \Pi(m); \quad \dot{v} = -v + \Pi(m).
\]

By Lemma 6.5, \( \omega(\mu_t, t \geq 0) \times \tilde{A} \) is attractor-free for the preceding semiflow restricted to \( \mathcal{P}(\mathbb{R}^2; V) \times \mathbb{R}^2 \). The dynamics on \( \omega(\mu_t, t \geq 0) \times \tilde{A} \) is given by

\[
\dot{\sigma} = \tan\theta; \quad \dot{v} = -v + f(\sigma) = -v + \mu^v_{\infty}.
\]

As the set \( \omega(\mu_t, t \geq 0) \times \tilde{A} \) is (weakly) compact and invariant in \( \mathcal{P}(\mathbb{R}^2; V) \times \mathbb{R}^2 \), we conclude similarly to [5], Theorem 4.11. \(\square\)
Some self-interacting diffusions on $\mathbb{R}^d$

Acknowledgments

This work has partially been supported by the Swiss National Foundation Grants 200020-112316/1 and PBNE2-119027. The author is glad to thank M. Benaïm for introducing her to the subject, P. Cattiaux for an introduction to the ultracontractivity property, V. Kleptsyn and P. Tarrès for helpful comments, and J.-F. Jouanin for valuable discussions. I also wish to thank anonymous referees for careful reading and comments, which improved this paper.

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