Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2010, Vol. 46, No. 2, 537–574
DOI: 10.1214/09-AIHP325

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Marking $(1,2)$ points of the Brownian web and applications[1]

C. M. Newman[a], K. Ravishankar[b] and E. Schertzer[c]

[a] *Courant Institute of Mathematical Sciences, NYU, New York, NY 10012, USA*
[b] *Department of Mathematics, SUNY College at New Paltz, New Paltz, NY 12561, USA*
[c] *Department of Mathematics, Columbia University, New York, NY 10027, USA*

Received 6 June 2008; revised 27 May 2009; accepted 22 June 2009

Abstract. The Brownian web (BW), which developed from the work of Arratia and then Tóth and Werner, is a random collection of paths (with specified starting points) in one plus one dimensional space–time that arises as the scaling limit of the discrete web (DW) of coalescing simple random walks. Two recently introduced extensions of the BW, the Brownian net (BN) constructed by Sun and Swart, and the dynamical Brownian web (DyBW) proposed by Howitt and Warren, are (or should be) scaling limits of corresponding discrete extensions of the DW – the discrete net (DN) and the dynamical discrete web (DyDW). These discrete extensions have a natural geometric structure in which the underlying Bernoulli left *or* right "arrow" structure of the DW is extended by means of branching (i.e., allowing left *and* right simultaneously) to construct the DN or by means of switching (i.e., from left to right and vice-versa) to construct the DyDW. In this paper we show that there is a similar structure in the continuum where arrow direction is replaced by the left or right parity of the $(1,2)$ space–time points of the BW (points with one incoming path from the past and two outgoing paths to the future, only one of which is a continuation of the incoming path). We then provide a complete construction of the DyBW and an alternate construction of the BN to that of Sun and Swart by proving that the switching or branching can be implemented by a Poissonian marking of the $(1,2)$ points.

Résumé. Le réseau Brownien (BW) construit à partir des travaux de Arratia, de Tòth et de Werner est une collection aléatoire de chemins (avec des points de depart determinés) dans un espace deux-dimensionnel (une dimension en temps et une autre en espace), qui est la limite d'échelle d'un réseau discret (DW) de marches aléatoires coalescentes. Récemment, deux extensions du BW ont été introduites: le filet Brownien (BN), construit par Sun et Swart, et le réseau Brownien dynamique (DyBW), proposé par Howitt et Warren. Ces deux objets sont (ou devraient être) la limite d'échelle de deux extensions naturelles du réseau discret – le filet discret (DN) et le réseau dynamique discret (DyDW). Le DN et le DyDW sont obtenus par une modification de la configuration des "flèches" droites *ou* gauches qui composent le réseau discret. Pour le DN, un mécanisme de ramification est introduit (en permettant des flèches droites *et* gauches simultanément) alors que pour le DyDW, la direction des flèches est modifiée (de droite à gauche et vice-versa). Dans cet article, nous montrons qu'il existe une structure géométrique analogue dans le cas continu. Plus précisément, la direction des flèches dans le cas discret est remplacée par la direction des points $(1,2)$ du réseau Brownien (en un point $(1,2)$ se trouvent un chemin entrant et deux chemins sortants, l'un d'eux étant la continuation du chemin entrant). Nous montrons que les ramifications et changements de direction peuvent être introduits dans le cas continu par un marquage de type Poisson des points $(1,2)$. Par l'intermédiaire de ce marquage, nous donnons une construction complète du DyBW et une construction alternative à celle de Sun et Swart du BN.

MSC: Primary 60K35; secondary 60J35; 60J70

Keywords: Brownian web; Brownian net; Dynamical Brownian web; Coalescing random walks; Poissonian marking; Nucleation on boundaries; Sticky Brownian motion

[1] Supported in part by NSF Grants DMS-01-04278 and DMS-06-06696.

1. Introduction

In [9], the present authors and L. R. Fontes obtained some results about exceptional times for a dynamical model of coalescing one-dimensional random walks (the "dynamical discrete web" (DyDW)). Underlying those results was the idea that there should be a natural continuum limit of the DyDW, the "dynamical Brownian web" (DyBW) for which corresponding results would be valid, provided such a continuum system actually exists. The DyBW was also proposed in a paper of Howitt and Warren [14], where the DyDW was first discussed, and some of its properties were analyzed, assuming its existence.

The main purpose of the present paper is to develop a Poissonian marking of certain nongeneric points (called $(1,2)$ points, as we will explain) in the (static) Brownian web (BW) which we then use to give the first complete construction of the DyBW. In a revised version [10] of [9], this construction will be used to argue that exceptional time results derived earlier for the DyDW should extend to the DyBW. As we shall see, this marking technology is natural and has other applications besides the DyBW. One of those, which we explore in detail in this paper, is an alternative construction of the "Brownian net" (BN) of Sun and Swart [23]. A future application [16], which we discuss briefly in Section 1.2 below, is to scaling limits of one-dimensional voter models in which there is "nucleation along boundaries." That will extend, in a nontrivial way, earlier work [8] on scaling limits in which nucleation "in the bulk" was treated by using marking of nongeneric $(0,2)$ points of the BW, which are simpler to deal with than $(1,2)$ points. Another model closely related to the marking of the Brownian web is a class of stochastic flows of kernels introduced by Howitt and Warren [14]. This is the subject of ongoing work [21].

In addition to direct applications of Poissonian markings of BW $(1,2)$ points, we believe that these constructions are of interest as special examples of an approach that is relevant beyond the Brownian web setting. Indeed, the idea of using Poissonian marking of nongeneric double points in the context of the Schramm–Loewner Evolution $SLE(6)$, was proposed in [4,5] as an approach to the continuum scaling limits of both "near-critical" and dynamical two-dimensional percolation models. In that setting, the critical scaling limit is analogous to the BW, dynamical percolation to the DyDW and near-critical percolation to a discrete web with small nonzero drift. Progress in applying that approach has been reported by Garban, Pete and Schramm [11,12]; for other results on scaling limits of near-critical percolation, see [6,17,18].

1.1. *Arrows, switching and branching*

The discrete web

The discrete web is a collection of coalescing one-dimensional simple random walks starting from every point in the discrete space–time domain $\mathbb{Z}^2_{\text{even}} = \{(x,t) \in \mathbb{Z}^2 : x+t \text{ is even}\}$. The Bernoulli percolation-like structure is highlighted by defining $\xi_{x,t}$ for $(x,t) \in \mathbb{Z}^2_{\text{even}}$ to be the increment of the random walk at location x at time t. These Bernoulli variables are symmetric and independent and the paths of all the coalescing random walks can be reconstructed by assigning to each point (x,t) an arrow from (x,t) to $\{x+\xi_{x,t}, t+1\}$ and considering all the paths starting from arbitrary points in $\mathbb{Z}^2_{\text{even}}$ that follow the arrow configuration $\aleph$. We note that there is also a set of dual (or backward) paths defined by the same $\xi_{x,t}$'s but with arrows from $(x,t+1)$ to $(x-\xi_{x,t},t)$. The collection of all dual paths is a system of backward (in time) coalescing random walks that do not cross any of the forward paths (see Fig. 1).

There are two natural variants of the discrete web; one is the dynamical discrete web (DyDW) which involves *switching* of arrows and the other is the discrete net (DN) which involves *branching* (or equivalently, adding) of arrows. Each of these is constructed by a straightforward modification of the arrow structure in the standard discrete web. The essence of this paper is a construction of analogous modifications in the continuum space–time setting.

The dynamical discrete web

In the DyDW, there is, in addition to the random walk discrete time parameter, an additional (continuous) dynamical time parameter τ. The system starts at $\tau=0$ as an ordinary DW and then evolves in τ by randomly switching the direction of each arrow at a fixed rate (say λ), independently of all other arrows. This naturally defines a dynamical arrow configuration $\tau \rightsquigarrow \aleph(\tau)$. If one follows the arrows starting from the (space–time) origin at $(0,0)$, this begins at $\tau=0$ as a simple symmetric random walk and then evolves dynamically in τ in a different way than the "dynamical random walks" studied in [2]. As noted in [9], the nature of exceptional dynamical times is quite different in this situation than in that of [2]. For example, the dynamical random walk constructed from $\aleph(\tau)$ violates the law of the iterated logarithm on a set of τ's of Hausdorff dimension one.

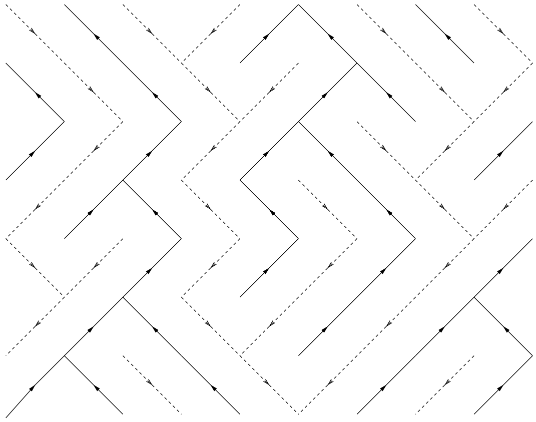

Fig. 1. Forward coalescing random walks (full lines) and their dual backward walks (dashed lines).

The discrete net
In the DN, space–time points have at any point arrows of *both* directions with probability $p \in [0, 1]$, independently of other points – i.e., individual points have either both directions (with probability p), corresponding to points where there is branching of paths, or only a left arrow (with probability $(1-p)/2$) or only a right arrow (with probability $(1-p)/2$). The DyDW and DN models may be coupled together by taking the DyDW, declaring that there are both arrows at a point if at least one switch occurred up to dynamical time τ and otherwise declaring that there is only one arrow whose direction is that of the DyDW at dynamical time 0. This yields the DN with $p = 1 - e^{-\lambda\tau}$.

Under diffusive scaling, individual random walk paths converge to Brownian motions and the entire collection of discrete paths in the DW converges in an appropriate sense (see [7]) to the continuum Brownian web (BW). We review in Section 2 some of the basic features of the BW, which developed from the work of Arratia [1] and of Tóth and Werner [24], but meanwhile we briefly comment on its structure. The BW is a random collection of paths (with specified starting points) in continuum space–time with one or more paths starting from every point. Furthermore, although generic (e.g., deterministic) space–time points have only $m_{\text{out}} = 1$ outgoing (to later times) paths from that point and $m_{\text{in}} = 0$ incoming paths passing through that point (from earlier times), there are non-generic points with other values of $(m_{\text{in}}, m_{\text{out}})$. In this paper, a dominant role is played by the $(1, 2)$ points as we shall explain.

It is natural that there should also exist scaling limits of the DyDW (including of the random walk from the origin evolving in τ) and of the DN (with appropriate scaling of τ and p along with space–time). Indeed, this has been studied by Sun and Swart [23] for the case of the net and by Howitt and Warren [14] for the case of the dynamical web. The focus of this paper is on how to construct these continuum objects directly from the BW in a way that parallels the discrete construction. A priori, this appears difficult since the discrete construction is entirely based on modifying the discrete arrow structure of the DW, while in the BW it is unclear whether there even is any arrow structure to modify.

The main themes of this paper are thus: "Where is the arrow structure of the BW?" and "How is it modified to yield the BN and the DyBW (including a dynamically evolving Brownian motion from the origin)?." As we will see, the answer to the first question is that the arrow structure of the BW comes from the $(1, 2)$ points, each of which is equipped with a left or right parity according to which of the two outgoing paths is the continuation of the single incoming path – see Fig. 2. The answer to the second question is based on a Poissonian marking of the $(1, 2)$ points, which can then be used either to create branching or to switch parity at marked points.

1.2. *Nucleation on boundaries*

The discrete-time one-dimensional voter model starts at time zero with colors assigned to each odd integer site and then evolves in time by assigning a color to the space–time point $(i, j+1)$ with $i + j + 1$ odd as that of $(i-1, j)$ or $(i+1, j)$ with probability $1/2$ each, independently of other space–time points. The genealogy of colors (looked at backwards in time) is described by coalescing random walks (on these odd space–time points) regardless of the initial state of the system. One often considers the case where there are q possible colors ($q = 2, 3, \ldots$); then the

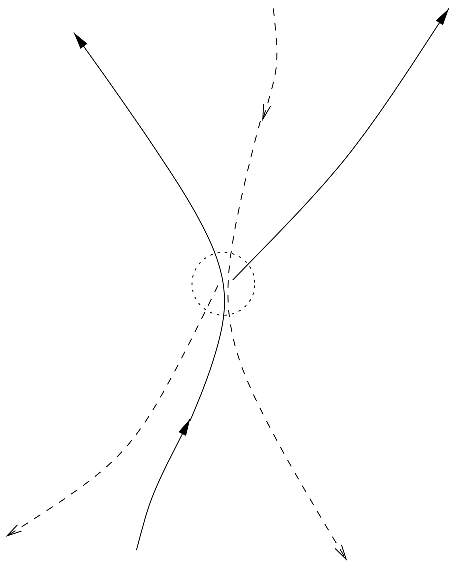

Fig. 2. A schematic diagram of a left $(m_{\mathrm{in}}, m_{\mathrm{out}}) = (1, 2)$ point with necessarily also $(\hat{m}_{\mathrm{in}}, \hat{m}_{\mathrm{out}}) = (1, 2)$. In this example the incoming forward path connects to the leftmost outgoing path (with a corresponding dual connectivity for the backward paths), the right outgoing path is a newly born path.

boundaries between sites of different colors evolve forward in time (on the even space–time points) — in the case $q = 2$ as *annihilating* random walks, as mixed annihilating-coalescing walks for $3 \le q < \infty$ and in the limit $q \to \infty$ (with each site having its own unique color at time zero) as coalescing random walks. Since the finite q case can essentially be recovered from the $q = \infty$ model by projection, one can restrict attention to the case of both forward and backward coalescing random walks.

Naturally, the continuum scaling limit of voter models is described by the BW. Indeed, in the voter model as just described, it suffices to consider (as did Arratia [1]) the collection of all outgoing BW paths from time zero. However, if one modifies the voter model to allow for small noise, i.e., at each space–time point there is a probability p that rather than take on the color of a neighboring spatial point one time step earlier, a random color (out of q possibilities, or a wholly new color for $q = \infty$) is chosen (or nucleated), then much more of the BW structure comes into play in the scaling limit (in which also p is properly scaled). As analyzed in [8], this model in the scaling limit is one in which new colors are nucleated on $(0, 2)$ points of the BW and it can be constructed by means of a Poissonian marking of those points. The reason $(0, 2)$ points are relevant is because a newly nucleated color in the voter model inside a cluster of some other color creates two new boundaries which need to persist for a macroscopic amount of time before coalescing in order to be seen in the scaling limit.

There are natural settings, namely the so-called q-state stochastic Potts models of Statistical Physics, such that for $q \ge 3$ (we recall that $q = 2$ corresponds to the Ising model) one needs to consider a more complex noise structure in which the probability of nucleation of new colors may depend on the color of the site in question and its neighbors. For example, one may require for nucleation that a site have a different color than its left (respectively, right) neighbor. For that type of noise, it turns out that the construction of the scaling limit naturally involves the Poissonian marking of left (respectively, right) $(1, 2)$ points. The reason $(1, 2)$ points are relevant here is that the newly nucleated color in the voter model is just to the right (or left) of a previously existing boundary and creates a new boundary that needs to persist in the scaling limit. This type of application of our marking of $(1, 2)$ points will be carried out in a future paper [16].

1.3. *Outline of the paper*

The remainder of the paper is organized as follows. In Section 2, we give a review of the basic structure of the Brownian web and its dual (or backward) web, with special emphasis on the $(1, 2)$ points. In Section 3, we explain precisely how to mark $(1, 2)$ points, which are points where backward and forward BW paths touch, by first defining for finitely many backward and forward paths a local time measure for touching to serve as a Poisson intensity measure. The overall marking process is then the limit as the number of forward and backward paths tends to infinity. In Section 3.3, we give a preliminary explanation of how the marking process will be used to construct the BN and the DyBW.

In Section 4, we consider the special marking process (and resulting modified Brownian web path) constructed from a *single* forward BW path and *all* backward paths that touch it from the right. In particular, we show that the resulting modified forward path is related to the original BW path by sticky reflection. Brownian motions with a sticky interaction will also play an important role in later sections as they do in [23] and [14]. In Section 5, we review the construction from [23] of the BN and then prove that our alternate construction using marked $(1,2)$ points is equivalent. In Section 6, we construct the DyBW and prove some elementary properties of this object. Section 7 contains the proofs of many of the results stated in previous sections along with some propositions and lemmas that are needed for those proofs. We note in particular that Section 7.3 contains a number of key results about the structure of excursions in the Brownian web from a single web path.

2. The Brownian web

2.1. *The forward Brownian web*

The (forward) Brownian web is the scaling limit of the discrete web under diffusive space–time scaling; it is a random collection of paths with specified starting points in space–time. The (continuous) paths take values in a metric space $(\bar{\mathbb{R}}^2, \rho)$ which is a compactification of $\mathbb{R}^2$. (Π, d) denotes the space whose elements are paths with specific starting points. The metric d is defined as the maximum of the sup norm of the distance between two paths and the distance between their respective starting points. The Brownian web takes values in a metric space $(\mathcal{H}, d_{\mathcal{H}})$, whose elements are compact collection of paths in (Π, d) with $d_{\mathcal{H}}$ the induced Hausdorff metric. Thus the Brownian web is an $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable, where $\mathcal{F}_{\mathcal{H}}$ is the Borel σ-field associated to the metric $d_{\mathcal{H}}$. The next theorem, taken from [7], gives some of the key properties of the BW.

Theorem 2.1. *There is an $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable $\mathcal{W}$ whose distribution is uniquely determined by the following three properties*:

(o) *From any deterministic point (x,t) in $\mathbb{R}^2$, there is almost surely a unique path $B_{(x,t)}$ starting from (x,t).*
(i) *For any deterministic, dense countable subset $\mathcal{D}$ of $\mathbb{R}^2$, almost surely, $\mathcal{W}$ is the closure in $(\mathcal{H}, d_{\mathcal{H}})$ of $\{B_{(x,t)} : (x,t) \in \mathcal{D}\}$.*
(ii) *For any deterministic n and $(x_1,t_1), \dots, (x_n,t_n)$, the joint distribution of $B_{(x_1,t_1)}, \dots, B_{(x_n,t_n)}$ is that of coalescing Brownian motions (with unit diffusion constant).*

Note that (i) provides a practical construction of the Brownian web. For $\mathcal{D}$ as defined above, construct coalescing Brownian motion paths starting from $\mathcal{D}$. This defines a *skeleton* for the Brownian web. $\mathcal{W}$ is simply defined as the closure of this precompact set of paths.

2.2. *The backward (dual) Brownian web*

We have considered in Section 1.1 the backward discrete web as the set of all coalescing random walks starting from $\mathbb{Z}^2_{\text{odd}}$ running backward in time without crossing the forward discrete web paths. The backward (dual) BW $\hat{\mathcal{W}}$ may be defined analogously as a functional of the (forward) BW $\mathcal{W}$. More precisely for a countable dense deterministic set of space–time points, the backward BW path from each of these is the (almost surely) unique continuous curve (going backwards in time) from that point that does not cross (but may touch) any of the (forward) BW paths; $\hat{\mathcal{W}}$ is then the closure of that collection of paths. The first part of the next proposition states that the "double BW," i.e., the pair $(\mathcal{W}, \hat{\mathcal{W}})$, is the diffusive scaling limit of the corresponding discrete pair $(W^\delta, \hat{W}^\delta)$ (as the scale parameter $\delta \to 0$). Convergence in the sense of weak convergence of probability measures on $(\mathcal{H}, \mathcal{F}_{\mathcal{H}}) \times (\hat{\mathcal{H}}, \hat{\mathcal{F}}_{\mathcal{H}})$ was proved in [7]; convergence of finite dimensional distributions and the second part of the proposition were already contained in [24].

Proposition 2.2.

1. *Invariance principle*: $(W^\delta, \hat{W}^\delta) \to (\mathcal{W}, \hat{\mathcal{W}})$ *as* $\delta \to 0$.
2. *For any* (*deterministic*) *pair of points* (x,t) *and* $(\hat{x},\hat{t})$ *there is almost surely a unique forward path* B *starting from* (x,t) *and a unique backward path* $\hat{B}$ *starting from* $(\hat{x},\hat{t})$.

The next proposition, from [22], which gives the joint distribution of a single forward and single backward BW path, has an extension to the joint distribution of finitely many forward and backward paths. We remark that that extension can be used to give a characterization (or construction) of the double Brownian web $(\mathcal{W}, \hat{\mathcal{W}})$ analogous to the one for the (forward) BW from Theorem 2.1 – see [8,22] for more details.

Proposition 2.3.

1. Distribution of $(B,\hat{B})$: *Let* $(B_{\text{ind}}, \hat{B}_{\text{ind}})$ *be a pair of independent forward and backward Brownian motions starting at* (x,t) *and* $(\hat{x},\hat{t})$ *and let* $(R_{\hat{B}_{\text{ind}}}(B_{\text{ind}}), \hat{B}_{\text{ind}})$ *be the pair obtained after reflecting* (*in the Skorohod sense*) B_{ind} *on* $\hat{B}_{\text{ind}}$, *i.e.*, $R_{\hat{B}_{\text{ind}}}(B_{\text{ind}})$ *is the following function of* $u \in [t,\hat{t}]$:

$$R_{\hat{B}_{\text{ind}}}(B_{\text{ind}}) = \begin{cases} B_{\text{ind}}(u) - 0 \wedge \min_{t \le v \le u}\big(B_{\text{ind}}(v) - \hat{B}_{\text{ind}}(v)\big) & \text{on } \{B_{\text{ind}}(t) \ge \hat{B}_{\text{ind}}(t)\}, \\ B_{\text{ind}}(u) - 0 \vee \max_{t \le v \le u}\big(B_{\text{ind}}(v) - \hat{B}_{\text{ind}}(v)\big) & \text{on } \{B_{\text{ind}}(t) < \hat{B}_{\text{ind}}(t)\}. \end{cases} \tag{2.1}$$

Then

$$\big(R_{\hat{B}_{\text{ind}}}(B_{\text{ind}}), \hat{B}_{\text{ind}}\big) = (B, \hat{B}) \quad \text{in law}, \tag{2.2}$$

where B *is the path in* $\mathcal{W}$ *starting at* (x,t) *and* $\hat{B}$ *is the path in* $\hat{W}$ *starting at* $(\hat{x},\hat{t})$.
2. *Similarly*,

$$\big(B_{\text{ind}}, R_{B_{\text{ind}}}(\hat{B}_{\text{ind}})\big) = (B, \hat{B}) \quad \text{in law}. \tag{2.3}$$

2.3. $(1,2)$ *points of the Brownian web*

While there is only a single path from any deterministic point in $\mathbb{R}^2$ in both the forward and backward webs, there exist random points $z \in \mathbb{R}^2$ with more than one path passing through or starting from z.

We now describe the "types" of points $(x,t) \in \mathbb{R}^2$, whether deterministic or not. We say that two paths $B, B' \in \mathcal{W}$ are equivalent paths entering $z = (x,t)$, denoted by $B =_{\text{in}}^{z} B'$, iff $B = B'$ on $[t-\varepsilon, t]$ for some $\varepsilon > 0$. The relation $=_{\text{in}}^{z}$ is a.s. an equivalence relation on the set of paths in $\mathcal{W}$ entering the point z and we define $m_{\text{in}}(z)$ as the number of those equivalence classes. ($m_{\text{in}}(z) = 0$ if there are no paths entering z.) $m_{\text{out}}(z)$ is defined as the number of distinct paths starting from z. For $\hat{\mathcal{W}}$, $\hat{m}_{\text{in}}(z)$ and $\hat{m}_{\text{out}}(z)$ are defined similarly.

Definition 2.1. *The type of* z *is the pair* $(m_{\text{in}}(z), m_{\text{out}}(z))$.

The following results from [24] (see also [8]) specify what types of points are possible in the Brownian web.

Theorem 2.4. *For the Brownian web*, *almost surely*, *every* (x,t) *has one of the following types*, *all of which occur*: $(0,1)$, $(0,2)$, $(0,3)$, $(1,1)$, $(1,2)$, $(2,1)$.

Proposition 2.5. *For the Brownian web*, *almost surely for every* z *in* $\mathbb{R}^2$, $\hat{m}_{\text{in}}(z) = m_{\text{out}}(z) - 1$ *and* $\hat{m}_{\text{out}}(z) = m_{\text{in}}(z) + 1$. *See Fig.* 2.

It is important to realize that points of type $(1,2)$ can be characterized in two ways, both of which will play a crucial role in our construction of the DyBW and BN. (1) $z \in \mathbb{R}^2$ is of type $(1,2)$ precisely if both a forward and a backward path pass through z. (2) A single incident path continues along exactly one of the two outward paths – with the choice determined intrinsically. It is either left-handed or right-handed according to whether the continuing path is to the left or the right of the incoming (from later time) backward path. For a left $(1,2)$ point z, the right (resp., left) outgoing path will be referred to as the *newly born path* starting from z. See Fig. 2 for a schematic diagram of the "left-handed" case. Both varieties occur and it is known [8] that each of the two varieties, as a subset of $\mathbb{R}^2$, has Hausdorff dimension 1. As noted in Section 1, the two varieties of $(1,2)$ points play the same role in the continuum that left and right arrows play in the discrete setting. In particular, one can change the direction of the "continuum" arrow at a given $(1,2)$ point z by simply connecting the incoming path to the newly born path starting from z. In the discrete picture, this amounts to changing the direction of an arrow whose switching induces a "macroscopic" effect in the web.

3. Marked (1, 2) points on the Brownian web

3.1. *The local time measure*

Recall that the ϕ-Hausdorff outer measure of an arbitrary subset E of $\mathbb{R}$ for $\phi:(0,\infty)\to(0,\infty)$ is defined as

$$m_\phi(E)=\lim_{\delta\downarrow 0}\inf\left\{\sum\phi\bigl(|b_i-a_i|\bigr)\middle|E\subset\bigcup_i[a_i,b_i],\,|b_i-a_i|<\delta\right\}. \tag{3.1}$$

In the following, we set $\phi(t)=\sqrt{2t\log(|\log(t)|)}$ and we denote the Lebesgue measure of E by $|E|$. Restricted to Borel subsets E of $\mathbb{R}$, m_ϕ is a measure.

Proposition 3.1.

1. *Let $(B,\hat{B})$ be defined as in Proposition 2.3. For almost every realization of $\mathcal{W}$, for every $t\le u\le\hat{t}$*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\left|\left\{v:\ t\le v\le u,\ \frac{|B(v)-\hat{B}(v)|}{\sqrt{2}}\le\varepsilon\right\}\right| \tag{3.2}$$

   exists and will be denoted by $L_{B,\hat{B}}(u)$.

2. *For a Borel set $A\subset\mathbb{R}$*

$$\int_{u\in A}\mathrm{d}L_{B,\hat{B}}(u)=m_\phi\bigl(\{u\in A\mid B(u)=\hat{B}(u)\}\bigr). \tag{3.3}$$

3. Distribution of $L_{B,\hat{B}}$: *$L_{B,\hat{B}}$ is a stochastic process on $[t,\hat{t}]$ which is identical in law to $\bar{L}_{B,\hat{B}}$ defined as follows:*

$$\bar{L}_{B,\hat{B}}(u)=\begin{cases}-0\wedge\min_{t\le v\le u}\bigl(B_{\text{ind}}(v)-\hat{B}_{\text{ind}}(v)\bigr)/\sqrt{2} & \text{on }\{B_{\text{ind}}(t)\ge\hat{B}_{\text{ind}}(t)\},\\ 0\vee\max_{t\le v\le u}\bigl(B_{\text{ind}}(v)-\hat{B}_{\text{ind}}(v)\bigr)/\sqrt{2} & \text{on }\{B_{\text{ind}}(t)<\hat{B}_{\text{ind}}(t)\},\end{cases} \tag{3.4}$$

   where $(B_{\text{ind}},\hat{B}_{\text{ind}})$ are defined as in Proposition 2.3.

Note that the third statement is analogous to the famous property discovered by Lévy that the local time (at the origin) of a one-dimensional Brownian motion is identical in law with its record time process (see, e.g., [15]). Statement 2 is analogous to the fact that the measure induced by the local time at 0 of a standard Brownian motion coincides with the ϕ-Hausdorff measure of its zero-set (see Theorem 1 in [19]).

Let us consider a family of n forward paths $\{B_i\}_{i=0}^{n-1}$ and a family of m backward paths $\{\hat{B}_j\}_{j=0}^{m-1}$. We will generally choose theses paths so that B_i and $\hat{B}_i$ have the same starting point z_i with $\mathcal{D}=\{z_i\}_{i=0}^{\infty}$ some dense deterministic set of points in $\mathbb{R}^2$ as defined in Section 2.1; also for consistency with other notation, we will generally assume that z_0 is the origin in $\mathbb{R}^2$. In non-ambiguous contexts, $\{B_i\}_{i=0}^{n-1}$ and $\{\hat{B}_j\}_{j=0}^{m-1}$ will also refer to the union of their respective traces in $\mathbb{R}^2$.

The expression for $L_{B,\hat{B}}$ given in (3.3) can be easily generalized to the family $\{B_i\}_{i=0}^{n-1}$ and $\{\hat{B}_j\}_{j=0}^{m-1}$. E.g., for a Borel $A \subset \mathbb{R}$, we simply define $L_{n,m}(A)$ by

$$\int_{u\in A} \mathrm{d}L_{n,m}(u) = m_\phi\big(\big\{t \in A \mid \exists x \in \mathbb{R} \text{ s.t. } (x,t) \in \{B_i\}_{i=0}^{n-1} \cap \{\hat{B}_j\}_{j=0}^{m-1}\big\}\big)$$
$$= m_\phi\big(A \cap \mathcal{P}\big(\{B_i\}_{i=0}^{n-1} \cap \{\hat{B}_j\}_{j=0}^{m-1}\big)\big),$$

where $\mathcal{P}$ denotes the projection onto the t-axis.

Finally, we can extend $L_{n,m}$ to be a measure acting on $\mathbb{R}^2$ in the following way, which implicitly uses the a.s. property of $\mathcal{W}$ that if a forward and a backward family meet at some t, they do so only at a single value of x.

Definition 3.1 (Local time measure). *For the forward family $\{B_i\}_{i=0}^{n-1}$ and the backward family $\{\hat{B}_j\}_{j=0}^{m-1}$, we define the local time (outer) measure $\mathcal{L}_{n,m}$ on $\mathbb{R}^2$ as follows. For a general space–time domain O,*

$$\mathcal{L}_{n,m}(O) = m_\phi\big(\mathcal{P}\big(\{B_i\}_{i=0}^{n-1} \cap \{\hat{B}_j\}_{j=0}^{m-1} \cap O\big)\big). \tag{3.5}$$

In particular, $\mathcal{L}_{n,m}$ is supported on the space–time points where the forward family touches the backward family. Finally, we define an outer measure

$$\mathcal{L}(O) = m_\phi\big(\mathcal{P}\big(\{B_i\}_{i=0}^{\infty} \cap \{\hat{B}_j\}_{j=0}^{\infty} \cap O\big)\big). \tag{3.6}$$

$\mathcal{L}(O)$ will be referred to as the local time outer measure of O.

Both $\mathcal{L}_{n,m}$ and $\mathcal{L}$ are measures when restricted to Borel sets but may take the value ∞. We note that for any open set $O \subset \mathbb{R}^2$, $\mathcal{L}(O) = \infty$. However, we will later encounter (see e.g., Section 7.7) some very natural subsets $O \subset \mathbb{R}^2$ with finite $\mathcal{L}$-measure. See Section 4i of [25] for a similar discussion.

3.2. *The marking process*

Let us consider the Poisson point process on $\mathbb{R}^2 \times \mathbb{R}^+$ with intensity measure

$$I_{n,m}\big(O \times [0,\tau]\big) = \sqrt{2}\mathcal{L}_{n,m}(O) \cdot \tau,$$

where O is any open subset of $\mathbb{R}^2$. We define the *partial marking* process $\tau \to \mathcal{M}_{n,m}(\tau)$ as

$$\mathcal{M}_{n,m}(\tau) = \big\{z \in \mathbb{R}^2\colon\ (z,u) \text{ is a Poisson point for some } u \le \tau\big\}. \tag{3.7}$$

Heuristically, $\mathcal{M}_{n,m}(\tau)$ consists of the locations of the switching (in the DyBW) between dynamical times 0 and τ if one restricts the dynamics to the "arrows" at the intersection of the forward family $\{B_i\}_{i=0}^{n-1}$ and the backward family $\{\hat{B}_j\}_{j=1}^{m-1}$, while other arrows remain frozen. In order to introduce a "full dynamics" we will couple the sequences $\{\mathcal{M}_{n,m}(\tau)\}_{n,m}$ in such way that for $n' \ge n$ and $m' \ge m$, $\mathcal{M}_{n,m}(\tau) \subseteq \mathcal{M}_{n',m'}(\tau)$. To achieve this, we define the point process $\mathbb{M}$ as follows:

Definition 3.2. *$\mathbb{M}$ is the four-dimensional Poisson point process on $\mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{N} \times \mathbb{N}$ with (locally finite and random) intensity measure I defined by*

$$I\big(O \times [0,\tau] \times \{0,\dots,n-1\} \times \{0,\dots,m-1\}\big) = \sqrt{2}\mathcal{L}_{n,m}(O) \cdot \tau, \tag{3.8}$$

where O is any open subset of $\mathbb{R}^2$.

We can then define $\mathcal{M}(\tau)$ as

$$\mathcal{M}(\tau) = \big\{z\colon\ \big(z,s,n',m'\big) \text{ is in } \mathbb{M} \text{ for some } n',m' \text{ and some } s \le \tau\big\} \tag{3.9}$$

and $\mathcal{M}_{n,m}(\tau)$ is simply obtained by adding the restriction to (3.9) that $n' \le n-1$ and $m' \le m-1$.

Informally, $\{\mathcal{M}(\tau)\}_{\tau\ge 0}$ can be seen as a Poisson Point Process on $\mathbb{R}^2 \times \mathbb{R}$ with intensity measure $\sqrt{2}\mathcal{L}(\mathrm{d}z) \times \mathrm{d}\tau$. In particular, for a Borel $O \subset \mathbb{R}^2$ with $\mathcal{L}(O) < \infty$, $\mathcal{M}(\tau) \cap O$ is a Poisson point Process on $\mathbb{R}^2 \times \mathbb{R}$ with intensity measure $\sqrt{2}1_{z\in O}\mathcal{L}(\mathrm{d}z) \times \mathrm{d}\tau$.

3.3. *Modifying the web using marking*

3.3.1. *Constructing the Brownian net*

Let $\tau > 0$. We define a partial Brownian net $\mathcal{N}_{n,m}(\tau)$ by having branching at the points of the partial marking $\mathcal{M}_{n,m}(\tau)$. (Later we will write $\mathcal{N}_n(\tau)$ for $\mathcal{N}_{n,n}(\tau)$.) For example, if the $(1,2)$ point in Fig. 2 is marked, then the Brownian net will include not only paths that connect to the left outgoing path (as in the original web) but also ones that connect to the right outgoing path. More formally, the set of paths in $\mathcal{N}_{n,m}(\tau)$ starting from $z \in \mathbb{R}^2$ is the set of paths interpolating the set $\mathcal{S}$ of points $\mathcal{M}_{n,m}(\tau) \cup \{z\} \cup \{+\infty\}$ with paths in $\mathcal{W}$ – i.e., between any *consecutive* pair of points in $\pi \cap \mathcal{S}$, π coincides with a path in $\mathcal{W}$.

Finally, we define $\mathcal{N}_{\mathrm{mark}}(\tau)$ as the closure of $\bigcup_{n,m=1}^{\infty} \mathcal{N}_{n,m}(\tau)$. In other words, $\mathcal{N}_{\mathrm{mark}}(\tau)$ is defined by allowing branching at every marked $(1,2)$ point in the Brownian web $\mathcal{W}$. Analogously, we can define a backward partial Brownian net $\hat{\mathcal{N}}_{n,m}(\tau)$ by allowing branching at the points $\mathcal{M}_{n,m}(\tau)$ in the dual web $\hat{\mathcal{W}}$ and define $\hat{\mathcal{N}}_{\mathrm{mark}}(\tau)$ as the closure of $\bigcup_{n,m=1}^{\infty} \hat{\mathcal{N}}_{n,m}(\tau)$. In Section 5, we prove the equivalence of $\mathcal{N}_{\mathrm{mark}}(\tau)$ to the Brownian net construction of Sun and Swart [23], which by their results (see Theorem 1.1 in [23]) then implies convergence of the properly rescaled discrete net to $\mathcal{N}_{\mathrm{mark}}(\tau)$ in an appropriate topology.

3.3.2. *Constructing the dynamical Brownian web*

We can construct a partial dynamical Brownian web $\mathcal{W}_{n,m}(\tau)$, at dynamical time τ, to replace the original $\mathcal{W}$ by switching the direction of all the marked $(1,2)$ points in $\mathcal{M}_{n,m}(\tau)$. Formally, π is in $\mathcal{W}_{n,m}(\tau)$ iff π is in the the partial net $\mathcal{N}_{n,m}(\tau)$ and at each time $t = \bar{t}_i$ that π hits a point $(\bar{x}_i, \bar{t}_i) \in \mathcal{M}_{n,m}(\tau)$, it then follows B^i_{new}, the newly born path of $\mathcal{W}$ starting from $(\bar{x}_i, \bar{t}_i)$, on $[\bar{t}_i, \bar{t}_i + a]$ for some $a > 0$. A nontrivial question is the existence of a limit for $\mathcal{W}_{n,m}(\tau)$ as $n, m \to \infty$. It will be shown in Section 6 that for almost all realizations of the web and its marking, a limit $\mathcal{W}(\tau)$ exists for every τ (see Proposition 6.1).

4. Sticky Brownian motion by marking a single path

From here through Section 6, τ will denote a fixed deterministic number and the marking will refer to $\mathcal{M}(\tau)$.

We first recall the definition of a one-dimensional sticky (at the origin) Brownian motion.

Definition 4.1. *$B_{\mathrm{stick},x}$ is a $(1/\bar{\tau})$-sticky Brownian motion starting at x iff there exists a one-dimensional standard Brownian motion B s.t.*

$$\forall t \ge 0 \quad \mathrm{d}B_{\mathrm{stick},x}(t) = 1_{B_{\mathrm{stick},x}(t)\ne 0}\,\mathrm{d}B(t) + \bar{\tau}1_{B_{\mathrm{stick},x}(t)=0}\,\mathrm{d}t \tag{4.1}$$

and B is constrained to stay positive as soon it first hits zero.

It is known that (4.1) has a unique (weak) solution. Furthermore, for $x = 0$ this solution can be constructed from a time-changed reflected Brownian motion. More precisely, consider

$$t \rightsquigarrow |\bar{B}|\big(C(t)\big) \quad \text{with } C^{-1}(t) = t + \frac{1}{\bar{\tau}} L_0(t),$$

where $|\bar{B}|$ is a reflected Brownian motion and L_0 is its local time at the origin. Then there exists a Brownian motion B such that $(|\bar{B}|(C(\cdot)), B)$ is a solution of (4.1) (see, e.g., [26]). In words, the sticky Brownian motion is obtained from the reflected one by "transforming" local time into real time. In particular, it spends a positive Lebesgue measure of time at the origin and the larger the "degree of stickiness" $1/\bar{\tau}$ is, the more the path sticks to the origin.

In this section we consider the path ${}_{[1]}r_z$ starting at $z \in \mathcal{D}$ and constructed by switching only the direction of the *left* $(1,2)$ points in $\mathcal{M}(\tau)$ on B_0, the path of $\mathcal{W}$ starting from the origin. As we shall see, unlike in the complete DyBW, it is not difficult to construct ${}_{[1]}r_z$ and the law of the pair $({}_{[1]}r_z, B_0)$ can be characterized explicitly. In particular, if we set ${}_{[1]}r_0 \equiv {}_{[1]}r_z$ for $z = (0, 0)$ then it readily follows from Proposition 4.1 below that $({}_{[1]}r_0 - B_0)/\sqrt{2}$ is a $(\sqrt{2}/\tau)$-sticky Brownian motion. This will be very useful in the rest of the paper (see Sections 5 and 6) where the analysis of paths that result from switching left and right $(1, 2)$ points is a direct extension of the analysis here. Our construction of a sticky Brownian motion using the marked excursions defined next is similar to Warren's construction in [27] using the excursions of a single Brownian motion.

Definition 4.2 (Excursions). *Let B_{new} be the newly born path emerging from a $(1, 2)$ point $z = (x, t)$ on any path $B \in \mathcal{W}$. The segment of B_{new} before coalescence with B is called an excursion from B.*

$D(e)$ is the time duration of the excursion e, $|e| \equiv \sup\{|B - e|(s)\colon t \le s \le t + D(e)\}$ is its diameter, $T(e) \equiv t$ its starting time, $(T(e), T(e) + D(e))$ its lifespan.

If an excursion e starts from a marked point, e is called a marked excursion.

A right marked excursion e is called nested iff there exists another right marked excursion e' s.t. $T(e)$ belongs to the lifespan of e'. An analogous definition holds for left marked excursions.

If a marked excursion e is not nested, e is said to be a maximal excursion.

${}_{[1]}r_0$ may be defined as the path obtained after joining together all the right maximal excursions from B_0. Stated differently, ${}_{[1]}r_0$ is the path whose excursions (in the standard sense) from B_0 coincide with the right maximal excursions from B_0 in the marked Brownian web. We note that every time ${}_{[1]}r_0$ hits a left $(1, 2)$ point on B_0 it then follows the newly born path starting from it. (Among all the marked left $(1, 2)$ points ${}_{[1]}r_0$ only hits the starting points of maximal excursions since nested excursions are "straddled" by some maximal excursions). Thus ${}_{[1]}r_0$ is consistent with the informal definition in terms of switching given earlier in this section.

Next, we recall that for any deterministic point $z \in \mathbb{R}^2$, $B_z \in \mathcal{W}$ is the path starting from z. We define ${}_{[1]}r_z$ as the path starting from z obtained by switching all the left marked $(1, 2)$ points on $B_0 \cap B_z$. (This informal definition may be made precise as was done for ${}_{[1]}r_0$ by using the right maximal excursions from $B_{z'}$, where z' is the coalescing point between B_0 and B_z.) Note that ${}_{[1]}r_z$ is a continuous path. To prove that, it is clearly enough to show that for fixed $T, \varepsilon \in (0, \infty)$ the process ${}_{[1]}r_z$ only performs finitely many excursions of diameter $\ge \varepsilon$ away from B_0 on the interval $[0, T]$. If that were not the case, there would exist a sequence of marked excursions $\{e_k\}$ from B_0 such that e_k would make an excursion away from B_0 with diameter ε and duration t_k, with $t_k \to 0$. But that would violate the compactness of $\mathcal{W}$.

We now set up some notation. For a path π in (Π, d) starting from z, we denote by t_π, the starting time of π. For two paths π_1, π_2, $T_{\pi_1,\pi_2} \equiv \inf\{t > t_{\pi_1} \vee t_{\pi_2}\colon \pi_1(t) = \pi_2(t)\}$ denotes the first meeting time of π_1 and π_2, which may be $+\infty$. In Section 7.4 we show the following proposition.

Proposition 4.1. *For any deterministic $z \in \mathbb{R}^2$, almost surely, there exists $B_z^{(1)}$, a standard Brownian motion starting at z so that ${}_{[1]}r_z$ satisfies the following SDE.*

$$\begin{aligned} &\mathrm{d}{}_{[1]}r_z(t) = \mathrm{d}B_z^{(1)}(t) + 1_{{}_{[1]}r_z(t)=B_0(t)}\tau\,\mathrm{d}t,\\ &\mathrm{d}B_0(t)\,\mathrm{d}B_z^{(1)}(t) = 1_{{}_{[1]}r_z(t)=B_0(t)}\,\mathrm{d}t,\\ &\forall t \ge T_{{}_{[1]}r_z,B_0},\quad {}_{[1]}r_z(t) \ge B_0(t). \end{aligned} \tag{4.2}$$

Here $\mathrm{d}B_0(t)\,\mathrm{d}B_z^{(1)}(t)$ denotes $\mathrm{d}\langle B_0, B_z^{(1)}\rangle(t)$, where $\langle B_0, B_z^{(1)}\rangle(t)$ is the cross-variation process of B_0 and $B_z^{(1)}$ at time t. The second part of Eq. (4.2) amounts to saying that away from the diagonal $\{t : {}_{[1]}r_z(t) = B(t)\}$, B_0 and ${}_{[1]}r_z$ evolve independently while on the diagonal they are perfectly correlated. In particular, without the drift on the diagonal to "unstick" ${}_{[1]}r_z$ from B_0, ${}_{[1]}r_z$ and B_0 would coalesce rather than stick when they meet.

Adopting the usual terminology, we will say that ${}_{[1]}r_z$ is distributed as a Brownian motion stickily reflected off B_0 with a degree of stickiness $1/\tau$. In particular, for $z = (x, 0)$ the process $\{({}_{[1]}r_z - B_0)/\sqrt{2}\}$ is a $(\sqrt{2}/\tau)$-sticky Brownian motion (see Definition 4.1).

In [23], Sun and Swart studied a similar equation but with the difference that ${}_{[1]}r_0$ (resp., B_0) is replaced by a right (resp., left) drifting Brownian motion (see Eq. (5.1)). For that equation, they established existence and uniqueness of a weak solution (see Proposition 2.1 in [23]). Since (4.2) and (5.1) only differ by their drift terms, existence and uniqueness for (4.2) follows from their result and the Girsanov theorem. In particular, a (weak) solution $({}_{[1]}r_0, B_0)$ of (4.2) is a strong Markov process.

5. The Brownian net by marking

In Section 5.1, we outline the construction of the Brownian net given by Sun and Swart [23] and state some related results. The presentation we give of that construction is taken from [20]. As will be seen, this construction of the Brownian net is different in spirit to the one using marking given in Section 3.3.1. However, we will show in Theorem 5.5 that the two constructions lead to the same object.

5.1. *The Brownian net as introduced by Sun and Swart*

We now recall the *left–right Brownian web* $(\mathcal{W}_l, \mathcal{W}_r)$, which is the key intermediate object in the construction of the Brownian net in [23]. Following [23], we call $(l_1, \dots, l_m; r_1, \dots, r_n)$ a collection of *left–right coalescing Brownian motions*, if $(l_1, \dots, l_m)$ is distributed as coalescing Brownian motions each with drift $-\tau$, $(r_1, \dots, r_n)$ is distributed as coalescing Brownian motions each with drift $+\tau$, paths in $(l_1, \dots, l_m; r_1, \dots, r_n)$ evolve independently when they are apart, and the interaction between l_i and r_j when they meet is a form of sticky reflection. More precisely, for any $L \in \{l_1, \dots, l_m\}$ and $R \in \{r_1, \dots, r_n\}$, the joint law of (L, R) at times $t > t_L \vee t_R$ is characterized as the unique weak solution of

$$\begin{aligned} &\mathrm{d}L(t) = \mathrm{d}B_l - \tau\,\mathrm{d}t,\\ &\mathrm{d}R(t) = \mathrm{d}B_r + \tau\,\mathrm{d}t,\\ &\mathrm{d}\langle B_l, B_r\rangle(t) = 1_{L(t)=R(t)}\,\mathrm{d}t,\\ &\forall t \ge T_{R,L}, \quad R(t) \ge L(t), \end{aligned} \tag{5.1}$$

where B^l, B^r are two standard Brownian motions. We then have the following characterization of the left–right Brownian web from [23].

Characterization of the left–right Brownian web. *There exists an* $(\mathcal{H}^2, \mathcal{F}_{\mathcal{H}^2})$*-valued random variable* $(\mathcal{W}_l, \mathcal{W}_r)$, *called the standard left–right Brownian web* (*with parameter* $\tau > 0$), *whose distribution is uniquely determined by the following two properties*:

(a) $\mathcal{W}_l$, *resp.* $\mathcal{W}_r$, *is distributed as the standard Brownian web*, *except tilted with drift* $-\tau$, *resp.* $+\tau$.
(b) *For any finite deterministic set* $z_1, \dots, z_m, z'_1, \dots, z'_n \in \mathbb{R}^2$, *the subset of paths in* $\mathcal{W}_l$ *starting from* $z_1, \dots, z_m$, *and the subset of paths in* $\mathcal{W}_r$ *starting from* $z'_1, \dots, z'_n$, *are jointly distributed as a collection of left–right coalescing Brownian motions*.

Similar to the Brownian web, *the left–right Brownian web* $(\mathcal{W}_l, \mathcal{W}_r)$ *admits a natural dual* $(\hat{\mathcal{W}}_l, \hat{\mathcal{W}}_r)$ *which is equidistributed with* $(\mathcal{W}_l, \mathcal{W}_r)$ *modulo a rotation by* 180° *of* $\mathbb{R}^2$. *In particular,* $(\mathcal{W}_l, \hat{\mathcal{W}}_l)$ *and* $(\mathcal{W}_r, \hat{\mathcal{W}}_r)$ *are pairs of tilted double Brownian webs*.

Based on the left–right Brownian web, [23] gave three equivalent characterizations of the Brownian net, which are called respectively the *hopping, wedge, and mesh characterizations*. We first recall what is meant by hopping, wedges and meshes.

Hopping. *Given two paths* $\pi_1, \pi_2 \in \Pi$, *let* t_1 *and* t_2 *be the starting times of those paths. For* $t > t_1 \vee t_2$ (*note the strict inequality*), t *is called an intersection time of* π_1 *and* π_2 *if* $\pi_1(t) = \pi_2(t)$. *By hopping from* π_1 *to* π_2, *we mean the construction of a new path by concatenating together the piece of* π_1 *before and the piece of* π_2 *after an intersection time. Given the left–right Brownian web* $(\mathcal{W}_l, \mathcal{W}_r)$, *let* $H(\mathcal{W}_l \cup \mathcal{W}_r)$ *denote the set of paths constructed by hopping a finite number of times between paths in* $\mathcal{W}_l \cup \mathcal{W}_r$.

Wedges. *Let $(\hat{\mathcal{W}}_l, \hat{\mathcal{W}}_r)$ be the dual left–right Brownian web almost surely determined by $(\mathcal{W}_l, \mathcal{W}_r)$. For a path $\hat{\pi} \in \hat{\Pi}$, let $t_{\hat{\pi}}$ denote its (backward) starting time. Any pair $\hat{l} \in \hat{\mathcal{W}}_l$, $\hat{r} \in \hat{\mathcal{W}}_r$ with $\hat{r}(t_{\hat{l}} \wedge t_{\hat{r}}) < \hat{l}(t_{\hat{l}} \wedge t_{\hat{r}})$ defines an open set*

$$W(\hat{r}, \hat{l}) = \{(x, u) \in \mathbb{R}^2 \colon T < u < t_{\hat{l}} \wedge t_{\hat{r}}, \hat{r}(u) < x < \hat{l}(u)\}, \tag{5.2}$$

where $T = \sup\{t < t_{\hat{l}} \wedge t_{\hat{r}} \colon \hat{r}(t) = \hat{l}(t)\}$ is the first (backward) hitting time of $\hat{r}$ and $\hat{l}$, which might be $-\infty$. Such an open set is called a wedge of $(\hat{\mathcal{W}}_l, \hat{\mathcal{W}}_r)$.

Meshes. *By definition, a mesh of $(\mathcal{W}_l, \mathcal{W}_r)$ is an open set of the form*

$$M = M(r, l) = \{(x, t) \in \mathbb{R}^2 \colon t_l < t < T_{l,r}, r(t) < x < l(t)\}, \tag{5.3}$$

where $l \in \mathcal{W}_l$, $r \in \mathcal{W}_r$ are paths such that $t_l = t_r$, $l(t_l) = r(t_r)$ and $r(s) < l(s)$ on $(t_l, t_l + \varepsilon)$ for some $\varepsilon > 0$. We call $(l(t_l), t_l)$ the bottom point, t_l the bottom time, $(l(T_{l,r}), T_{l,r})$ the top point, $T_{l,r}$ the top time, r the left boundary, and l the right boundary of M.

Given an open set $A \subset \mathbb{R}^2$ and a path $\pi \in \Pi$, we say π *enters* A if there exist $t_\pi < s < t$ such that $\pi(s) \notin A$ and $\pi(t) \in A$. We say π *enters A from outside* if there exists $t_\pi < s < t$ such that $\pi(s) \notin \bar{A}$, the closure of A, and $\pi(t) \in A$. We now recall the following characterization of the Brownian net from [23].

Theorem 5.1 (Characterization of the Brownian net). *There exists an $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable $\mathcal{N}$, the standard Brownian net (with parameter τ), whose distribution is uniquely determined by property* (a) *and any of the three equivalent properties* (b1)–(b3) *below*:

(a) *There exist $\mathcal{W}_l, \mathcal{W}_r \subset \mathcal{N}$ such that $(\mathcal{W}_l, \mathcal{W}_r)$ is distributed as the left–right Brownian web.*
(b1) *Almost surely, $\mathcal{N}$ is the closure of $H(\mathcal{W}_l \cup \mathcal{W}_r)$ in (Π, d).*
(b2) *Almost surely, $\mathcal{N}$ is the set of paths in Π which do not enter any wedge of $(\hat{\mathcal{W}}_l, \hat{\mathcal{W}}_r)$ from outside.*
(b3) *Almost surely, $\mathcal{N}$ is the set of paths in Π which do not enter any mesh of $(\mathcal{W}_l, \mathcal{W}_r)$.*

As pointed out in [20], the construction of the Brownian net from the left–right Brownian web can be regarded as an outside-in approach because $\mathcal{W}_l$ and $\mathcal{W}r$ are the "outermost" paths among all paths in $\mathcal{N}$. On the other hand, the marking construction of the Brownian net can be regarded as an inside-out approach. We start from a standard Brownian web, which consist of the "innermost" paths in the Brownian net, and construct the rest of the Brownian net paths by allowing branching at a Poisson set of marked points in the Brownian web.

5.2. *Equivalence of the constructions*

The main ingredient in the construction we just described is the pair $(\mathcal{W}_r, \mathcal{W}_l)$. In order to prove the equivalence between the two constructions we first prove that the sets of leftmost and rightmost paths of $\mathcal{N}_{\mathrm{mark}}$ (as defined in Section 3.3.1) are distributed as such a pair (see Proposition 5.4).

In Section 4, ${}_{[1]}r_z$ was constructed from B_z by switching all the marked left $(1, 2)$ points on B_0, the path of $\mathcal{W}$ starting from the origin. Analogously, we can define ${}_{[n]}r_z$ after switching all the marked left $(1, 2)$ points on $B_0, B_1, \ldots, B_{n-1}$, where B_k is the path starting from z_k. As can easily be seen, the interaction between ${}_{[n]}r_z$ and the family $\{B_i\}_{i=0}^{n-1}$ is local. Hence, Proposition 4.1 implies that ${}_{[1]}r_z$ evolves like an independent Brownian motion away from $\{B_i\}_{i=0}^{n-1}$ and the interaction between ${}_{[n]}r_z$ and B_i when they meet is a sticky reflection. More precisely, we have the following immediate generalization of Proposition 4.1.

Proposition 5.2. *For any deterministic z, there exists $B_z^{(n)}$, a standard Brownian motion starting at z, so that ${}_{[n]}r_z, \{B_k\}_{k=0}^{n-1}$ satisfy the following SDE.*

$$\begin{aligned} &\mathrm{d}\,{}_{[n]}r_z = \mathrm{d}B_z^{(n)}(t) + 1_{\bigcup_{k=0}^{n-1}\{{}_{[n]}r_z(t) = B_k(t)\}}\tau\,\mathrm{d}t, \\ &\mathrm{d}B_k(t)\,\mathrm{d}B_z^{(n)}(t) = 1_{{}_{[n]}r_z(t) = B_k(t)}\,\mathrm{d}t, \\ &\forall t \geq T_{{}_{[n]}r_z, B_k}, \quad {}_{[n]}r_z(t) \geq B_k(t). \end{aligned} \tag{5.4}$$

We now motivate the next proposition. As $n \to \infty$, $\{B_k\}_{k=0}^{n-1}$ "fills" more and more space of $\mathbb{R}^2$ and because ${}_{[n]}r_z$ sticks to the family $\{B_k\}_{k=0}^{n-1}$ it is intuitively clear that $1_{\bigcup_{k=0}^{n-1}\{{}_{[n]}r_z=B_k\}} \approx 1$ (see Lemma 7.8 for a precise version of this statement). Hence for large n, the first part of (5.4) becomes

$$\mathrm{d}_{[n]}r_z(t) = \mathrm{d}B_z^{(n)}(t) + 1_{\bigcup_{k=0}^{n-1}\{{}_{[n]}r_z=B_k\}}\tau\,\mathrm{d}t \tag{5.5}$$

$$\approx \mathrm{d}B_z^{(n)}(t) + \tau\,\mathrm{d}t. \tag{5.6}$$

Hence, for any $k \in \mathbb{N}$, we expect $({}_{[n]}r_z, B_k)$ to converge as $n \to \infty$ in distribution to a pair (r_z, B_k) satisfying the following SDE.

$$\begin{aligned}&\mathrm{d}r_z = \mathrm{d}B_z^r + \tau\,\mathrm{d}t,\\ &\mathrm{d}B_k(t)\,\mathrm{d}B_z^r(t) = 1_{r_z(t)=B_k(t)}\,\mathrm{d}t,\\ &\forall t \geq T_{r_z,B_k}, \quad r_z \geq B_k,\end{aligned} \tag{5.7}$$

where B_z^r is a Brownian motion starting from z.

We recall that $\{z_i\}_{i=0}^{\infty}$ is a dense deterministic subset of $\mathbb{R}^2$. Let $i \in \mathbb{N}$. In the following, we write ${}_{[n]}r_i$ for ${}_{[n]}r_{z_i}$. Since $\{{}_{[n]}r_i\}_n$ is clearly increasing in n, the sequence $\{{}_{[n]}r_i\}_n$ actually converges *pathwise* and the limit is a drifting Brownian motion. (Although it is not even clear a priori that the sequence of paths is bounded, this will follow from the fact, as motivated by (5.5)–(5.7), that there is convergence in distribution.) This pathwise limit will be referred to as r_i; it corresponds to the rightmost path of the net $\mathcal{N}_{\mathrm{mark}}$ starting from z_i. In particular, any path of any partial net $\mathcal{N}_n(=\mathcal{N}_{n,n})$ starting at z_i is always to the left of r_i (i.e., $\leq r_i$). This motivates the following proposition, whose proof is given in Section 7.5.

Proposition 5.3. *${}_{[n]}r_i$ converges pointwise to a continuous path r_i starting from z_i with (r_i, B_k) satisfying the three-part SDE (5.7).*

Analogously, using the set of marked *right* $(1, 2)$ points of $\mathcal{W}$, we can define $\{l_j\}_j$ a family of left-drifting Brownian motions reflected in a sticky way on the paths of $\mathcal{W}$. In Section 7.5 we prove the following extension of Proposition 5.3.

Proposition 5.4. *$\{r_j\}_j$ (resp. $\{l_j\}_j$) is a family of coalescing right- (resp., left-) drifting Brownian motions with drift τ (resp., $-\tau$). The pair $(\mathcal{W}_l, \mathcal{W}_r)$, defined as the closures of $\{l_j\}_j, \{r_j\}_j$ respectively, is distributed as a left–right Brownian web.*

Now, let $\mathcal{N}_{\mathrm{hop}}$ denote the net obtained from $(\mathcal{W}_r, \mathcal{W}_l)$ by the hopping construction given in Section 5.1. In Section 7.5, we prove

Theorem 5.5.

$$\mathcal{N}_{\mathrm{hop}} = \mathcal{N}_{\mathrm{mark}}. \tag{5.8}$$

6. The dynamical Brownian web

In order to describe the dynamical web, we will need the following notion of stickiness.

Definition 6.1 (Stickiness). *Let π_1, π_2 be in the net $\mathcal{N}$ with $x = \pi_1(t) = \pi_2(t)$. We say that π_1 sticks to π_2 at $z = (x, t)$, or equivalently $\pi_2 \sim^z \pi_1$, iff for any $\varepsilon > 0$,*

$$\int_t^{t+\varepsilon} 1_{\pi_1(u)=\pi_2(u)}\,\mathrm{d}u > 0 \quad \textit{and} \quad \int_{t-\varepsilon}^{t} 1_{\pi_1(u)=\pi_2(u)}\,\mathrm{d}u > 0.$$

We now set up some notation. We say that a path enters a point $z=(x,t)$ if $t_\pi < t$ and $\pi(t)=x$. Let z be a $(1,2)$ point in $\mathcal{N}_{\text{mark}}$. For any $B \in \mathcal{W}$ entering z, we denote by B_{switch} the path obtained from B after switching the direction of z. Since for any paths $\pi \in \mathcal{N}_{\text{mark}}$ and $\bar{B}, B \in \mathcal{W}$ entering z, $\pi \sim^z B$ iff $\pi \sim^z \bar{B}$, we will sometimes write $\pi \sim^z B$ without specifying B to mean that there exists a $B \in \mathcal{W}$ such that $\pi \sim^z B$. Analogously, we will write $\pi \sim^z B_{\text{switch}}$, without specifying the path B from which B_{switch} was constructed.

Recall the partial dynamical web $\{\mathcal{W}_{n,m}(\tau)\}_{\tau\geq 0}$ given in Section 3.3.2. In the following, $\mathcal{N}_{\text{mark}}(\tau)$ is the net constructed from $\mathcal{M}(\tau)$. The proof of the next proposition is given in Section 7.7.1. That proof makes clear that the three parts of Proposition 6.1 correspond to three alternative constructions of the dynamical Brownian web.

Proposition 6.1.

(1) *There exists* $\{\mathcal{W}(\tau)\}_{\tau\geq 0}$ *in* $(\mathcal{H}, d_{\mathcal{H}})$ *s.t. almost surely*

$$\forall \tau \geq 0 \quad \lim_{n,m\uparrow\infty} d_{\mathcal{H}}\big(\mathcal{W}_{n,m}(\tau), \mathcal{W}(\tau)\big) = 0.$$

(2) $\mathcal{W}(\tau) = \{\pi \in \mathcal{N}_{\text{mark}}(\tau)\colon$ *every time* π *enters a point* z *in* $\mathcal{M}(\tau)$, $\pi \sim^z B_{\text{switch}}\}$.
(3) *Almost surely*, $\mathcal{W}(\tau)$ *satisfies the two following conditions* (*of Theorem* 2.1) *for every* $\tau \geq 0$.
  (o) *From any deterministic point* z *in* $\mathbb{R}^2$, *there is a unique path* $B_z^\tau \in \mathcal{W}(\tau)$ *starting from* z.
  (i) $\mathcal{W}(\tau)$ *is the closure in* $(\mathcal{H}, d_{\mathcal{H}})$ *of* $\{B_i^\tau\}$ *where* B_i^τ *is the unique path in* $\mathcal{W}(\tau)$ *starting from* $z_i \in \mathcal{D}$.

To motivate item (2), note that in the partial dynamical web $\mathcal{W}_{n,m}(\tau)$, any path π entering a point $z \in \mathcal{M}_{n,m}(\tau)$ locally coincides with any path $B \in \mathcal{W}$ entering z and then connects to the newly born path starting from z. Hence, π locally coincides with B_{switch} and therefore obviously sticks to it. However, if z belongs to $\mathcal{M}(\tau) \setminus \mathcal{M}_{n,m}(\tau)$, then $\pi \sim^z B$. In the limit $n, m \to \infty$, $\pi \sim^z B_{\text{switch}}$ for every z in $\mathcal{M}(\tau)$.

We now turn to the description of some properties of the dynamical Brownian web. We start with a definition.

Definition 6.2. (B, B') *is a* $(1/\tau)$*-sticky pair of Brownian motions iff*:

1. B *and* B' *are both Brownian motions starting at* (x_B, t_B) *and* $(x_{B'}, t_{B'})$ *that move independently when they do not coincide*.
2. *For* $t \geq 0$, *define* $B_{\text{stick}}(t) \equiv |B - B'|(t + t_B \vee t_{B'})/\sqrt{2}$. *Conditioned on* $x = B_{\text{stick}}(0)$, $\{B_{\text{stick}}(t)\}_{t\geq 0}$ *is a* $(\sqrt{2}/\tau)$*-sticky Brownian motion* (*see Definition* 4.1).

We call $(B_1, \ldots, B_m; B'_1, \ldots, B'_n)$ a collection of $(1/\tau)$-*sticking-coalescing Brownian motions*, if $(B_1, \ldots, B_m)$ and $(B'_1, \ldots, B'_n)$ are each distributed as a set of coalescing Brownian motions and for any $B \in \{B_1, \ldots, B_m\}$ and $B' \in \{B'_1, \ldots, B'_n\}$, (B, B') is a $(1/\tau)$-sticky pair of Brownian motions.

We will say that $(\mathcal{W}, \mathcal{W}')$ is a $1/\tau$-*sticky pair of Brownian webs* iff $(\mathcal{W}, \mathcal{W}')$ satisfies the following properties

(a) $\mathcal{W}$, resp. $\mathcal{W}'$, is distributed as the standard Brownian web.
(b) For any finite deterministic set $z_1, \ldots, z_m, z'_1, \ldots, z'_n \in \mathbb{R}^2$, the subset of paths in $\mathcal{W}$ starting from $z_1, \ldots, z_m$, and the subset of paths in $\mathcal{W}'$ starting from $z'_1, \ldots, z'_n$, are jointly distributed as a collection of $(1/\tau)$-sticking–coalescing Brownian motions.

Note that $(\mathcal{W}, \mathcal{W}')$ is defined in a similar way as $(\mathcal{W}_l, \mathcal{W}_r)$ except that in (a) there is no drift and in (b) the collection of left–right coalescing Brownian motions is replaced by the collection of $(1/\tau)$-sticking–coalescing Brownian motions. We are now ready to state the main result of this section whose proof is postponed to Section 7.7.

Theorem 6.2.

(a) $(\mathcal{W}, \mathcal{W}(\tau))$ *is a* $1/(2\tau)$*-sticky pair of Brownian webs*.
(b) (*A Markov property*). *For* $\tau_1 \leq \tau_2$ *and given* $(\mathcal{W}, \{\mathcal{M}(\tau)\}_{\tau\leq\tau_1})$, *the distribution of the pair* $(\mathcal{W}(\tau_1), \mathcal{W}(\tau_2))$ *only depends on* $\mathcal{W}(\tau_1)$.
(c) (*Stationarity*). *For* $\tau_1 \leq \tau_2$, $(\mathcal{W}(\tau_1), \mathcal{W}(\tau_2))$ *and* $(\mathcal{W}, \mathcal{W}(\tau_2 - \tau_1))$ *are equidistributed*.
(d) *For any fixed deterministic time* $t_0 > 0$, *the process* $\tau \to B_0^\tau(t_0)$ *is piecewise constant*.

We remark that existence of a consistent family of finite dimensional distributions for the process $\mathcal{W}(\tau)$ follows from the results of [14] – see in particular Theorem 9 there.

7. Proofs

This section is organized as follows. In Section 7.1, we recall some useful properties of the Brownian web. In Section 7.2, we complete the construction of the local time measure outlined in Section 3.1. In Section 7.3, we carefully study some quantities related to the marked excursions of the web. Those results, whose proofs can be skipped at first reading, will be the key ingredients in the proofs of Proposition 4.1 (in Section 7.4) and Theorem 6.2 (in Section 7.7). In Section 7.5, we provide a proof of the results from Section 5 on the equivalence between the marking and the hopping constructions of the Brownian net. In Section 7.6, we give a proof of a basic fact relating the BN to $(1,2)$ points of the BW — that every "point of separation" in the BN is (in our coupling of the BW and BN) also a $(1,2)$ point of the BW. We study some elementary properties of the separation points in the Brownian net, and apply those results in Section 7.7 to prove Proposition 6.1 about the existence of the dynamical Brownian web. We note that the results about separation points of the Brownian net had already been derived by one of us (E. S.) jointly with Sun and Swart and appears in a paper [20] by those three authors.

7.1. *Some results about the Brownian web*

We start by defining the age of a point (x,t) as

$$\sup\{t - t_B : B \in \mathcal{W} \text{ and } B(t) = x\}. \tag{7.1}$$

The γ-age truncation of the Brownian web is the set of paths obtained after shortening every path of $\mathcal{W}$ by removing (if necessary) the initial segment consisting of those points of age less than γ. In [8] it was proved that:

Proposition 7.1. *The γ-age truncation of $\mathcal{W}$ is "locally sparse" in the sense that for every bounded set U, the intersection between U and the γ-age truncation of $\mathcal{W}$ only consists of finitely many path segments.*

Two corollaries of that proposition can be formulated as follows:

Corollary 7.1. *Given B and $\{B_n\}$ in $\mathcal{W}$ so that $B_n \to B$ (in (Π, d)) then the coalescence time of B_n and B converges to the starting time of B.*

Proof. Let t be the starting time of B and take any $\bar{t} > t$. Let us consider the points z_n (resp., z) where B_n (resp., B) intersect the line $\mathbb{R} \times \{\bar{t}\}$. The toplogy of (Π, d) (see [7]) implies that the starting time of B_n converges to t. Hence, for $n \geq n_0$ with n_0 large enough, z_n has an age larger than $(\bar{t} - t)/2 > 0$. Moreover, since $z_n \to z$, the sequence $\{z_n\}$ belongs to a bounded segment of the line. By Proposition 1, we get that $\{z_n\}_{n \geq n_0}$ consist of only finitely many points. Therefore, z_n is fixed after a certain n and B_n coincides with B at $\bar{t}$. Since this is valid for any $\bar{t} > t$, the claim of Corollary 7.1 follows. □

Corollary 7.2. *Let B be a path in $\mathcal{W}$ starting at t_0. For any $\mathcal{D}$ as in Theorem 2.1 and $t > t_0$, on $[t, \infty)$ the path B coincides with a path of the skeleton (determined by $\mathcal{D}$).*

Proof. By definition, there exists B_n in the skeleton converging to B. The conclusion immediately follows from the previous corollary. □

7.2. *Existence of the local time measure*

In this section, we prove Proposition 3.1 on which is based the construction of the local time measure. For simplicity of notation, we assume $(x,t) = (0,0)$.

Let $(\bar{B}_1, \bar{B}_2)$ be two independent standard Brownian motion paths starting at $(0,0)$. We define $(B_{\mathrm{ind}}, \hat{B}_{\mathrm{ind}})$ as

$$\begin{aligned} &B_{\mathrm{ind}}(u) = \bar{B}_1(u), \\ &\hat{B}_{\mathrm{ind}}(u) = \hat{x} + \bar{B}_2(u) - \bar{B}_2(\hat{t}) \quad \text{for } u \in [0, \hat{t}\,]. \end{aligned} \tag{7.2}$$

Clearly, $(B_{\mathrm{ind}}, \hat{B}_{\mathrm{ind}})$ is a pair of independent forward and backward Brownian motions and we construct the system of refelected paths $(B, \hat{B})$ as in Proposition 2.3, i.e. $(B, \hat{B}) = (R_{\hat{B}_{\mathrm{ind}}}(B_{\mathrm{ind}}), \hat{B}_{\mathrm{ind}})$.

In the following, we will assume that $\hat{B}(0)(= \hat{B}_{\mathrm{ind}}(0) = \hat{x} - \bar{B}_2(\hat{t})) < 0$. The case $\hat{B}(0) > 0$ can be treated analogously, and $\hat{B}(0) = 0$ has zero probability. Let $R_0(\bar{B}_1 - \bar{B}_2)$ (resp., $R_0(B_{\mathrm{ind}} - \hat{B}_{\mathrm{ind}})$) be the Skorohod reflection of $\bar{B}_1 - \bar{B}_2$ (resp., $B_{\mathrm{ind}} - \hat{B}_{\mathrm{ind}}$) at zero, i.e.,

$$R_0(\bar{B}_1 - \bar{B}_2)(u) = (\bar{B}_1 - \bar{B}_2)(u) - \min_{[0,u]}(\bar{B}_1 - \bar{B}_2), \tag{7.3}$$

$$R_0(B_{\mathrm{ind}} - \hat{B}_{\mathrm{ind}})(u) = (B_{\mathrm{ind}} - \hat{B}_{\mathrm{ind}})(u) - 0 \wedge \min_{[0,u]}(B_{\mathrm{ind}} - \hat{B}_{\mathrm{ind}}) \tag{7.4}$$

$$= (B - \hat{B})(u). \tag{7.5}$$

Let T_0 be the first time $(B_{\mathrm{ind}} - \hat{B}_{\mathrm{ind}})$ hits 0. Since $(B_{\mathrm{ind}} - \hat{B}_{\mathrm{ind}})$ is a translation of $\bar{B}_1 - \bar{B}_2$ by $-\hat{B}_{\mathrm{ind}}(0) > 0$, Eq. (7.3) immediately implies that

$$R_0\big(B_{\mathrm{ind}}(u) - \hat{B}_{\mathrm{ind}}\big)(u) = R_0(\bar{B}_1 - \bar{B}_2)(u) \quad \forall u \ge T_0, \tag{7.6}$$

$$R_0\big(B_{\mathrm{ind}}(u) - \hat{B}_{\mathrm{ind}}\big)(u) \ne 0 \quad \forall u < T_0. \tag{7.7}$$

$(\bar{B}_1 - \bar{B}_2)/\sqrt{2}$ is a standard Brownian motion and it is a well known result (see, e.g., [15]) that $R_0(\bar{B}_1 - \bar{B}_2)/\sqrt{2}$ is distributed as the absolute value of a Brownian motion and its local time at 0, defined as

$$L(u) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \left| \left\{ v \le u \colon \frac{R_0(\bar{B}_1 - \bar{B}_2)(v)}{\sqrt{2}} < \varepsilon \right\} \right| \tag{7.8}$$

is equal to $-\min_{[0,u]}(\bar{B}_1 - \bar{B}_2)/\sqrt{2}$. This implies that the quantity

$$L_{B,\hat{B}}(u) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \left| \left\{ v \le u \colon \frac{1}{\sqrt{2}} R_0(B_{\mathrm{ind}} - \hat{B}_{\mathrm{ind}})(v) = \frac{1}{\sqrt{2}}(B - \hat{B})(v) < \varepsilon \right\} \right| \tag{7.9}$$

is well defined and moreover

$$L_{B,\hat{B}}(u) = L(u \vee T_0) - L(T_0) \tag{7.10}$$

$$= -\min_{[0,u\vee T_0]} \frac{\bar{B}_1 - \bar{B}_2}{\sqrt{2}} + \min_{[0,T_0]} \frac{\bar{B}_1 - \bar{B}_2}{\sqrt{2}} \tag{7.11}$$

$$= -0 \wedge \frac{\min_{[0,u]}(B_{\mathrm{ind}} - \hat{B}_{\mathrm{ind}})}{\sqrt{2}}. \tag{7.12}$$

This completes the proof of items 1 and 3 of Proposition 3.1.

Finally, item 2 follows from the fact (see Theorem 1 in [19]) that almost surely, the local time measure at zero of a Brownian motion is the ϕ-Hausdorff measure of its zero-set.

7.3. *Excursions*

To motivate this section, let us consider the pair $({}_{[1]}r_0, B_0)$ (see Section 4). On any interval of $\{t\colon B_0(t) \ne {}_{[1]}r_0(t)\}$, ${}_{[1]}r_0$ coincides with some path of the Brownian web other than B_0. Therefore, away from B_0, ${}_{[1]}r_0$ evolves as a

Brownian motion independent of B_0 (this is part of the proof in Section 7.4 below of Proposition 4.1, which describes the distribution of $({}_{[1]}r_0, B_0)$). Hence, to determine the distribution of $({}_{[1]}r_0, B_0)$, we will need to analyze how ${}_{[1]}r_0$ escapes from the diagonal $\{t\colon {}_{[1]}r_0(t) = B_0(t)\}$.

Let us define $t_\varepsilon^r = \inf\{s\colon ({}_{[1]}r_0 - B_0)(s) = \sqrt{2}\varepsilon\}$, the first time the pair $({}_{[1]}r_0, B_0)$ escapes from the $\sqrt{2}\varepsilon$-neighborhood of the diagonal. By construction, t_ε^r is also the first time any right marked excursion is at a spatial distance $\sqrt{2}\varepsilon$ from B_0. In Section 7.3.1, we give an explicit expression for the distribution of t_ε^r. In Section 7.3.2, we obtain asymptotics for $\mathbb{E}(t_\varepsilon^r)$ for small ε. This will be used to prove Proposition 4.1. Finally, we present Proposition 7.4 in Section 7.3.3 – a result relating left and right excursions from B_0. It will be used to prove Theorem 6.2(a) which describes the joint distribution of the dynamical Brownian web at two different dynamical times.

7.3.1. *Distribution of* t_ε^r

In this section, we will prove the following proposition.

Proposition 7.2. *Let* $|B|_0^\varepsilon(t)$ *be a Brownian motion on* $[0, \varepsilon]$, *starting at* 0 *and reflected at* 0 *and* ε *and let* $l_\varepsilon(t)$ *be its local time at level* ε. *Then*

$$\mathbb{P}\big(t_\varepsilon^r \le t\big) = \mathbb{P}\left(l_\varepsilon(t) \ge \operatorname{Exp}\left[\frac{1}{\sqrt{2}\tau}\right]\right), \tag{7.13}$$

where $\operatorname{Exp}[1/(\sqrt{2}\tau)]$ *is an exponential random variable with mean* $1/(\sqrt{2}\tau)$, *independent of* $|B|_0^\varepsilon$.

By definition, $t_\varepsilon^r \le t$ iff a marked excursion enters the region

$$U_{\varepsilon,t} = \big\{(x,u)\colon 0 \le u \le t, B_0(u) + \sqrt{2}\varepsilon \le x\big\}. \tag{7.14}$$

Equivalently, this condition can be re-expressed using the dual Brownian web.

Lemma 7.1. $t_\varepsilon^r \le t$ *iff there exists a backward path* $\hat{B}$ *starting from* $U_{\varepsilon,t}$ *and hitting* B_0 *at a marked point*.

Proof. To show the only if part of the lemma, assume there exists a right marked excursion e_r from B_0 and $0 \le s \le t$ such that $(e_r(s), s) \in U_{\varepsilon,t}$. One can then construct a sequence $\{\hat{B}_n\}$ in $\hat{\mathcal{W}}$ such that $\hat{B}_n$ starts at $(\hat{x}_n, \hat{t}_n)$ with $B_0(\hat{t}_n) < \hat{x}_n < e_r(\hat{t}_n)$ and $(\hat{x}_n, \hat{t}_n) \to (e_r(s), s)$. Since paths of the web and its dual do not cross, $\hat{B}_n$ is squeezed between e_r and B_0 and thus enters the marked starting point z of e_r. By compactness of $\hat{\mathcal{W}}$, $\hat{B}_n$ converges (along a subsequence) to some path $\hat{B} \in \hat{\mathcal{W}}$ starting at $(e_r(s), s) \in U_{\varepsilon,t}$ and entering the point z. The converse argument to prove the if part of the lemma is similar. □

We denote by $L_{\varepsilon,t}([t_1, t_2])$ the local time measure of all the points in $\mathbb{R} \times [t_1, t_2]$ where B_0 meets a backward path starting from $U_{\varepsilon,t}$. This naturally defines a measure $L_{\varepsilon,t}$ on $\mathbb{R}$ and we set $L_{\varepsilon,t}([0,t]) \equiv \tilde{l}_\varepsilon(t)$. By definition, the set of marked points at the intersection between B_0 and the set of backward paths starting from $U_{\varepsilon,t}$ is a Poisson point process with intensity $\sqrt{2}\tau\tilde{l}_\varepsilon(t)$. Hence,

$$\mathbb{P}\big(t_\varepsilon^r \le t\big) = \mathbb{P}\left(\tilde{l}_\varepsilon(t) \ge \operatorname{Exp}\left[\frac{1}{\sqrt{2}\tau}\right]\right), \tag{7.15}$$

where $\operatorname{Exp}[1/(\sqrt{2}\tau)]$ is independent of $\mathcal{W}$.

To study the measure $L_{\varepsilon,t}$, we introduce the (backward) process I_ε^t (see Fig. 3) defined as

$$\forall s \in [0,t] \quad I_\varepsilon^t(s) = \inf\big\{\hat{B}(s)\colon \hat{B} \in \hat{\mathcal{W}}, z(\hat{B}) \in U_{\varepsilon,t}\big\}, \tag{7.16}$$

where $z(\hat{B})$ denotes the starting point of $\hat{B}$.

Not surprisingly, the set of times when I_ε^t and B_0 coincide is the support of $L_{\varepsilon,t}$. This claim can be verified as follows. Because of the compactness of $\hat{\mathcal{W}}$, the time it takes for a path in $\hat{\mathcal{W}}$ starting from $U_{\varepsilon,t}$ to reach the curve B_0

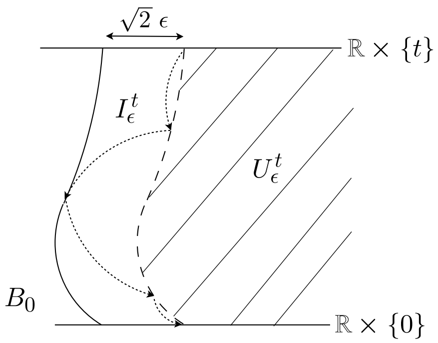

Fig. 3. The process I^t_ε is the left envelope of all the backward paths starting from the region U^t_ε.

is uniformly bounded away from 0. This means that the (backward) age of those paths (see (7.1)) is strictly positive and the claim follows directly from Proposition 7.1. Proposition 7.2 directly follows from (7.15) and the following lemma.

Lemma 7.2. *The process* $|B|_0^\varepsilon$ *defined on* $[0,t]$ *by*

$$|B|_0^\varepsilon(s) \equiv -\frac{1}{\sqrt{2}}\big(I^t_\varepsilon(t-s) - B_0(t-s) - \sqrt{2}\varepsilon\big)$$

is a Brownian motion on $[0,\varepsilon]$ *starting at* 0 *and reflected at* 0 *and* ε.

Proof. Let $\{\hat{B}_{k,n}\}_{n\in\mathbb{N},k\in\{0,\dots,2^n\}}$ be the family of backward paths starting from points of the form $z_{k,n} = (B_0(kt/2^n) + \sqrt{2}\varepsilon, kt/2^n)$. We define

$$\{\pi_{k,n}\} = \left\{\frac{-1}{\sqrt{2}}\big(\hat{B}_{k,n}(t-s) - B_0(t-s) - \sqrt{2}\varepsilon\big)\colon\ t - kt/2^n \le s \le t\right\}. \tag{7.17}$$

Clearly, $\{\pi_{k,n}\}$ starts from $\{(0, t - kt/2^n)\}$ and is identical in law with a family of forward coalescing Brownian motions Skorohod reflected at ε.

As can easily be seen, the process

$$_n|B|_0^\varepsilon(u) \equiv \sup\big\{\pi_{k,n}(u)\colon\ k \in \{0,\dots,2^n\}\big\} \tag{7.18}$$

converges pointwise to $|B|_0^\varepsilon$ as n goes to ∞.

Now, let us decompose the process $|B|_0^\varepsilon$ into its up and downcrossings (the first upcrossing is the section of the path on $[0, t^1_\varepsilon]$, where t^1_ε is the first time $|B|_0^\varepsilon$ hits ε; the first downcrossing is the section of the path between t^ε_1 and its return time to 0). We aim to prove that an upcrossing (resp., downcrossing) is a copy of an independent Brownian motion starting at 0 (resp., ε), reflected at 0 (resp., ε) and stopped when it hits ε (resp., 0). It is straightforward to show the equidistribution and independence of the up and downcrossings. The downcrossings have the required distribution because $|B|_0^\varepsilon$ coincides with $\pi_{k,n}$ for some n and k during a downcrossing. It remains to determine the law of the upcrossings. Let u_1 (resp., $u_{1,n}$) be the first upcrossing of the process $|B|_0^\varepsilon$ (resp., $_n|B|_0^\varepsilon$). $u_{1,n}$ is simply made of pieces of Brownian motions stopped if they hit ε. Let B (which depends on n) be the continuous process starting at 0 and obtained by gluing those pieces at their endpoints (see Fig. 4). By a simple induction, it is easy to see that

$$\forall t \in \big[k/2^n, (k+1)/2^n\big), \quad u_{1,n}(t) = B(t) - \inf\left\{B\left(\frac{j}{2^n}\right)\colon\ j = 0, 1/2^n, \dots, k/2^n\right\} \tag{7.19}$$

and by the Markov property, B is a Brownian motion stopped when $u_{1,n}$ hits ε. As $n \to \infty$, the right hand side of (7.19) converges in law to

$$B(t) - \inf_{[0,t]} B, \tag{7.20}$$

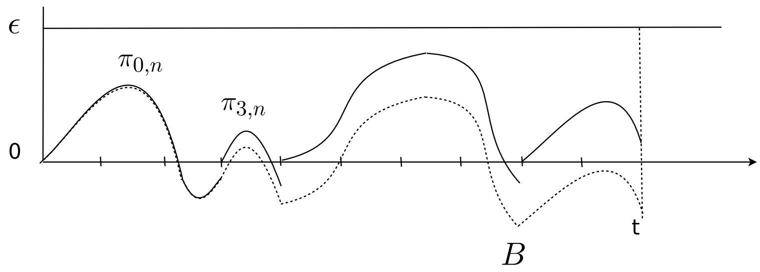

Fig. 4. The continuous dashed path B is constructed from the plain path ${}_n|B|_0^\varepsilon$.

where B is a Brownian motion stopped when $B(t) - \inf_{[0,t]} B$ hits ε. On the other hand, the left-hand side of (7.19) converges almost surely to u_1. Hence, the first upcrossing of $|B|_0^\varepsilon(s)$ is identical in distribution with that of a Brownian motion starting at 0, Skorohod reflected at 0 and stopped when it hits 1. □

7.3.2. *Rate of excursions from* B_0

In this subsection, we prove

Proposition 7.3. $\lim_{\varepsilon\downarrow 0} \mathbb{E}(t_\varepsilon^r)/\varepsilon = \frac{\sqrt{2}}{\tau}$ *and* $\mathbb{E}([t_\varepsilon^r]^2) = o(\varepsilon)$ *as* $\varepsilon \downarrow 0$.

We only prove the first claim. The second one can be proved along the same lines.

Let $\mathbb{P}_{\mathcal{W}}$ denote the probability distribution of the marked Brownian web conditioned on the web $\mathcal{W}$. By Proposition 7.2, we have the following.

$$\begin{aligned}\mathbb{E}(t_\varepsilon^r)/\varepsilon &= \int_0^\infty \mathbb{P}\big(t_\varepsilon^r > \varepsilon t\big)\,\mathrm{d}t = \int_0^\infty \mathbb{E}\big(\mathbb{P}_{\mathcal{W}}\big(t_\varepsilon^r > \varepsilon t\big)\big)\,\mathrm{d}t \\ &= \int_0^\infty \mathbb{E}\left(\mathbb{P}_{\mathcal{W}}\left(l_\varepsilon(\varepsilon t) < \mathrm{Exp}\left[\frac{1}{\sqrt{2}\tau}\right]\right)\right)\mathrm{d}t = \int_0^\infty \mathbb{E}\big(\exp\big[-\sqrt{2}\tau\cdot l_\varepsilon(\varepsilon t)\big]\big)\,\mathrm{d}t.\end{aligned}$$

To take the limit as $\varepsilon \to 0$, we will use the following lemma.

Lemma 7.3. *Let* $t, \gamma > 0$. *There exist* $c, C \in (0, \infty)$ *such that*

$$\mathbb{P}\left(\left|l_\varepsilon(\varepsilon t) - \frac{t}{2}\right| > \gamma t\right) \le C\exp\left(-c\frac{t\gamma}{\varepsilon}\right). \tag{7.21}$$

Hence, $l_\varepsilon(\varepsilon t)$ *converges in probability to* $t/2$ *as* $\varepsilon \to 0$.

Proof. We need to show that

$$\mathbb{P}\left(l_\varepsilon(\varepsilon t) - \frac{t}{2} > t\gamma\right) \le C\exp\left(-c\frac{t\gamma}{\varepsilon}\right), \tag{7.22}$$

$$\mathbb{P}\left(\frac{t}{2} - l_\varepsilon(\varepsilon t) > t\gamma\right) \le C\exp\left(-c\frac{t\gamma}{\varepsilon}\right). \tag{7.23}$$

We only prove the first inequality. The second one can be obtained using analogous arguments. Using the scaling invariance of Brownian motion, the first inequality reduces to

$$\mathbb{P}\left(\varepsilon l_1(t/\varepsilon) - \frac{t}{2} > t\gamma\right) \le C\exp\left(-c\frac{t\gamma}{\varepsilon}\right), \tag{7.24}$$

where $l_1(u)$ is identical in distribution to the local time accumulated on the set $\{x = 2j+1\}_{j\in\mathbb{Z}}$ at time u by a standard Brownian motion B. Define $t_0 = \inf\{s\colon\ B(s) = \pm 1\}$ and for $k \ge 1$, $t_k = \inf\{t \ge t_{k-1}\colon\ |B(t) - B(t_{k-1})| = 2\}$.

$\Delta t_k = t_{k+1} - t_k$ has mean 4. Furthermore, by excursion theory, the local times Δl_k accumulated on $\{x = 2j+1\}_{j\in\mathbb{Z}}$ during the time intervals $[t_k, t_{k+1}]$, for $k \geq 0$ are independent exponential random variables with mean 2.

Define $N_\varepsilon(t) = \inf\{k\colon\ t_k \geq t/\varepsilon\}$. Then, if we set $\gamma' = (1+\gamma)$ and $n = \frac{t}{4\varepsilon}(1+\gamma) = \frac{t\gamma'}{4\varepsilon}$,

$$\begin{aligned}\mathbb{P}\biggl(\varepsilon l_1(t/\varepsilon) - \frac{t}{2} > t\gamma\biggr) &\leq \mathbb{P}\bigl(N_\varepsilon(t) > n\bigr) + \mathbb{P}\biggl(\biggl[\varepsilon\sum_{k\leq n}\Delta l_k\biggr] - \frac{t}{2} > t\gamma\biggr)\\ &\leq \mathbb{P}\biggl(\sum_{k\leq n}\Delta t_k < \frac{t}{\varepsilon}\biggr) + \mathbb{P}\biggl(\varepsilon\sum_{k\leq n}[\Delta l_k - 2] > \frac{t\gamma}{2}\biggr)\\ &\leq \mathbb{P}\biggl(\frac{1}{n}\sum_{k\leq n}[4 - \Delta t_k] \geq \frac{4\gamma}{\gamma'} - \frac{16\varepsilon}{t\gamma'}\biggr) + \mathbb{P}\biggl(\frac{1}{n}\sum_{k\leq n}[\Delta l_k - 2] > \frac{2\gamma}{\gamma'}\biggr).\end{aligned}$$

Eq. (7.24) follows by classical large deviation estimates. □

To complete the analysis of $\lim_{\varepsilon\to 0}\mathbb{E}(t_\varepsilon^r)/\varepsilon$, we use Lemma 7.3 with $\gamma = 1/4$ to see that

$$\mathbb{E}\bigl(\exp\bigl(-\sqrt{2}\tau\cdot l_\varepsilon(\varepsilon t)\bigr)\bigr) \leq \exp\biggl(-\tau\frac{\sqrt{2}t}{4}\biggr) + \mathbb{P}\biggl(l_\varepsilon(\varepsilon t) \leq \frac{t}{4}\biggr) \tag{7.25}$$

$$\leq \exp\biggl(-\tau\frac{\sqrt{2}t}{4}\biggr) + C\exp\biggl(-c\frac{t}{4\varepsilon}\biggr). \tag{7.26}$$

It follows that the family $\{\mathbb{P}(t_\varepsilon^r \geq \varepsilon\cdot)\}_{\varepsilon\leq 1}$ is uniformly integrable. Therefore, by Lemma 7.3

$$\lim_{\varepsilon\downarrow 0}\int_0^\infty \mathbb{E}\bigl(\exp\bigl(-\sqrt{2}\tau\cdot l_\varepsilon(\varepsilon u)\bigr)\bigr)\,\mathrm{d}u = \int_0^\infty \lim_{\varepsilon\downarrow 0}\mathbb{E}\bigl(\exp\bigl(-\sqrt{2}\tau\cdot l_\varepsilon(\varepsilon u)\bigr)\bigr)\,\mathrm{d}u = \int_0^\infty \mathrm{e}^{-\sqrt{2}\tau\cdot u/2}\,\mathrm{d}u = \frac{\sqrt{2}}{\tau}.$$

This completes the proof of Proposition 7.3.

7.3.3. *Marked right and left excursions*

Let e_l be a left marked excursion from B_0. We say that $T(e_l)$ (the starting time of e_l) is straddled by the right excursion e_r iff $T(e_r) < T(e_l) < T(e_r) + D(e_r)$. In this subsection, we prove the following proposition.

Proposition 7.4. *Let $e_{l,\varepsilon}$ be the first left marked excursion from B_0 with diameter* (*see Definition* 4.2) *greater or equal to $\sqrt{2}\varepsilon$. Then*

$$\mathbb{P}\bigl(T(e_{l,\varepsilon}) \text{ is straddled by some right marked excursion } e_r\bigr) \to 0 \quad \text{as } \varepsilon\downarrow 0. \tag{7.27}$$

Define $A_\varepsilon \equiv \{T(e_{l,\varepsilon})$ is straddled by some right marked excursion $e_r\}$ and t_ε^l as the left analog of t_ε^r (so that t_ε^l is the first time t that $B_0(t) - e_{l,\varepsilon}(t) = \sqrt{2}\varepsilon$ and therefore $t_\varepsilon^l \geq T(e_{l,\varepsilon})$).

For any $H > 0$,

$$\begin{aligned}\limsup_{\varepsilon\downarrow 0}\mathbb{P}(A_\varepsilon) &\leq \limsup_{\varepsilon\downarrow 0}\mathbb{P}\bigl(A_\varepsilon, t_\varepsilon^l \leq \varepsilon H\bigr) + \limsup_{\varepsilon\downarrow 0}\mathbb{P}\bigl(t_\varepsilon^l > \varepsilon H\bigr)\\ &\leq \limsup_{\varepsilon\downarrow 0}\mathbb{P}\bigl(A_\varepsilon, t_\varepsilon^l \leq \varepsilon H\bigr) + \limsup_{\varepsilon\downarrow 0}\frac{\mathbb{E}(t_\varepsilon^l)}{\varepsilon H}\\ &= \limsup_{\varepsilon\downarrow 0}\mathbb{P}\bigl(A_\varepsilon, t_\varepsilon^l \leq \varepsilon H\bigr) + \frac{\sqrt{2}}{\tau H},\end{aligned}$$

where the equality follows from Proposition 7.3 and the the identity $\mathbb{E}(t_\varepsilon^r) = \mathbb{E}(t_\varepsilon^l)$. Since H can be made arbitrarily large, in order to prove Proposition 7.4 it suffices to show that for any $H > 0$,

$$\limsup_{\varepsilon\downarrow 0} \mathbb{P}\big(A_\varepsilon, t_\varepsilon^l \le \varepsilon H\big) = 0. \tag{7.28}$$

Let $\varepsilon_n = (\varepsilon H)/2^n$ for $n \ge 0$ and let $\varepsilon_{-1} = +\infty$. Breaking up A_ε accordingly to the duration of the excursion e_r straddling $T(e_{l,\varepsilon})$, we have

$$\begin{aligned} \mathbb{P}\big(A_\varepsilon, t_\varepsilon^l \le \varepsilon H\big) &= \mathbb{P}\big(\exists \text{ a right marked excursion } e_r \text{ with} \qquad (7.29)\\ &\qquad T(e_r) \le T(e_{l,\varepsilon}) \le T(e_r) + D(e_r), t_\varepsilon^l \le \varepsilon H\big) \\ &= \sum_{n\ge -1} \mathbb{P}\big(C_n'\big) \le \sum_{n\ge -1} \mathbb{P}(C_n), \qquad (7.30)\end{aligned}$$

where

$$\begin{aligned} C_n' &= \big\{\exists \text{ a right marked excursion } e_r \text{ with } D(e_r) \in [\varepsilon_{n+1}, \varepsilon_n) \text{ s.t.} \\ &\qquad T(e_r) \le T(e_{l,\varepsilon}) \le T(e_r) + \varepsilon_n, t_\varepsilon^l \le \varepsilon H\big\}, \qquad (7.31)\\ C_n &= \big\{\exists \text{ a right marked excursion } e_r \text{ with } D(e_r) \ge \varepsilon_{n+1} \text{ s.t.} \\ &\qquad T(e_r) \le T(e_{l,\varepsilon}) \le T(e_r) + \varepsilon_n, T(e_{l,\varepsilon}) \le \varepsilon H\big\}. \qquad (7.32)\end{aligned}$$

Let $\mathbb{P}_{L,\mathcal{W}}$ be the probability distribution of the marked Brownian web conditioned on $\mathcal{W}$ and the marking of the left $(1,2)$ points. Since given the Brownian web, the markings of the left and the right $(1,2)$ points are independent, we get

$$\begin{aligned} \mathbb{P}_{L,\mathcal{W}}(C_n) &= \sqrt{2}\tau \mathcal{L}_{D(e_r)\ge\varepsilon_{n+1}}\big(\mathbb{R} \times \big[T(e_{l,\varepsilon}) - \varepsilon_n, T(e_{l,\varepsilon})\big]\big) \qquad (7.33)\\ &= \sqrt{2}\tau L_{D(e_r)\ge\varepsilon_{n+1}}\big(\big[T(e_{l,\varepsilon}) - \varepsilon_n, T(e_{l,\varepsilon})\big]\big), \qquad (7.34)\end{aligned}$$

where $\mathcal{L}_{D(e_r)\ge\varepsilon_{n+1}}$ is the local time measure on the possible starting points in $\mathbb{R}^2$ of a right excursion from B_0 with $D(e_r) \ge \varepsilon_{n+1}$, and $L_{D(e_r)\ge\varepsilon_{n+1}}$ is the projection of that measure along the t-axis. Let $n \ge 0$. Since $T(e_{l,\varepsilon}) \in [0, \varepsilon H]$, there exists $k \in \{-1, 0, \dots, 2^n - 2\}$ such that

$$\big[T(e_{l,\varepsilon}) - \varepsilon_n, T(e_{l,\varepsilon})\big] \subset T_{k,n} \quad \text{with } T_{k,n} = \big[k\varepsilon_n, (k+2)\varepsilon_n\big]. \tag{7.35}$$

Hence,

$$\begin{aligned} \mathbb{P}_{L,\mathcal{W}}(C_n) &\le \sqrt{2}\tau \max_{-1\le k\le 2^n-2} L_{D(e_r)\ge\varepsilon_{n+1}}(T_{k,n}) \qquad (7.36)\\ &\le \sqrt{2}\tau \max_{0\le k\le 2^n} L_{D(e_r)\ge\varepsilon_{n+1}}(T_{k,n}), \qquad (7.37)\end{aligned}$$

where we used the equality $\mathcal{L}(\mathbb{R} \times T_{k,n}) = \mathcal{L}(\mathbb{R} \times [k\varepsilon_n \vee 0, (k+2)\varepsilon_n \vee 0])$ to deduce the second inequality. Note that with the convention that $T_{0,-1} = [0, \varepsilon H]$, the formula above also remains valid for $n = -1$. Averaging over the realizations of $\mathcal{W}$ and the marking of left $(1,2)$ points, we obtain that for any $p \ge 1$,

$$\begin{aligned} \mathbb{P}(C_n) &\le \sqrt{2}\tau \mathbb{E}\Big(\max_{0\le k\le 2^n} L_{D(e_r)\ge\varepsilon_{n+1}}(T_{k,n})\Big) \qquad (7.38)\\ &\le C_p \Big|\max_{0\le k\le 2^n} L_{D(e_r)\ge\varepsilon_{n+1}}(T_{k,n})\Big|_p, \qquad (7.39)\end{aligned}$$

where C_p is a finite positive constant and $|X|_p$ denotes the L_p norm of X w.r.t. $\mathbb{P}$.

Lemma 7.4. *For any $p \geq 1$ there exists $K < \infty$ s.t. for $n \geq -1$,*

$$\Big| \max_{0 \leq k \leq 2^n} L_{\{D(e_r) \geq \varepsilon_{n+1}\}}(T_{k,n}) \Big|_p \leq K 2^{n(1/p-1/2)} \sqrt{\varepsilon H}. \tag{7.40}$$

Proof. We prove the lemma for $n \geq 0$. The case $n = -1$ (where $T_{0,-1} = [0, \varepsilon H]$) can be treated analogously. By translation invariance of the marked Brownian web,

$$\mathbb{P}\big[L_{\{D(e_r) \geq \varepsilon_{n+1}\}}\big(\big[k\varepsilon_n, (k+2)\varepsilon_n\big]\big) > x\big] = \mathbb{P}\big[L_{\{D(e_r) \geq \varepsilon_{n+1}\}}\big([0, 2\varepsilon_n]\big) > x\big]. \tag{7.41}$$

Therefore,

$$\mathbb{P}\Big(\max_{0 \leq k \leq 2^n} L_{\{D(e_r) \geq \varepsilon_{n+1}\}}(T_{k,n}) > x\Big) \leq 2^{n+1} \mathbb{P}\big(L_{\{D(e_r) \geq \varepsilon_{n+1}\}}\big([0, 2\varepsilon_n]\big) > x\big).$$

The scaling invariance of the Brownian web under the mapping on paths, $B \rightsquigarrow \lambda^{-1/2} B(\lambda t)$, yields (for $a_0, b_0 \geq 0$) the equidistribution of $L_{\{D(e_r) \geq a_0 \lambda\}}([0, b_0 \lambda])$ and $\sqrt{\lambda} L_{\{D(e_r) \geq a_0\}}([0, b_0])$. Hence

$$L_{\{D(e_r) \geq \varepsilon_{n+1}\}}\big([0, 2\varepsilon_n]\big) =_d \sqrt{\frac{\varepsilon H}{2^n}} L_{\{D(e_r) \geq 1/2\}}\big([0, 2]\big),$$

which, using the standard identity that $(|X|_p)^p$ equals $\int_0^\infty p x^{p-1} \mathbb{P}(|X| > x)\, \mathrm{d}x$, implies that

$$\begin{aligned} &\Big| \max_{0 \leq k \leq 2^n} L_{\{D(e_r) \in T_n\}}(T_{k,n}) \Big|_p \\ &\quad \leq 2^{1/p} 2^{n/p} \left(p \int_0^\infty x^{p-1} \mathbb{P}\left[L_{\{D(e_r) > 1/2\}}\big([0, 2]\big) > x \sqrt{\frac{2^n}{\varepsilon H}} \right] \mathrm{d}x \right)^{1/p} \\ &\quad = 2^{1/p} 2^{n(1/p-1/2)} \sqrt{\varepsilon H} \big| L_{\{D(e_r) > 1/2\}}\big([0, 2]\big) \big|_p. \end{aligned}$$

To complete the proof, we need to show that for any $p \geq 1$, $|L_{\{D(e_r) > 1/2\}}([0, 2])|_p < \infty$.

We use the fact (see, e.g., [7]) that for any $s > 0$, there are two distinct dual Brownian paths starting from $(B_0(s), s)$, those two paths being separated by the path B_0. In order for $s \in [0, 2]$ to be in the support of $L_{\{D(e_r) > 1/2\}}$, B_0 must be hit by a (dual) path of $\hat{\mathcal{W}}$ starting in the region $\{(x, t)\colon x \geq B_0(t), t \geq s + 1/2\}$. At any such time s, there must be an integer k in $\{1, \ldots, 10\}$ such that B_0 is hit by $\hat{B}_{k/4}$, the dual path starting at $(B_0(k/4), k/4)$ and located to the right of B_0. This implies that $L_{\{D(e_r) > 1/2\}}$ is bounded above by the local time measure induced by the finite family of backward paths $\{\hat{B}_{k/4}\}_{k \leq 10}$. From [22] (see Proposition 2.3 above), the process

$$s \to \hat{B}_{k/4}(k/4 - s) - B_0(k/4 - s) \tag{7.42}$$

defined on $[0, k/4]$ is a Brownian motion reflected at 0 and the local time measure $L_{B_0, \hat{B}_{k/4}}$ is just the usual local time measure at the origin of that reflecting Brownian motion. It is a standard fact that local time at the origin has all moments and Lemma 7.4 follows. □

Combining (7.29)–(7.30), (7.38), (7.39) and Lemma 7.4 for any $p > 2$, there exists $C'_p < \infty$ s.t.

$$\mathbb{P}\big(A_\varepsilon, t_\varepsilon^l \leq \varepsilon H\big) \leq C'_p \sqrt{\varepsilon H}, \tag{7.43}$$

so that (7.28) and hence Proposition 7.4 follow.

7.4. *Distribution of* $(B_0, {}_{[1]}r_z)$ (*proof of Proposition* 4.1)

First, we prove the following lemma.

Lemma 7.5. *The family* $\{(B_0, {}_{[1]}r_z)\}_{z\in\mathbb{R}\times\{0\}}$ *of random pairs of continuous paths is a family of strong Markov processes with stationary transition probabilities.*

More precisely, for any stopping time T, *conditioned on the past of the paths up to* T, *i.e., conditioned on* $\{\mathcal{F}_T\}$ (*where* $\mathcal{F}_t$ *is the* σ*-field generated by* $\{(B_0(s), {}_{[1]}r_0(s))\}_{s\le t}$ *and* $\{\mathcal{F}_T\}$ *is defined accordingly*),

$$\big(B_0(t+T) - B_0(T), {}_{[1]}r_z(t+T) - B_0(T)\big)_{t\ge 0}$$

is distributed like $(B_0, {}_{[1]}r_{z(T)})$ *with* $z(T) = {}_{[1]}r_z(T) - B_0(T)$.

Proof. We take $z = (0, 0)$, first prove the weak Markov property and then the strong Markov property. The proof can trivially be extended to any deterministic z.

Weak Markov property: Recall that $\mathcal{L}_{1,n}$ is the natural local time measure on the set E_n defined as

$$E_n = B_0 \cap \left(\bigcup_{i=0}^{n-1} \hat{B}_i\right).$$

For the time being, $T > 0$ is deterministic. In the following, E_n^- will denote the subset of $E_n \cap \{t \le T\}$ consisting of all the points on B_0 hit by a path $\hat{B}_i$ starting from $z_i = (x_i, t_i)$ with $i \le n-1$ and $t_i \le T$. E_n^+ will refer to $E_n \cap \{t \ge T\}$. Finally, we define ${}_{(1,n)}\bar{r}_0$ as the path constructed from B_0 by switching the direction of the marked left $(1, 2)$ points in $E_n^+ \cup E_n^-$.

Let $\mathcal{L}^+_{(1,n)}$ (resp., $\mathcal{L}^-_{(1,n)}$) be the measure $\mathcal{L}_{(1,n)}$ restricted to E_n^+ (resp., E_n^-). First, conditioned on the Brownian web, the markings of E_n^+ and E_n^- are two independent Poisson point processes with respective intensity measures $\mathcal{L}^+_{(1,n)}$ and $\mathcal{L}^-_{(1,n)}$. Second, $(\mathcal{L}^+_{(1,n)}, \{{}_{(1,n)}\bar{r}_0(t)\}_{t\ge T})$ (resp., $\mathcal{L}^-_{(1,n)}$) is measurable w.r.t.

$$\big(\mathcal{W}_{[T,\infty]}, B_0(T), {}_{(1,n)}\bar{r}_0(T)\big) \quad \text{(resp., } \mathcal{W}_{[-\infty,T]}),$$

where $\mathcal{W}_{[t_1,t_2]}$ is the set of paths in $\mathcal{W}$ starting in the window $[t_1, t_2]$ and stopped at t_2. By independence of $\mathcal{W}_{[T,\infty)}$ and $\mathcal{W}_{[-\infty,T]}$, the future evolution of $(B_0, {}_{(1,n)}\bar{r}_0)$ is independent of its past given $(B_0(T), {}_{(1,n)}\bar{r}_0(T))$. Assuming momentarily that ${}_{(1,n)}\bar{r}_0$ converges pointwise to ${}_{[1]}r_0$, it is straightforward to show that ${}_{[1]}r_0$ also continues afresh at T provided that the distribution of $(B_0, {}_{[1]}r_{\bar{z}})$, with $\bar{z} = (\bar{x}, 0)$, is continuous with respect to $\bar{x}$. This we will do next. The stationarity of transition probabilities in Lemma 7.5 then simply follows from the translation invariance of the marked Brownian web.

We now prove that $(B_0, {}_{[1]}r_{(\bar{x},0)})$ is continuous with respect to $\bar{x}$. Let $\bar{z}_n = (\bar{x}_n, 0) \to \bar{z}$. We distinguish between two cases:

1. $\bar{z} = (\bar{x}, 0)$ with $\bar{x} \ne 0$. Before meeting B_0, ${}_{[1]}r_{\bar{z}}$ (resp., ${}_{[1]}r_{\bar{z}_n}$) follows $B_{\bar{z}}$ (resp., $B_{\bar{z}_n}$), the path in $\mathcal{W}$ starting from $\bar{z}$ (resp., $\bar{z}_n$). For n large enough, $B_{\bar{z}}$ and $B_{\bar{z}_n}$ coalesce at some time μ_n before either of those paths meets B_0. Hence, ${}_{[1]}r_{\bar{z}}$ and ${}_{[1]}r_{\bar{z}_n}$ coalesce at time μ_n with $\mu_n \to 0$ as $n \uparrow \infty$.
2. $\bar{z} = (0, 0)$. For any $\gamma > 0$, we can always find a marked left $(1, 2)$ point at $(B_0(t), t)$ for some $t \in [0, \gamma]$. Let $\hat{B} \in \hat{\mathcal{W}}$ pass through that mark and let (x_M, t_M) be the earliest of the marks along $\hat{B}$. Since almost surely $(0, 0)$ is not a $(1, 2)$ point, t_M is strictly positive and for n large enough $0 < x_n < \hat{B}(0)$. For n large enough, $B_{\bar{z}_n}$ coalesces with B_0 before t_M. By construction, ${}_{[1]}r_{z_n}$ and ${}_{[1]}r_0$ can only cross $\hat{B}$ at a marked point on $B_0 \cap \hat{B}$. Since t_M is the earliest marked point on $\hat{B}$, ${}_{[1]}r_{\bar{z}_n}$ and ${}_{[1]}r_0$ are squeezed between B_0 and $\hat{B}$ on $[0, t_M]$ and thus they meet (and coalesce) by $t_M \le \gamma$.

For the weak Markov property, it remains to prove that ${}_{(1,n)}\bar{r}_0$ converges to ${}_{[1]}r_0$. Recall that the excursions of ${}_{[1]}r_0$ from B_0 coincide with the maximal excursions from B_0 (see Definition 4.2). First, let e be a maximal excursion starting at some z. For n large enough, it is clear that z belongs to $E_n^+ \cup E_n^-$. By definition of a maximal excursion, ${}_{(1,n)}\bar{r}_0$ hits z and then follows e. Second, let z' be the starting point of a marked excursion e' which is not maximal and hence is nested in some maximal excursion. For n large enough, ${}_{(1,n)}\bar{r}_0$ follows that maximal excursion and therefore misses the excursion e'. Hence, in the limit, the excursions of ${}_{(1,n)}\bar{r}_0$ coincide with the maximal excursions from B_0, and thus ${}_{(1,n)}\bar{r}_0$ converges pointwise to ${}_{[1]}r_0$.

Strong Markov property: Now let T be a stopping time with respect to the right-continuous filtration $\{\mathcal{F}_t\}$ and let T_n be the following discrete approximation of T:

$$\text{if } T \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right), \quad T_n = \frac{k+1}{2^n}. \tag{7.44}$$

T_n is a discrete stopping time and the weak Markov property implies that Lemma 7.5 is also valid for T_n. $\{(B_0(t+T_n) - B_0(T_n), {}_{[1]}r_0(t+T_n)) - B_0(T_n)\}_{t\geq 0}$ converges pathwise to $\{(B_0(t+T) - B_0(T), {}_{[1]}r_0(t+T) - B_0(T))\}_{t\geq 0}$ as $n \to \infty$. The result now follows from the distributional continuity of $(B_0, {}_{[1]}r_{(\bar{x},0)})$ with respect to $\bar{x}$ that we have already established. □

Next, we claim that the pair $(B_0, {}_{[1]}r_z)$ satisfies the three following properties:

(1) B_0 is a standard Brownian path starting at $(0,0)$. ${}_{[1]}r_z$ starts at z.
(2) Away from the diagonal $\{t\colon {}_{[1]}r_z(t) = B_0(t)\}$, the two processes evolve as two independent Brownian motions.
(3) Defining $t^r_\varepsilon \equiv \inf\{t > 0\colon |{}_{[1]}r_0 - B_0|(t) = \sqrt{2}\varepsilon\}$ satisfies:
 (i) $\mathbb{P}(({}_{[1]}r_0 - B_0)(t^r_\varepsilon) = +\sqrt{2}\varepsilon) = 1$,
 (ii) $\lim_{\varepsilon\downarrow 0} \mathbb{E}(t^r_\varepsilon)/\varepsilon = \sqrt{2}/\tau$ and $\mathbb{E}([t^r_\varepsilon]^2) = o(\varepsilon)$ as $\varepsilon \downarrow 0$.

In words, (1) and (2) describe the pair $(B_0, {}_{[1]}r_z)$ away from the diagonal. (3) describes the splitting mechanism when $(B_0, {}_{[1]}r_z)$ is on the diagonal. 3(i) says that ${}_{[1]}r_z$ always escapes the diagonal to the right. (Note that the definition of t^r_ε given in (3) is consistent with the one given in Section 7.4 as the first time ${}_{[1]}r_0 - B_0$ hits $+\sqrt{2}\varepsilon$.) 3(ii) specifies the rate at which $(B_0, {}_{[1]}r_z)$ escapes the diagonal. We note that this approach is very similar to the one in [13].

We now turn to the verification of (1)–(3) for $(B_0, {}_{[1]}r_z)$. Property (1) is obviously satisfied. Property (2) follows directly from Lemma 7.5 and the definition of ${}_{[1]}r_z$. Property (3)(i) is obvious. Property 3(ii) is given by Proposition 7.3 above.

Next, we verify that if $(\bar{B}_0, {}_{[1]}\bar{r}_z)$ is a solution of the SDE (4.2), it also satisfies conditions (1)–(3).

Lemma 7.6. *Let $(\bar{B}_0, {}_{[1]}\bar{r}_z)$ be a solution of the SDE* (4.2). *Then $(\bar{B}_0, {}_{[1]}\bar{r}_z)$ is a strong Markov process with stationary transition probabilities and it satisfies conditions* (1)–(3).

Proof. As discussed in Section 4, the SDE (4.2) has a unique weak solution which implies that $(\bar{B}_0, {}_{[1]}\bar{r}_z)$ is a strong Markov process. The stationarity property is obvious and $(\bar{B}_0, {}_{[1]}\bar{r}_z)$ obviously satisfies properties (1), (2) and (3)(i). It remains to verify 3(ii).

Since $B_{\text{stick}} \equiv ({}_{[1]}\bar{r}_0 - \bar{B}_0)/\sqrt{2}$ is a $(\sqrt{2}/\tau)$-sticky Brownian motion, it is identical in law with

$$t \rightsquigarrow |B|\bigl(C(t)\bigr), \quad \text{where } C^{-1}(t) = t + \frac{\sqrt{2}}{\tau} L_0(t),$$

where $|B|$ is a reflected Brownian motion and L_0 is its local time at the origin. Therefore, t^r_ε (for $({}_{[1]}\bar{r}_0 - \bar{B}_0)$) is distributed like

$$\frac{\sqrt{2}}{\tau} L_0(T_\varepsilon) + T_\varepsilon,$$

where T_ε is the first time $|B|$ hits ε. By excursion theory, $L_0(T_\varepsilon)$ is an exponential random variable with mean ε. Since the distribution of T_ε is that of $\varepsilon^2 T_1$, we indeed get

$$\mathbb{E}\bigl(t^r_\varepsilon\bigr)/\varepsilon \to \frac{\sqrt{2}}{\tau} \quad \text{and} \quad \mathbb{E}\bigl(\bigl[t^r_\varepsilon\bigr]^2\bigr) = o(\varepsilon). \tag{7.45}$$

□

Finally, we prove the following uniqueness result which is the last ingredient needed to prove Proposition 4.1. This result is analog to Proposition 16 in [13].

Lemma 7.7. *Let* $\{(B_0, {}_{[1]}r_z)\}_{z\in\mathbb{R}\times\{0\}}$ *and* $\{(\bar{B}_0, {}_{[1]}\bar{r}_z)\}_{z\in\mathbb{R}\times\{0\}}$ *be two families of strong Markov processes, with stationary transition probabilities, satisfying properties* (1)–(3). *For* $z=(x,0)$, $B_{\text{stick},x}\equiv(B_0-{}_{[1]}r_z)/\sqrt{2}$ *and* $\bar{B}_{\text{stick},x}\equiv(\bar{B}_0-{}_{[1]}\bar{r}_z)/\sqrt{2}$ *are equidistributed.*

Proof. By stationarity of the transition probabilities and the Markov property, $B_{\text{stick},x}$ or $\bar{B}_{\text{stick},x}$ can be decomposed into two independent parts. The first part is a Brownian motion stopped when it hits zero while the second one is distributed like $B_{\text{stick},0}$ or $\bar{B}_{\text{stick},0}$. Hence, it is enough to show that $B_{\text{stick}}\equiv B_{\text{stick},0}$ and $\bar{B}_{\text{stick}}\equiv\bar{B}_{\text{stick},0}$ are equidistributed. Also by the Markov property and the stationarity of the transition probabilities, it is enough to show that for any $s\ge 0$, $B_{\text{stick}}(s)$ and $\bar{B}_{\text{stick}}(s)$ are equidistributed. We also note that by property (3)(i), B_{stick} and $\bar{B}_{\text{stick}}$ are ≥ 0.

In the following, X denotes either B_{stick} or $\bar{B}_{\text{stick}}$ and f is a positive bounded continuous function vanishing on the interval $[0,\varepsilon_0]$, with $\varepsilon_0>0$. For any $\varepsilon<\varepsilon_0$, define $t_\varepsilon^0=0$ and, for any $k\ge 0$,

$$t_\varepsilon^{2k+1}\equiv\inf\{t>t_\varepsilon^{2k}:\ |X|(t)=\varepsilon\},\qquad t_\varepsilon^{2k+2}\equiv\inf\{t>t_\varepsilon^{2k+1}:\ X(t)=0\}. \tag{7.46}$$

We have

$$\mathbb{E}\left(\int_0^\infty \mathrm{e}^{-\lambda s} f(X(s))\,\mathrm{d}s\right)=\sum_{k=1}^\infty \mathbb{E}\left(\int_{t_\varepsilon^{2k-1}}^{t_\varepsilon^{2k}} f(X(s))\mathrm{e}^{-\lambda s}\,\mathrm{d}s\right) \tag{7.47}$$

$$=\mathbb{E}\left(\int_{t_\varepsilon^1}^{t_\varepsilon^2} f(X(s))\mathrm{e}^{-\lambda s}\,\mathrm{d}s\right)\sum_{k=0}^\infty \mathbb{E}\left(\mathrm{e}^{-\lambda t_\varepsilon^{2k}}\right). \tag{7.48}$$

Next,

$$t_\varepsilon^{2k}=\sum_{i=0}^{k-1}\left(\left[t_\varepsilon^{2i+2}-t_\varepsilon^{2i+1}\right]+\left[t_\varepsilon^{2i+1}-t_\varepsilon^{2i}\right]\right).$$

By stationarity and the Markov property, we get that

$$\mathbb{E}\left(\mathrm{e}^{-\lambda t_\varepsilon^{2k}}\right)=\left(\mathbb{E}\left(\mathrm{e}^{-\lambda t_\varepsilon^1}\right)\right)^k\left(\mathbb{E}\left(\mathrm{e}^{-\lambda[t_\varepsilon^2-t_\varepsilon^1]}\right)\right)^k. \tag{7.49}$$

This implies that

$$\mathbb{E}\left(\int_0^\infty \mathrm{e}^{-\lambda s} f(X(s))\,\mathrm{d}s\right)=\mathbb{E}\left(\int_{t_\varepsilon^1}^{t_\varepsilon^2} f(X(s))\mathrm{e}^{-\lambda s}\,\mathrm{d}s\right)\frac{1}{1-\mathbb{E}(\mathrm{e}^{-\lambda t_\varepsilon^1})\cdot\mathbb{E}(\mathrm{e}^{-\lambda[t_\varepsilon^2-t_\varepsilon^1]})}. \tag{7.50}$$

Moreover, since

$$\lim_{\varepsilon\downarrow 0}\frac{\mathbb{E}(t_\varepsilon^r)}{\varepsilon}=\frac{\sqrt{2}}{\tau}\quad\text{and}\quad \mathbb{E}\left(\left[t_\varepsilon^r\right]^2\right)=\mathrm{o}(\varepsilon) \tag{7.51}$$

and $t_\varepsilon^1=t_\varepsilon^r$, it follows that

$$\mathbb{E}\left(\mathrm{e}^{-\lambda t_\varepsilon^1}\right)=1-\frac{\sqrt{2}\lambda}{\tau}\varepsilon+\mathrm{o}(\varepsilon). \tag{7.52}$$

During $[t_\varepsilon^1,t_\varepsilon^2]$, the process coincides with a Brownian motion starting at ε and stopped when it hits 0. By standard computations, we get that

$$\mathbb{E}\left(\mathrm{e}^{-\lambda[t_\varepsilon^2-t_\varepsilon^1]}\right)=\mathrm{e}^{-\sqrt{2\lambda}\varepsilon}. \tag{7.53}$$

Combining equations (7.50), (7.52) and (7.53), we obtain

$$\mathbb{E}\biggl(\int_0^\infty \mathrm{e}^{-\lambda s} f\bigl(X(s)\bigr)\,\mathrm{d}s\biggr) = \frac{\mathbb{E}(\int_{t_\varepsilon^1}^{t_\varepsilon^2} f(X(s))\mathrm{e}^{-\lambda s}\,\mathrm{d}s)}{\varepsilon}\sqrt{2}\biggl(\sqrt{\lambda}+\frac{\lambda}{\tau}+o(1)\biggr)^{-1}. \tag{7.54}$$

Since the left-hand side of the equality does not depend on ε,

$$\frac{\mathbb{E}(\int_{t_\varepsilon^1}^{t_\varepsilon^2} f(X(s))\mathrm{e}^{-\lambda s}\,\mathrm{d}s)}{\varepsilon} \tag{7.55}$$

has a limit $l(X)$, depending on f, as $\varepsilon \to 0$ and

$$\mathbb{E}\biggl(\int_0^\infty \mathrm{e}^{-\lambda s} f\bigl(X(s)\bigr)\,\mathrm{d}s\biggr) = \int_0^\infty \mathrm{e}^{-\lambda s}\mathbb{E}\bigl(f\bigl(X(s)\bigr)\bigr)\,\mathrm{d}s = \sqrt{2}l(X)\biggl(\sqrt{\lambda}+\frac{\lambda}{\tau}\biggr)^{-1}. \tag{7.56}$$

Futhermore, using the various defining properties of B_{stick} and $\bar{B}_{\text{stick}}$,

$$\begin{aligned} l(X) &= \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\mathbb{E}\biggl(\mathrm{e}^{-\lambda t_\varepsilon^1}\int_0^{t_\varepsilon^2-t_\varepsilon^1} f\bigl(X\bigl(u+t_\varepsilon^1\bigr)\bigr)\mathrm{e}^{-\lambda u}\,\mathrm{d}u\biggr) \\ &= \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\mathbb{E}\bigl(\mathrm{e}^{-\lambda t_\varepsilon^1}\bigr)\mathbb{E}\biggl(\int_0^{t_\varepsilon^2-t_\varepsilon^1} f\bigl(X\bigl(u+t_\varepsilon^1\bigr)\bigr)\mathrm{e}^{-\lambda u}\,\mathrm{d}u\biggr) \\ &= \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\mathbb{E}\biggl(\int_0^T f\bigl(\varepsilon+B(v)\bigr)\mathrm{e}^{-\lambda v}\,\mathrm{d}v\biggr), \end{aligned}$$

where B is a standard Brownian motion and T is the first time it hits $-\varepsilon$. (The second equality follows from the strong Markov property while the third one follows from $\lim_{\varepsilon\downarrow 0}\mathbb{E}(\mathrm{e}^{-\lambda t_\varepsilon^1}) = 1$ and also from the fact that on $[t_\varepsilon^1, t_\varepsilon^2]$, X evolves like a Brownian motion.)

Thus $l(B_{\text{stick}}) = l(\bar{B}_{\text{stick}})$ and therefore

$$\int_0^\infty \mathrm{e}^{-\lambda s}\mathbb{E}\bigl(f\bigl(B_{\text{stick}}(s)\bigr)\bigr)\,\mathrm{d}s = \int_0^\infty \mathrm{e}^{-\lambda s}\mathbb{E}\bigl(f\bigl(\bar{B}_{\text{stick}}(s)\bigr)\bigr)\,\mathrm{d}s.$$

Inverting the Laplace transform yields that for every s and every positive bounded continuous function f vanishing on the interval $[0, \varepsilon_0]$, $\mathbb{E}(f(B_{\text{stick}}(s))) = \mathbb{E}(f(\bar{B}_{\text{stick}}(s)))$. By the monotone convergence theorem, we can remove the constraint $f(x) = 0$ for $x \in [0, \varepsilon_0]$ which implies that $B_{\text{stick}}(s)$ and $\bar{B}_{\text{stick}}(s')$ are equidistributed. □

Lemma 7.7 shows that the distribution of $(B_0 - {}_{[1]}r_z)/\sqrt{2}$ is determined by the three properties stated above. By Lemma 7.6, it follows that $({}_{[1]}r_0 - B_0)/\sqrt{2}$ is a $(\sqrt{2}/\tau)$-sticky Brownian motion. The proof of Proposition 4.1 is a consequence of the following observation. Let $z = (x, 0)$. A pair $(B_0, {}_{[1]}r_z)$ satisfying properties (1) and (2) and such that $({}_{[1]}r_z - B_0)/\sqrt{2}$ is a $(\sqrt{2}/\tau)$-sticky Brownian motion satisfies the SDE (4.2).

Proposition 4.1 being now established, we end this section with a possibly surprising theorem about the exit time of a sticky Brownian motion. Combining Proposition 7.2 and Proposition 4.1, we have

Theorem 7.5. *Let B_{stick} be a Brownian motion starting at* 0 *and stickily reflected at* 0 *with an amount of stick $\bar{\tau}$. If t_ε is the first ε hitting time of B_{stick}*:

$$\mathbb{P}(t_\varepsilon \le t) = \mathbb{P}\biggl(l_\varepsilon(t) \ge \operatorname{Exp}\biggl[\frac{1}{2\bar{\tau}}\biggr]\biggr),$$

where l_ε is the local time at level ε at time t of a Brownian motion on $[0, \varepsilon]$, *starting at* 0, *reflected at* 0 *and* ε, *and* $\operatorname{Exp}(1/(2\bar{\tau}))$ *is an independent exponential random variable with mean* $1/(2\bar{\tau})$.

7.5. *The Brownian net by marking*

The heuristics described in Section 5.2 are made rigorous in this subsection.

Proof of Proposition 5.3. We set $i = 0$ (with $z_i = 0$) as the proof for general i is essentially the same. Recall that ${}_{[n]}r_0$ and $\{B_k\}_{k\le n-1}$ are coupled via the SDE (5.4). We start by proving that for such a coupling we have

Lemma 7.8. $\forall t \ge 0, \mathbb{P}(t \in \bigcap_{j=0}^{n-1}\{s\colon r_0(s) \ne B_j(s)\}) \to 0$ *as* $n \to \infty$.

Proof. Let ε be a fixed positive number. We define

$$x_n^\varepsilon = \sup\bigl\{B_j(t-\varepsilon)\colon\ B_j(t-\varepsilon) \le r_0(t-\varepsilon) \text{ for } j \le n-1\bigr\}.$$

Let B^ε be the path in $\{B_i\}_{i=0}^{n-1}$ such that $B^\varepsilon(t-\varepsilon) = x_n^\varepsilon$. For any $s \ge t-\varepsilon$, we define

$$\Delta^\varepsilon(s) = \frac{1}{\sqrt{2}}\bigl(r_0 - B^\varepsilon\bigr)(s).$$

By (5.7), conditioned on the past of $(r_0, B_0, \dots, B_{n-1})$ up to time $t-\varepsilon$, Δ^ε solves the following SDE, where B is a standard Brownian motion.

$$\mathrm{d}\Delta^\varepsilon(s) = 1_{\Delta^\varepsilon \ne 0}\,\mathrm{d}B(s) + \frac{\tau}{\sqrt{2}}\,\mathrm{d}s, \quad \Delta^\varepsilon(t-\varepsilon) = x_n^\varepsilon. \tag{7.57}$$

Δ^ε is a drifting Brownian motion stickily reflected at 0 and

$$\mathbb{P}\Biggl(t \in \bigcap_{j=0}^{n-1}\bigl\{s\colon\ r_0(s) \ne B_j(s)\bigr\}\Biggr) \le \mathbb{P}\bigl(r_0(t) \ne B^\varepsilon(t)\bigr) = \mathbb{P}\bigl(\Delta^\varepsilon(t) \ne 0 | \Delta^\varepsilon(t-\varepsilon) = x_n^\varepsilon\bigr).$$

Since $x_n^\varepsilon \to 0$ as $n \to \infty$,

$$\limsup_{n\to\infty} \mathbb{P}\Biggl(t \in \bigcap_{j=0}^{n-1}\bigl\{s\colon\ r_0(s) \ne B_j(s)\bigr\}\Biggr) \le \mathbb{P}\bigl(\Delta^\varepsilon(t) \ne 0 | \Delta^\varepsilon(t-\varepsilon) = 0\bigr) = \mathbb{P}\bigl(\Delta^\varepsilon(\varepsilon) \ne 0 | \Delta^\varepsilon(0) = 0\bigr).$$

Note that the process $\tilde{\Delta}^\varepsilon$ defined by $\mathrm{d}\tilde{\Delta}^\varepsilon = \mathrm{d}\Delta_\varepsilon - \frac{\tau}{\sqrt{2}} 1_{\Delta_\varepsilon \ne 0}\,\mathrm{d}t$ is a $(\sqrt{2}/\tau)$-sticky Brownian motion. For such a process, it is known (see e.g., [3]) that $\mathbb{P}(\tilde{\Delta}^\varepsilon(\varepsilon) \ne 0 | \tilde{\Delta}^\varepsilon(0) = 0) \to 0$ as $\varepsilon \to 0$. By a straightforward application of the Girsanov theorem, we see that $\mathbb{P}(\Delta^\varepsilon(\varepsilon) \ne 0 | \Delta^\varepsilon(0) = 0) \to 0$ as $\varepsilon \to 0$ and Lemma 7.8 follows. □

Let $t > 0$ and $\mathbb{P}^t_{{}_{[n]}r_0, B_k}$ be the probability measure induced by the pair $({}_{[n]}r_0, B_k)$ on the space of continuous functions on $[0,t]$ endowed with its usual Borel σ-algebra. $\mathbb{P}^t_{r_0, B_k}$ is defined analogously as the distribution of the pair satisfying (5.7). We first prove that

$$\mathbb{P}^t_{{}_{[n]}r_0, B_k} \Longrightarrow \mathbb{P}^t_{r_0, B_k} \quad \text{as } n \to \infty. \tag{7.58}$$

We define ${}_n\chi(t) = 1_{t \in \bigcap_{j=0}^{n-1}\{s\colon r_0(s) \ne B_j(s)\}}$. Lemma 7.8 above and Fubini's theorem imply that

$$\mathbb{E}\biggl(\int_0^t {}_n\chi\bigl(t'\bigr)\,\mathrm{d}t'\biggr) \to 0. \tag{7.59}$$

For $n \ge k$, the SDEs (5.4) and (5.7) only differ by their drift term. By the Girsanov theorem, $\mathbb{P}^t_{{}_{[n]}r_0, B_k}$ is absolutely continuous with respect to $\mathbb{P}^t_{r_0, B_k}$ and

$$\mathrm{d}\mathbb{P}^t_{{}_{[n]}r_0, B_k} = \mathrm{d}\mathbb{P}^t_{r_0, B_k} \exp\biggl(-\tau \int_0^t {}_n\chi(t')\,\mathrm{d}r_0(t') + \frac{\tau^2}{2}\int_0^t {}_n\chi(t')\,\mathrm{d}t'\biggr). \tag{7.60}$$

Since r_0 is a (drifting) Brownian motion, (7.59) and standard arguments imply that the term in the exponential tends to zero in probability. It follows that

$$\mathbb{P}^t_{[n]r_0, B_k} \Longrightarrow \mathbb{P}^t_{r_0, B_k} \quad \text{as } n \to \infty. \tag{7.61}$$

The pointwise convergence of ${}_{[n]}r_0$ to r_0 was already explained in Section 5.2 by the fact that ${}_{[n]}r_0$ is monotonic in n. This completes the proof of Proposition 5.3. □

Proof of Proposition 5.4. It is easy to see from Proposition 5.3 that $\mathcal{W}_r$ (resp., $\mathcal{W}_l$) is a right-drifting (resp., left-drifting) Brownian web (it is enough to check that two paths in $\mathcal{W}_r$ evolve independently when they are apart; this can been done by simple locality arguments). It remains to prove that $\mathcal{W}_r$ and $\mathcal{W}_l$ interact in the sticky way of a left–right Brownian web (see [23] and Section 5.1 above). This boils down to proving that (r_i, l_j) satisfies the four-part SDE (5.1). For simplicity, let us take $i = j = 0$. Other cases can be treated similarly. We already know that r_0 and l_0 satisfy

$$\mathrm{d}r_0 = \mathrm{d}B_0^r + \tau\,\mathrm{d}t, \tag{7.62}$$

$$\mathrm{d}l_0 = \mathrm{d}B_0^l - \tau\,\mathrm{d}t, \tag{7.63}$$

and that $r_0 \ge l_0$. It remains to show that $\mathrm{d}\langle B_0^r, B_0^l\rangle(t) = 1_{r_0=l_0}(t)\,\mathrm{d}\langle B_0, B_0\rangle(t) = 1_{r_0=l_0}(t)\,\mathrm{d}t$. As can easily be seen, r_0 and l_0 evolve independently away from each other. Therefore,

$$\begin{aligned} \mathrm{d}\langle B_0^r, B_0^l\rangle(t) &= 1_{r_0=l_0}(t)\,\mathrm{d}\langle B_0^r, B_0^l\rangle(t) + 1_{r_0\neq l_0}(t)\,\mathrm{d}\langle B_0^r, B_0^l\rangle(t) && (7.64)\\ &= 1_{r_0=l_0}(t)\,\mathrm{d}\langle B_0^r, B_0^l\rangle(t). && (7.65) \end{aligned}$$

B_0 is squeezed between r_0 and l_0. Hence, $r_0(t) = l_0(t)$ implies that $B_0(t) = r_0(t) = l_0(t)$. Since, by Proposition 5.3,

$$\mathrm{d}\langle B_0, B_0^r\rangle(t) = 1_{r_0(t)=B_0(t)}\,\mathrm{d}t, \tag{7.66}$$

$$\mathrm{d}\langle B_0, B_0^l\rangle(t) = 1_{l_0(t)=B_0(t)}\,\mathrm{d}t, \tag{7.67}$$

(7.64) implies, as desired, that

$$\mathrm{d}\big\langle B_0^r, B_0^l\big\rangle(t) = 1_{r_0=l_0}(t)\,\mathrm{d}\langle B_0, B_0\rangle(t) = 1_{r_0=l_0}(t)\,\mathrm{d}t. \tag{7.68}$$

□

Proof of Theorem 5.5. In the proof we will also consider $\mathcal{N}_{\text{wedge}}$, the net obtained from $(\mathcal{W}_r, \mathcal{W}_l, \hat{\mathcal{W}}_r, \hat{\mathcal{W}}_l)$ by the wedge construction of Section 5.1. Here $\hat{\mathcal{W}}_r$ and $\hat{\mathcal{W}}_l$ are respectively the dual (backward) webs of $\mathcal{W}_r, \mathcal{W}_l$ (constructed by marking) which can be constructed using the dual versions of Propositions 5.3 and 5.4.

Since $\mathcal{N}_{\text{hop}} = \mathcal{N}_{\text{wedge}}$ (see Theorem 5.1), it suffices to show that (i) $\mathcal{N}_{\text{mark}} \supset \mathcal{N}_{\text{hop}}$ and (ii) $\mathcal{N}_{\text{mark}} \subset \mathcal{N}_{\text{wedge}}$.

In order to prove (i), we need to show that a path obtained by hopping from $\mathcal{W}_r$ to $\mathcal{W}_l$ (or $\mathcal{W}_l$ to $\mathcal{W}_r$) is still in $\mathcal{N}_{\text{mark}}$. Take two paths r_i and l_j intersecting at time t; we need to show that the concatenation of r_i (before t) with l_j (after t) is in $\mathcal{N}_{\text{mark}}$ and similarly for the other concatenation. First, if we consider the analogous question in a partial net $\mathcal{N}_n$ the result is obviously true. Indeed, if ${}_nr_i$ and ${}_nl_j$ are respectively the right- and left-most paths of $\mathcal{N}_n$ starting from z_i and z_j, the path constructed by hopping from one path to the other at some meeting point is in $\mathcal{N}_n$. Let $\varepsilon > 0$ be fixed. Almost surely, there is some $u \in [t, t+\varepsilon]$ such that $r_i(u) > l_j(u)$. Taking n large enough, we get ${}_nr_i(u) > {}_nl_j(u)$. On the other hand, ${}_nr_i \le r_i$ and ${}_nl_j \ge l_j$ so that ${}_nr_i(t) \le {}_nl_i(t)$. Consequently, there exists $v \in [t, u]$ where ${}_nl_j$ and ${}_nr_i$ intersect. Now consider the path obtained by hopping from ${}_nr_i$ to ${}_nl_j$ at time v. This path is in $\mathcal{N}_n$ and approximates the one obtained by hopping from r_i to l_j at time t except on $[t, t+\varepsilon]$. Since ε is arbitrary, the latter path is approximated by paths in $\bigcup_n \mathcal{N}_n$ and therefore it also belongs to $\mathcal{N}_{\text{mark}}$.

We now prove (ii). Consider a wedge constructed from a pair $(\hat{r}_i, \hat{l}_j)$ starting at $((x_i, t), (x_j, t))$ with $x_i < x_j$ and let us assume there exists $\pi \in \mathcal{N}_{\text{mark}}$ entering this wedge from outside and show that this leads to a contradiction. Again, we can approximate $(\hat{r}_i, \hat{l}_j)$ by $({}_n\hat{r}_i, {}_n\hat{l}_j) \in \hat{\mathcal{N}}_n \times \hat{\mathcal{N}}_n$ and π by $\pi_n \in \mathcal{N}_n$. Since ${}_n\hat{r}_i \ge \hat{r}_i$ and ${}_n\hat{l}_j \le \hat{l}_j$, the pair $({}_n\hat{r}_i,\ {}_n\hat{l}_j)$ forms a "partial wedge" approximating the original wedge from inside. Hence, for n large enough, π_n would enter this partial wedge from outside. By considering separately the cases where the putative entering is at a marked $(1, 2)$ point of $\mathcal{M}_n$ or not, such an entry is seen to be impossible. □

7.6. *Separation points in the Brownian net*

In Section 3.3 we defined the dynamical Brownian web as the limit of partial dynamical webs. In this subsection, we give a series of results which will guarantee the existence of such a limit. These are essentially identical to results in [20]. However, in [20] the proofs rely on the hopping construction of the Brownian net, while in this paper we show the results by using the marking approach. As we shall see, the two points of view are rather different. We start with a definition.

Definition 7.1 (Separation points). *Two paths π_1 and π_2 in $\mathcal{N}$ starting respectively at (x_1, t_1) and (x_2, t_2) separate at $z = (x, t)$ iff $t > t_1 \vee t_2$ with $\pi_1(t) = \pi_2(t)$ and there exists $a > 0$ such that π_1, π_2 do not touch on $(t, t + a]$. A point z is called a separation point of $\mathcal{N}$ iff there is some $\pi_1, \pi_2 \in \mathcal{N}$ that separate at z.*

Note that in the partial Brownian net $\mathcal{N}_n$ paths separate at marked $(1, 2)$ points. That remains valid in the Brownian net (i.e. when $n \to \infty$). Indeed, in Section 7.6.1, we prove the following result.

Proposition 7.6. *The set of separation points in $\mathcal{N}_{\text{mark}}$ and the set of marked $(1, 2)$ points of the Brownian web coincide.*

Furthermore, in Section 7.6.2 we prove the following proposition, which uses the notation $\pi \sim^z B$ and $\pi \sim^z B_{\text{switch}}$ introduced in Section 6.

Proposition 7.7. *Let $z = (x, t)$ be a separation point in $\mathcal{N}_{\text{mark}}$, B be any path of $\mathcal{W}$ passing through z, and $\mathcal{N}_{\le t-\varepsilon}$ be the set of paths in $\mathcal{N}_{\text{mark}}$ starting before or at time $t - \varepsilon$. For any $\varepsilon \ge 0$, define the following (which will not depend on the choice of $B \in \mathcal{W}$).*

$$[\sim^z_\varepsilon B_{\text{switch}}] = \{\pi \in \mathcal{N}_{\le t-\varepsilon} \colon \pi \text{ enters } z \text{ and } \pi \sim^z B_{\text{switch}}\},$$

$$[\sim^z_\varepsilon B] = \{\pi \in \mathcal{N}_{\le t-\varepsilon} \colon \pi \text{ enters } z \text{ and } \pi \sim^z B\},$$

$$\nparallel^z_\varepsilon = \{\pi \in \mathcal{N}_{\le t-\varepsilon} \colon \pi \text{ does not enter } z\}.$$

1. *Let E_z be the set of paths in $\mathcal{N}_{\text{mark}}$ entering z. $\sim^z$ is an equivalence relation on E_z, and E_z can be decomposed into the two equivalence classes $[\sim^z_0 B_{\text{switch}}]$ and $[\sim^z_0 B]$.*
2. *For $\varepsilon > 0$ (note the strict inequality), $[\sim^z_\varepsilon B_{\text{switch}}]$, $[\sim^z_\varepsilon B]$ and $\nparallel^z_\varepsilon$ are disjoint elements of $\mathcal{H}$.*
3. *$\exists \bar{z} \in \mathbb{R}^2$ and $\varepsilon > 0$ s.t. every path of $\mathcal{W}$ starting in the ball $B(\bar{z}, \varepsilon)$ enters z.*

We note that in the partial net, each path entering a marked point z coincides either with B or B_{switch} for a positive interval of time. In the full net limit, a path coincides either with B or B_{switch} for a positive Lebesgue measure of time.

7.6.1. *Proof of Proposition 7.6*
By construction, marked points are separation points so we only need to prove the converse.

Definition 7.2 ((T_1, T_2) separation points). *(x, t) with $T_1 < t < T_2$ is said to be a (T_1, T_2) separation point iff there are two paths π_1 and π_2 in the net starting from $\mathbb{R} \times \{T_1\}$ and separating at (x, t) which do not touch on $(t, T_2]$.*

Let T_1, T_2 be two rational numbers. It suffices to prove that if (x, t) is a (T_1, T_2) separation point of $\mathcal{N}_{\text{mark}}$, then it is a marked $(1, 2)$ point. Let π_1 and π_2 be two paths as described in Definition 7.2. Since the net is closed under hopping, we can assume without loss of generality that π_1 and π_2 have been chosen to coincide up to t.

By construction, there exist $\{\pi_i^n\}_{i=1,2}$ with π_i^n in the partial net $\mathcal{N}_n (= \mathcal{N}_{n,n}$; see Section 3.3.1) so that $\{\pi_i^n\}$ converges to π_i. Let us take two numbers $T_1 < q_1 < q_2 \le t$ where q_2 is arbitrarily close to t. Proposition 7.8 below (for $S = q_1$ and $T = q_2$), implies that $\pi_1^n(q_2) = \pi_2^n(q_2)$ for large enough n.

Hence, for large enough n, π_1^n and π_2^n start below $\mathbb{R} \times \{q_1\}$ and separate at a point arbitrarily close to (x,t). Since the set of (q_1, T_2) separation points is locally finite (see Proposition 7.9 below), π_1^n and π_2^n separate at (x,t) for large enough n. By construction, π_1^n and π_2^n only separate at marked points and Proposition 7.6 follows.

Proposition 7.8 ([23]). *For any S, T with $S < T$, the set of intersection points between the line $\mathbb{R} \times T$ and the set paths of $\mathcal{N}$ starting on or below $\mathbb{R} \times \{S\}$ is (almost surely) locally finite.*

Proposition 7.9 ([20]). *For any S, T with $S < T$, the set of (S,T)-separation points is (almost surely) locally finite.*

7.6.2. *Proof of Proposition 7.7*

In the following, for any paths π_1, π_2 in (Π, d) entering a point z, we will write $\pi_1 \sim^z_{\text{out}} \pi_2$ (resp., $\pi_1 \sim^z_{\text{in}} \pi_2$) iff for any $\varepsilon > 0$,

$$\int_t^{t+\varepsilon} 1_{\pi_1(u)=\pi_2(u)}\, \mathrm{d}u > 0 \quad \left(\text{resp., } \int_{t-\varepsilon}^{t} 1_{\pi_1(u)=\pi_2(u)}\, \mathrm{d}u > 0\right).$$

Note that $\pi_1 \sim^z \pi_2$ iff $\pi_1 \sim^z_{\text{out}} \pi_2$ and $\pi_1 \sim^z_{\text{in}} \pi_2$. In order to prove Proposition 7.7, we will use the following result from [20]. Since this result is part of a much larger theorem there, we provide a direct proof. For a "pictorial" representation of the result, see Fig. 5.

Theorem 7.10 ([20]). *Let $z = (x,t)$ be a separation point in $\mathcal{N}$ and let $\varepsilon > 0$. There exist three distinct meshes $M_l(r,l)$, $M_r(r',l')$ and $M_{\text{top}}(r'',l'')$ such that:*

1. *The bottom times of $M_l(r,l)$, $M_r(r',l')$ are in $(t-\varepsilon, t)$ and their top times are in (t,∞). Moreover, $l \le r'$ (at coexistence times of M_r and M_l), $l(t) = r'(t) = x$ and $l \sim^z_{\text{in}} r'$.*
2. *z is the bottom point of $M_{\text{top}}(r'',l'')$. $M_{\text{top}}(r'',l'')$ is squeezed between $M_l(r,l)$ and $M_r(r',l')$ (i.e., $l \le r''$ and $l'' \le r'$ at respective coexistence times). Moreover, $r'' \sim^z_{\text{out}} l$, $r' \sim^z_{\text{out}} l''$.*

Proof. In the following, we say that two paths π_1 and π_2 meet at time $\bar{t}$ iff $\pi_1(\bar{t}) = \pi_2(\bar{t})$ but $\pi_1 < \pi_2$ or $\pi_1 > \pi_2$ on $(\bar{t} - a, \bar{t})$ for some $a > 0$.

Construction of M_r and M_l. Recall that we constructed the Brownian net by marking a non-drifting Brownian web. There is an alternative marking construction of the net which can be described as follows. Start with a *left-drifting* Brownian web $\mathcal{W}_l$, with drift $-\tau$. Mark the $(1,2)$ points of $\mathcal{W}_l$ and construct $\mathcal{N}_l$ by branching at all the left $(1,2)$ points (of $\mathcal{W}_l$) in $\mathcal{M}_l(2\tau)$, the set of marks whose dynamical time coordinate is $\le 2\tau$ (the factor 2 compensates for

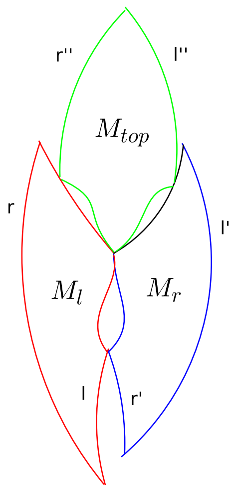

Fig. 5. Structure of meshes around a separation point.

the $-\tau$ drift in $\mathcal{W}_l$). On the one hand, repeating step by step what was done in Section 5 (see Theorem 5.5), one can show that $\mathcal{N}_l$ is identical in law to the usual $\mathcal{N}_{\text{hop}}$, as in Section 5.1. On the other hand, following the proof of Proposition 7.6, separation points of the net $\mathcal{N}_l$ must be marked left $(1,2)$ points of $\mathcal{W}_l$. Hence, separation points of the net $\mathcal{N}_{\text{hop}}$ are left $(1,2)$ points of $\mathcal{W}_l$ and symmetrically they are also right $(1,2)$ points of $\mathcal{W}_r$.

One consequence is that $z=(x,t)$ must be a separation point for two paths $\bar{l}\in\mathcal{W}_l$ and $\bar{r}\in\mathcal{W}_r$ starting from deterministic points. Lemma 6.5 in [23] analyzes meshes to the left of a path $\bar{l}\in\mathcal{W}_l$. Using that lemma and the fact that points on $\bar{l}$ where other paths from $\mathcal{W}_l$ coalesce with $\bar{l}$ from the left are dense in $\bar{l}$ (along with the analogous results for $\bar{r}$), it follows that there exists a mesh $M_l(r,l)$ (resp., $M_r(r',l')$) with bottom time in $(t-\varepsilon,t)$ and top time $>t$ such that $l(t)=\bar{l}(t)=x$ (resp., $r'(t)=\bar{r}(t)=x$).

By Corollary 7.2, l and r' coalesce with some paths l_i and r_j (in the skeleton of $\mathcal{W}_l$ and $\mathcal{W}_r$ respectively) before entering the point z. The pair (l_i,r_j) satisfies the SDE (5.1) and in particular, $l_i\le r_j$ from the first time they meet. It is clear that l_i and r_j do not meet and separate at the same point. Hence, there exists $a'>0$ so that $l_i\le r_j$ on $[t-a',\infty)$ and a sequence $t_n\uparrow t$ s.t. $l_i(t_n)=r_j(t_n)$. It immediately follows that there exists a'' with $a'\ge a''>0$ so that $l\le r'$ on $[t-a'',\infty)$ and a sequence $t'_n\uparrow t$ s.t. $l(t'_n)=r'(t'_n)$. Lemma 7.9 below then immediately implies that $l\sim^z_{\text{in}} r'$.

Construction of M_{top}. Up to reversal of the time coordinate, the backward Brownian net is distributed as the Brownian net (see Section 5.1). Hence, by what has been just proved, z is a separation point for two paths $(\hat{l},\hat{r})\in(\hat{\mathcal{W}}_l,\hat{\mathcal{W}}_r)$ and there exists $a>0$ such that $\hat{r}\le\hat{l}$ on $(-\infty,t+a]$. Let r'' (resp., l'') be the newly born path of $\mathcal{W}_r$ (resp., $\mathcal{W}_l$) starting from z. Since (x,t) is a right $(1,2)$ point for $\mathcal{W}_r$ and a left $(1,2)$ point for $\mathcal{W}_l$, we get that on $(t,t+a]$

$$r''\le\hat{r}\le\hat{l}\le l''\quad\text{and}\quad r''\le r',\, l\le l''. \tag{7.69}$$

M_{top} is defined as the mesh $M_{\text{top}}(r'',l'')$ formed by r'' and l''.

The second part of (7.69) implies that M_{top} is either squeezed between M_r and M_l or it contains either l or r'. Since paths of $\mathcal{N}$ do not enter meshes from outside, we get that on $(t,t+a]$

$$l\le r''\le\hat{r}\le\hat{l}\le l''\le r'. \tag{7.70}$$

Recall the construction of the net $\mathcal{N}_l$ (described at the beginning of this proof) based on the marking of a left-drifting Brownian web and let $(l,\hat{l})$ be a pair of paths in $(\mathcal{W}_l,\hat{\mathcal{W}}_l)$. As can be easily seen, the set

$$\big\{(x,t)\colon\ l(t)=\hat{l}(t)=x \text{ and } \exists a>0 \text{ s.t. } \forall s\in(t,t+a), l(s)<\hat{l}(s)\big\}$$

has zero local time measure. (By Proposition 2.3 and taking the difference between l and $\hat{l}$, this follows from the fact that, for a standard Brownian motion B, the set

$$\big\{t\colon\ B(t)=0 \text{ and } \exists a>0 \text{ s.t. } \forall s\in(t,t+a), \big|B(s)\big|>0\big\}$$

has zero local time measure.) Since in $\mathcal{N}_l$, separation points are left marked $(1,2)$ points, the argument just given implies that for every *marked* point, there exists $t_n\downarrow t$ such that $l(t_n)=\hat{l}(t_n)$. By (7.70), $l\le r''\le\hat{l}$, implying that $l(t_n)=r''(t_n)$. By Lemma 7.9 below we have that $l\sim^z_{\text{out}} r''$ and by a similar argument, we get $r'\sim^z_{\text{out}} l''$. □

Lemma 7.9. *Let $(l,r)\in(\mathcal{W}_l,\mathcal{W}_r)$ be such that for some $t>t_r\vee t_l$, $l(t)=r(t)$. For any $\varepsilon>0$, $\int_{t-\varepsilon}^{t+\varepsilon}1_{l(s)=r(s)}\,\mathrm{d}s>0$.*

Proof. Choose any t' with $t_r\vee t_l<t'<t-\varepsilon$. By Corollary 7.2, on $[t',\infty)$, the pair (l,r) coincides with a pair (L,R) of $(\mathcal{W}_l,\mathcal{W}_r)$, starting from deterministic points and satisfying the SDE (5.1). Lemma 7.9 then follows from the fact (see Proposition 3.1 in [23]) that the support of the measure μ, defined as $\mu([t_1,t_2])=|\{t\in[t_1,t_2]\colon\ L(t)=R(t)\}|$, coincides with $\{t\colon\ L(t)=R(t)\}$. □

We now prove the first two claims of Proposition 7.7 for a separation point $z=(x,t)$. Note that if claim 2 holds for a given ε, it immediately holds for any $\varepsilon'>\varepsilon$. Hence, w.l.o.g., we can take $\varepsilon>0$ small enough such that there is a path $B\in\mathcal{W}$ entering z and starting at $t'\le t-\varepsilon$. In the following, $\tilde{E}_\varepsilon$ will denote the subset of $\mathcal{N}_{\le t-\varepsilon}$ consisting of all the paths entering z.

Recall that paths of $\mathcal{N}$ do not enter meshes (see Theorem 5.1(b3) in Section 5.1). Hence, for any given mesh M with bottom time in $(t-\varepsilon,\infty)$, we can partition $\mathcal{N}_{\leq t-\varepsilon}$ into $\{R(M), L(M)\}$, where $R(M)$ (resp., $L(M)$) is the compact subset of $\mathcal{N}_{\leq t-\varepsilon}$ consisting of all the paths passing to the right (resp., left) of M. Let M_r, M_l and M_{top} be as in Theorem 7.10 and let us define

$$\tilde{E}^r_\varepsilon = \big[L(M_r)\cap R(M_l)\big]\cap R(M_{\text{top}}), \qquad \tilde{E}^l_\varepsilon = \big[L(M_r)\cap R(M_l)\big]\cap L(M_{\text{top}}), \tag{7.71}$$

$$\tilde{E}^c_\varepsilon = R(M_r)\cup L(M_l). \tag{7.72}$$

In particular $\{\tilde{E}^r_\varepsilon, \tilde{E}^l_\varepsilon\}$ (resp., $\{\tilde{E}^r_\varepsilon, \tilde{E}^l_\varepsilon, \tilde{E}^c_\varepsilon\}$) defines a natural partition of $\tilde{E}_\varepsilon$ (resp., $\mathcal{N}_{\leq t-\varepsilon}$) into elements of $\mathcal{H}$.

By definition, paths in $\tilde{E}^l_\varepsilon$ are squeezed between l and r' below z while they are squeezed between l and r'' above z. Hence, Theorem 7.10 immediately implies that for any two paths $\pi_1,\pi_2\in\tilde{E}^l_\varepsilon$, $\pi_1\sim^z\pi_2$. The same property holds for $\tilde{E}^r_\varepsilon$. Conversely, if $\pi_l\in\tilde{E}^l_\varepsilon$ and $\pi_r\in\tilde{E}^r_\varepsilon$, the two paths separate at z. This implies that $\sim^z$ is an equivalence relation on $\tilde{E}_\varepsilon$ and the corresponding equivalence classes are given by $\tilde{E}^r_\varepsilon$ and $\tilde{E}^l_\varepsilon$. Since B and B_{switch} separate at z they do not belong to the same equivalence class and claims 1 and 2 of Proposition 7.7 follow.

Next, we say that the ball $B(\bar{z},\bar{\varepsilon})$ with $\bar{z}=(\bar{x},\bar{t})$ is squeezed between between l and r' iff $\bar{t}-\bar{\varepsilon}\geq t_l\vee t_{r'}$ and for every $(x',t')\in B(\bar{z},\bar{\varepsilon})$, $l(t')\leq x'\leq r'(t')$. It is clear that one can find such a ball below the point z and that any path starting from that ball is squeezed between l and r' and so is forced to enter the point z. Claim 3 of Proposition of 7.7 follows.

7.7. *The dynamical Brownian web*

7.7.1. *Proof of Proposition* 6.1

In the following, we use the notation of Proposition 7.7.

By compactness of $\mathcal{N}(\tau)$, $\{\mathcal{W}_{(n,m)}(\tau)\}_{(n,m)}$ is a precompact subset of $\mathcal{H}$. Let $\mathcal{W}_1$ be any subsequential limit of $\{\mathcal{W}_{(n,m)}(\tau)\}_{(n,m)}$ as $n,m\to\infty$ and let

$$\mathcal{W}_2(\tau)=\big\{\pi\in\mathcal{N}_{\text{mark}}(\tau)\text{: every time } \pi \text{ enters a point } z \text{ in } \mathcal{M}(\tau), \pi\sim^z B_{\text{switch}}\big\}$$

be as in item (2) of Proposition 6.1. We first prove

(i) $\mathcal{W}_1(\tau)\subset\mathcal{W}_2(\tau)$.

Let $z=(x,t)\in\mathcal{M}(\tau)$, and let $\pi\in\mathcal{W}_1(\tau)$ start at $t-2\varepsilon$ with $\varepsilon>0$, and pass through z. By definition, there exists a sequence $\{\pi_N\}_{N\geq 0}$ so that π_N belongs to $\bigcup_{n,m>N}\mathcal{W}_{n,m}(\tau)$ and $\{\pi_N\}$ converges to π. Taking N large enough, we can assume w.l.o.g. that π_N belongs to $\mathcal{N}_{\leq t-\varepsilon}$ and $(x,t)\in\mathcal{M}_{n,m}(\tau)$ for $n,m>N$. By Proposition 7.7(2), π_N enters z for N large enough. By construction, $\pi_N\sim^z B_{\text{switch}}$ and since $\pi_N\to\pi$, Proposition 7.7(2) implies that $\pi\sim^z B_{\text{switch}}$. Hence, $\mathcal{W}_1(\tau)\subset\mathcal{W}_2(\tau)$.

Next, we prove that $\mathcal{W}_2(\tau)$ satisfies (3)(o). We first claim that when two paths of $\mathcal{W}_2(\tau)$ meet, they coalesce. Let $\pi_1,\pi_2\in\mathcal{W}_2(\tau)$ start at t_1,t_2 respectively and meet at $t'>t_1\vee t_2$ and let us assume that π_1 and π_2 separate at $z=(x,t)$ with $t\geq t'$. By Proposition 7.7(1), either $\pi_1\sim^z B$ or $\pi_2\sim^z B$. This contradicts the definition of $\mathcal{W}_2(\tau)$ and we conclude that $\mathcal{W}_2(\tau)$ is a coalescing collection of paths. Let $z_i\in\mathcal{D}$. Any path in $\mathcal{W}_2(\tau)$ starting at z_i is squeezed between r_i and l_i, the paths in $\mathcal{W}_r$ and $\mathcal{W}_l$ respectively starting from $z_i=(x_i,t_i)$. Since there exists a sequence $t'_n\downarrow t_i$ s.t. $l_i(t'_n)=r_i(t'_n)$ and since paths in $\mathcal{W}_2(\tau)$ coalesce, there must be a unique path in $\mathcal{W}_2(\tau)$ starting from z_i. We call this path B^τ_i and define $\mathcal{W}_3(\tau)$ as $\overline{\{B^\tau_i\}}$. We continue to prove:

(ii) $\mathcal{W}_2(\tau)\subset\mathcal{W}_3(\tau)$.

Let $\pi\in\mathcal{W}_2(\tau)$ start at (x',t') and let $\varepsilon>0$. We claim that π hits a path in $\mathcal{W}_r\cup\mathcal{W}_l$ in $(t',t'+\varepsilon]$. To see this, let $a\in(t',t'+\varepsilon)$ and let $\{r_n\}_n\subset\mathcal{W}_r$ (resp., $\{l_n\}_n\subset\mathcal{W}_r$) start at z^r_n (resp., z^l_n) with z^r_n (resp., z^l_n) converging to $(\pi(a),a)$ from the left (resp., from the right) of π. If there is not any path in $\{r_n,l_n\}$ meeting π on $(a,t'+\varepsilon)$, $\{r_n\}$ and $\{l_n\}$ converge (along a subsequence) to $r\in\mathcal{W}_r$ and $l\in\mathcal{W}_l$ respectively, both starting at $(\pi(a),a)$ and s.t. $r<\pi<l$ on $(a,t'+\varepsilon)$. In other words, π enters a mesh from outside, yielding a contradiction to Theorem 5.1.

$\mathcal{M}(\tau)$, or equivalently the set of separation points in $\mathcal{N}(\tau)$, is dense along any path π' in $\mathcal{W}_r\cup\mathcal{W}_l$. Since once π touches some π', they can only separate at a point in $\mathcal{M}(\tau)$, it follows that π enters some point $z\in\mathcal{M}(\tau)$ before $t+\varepsilon$. By virtue of Proposition 7.7(3), there exists a ball $B(\bar{z},\varepsilon')$ such that any path in $\mathcal{N}(\tau)$ starting in $B(\bar{z},\varepsilon')$ enters the point z. Hence, any path B_i^τ such that z_i belongs to $\mathcal{D} \cap B(\bar{z}, \varepsilon')$ hits z. It follows that π coalesces with some B_i^τ before time $t' + \varepsilon$. As a consequence, $\mathcal{W}_2(\tau) \subset \mathcal{W}_3(\tau)$. Finally, we prove:

(iii) $\mathcal{W}_3(\tau) \subset \mathcal{W}_1(\tau)$.

It is clear that there is at least one path $\pi_i \in \mathcal{W}_1(\tau)$ starting from z_i. Since $\mathcal{W}_1(\tau) \subset \mathcal{W}_2(\tau)$, property 3(o) for $\mathcal{W}_2(\tau)$ (which we have already proved) implies that $\pi_i = B_i^\tau$. Since $\mathcal{W}_1(\tau)$ is compact, it follows that $\mathcal{W}_3(\tau) \subset \mathcal{W}_1(\tau)$ and from (i), (ii) above, we get that $\mathcal{W}_1(\tau) = \mathcal{W}_2(\tau) = \mathcal{W}_3(\tau)$. This shows that all subsequence limits of $\{\mathcal{W}_{n,m}\}$ agree and Proposition 6.1 follows.

7.7.2. $(\mathcal{W}, \mathcal{W}(\tau))$ *is a* $1/(2\tau)$*-sticky pair of Brownian webs*

In the remaining subsections of the paper we prove the four parts of Theorem 6.2. In this subsection and the next, the term marking will refer to the set $\mathcal{M}(\tau)$. We already showed in the proof of Proposition 6.1 that $\mathcal{W}(\tau)$ is a coalescing set of paths. By a simple locality argument, it is not hard to see that for $i \neq j$, B_i^τ and B_j^τ move independently when they are apart. In the following, we prove that (B_i, B_j^τ) is a $1/(2\tau)$-sticky pair of Brownian motions. This ensures that each B_j^τ is a Brownian motion and since the paths of $\mathcal{W}(\tau)$ are coalescing, it follows that $\mathcal{W}(\tau)$ is a Brownian web and furthermore that the interaction between $\mathcal{W}$ and $\mathcal{W}(\tau)$ is $(1/2\tau)$-sticky as claimed.

We now prove that (B_i, B_j^τ) is a $1/(2\tau)$-sticky pair of Brownian motions. Since the distribution of the Brownian net is invariant under translation in the space time domain, Proposition 6.1(2) implies that $\mathcal{W}(\tau)$ is also translation invariant. Hence, it suffices to prove that (B_0, B_j^τ) is a $1/(2\tau)$-sticky pair of Brownian motions.

Define ${}_{(n,m)}B_j^\tau$ as the path obtained from B_j after switching the directions of the points in $\mathcal{M}_{(n,m)}(\tau)$. By parts (1) and (3)(o) of Proposition 6.1, we have

$$\lim_{n\uparrow\infty} \lim_{m\uparrow\infty} \mathrm{d}\big({}_{(n,m)}B_j^\tau, B_j^\tau\big) = 0. \tag{7.73}$$

In the following, we will denote by ${}_{[n]}B_j \equiv {}_{[n]}B_j^\tau$ the limit of ${}_{(n,m)}B_j^\tau$ as $m \to \infty$. Informally, ${}_{[n]}B_j$ is the path constructed from B_j after switching the direction of all the (left and right) $(1,2)$ points in $\mathcal{M}(\tau)$ that lie on $\{B_i\}_{i=0}^{n-1}$.

In order to prove that (B_0, B_j^τ) is a $1/(2\tau)$-sticky pair of Brownian motions, we claim that it is enough to prove the following lemma (which is done in Section 7.7.3 below).

Lemma 7.10. $(B_0, {}_{[1]}B_j)$ *is* $1/(2\tau)$*-sticky pair of Brownian motions.*

The sufficiency of Lemma 7.10 follows from the observation that the law of $(B_0, {}_{[n]}B_j)$ is identical to the one of $(B_0, {}_{[1]}B_j)$. For example, for $n = 2$, one may consider a revised marked Brownian web $\mathcal{W}^*$ in which all the marked $(1,2)$ points along the finite segment of B_1 before it coalesces with B_0 have been switched. In $\mathcal{W}^*$ the marks along $B_0^* (\equiv B_0)$ are the same as in the original web. The following lemma (for $k = 1$ and $l = 0$) states that this $\mathcal{W}^*$ is equidistributed with the original Brownian web. On the other hand, the pair $(B_0^*, {}_{[1]}B_j^*)$ for $\mathcal{W}^*$ is *identical* to the pair $(B_0, {}_{[2]}B_j)$ for the original marked web. Since ${}_{[n]}B_j$ almost surely converges to B_j^τ, (B_0, B_j^τ) is $1/(2\tau)$-sticky pair of Brownian motions.

Lemma 7.11. *Let* ${}_{[1]}\mathcal{W}$ *denote the web resulting from switching all the marked* $(1,2)$ *points in the original web* $\mathcal{W}$ *along* B_0; *then* ${}_{[1]}\mathcal{W}$ *is equidistributed as the original web. Similarly, if for some fixed* k, l *with* $k \neq l$, $\mathcal{W}^*$ *denotes the marked web resulting from switching the original web along the finite segment of* B_k *before it coalesces with* B_l. *Then* $\mathcal{W}^*$ *is equidistributed with* $\mathcal{W}$.

Proof. To prove the first part of the lemma, it suffices to show that $\{{}_{[1]}B_j\}$ are coalescing Brownian motions. Lemma 7.10 implies that each individual ${}_{[1]}B_j$ is a Brownian motion and their construction shows that they are independent before meeting. The proof that they coalesce upon meeting is basically the same as that given for the paths of $\mathcal{W}_2(\tau)$ in Section 7.7.1. For the second part of the lemma w.l.o.g., set $k = 0$. Then the paths $B_j^* \in \mathcal{W}^*$ starting from z_j coincide with ${}_{[1]}B_j$ for times before the coalescence time of B_0 and B_l and afterward coincide with paths in $\mathcal{W}$. It follows that $\{B_j^*\}$ are coalescing Brownian motions and thus that $\mathcal{W}^*$ is equidistributed with $\mathcal{W}$. □

7.7.3. *Proof of Lemma 7.10*

We prove the result for $j = 0$. The result can then be trivially extended to any j. Our proof follows along the lines of the proof of Proposition 4.1 given in Section 7.4, except of course that here both right and left marked $(1, 2)$ points along B_0 are switched leading to ${}_{[1]}B_0$ rather than ${}_{[1]}r_0$. Here, it is enough to prove that $(B_0, {}_{[1]}B_z)_{z\in\mathbb{R}\times\{0\}}$ is a family of strong Markov processes with stationary transition probabilities and that the pair $(B_0, {}_{[1]}B_z)$ satisfies the following three properties:

(1) B_0 is a standard Brownian path starting at $(0, 0)$. ${}_{[1]}B_z$ starts at z.
(2) Away from the diagonal $\{t: {}_{[1]}B_z(t) = B_0(t)\}$, the two processes evolve as two independent Brownian motions.
(3) Defining $t_\varepsilon = \inf\{t > 0: |{}_{[1]}B_0 - B_0|(t) = \sqrt{2}\varepsilon\}$, one has:
   (i) $\mathbb{P}(({}_{[1]}B_0 - B_0)(t_\varepsilon) = \sqrt{2}\varepsilon) = \frac{1}{2}$,
   (ii) $\lim_{\varepsilon\downarrow 0} \mathbb{E}(t_\varepsilon)/\varepsilon = \sqrt{2}/(2\tau)$ and $\mathbb{E}([t_\varepsilon]^2) = o(\varepsilon)$.

The strong Markov property and the stationarity of the transition probabilities can be shown as in Lemma 7.5. Those two properties and the definition of ${}_{[1]}B_z$ easily imply Properties (1) and (2). Property (3)(i) is clearly true by right–left symmetry. It remains to prove (3)(ii). Recall the definition of ${}_{[1]}r_0$ given in Section 4. We define ${}_{[1]}l_0$ analogously, i.e., ${}_{[1]}l_0$ is obtained from B_0 by switching all the marked right $(1, 2)$ points in $\mathcal{M}(\tau) \cap B_0$. We also define

$$t^l_\varepsilon = \inf\bigl\{t: {}_{[1]}l_0(t) = B_0(t) - \sqrt{2}\varepsilon\bigr\}, \tag{7.74}$$

$$t^r_\varepsilon = \inf\bigl\{t: {}_{[1]}r_0(t) = B_0(t) + \sqrt{2}\varepsilon\bigr\}. \tag{7.75}$$

t^r_ε, which was carefully studied in Section 7.3, (resp., t^l_ε) is the first time a right (resp., left) marked excursion away from B_0 hits $B_0 + \sqrt{2}\varepsilon$ (resp., $B_0 - \sqrt{2}\varepsilon$). In order to verify the first part of (3)(ii), we will prove that $\lim_{\varepsilon\downarrow 0} \mathbb{E}(t_\varepsilon)/\varepsilon$ coincides with $\lim_{\varepsilon\downarrow 0} \mathbb{E}(t^r_\varepsilon \wedge t^l_\varepsilon)/\varepsilon$ and that $\mathbb{E}(t^r_\varepsilon \wedge t^l_\varepsilon)/\varepsilon$ has the desired limit. The second part can be proved similarly. We first use the following lemma.

Lemma 7.12. *${}_{[1]}B_0$ is obtained by joining together marked excursions from B_0.*

Proof. Let z be a point at which ${}_{[1]}B_0$ separates from B_0. By Proposition 7.6, z is a marked point of the original Brownian web $\mathcal{W}$ and there is a marked excursion e from B_0 starting at z. By the structure of the separation points given in Proposition 7.7 and since ${}_{(1,n)}B^\tau_0 \to {}_{[1]}B_0$ as $n \to \infty$, we see that ${}_{(1,n)}B^\tau_0$ follows the excursion e for sufficiently large n. As a consequence, ${}_{[1]}B_0$ also follows e. Since this is true for every such z, the lemma follows. □

Lemma 7.12 immediately implies that

$$t_\varepsilon \ge t^r_\varepsilon \wedge t^l_\varepsilon. \tag{7.76}$$

Continuing with our proof of Property (3)(ii), we define

$$T_\varepsilon = \inf\bigl\{t \ge t^r_\varepsilon \wedge t^l_\varepsilon: \bigl|{}_{[1]}B_0(t) - B_0(t)\bigr| = 0\bigr\}.$$

Using $T^{(0)}_\varepsilon \equiv T_\varepsilon$ as a (first) stopping time increment, denoting the segments of $\{B_0, {}_{[1]}r_0, {}_{[1]}l_0, {}_{[1]}B_0\}$ up to time $T^{(0)}_\varepsilon$ by $\{B^{(0)}_0, {}_{[1]}r_0{}^{(0)}, {}_{[1]}l_0{}^{(0)}, {}_{[1]}B_0{}^{(0)}\}$ and then translating $(B_0(T_\varepsilon), T_\varepsilon)$ onto $(0, 0)$, we may inductively define

$$\bigl\{B^{(n)}_0, {}_{[1]}r_0{}^{(n)}, {}_{[1]}l_0{}^{(n)}, {}_{[1]}B_0{}^{(n)}, T^{(n)}_\varepsilon\bigr\},$$

which, as in the proof of Lemma 7.5, are i.i.d. Next, define

$$K_\varepsilon = \inf\bigl\{k: \exists t \in \bigl[0, T^{(k)}_\varepsilon\bigr], \bigl|{}_{[1]}B_0{}^{(k)} - B^{(k)}_0\bigr|(t) = \sqrt{2}\varepsilon\bigr\} \tag{7.77}$$

and also,

$$\tilde{T}^{(n)}_\varepsilon = T^{(n)}_\varepsilon \wedge \inf\bigl\{t \in \bigl[0, T^{(n)}_\varepsilon\bigr]: \bigl|{}_{[1]}B_0{}^{(n)}(t) - B^{(n)}_0(t)\bigr| = \sqrt{2}\varepsilon\bigr\}.$$

Then, (letting $\tilde{T}_\varepsilon \equiv \tilde{T}_\varepsilon^{(0)}$) we have

$$\mathbb{E}(t_\varepsilon) = \sum_{n\geq 0} \mathbb{E}\big(\tilde{T}_\varepsilon^{(n)} 1_{K_\varepsilon \geq n}\big) = \sum_{n\geq 0} \mathbb{E}(\tilde{T}_\varepsilon)\mathbb{P}(K_\varepsilon \geq n) \tag{7.78}$$

$$= \mathbb{E}(\tilde{T}_\varepsilon) \sum_{n\geq 0} \mathbb{P}\big(\forall t \in [0, T_\varepsilon], |{}_{[1]}B_0 - B_0|(t) < \sqrt{2}\varepsilon\big)^n \tag{7.79}$$

$$= \frac{\mathbb{E}(\tilde{T}_\varepsilon)}{\mathbb{P}(\exists t \in [0, T_\varepsilon], |{}_{[1]}B_0 - B_0|(t) = \sqrt{2}\varepsilon)} \tag{7.80}$$

$$\leq \frac{\mathbb{E}(\tilde{T}_\varepsilon)}{\mathbb{P}(t_\varepsilon^l \wedge t_\varepsilon^r = t_\varepsilon)}. \tag{7.81}$$

Next we prove the following three lemmas.

Lemma 7.13. $\mathbb{E}(\tilde{T}_\varepsilon - t_\varepsilon^r \wedge t_\varepsilon^l)/\varepsilon \to 0$ *as* $\varepsilon \downarrow 0$.

Proof. The path ${}_{[1]}B_0$ evolves like a Brownian motion when it is away from B_0. It follows that $\mathbb{E}(\tilde{T}_\varepsilon - t_\varepsilon^r \wedge t_\varepsilon^l) \leq \sup_{x\in[0,\varepsilon]}(\mathbb{E}(S_x))$ where S_x is the time a standard Brownian motion starting at x exits the interval $[0, \varepsilon]$. This yields the claimed result. □

Lemma 7.14. $\mathbb{E}(t_\varepsilon^r \wedge t_\varepsilon^l)/\varepsilon \to \sqrt{2}/(2\tau)$.

Proof. Conditioned on $\mathcal{W}$ (but not the marking $\mathcal{M}(\tau)$), t_ε^r and t_ε^l are independent. If we denote by $\mathbb{P}_\mathcal{W}$ the probability distribution of the marked Brownian web conditioned on a realization of the web $\mathcal{W}$, and by $\mathbb{E}$ expectation with respect to the distribution $\mathbb{P}$ of $\mathcal{W}$, we have

$$\mathbb{E}(t_\varepsilon^r \wedge t_\varepsilon^l)/\varepsilon = \int_0^\infty \mathbb{E}\big(\mathbb{P}_\mathcal{W}\big(t_\varepsilon^r \wedge t_\varepsilon^l \geq \varepsilon t\big)\big)\,\mathrm{d}t = \int_0^\infty \mathbb{E}\big(\mathbb{P}_\mathcal{W}\big(t_\varepsilon^r \geq \varepsilon t\big) \cdot \mathbb{P}_\mathcal{W}\big(t_\varepsilon^l \geq \varepsilon t\big)\big)\,\mathrm{d}t. \tag{7.82}$$

By Proposition 7.2,

$$\mathbb{P}_\mathcal{W}\big(t_\varepsilon^r \geq \varepsilon t\big) = \mathbb{P}_\mathcal{W}\big(L_{\varepsilon,\varepsilon t}\big([0, \varepsilon t]\big) \leq \mathrm{Exp}\big(1/\big(\sqrt{2}\tau\big)\big)\big) = \exp\big(-\sqrt{2}\tau l_\varepsilon(\varepsilon t)\big). \tag{7.83}$$

By Lemma 7.3, we know that in probability $l_\varepsilon(\varepsilon t) \to t/2$, implying that $\mathbb{P}_\mathcal{W}(t_\varepsilon^r \geq \varepsilon t) \to \mathrm{e}^{-\tau t/\sqrt{2}}$. By symmetry, $\mathbb{P}_\mathcal{W}(t_\varepsilon^l \geq \varepsilon t) \to \mathrm{e}^{-\tau t/\sqrt{2}}$. In Section 7.3.2, we showed that $\{\mathbb{P}(t_\varepsilon^r \geq \varepsilon\cdot) = \mathbb{E}(\mathbb{P}_\mathcal{W}(t_\varepsilon^r \geq \varepsilon\cdot))\}_{\varepsilon\leq 1}$ is uniformly integrable. Since

$$\mathbb{E}\big(\mathbb{P}_\mathcal{W}\big(t_\varepsilon^r \geq \varepsilon t\big)\mathbb{P}_\mathcal{W}\big(t_\varepsilon^l \geq \varepsilon t\big)\big) \leq \mathbb{E}\big(\mathbb{P}_\mathcal{W}\big(t_\varepsilon^r \geq \varepsilon t\big)\big) = \mathbb{P}\big(t_\varepsilon^r \geq \varepsilon t\big),$$

we have that

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\big(t_\varepsilon^r \wedge t_\varepsilon^l\big)/\varepsilon\,\mathrm{d}t = \int_0^\infty \lim_{\varepsilon\downarrow 0} \mathbb{E}\big(\mathbb{P}_\mathcal{W}\big(t_\varepsilon^r \geq \varepsilon t\big) \cdot \mathbb{P}_\mathcal{W}\big(t_\varepsilon^r \geq \varepsilon t\big)\big)\mathrm{d}t = \int_0^\infty \exp\big(-2t\tau/\sqrt{2}\big) = \frac{\sqrt{2}}{2\tau}. \tag{7.84}$$

□

Lemma 7.15. $\lim_{\varepsilon\downarrow 0} \mathbb{P}(t_\varepsilon^l \wedge t_\varepsilon^r \neq t_\varepsilon) = 0$.

Proof. By symmetry, it suffices to prove that

$$\lim_{\varepsilon\downarrow 0} \mathbb{P}\big(t_\varepsilon^l = t_\varepsilon^l \wedge t_\varepsilon^r, t_\varepsilon^l \neq t_\varepsilon\big) = 0. \tag{7.85}$$

Assume that $t_\varepsilon^l = t_\varepsilon^r \wedge t_\varepsilon^l$ and $t_\varepsilon \neq t_\varepsilon^l$. Then there exists a left marked excursion $e_{l,\varepsilon}$ from B_0 starting at $T(e_{l,\varepsilon})$ which is at a distance $\sqrt{2}\varepsilon$ from B_0 at time t_ε^l. Since $t_\varepsilon \neq t_\varepsilon^l$, ${}_{[1]}B_0$ avoids this excursion implying that ${}_{[1]}B_0$ follows a right marked excursion e_r during the time interval $[T(e_r), T(e_r) + D(e_r)]$ such that $T(e_r) < T(e_{l,\varepsilon}) < T(e_r) + D(e_r)$. In other words, $T(e_{l,\varepsilon})$ is straddled by a marked right excursion. The lemma follows from Proposition 7.4. □

By (7.78)–(7.81) and Lemmas 7.13, 7.14 and 7.15, we have $\limsup_{\varepsilon\downarrow 0} \mathbb{E}(t_\varepsilon)/\varepsilon \le \lim_{\varepsilon\downarrow 0} \mathbb{E}(t_\varepsilon^r \wedge t_\varepsilon^l)/\varepsilon = \sqrt{2}/(2\tau)$. By (7.76), Property (3)(ii) and hence Lemma 7.10 follow. We conclude that $(\mathcal{W}, \mathcal{W}(\tau))$ has the required distribution.

7.7.4. *Markov property and stationarity*

We continue with the second and third properties of Theorem 6.2. $\mathcal{W}(\tau_2)$ is constructed by modifying $\mathcal{W} = \mathcal{W}(\tau = 0)$ according to the marking $\mathcal{M}(\tau_2)$. In order to prove the Markov property and stationarity, it suffices to prove that this is distributionally equivalent to the following procedure: (1) construct $\mathcal{W}(\tau_1)$ from $(\mathcal{W}, \mathcal{M}(\tau_1))$; then (2) construct $\mathcal{W}(\tau_2)$ from $(\mathcal{W}(\tau_1), \mathcal{M}^{\tau_1}(\Delta\tau))$ where $\mathcal{M}^{\tau_1}(\Delta\tau)$ is a marking of $\mathcal{W}(\tau_1)$ with intensity $\Delta\tau \equiv \tau_2 - \tau_1$ which, given the past $(\mathcal{W}, \{\mathcal{M}(\tau)\}_{\tau \le \tau_1})$, only depends on $\mathcal{W}(\tau_1)$.

Recall that given $\mathcal{W}$:

(i) for any measurable subset $O \subset \mathbb{R}^2$ with $\mathcal{L}(O) < \infty$ (where $\mathcal{L}$ is the local time outer measure – see Definition 3.1), $[\mathcal{M}(\tau_2) \setminus \mathcal{M}(\tau_1)] \cap O$ is a Poisson Point Process on $\mathbb{R}^2$ with intensity measure $(\tau_2 - \tau_1)\mathcal{L}(\cdot \cap O)$, and
(ii) $\{\mathcal{M}(\tau)\}_{\tau \le \tau_1}$ and $\tilde{\mathcal{M}}(\Delta\tau) \equiv \mathcal{M}(\tau_2) \setminus \mathcal{M}(\tau_1)$ are independent.

$\tilde{\mathcal{M}}(\Delta\tau)$ induces a natural marking on $\mathcal{W}(\tau_1)$. Indeed, for every $n \ge 0$, we can define $\tilde{\mathcal{M}}_{n,n}^{\tau_1}(\Delta\tau)$ as $\tilde{\mathcal{M}}(\Delta\tau) \cap E_n$ where $E_n = \{B_i^{\tau_1}\}_{i=0}^{n-1} \cap \{\hat{B}_j^{\tau_1}\}_{j=0}^{n-1}$ and $\mathcal{M}^{\tau_1}(\Delta\tau) \equiv \lim_{n\uparrow\infty} \mathcal{M}_{n,n}^{\tau_1}(\Delta\tau)$. We will denote by $\mathcal{W}_{n,n}^{\tau_1}(\Delta\tau)$ the web obtained from $\mathcal{W}(\tau_1)$ by switching the direction of all the points in $\mathcal{M}_{n,n}^{\tau_1}(\Delta\tau)$.

We have already proved that $\mathcal{W}(\tau_1)$ is a Brownian web. Hence, $\mathcal{L}(E_n) < \infty$ and, by item (i) above, conditioned on $\mathcal{W}$, $\mathcal{M}_{n,n}^{\tau_1}(\Delta\tau)$ is a Poisson Point Process with intensity measure $(\tau_2 - \tau_1)\mathcal{L}(\cdot \cap E_n)$.

Lemma 7.16. *Let $\mathcal{L}_{n,n}^{\tau_1}$ be the local time measure on $\mathbb{R}^2$ induced by $\{B_i^{\tau_1}\}_{i=0}^{n-1} \cup \{\hat{B}_j^{\tau_1}\}_{j=0}^{n-1}$, i.e.,*

$$\mathcal{L}_{n,n}^{\tau_1}(O) = m_\phi\big(\mathcal{P}\big(\big\{B_i^{\tau_1}\big\}_{i=0}^{n-1} \cap \big\{\hat{B}_j^{\tau_1}\big\}_{j=0}^{n-1} \cap O\big)\big)$$

(*where $\mathcal{P}$ is the projection on the t-axis*). *Then $\mathcal{L}(O \cap E_n) = \mathcal{L}_{n,n}^{\tau_1}(O)$, where $\mathcal{L}$ is the usual local time measure of* (3.6).

Proof. For a web $\mathcal{W}'$, let $\mathcal{W}' \cap \hat{\mathcal{W}}'$ denote the set of $(1,2)$ points of $\mathcal{W}'$. By definition,

$$\mathcal{L}_{n,n}^{\tau_1}(O) = m_\phi\big(\mathcal{P}(E_n \cap O)\big),$$
$$\mathcal{L}(O \cap E_n) = m_\phi\big(\mathcal{P}\big([\mathcal{W} \cap \hat{\mathcal{W}}] \cap E_n \cap O\big)\big).$$

Hence, in order to prove our lemma it is sufficient to prove that for every Borel O

$$m_\phi\big(\mathcal{P}\big([E_n \cap O] \setminus [\mathcal{W} \cap \hat{\mathcal{W}}]\big)\big) = 0$$

which will follow if we can prove that

$$m_\phi\big(\mathcal{P}\big(\big[\mathcal{W}(\tau_1) \cap \hat{\mathcal{W}}(\tau_1)\big] \setminus [\mathcal{W} \cap \hat{\mathcal{W}}]\big)\big) = 0. \tag{7.86}$$

In order to prove (7.86) we prove

$$m_\phi\big(\mathcal{P}\big([\mathcal{W} \cap \hat{\mathcal{W}}] \setminus \big[\mathcal{W}(\tau_1) \cap \hat{\mathcal{W}}(\tau_1)\big]\big)\big) = 0 \tag{7.87}$$

instead. The lemma will follow from the equidistribution of $(\mathcal{W}, \mathcal{W}(\tau_1))$ and $(\mathcal{W}(\tau_1), \mathcal{W})$. (Recall that in Sections 7.7.2 and 7.7.3 we already proved that $(\mathcal{W}, \mathcal{W}(\tau_1))$ is a sticky pair of webs whose distribution is invariant under permutation of the two webs.)

We now prove (7.87). For a given realization of $(\mathcal{W}, \mathcal{W}(\tau_1))$, let us assume that

$$m_\phi\big(\mathcal{P}\big([\mathcal{W} \cap \hat{\mathcal{W}}] \setminus \big[\mathcal{W}(\tau_1) \cap \hat{\mathcal{W}}(\tau_1)\big]\big)\big) > 0$$

and find a contradiction. By construction of $\mathcal{M}(\tau_2)$, there would be strictly positive probability that $\mathcal{M}(\tau_2) \setminus [\mathcal{W}(\tau_1) \cap \hat{\mathcal{W}}(\tau_1)] \neq \varnothing$.

Let z be any point in $\mathcal{M}(\tau_2)$. Then z is a separation point of $\mathcal{N}(\tau_2)$. Proposition 7.7(3) directly implies that for some i the path $B_i^{\tau_1} \in \mathcal{W}(\tau_1)$, from z_i, enters z. Since up to a reversal of the t-axis $\mathcal{N}(\tau_2)$ and $\hat{\mathcal{N}}(\tau_2)$ are equidistributed, there is a path $\hat{B}_j^{\tau_1} \in \hat{\mathcal{W}}(\tau_1)$ meeting B^{τ_1} at z and hence z is in $\mathcal{W}(\tau_1) \cap \hat{\mathcal{W}}(\tau_1)$. It would follow that $\mathcal{M}(\tau_2) \subset \mathcal{W}(\tau_1) \cap \hat{\mathcal{W}}(\tau_1)$, yielding a contradiction. This ends the proof of the lemma. □

Lemma 7.16 implies that $\mathcal{M}_{n,n}^{\tau_1}(\Delta\tau)$ only depends on $\mathcal{W}(\tau_1)$. Moreover, $\mathcal{W}(\tau_1)$ being a Brownian web, we also have the distributional identities,

$$\big(\mathcal{W}(\tau_1), \mathcal{M}_{n,n}^{\tau_1}(\Delta\tau)\big) =_d \big(\mathcal{W}, \tilde{\mathcal{M}}_{n,n}(\Delta\tau)\big), \tag{7.88}$$

$$\big(\mathcal{W}(\tau_1), \mathcal{W}_{n,n}^{\tau_1}(\tau_2)\big) =_d \big(\mathcal{W}, \mathcal{W}_{n,n}(\Delta\tau)\big), \tag{7.89}$$

where $\mathcal{W}_{n,n}(\Delta\tau)$ and $\tilde{\mathcal{M}}_{n,n}(\Delta\tau)$ are defined as in Section 3. It remains to prove that $\mathcal{W}_{n,n}^{\tau_1}(\Delta\tau)$ converges (in $(\mathcal{H}, d_{\mathcal{H}})$) to $\mathcal{W}(\tau_2)$.

Lemma 7.17. *$\mathcal{M}^{\tau_1}(\Delta\tau)$ and $\tilde{\mathcal{M}}(\Delta\tau)$ coincide.*

Proof. By construction, $\mathcal{M}^{\tau_1}(\Delta\tau) \subset \tilde{\mathcal{M}}(\Delta\tau)$ since $\mathcal{M}^{\tau_1}(\Delta\tau)$ is the marking induced by $\tilde{\mathcal{M}}(\Delta\tau)$ on $\mathcal{W}(\tau_1)$. Analogously, we define $\mathcal{M}'(\Delta\tau)$ as the marking induced by $\mathcal{M}^{\tau_1}(\Delta\tau)(= \lim_{n\uparrow\infty} \mathcal{M}_{n,n}^{\tau_1}(\Delta\tau))$ on $\mathcal{W}$. We already proved that $(\mathcal{W}, \mathcal{W}(\tau_1))$ is a $(1/2\tau)$-sticky pair of webs. Therefore, $(\mathcal{W}, \mathcal{W}(\tau_1))$ is equidistributed with $(\mathcal{W}(\tau_1), \mathcal{W})$ and, by (7.88), $(\mathcal{W}, \mathcal{W}(\tau_1), \mathcal{M}^{\tau_1}(\Delta\tau))$ is equidistributed with $(\mathcal{W}(\tau_1), \mathcal{W}, \tilde{\mathcal{M}}(\Delta\tau))$. Thus,

$$\big(\mathcal{W}, \mathcal{M}'_{n,n}(\Delta\tau)\big) =_d \big(\mathcal{W}(\tau_1), \mathcal{M}_{n,n}^{\tau_1}(\Delta\tau)\big).$$

By (7.88), we conclude that $\mathcal{M}'_{n,n}(\Delta\tau)$ is distributed like $\tilde{\mathcal{M}}_{n,n}(\Delta\tau)$. Since by construction, $\mathcal{M}'_{n,n}(\Delta\tau) \subset \tilde{\mathcal{M}}_{n,n}(\Delta\tau)$, it follows that $\mathcal{M}'_{n,n}(\Delta\tau) = \tilde{\mathcal{M}}_{n,n}(\Delta\tau)$ and $\mathcal{M}'(\Delta\tau) = \tilde{\mathcal{M}}(\Delta\tau)$. Since $\mathcal{M}'(\tau) \subset \mathcal{M}^{\tau_1}(\Delta\tau)$, we deduce that $\tilde{\mathcal{M}}(\Delta\tau) \subset \mathcal{M}^{\tau_1}(\Delta\tau)$ and hence $\mathcal{M}^{\tau_1}(\Delta\tau) = \tilde{\mathcal{M}}(\Delta\tau)$. □

Let $\mathcal{W}'$ be any subsequential limit of $\{\mathcal{W}_{n,n}^{\tau_1}(\Delta\tau)\}$. We next prove that $\mathcal{W}' = \mathcal{W}(\tau_2)$ via two inclusions, which completes this section.

(i) $\mathcal{W}' \subseteq \mathcal{W}(\tau_2)$.

Let $z = (x, t) \in \mathcal{M}(\tau_2)$, and let $\pi \in \mathcal{W}'$ start at $t - 2\varepsilon$ with $\varepsilon > 0$, and pass through z. By Proposition 6.1, what we need to show is that $\pi \sim^z B_{\text{switch}}$. By construction, there exists a sequence $\{\pi_N\}_{N \geq 0}$ so that π_N belongs to $\bigcup_{n,m>M} \mathcal{W}_{n,m}^{\tau_1}(\Delta\tau)$ and $\{\pi_N\}$ converges to π. Taking N large enough, we can assume w.l.o.g. that π_N belongs to $\mathcal{N}_{\leq t-\varepsilon}$ and $(x, t) \in \mathcal{M}_{n,m}(\tau_2)$ for $n, m > N$. Moreover, by Proposition 7.7(2) we can also assume that π_N enters the point z. We distinguish between two cases.

1. $z \in \mathcal{M}(\tau_1)$. Here, $z \notin \mathcal{M}^{\tau_1}(\Delta\tau)$ and by construction, $\pi_N \sim^z B^{\tau_1}$. Since $B^{\tau_1} \sim^z B_{\text{switch}}$, Proposition 7.7(1) implies that $\pi_N \sim^z B_{\text{switch}}$. By Proposition 7.7(2), $\pi \sim^z B_{\text{switch}}$.
2. $z \in \tilde{\mathcal{M}}(\Delta\tau)$. Since $\mathcal{M}^{\tau_1}(\Delta\tau) = \tilde{\mathcal{M}}(\Delta\tau)$, we get that $\pi_N \sim^z B_{\text{switch}}^{\tau_1}$. We claim that $B_{\text{switch}}^{\tau_1} \sim^z B_{\text{switch}}$, implying that $\pi \sim^z B_{\text{switch}}$ as desired. The claim can be verified as follows. Let us assume that $B_{\text{switch}}^{\tau_1} \sim^z B$ (and show that this leads to a contradiction). Then $B^{\tau_1} \sim^z B_{\text{switch}}$, implying that B and B^{τ_1} separate at z, or equivalently that $z \in \mathcal{M}(\tau_1)$. Since $\mathcal{M}(\tau_1)$ and $\tilde{\mathcal{M}}(\Delta\tau) = \mathcal{M}(\tau_2) \setminus \mathcal{M}(\tau_1)$ are disjoint, the claim follows.

(ii) $\mathcal{W}' \supseteq \mathcal{W}(\tau_2)$.

There is at least one path B' in $\mathcal{W}'$ starting from z_i. By Proposition 6.1(3)(o), there is a unique path $B_i^{\tau_2} \in \mathcal{W}(\tau_2)$ starting from there. Since $\mathcal{W}' \subseteq \mathcal{W}(\tau_2)$ we get $B' = B_i^{\tau_2}$. Hence, $\mathcal{W}' \supseteq \overline{\{B_i^{\tau_2}\}} = \mathcal{W}(\tau_2)$ (see Proposition 6.1(3)(ii)).

7.7.5. $\tau \to B_0^\tau(t)$ *is piecewise constant*
For any $\tau \le \tau_0$, the path B_0^τ belongs to $\mathcal{N}(\tau_0)$. Given $\mathcal{N}(\tau_0)$, $B_0^\tau(t)$ only depends on the direction of the $(1,2)$ points of $\mathcal{W}(\tau)$ which are located at the $(0,t)$-separation points of $\mathcal{N}(\tau_0)$. Since the set of $(0,t)$-separation points is locally finite (see Proposition 7.9), $\tau \to B_0^\tau(t)$ is piecewise constant.

Acknowledgments

The authors thank Rongfeng Sun and Jan Swart for very useful communications and discussions concerning their work on the Brownian net. They also thank the anonymous referee for a number of very helpful suggestions which have led to a clearer presentation of some of our arguments.

References

[1] R. Arratia. Coalescing Brownian motions on ℝ and the voter model on ℤ. Unpublished partial manuscript, 1981. Available at http://almaak.usc.edu/~rarratia/.
[2] I. Benjamini, O. Häggstrom, Y. Peres and J. E. Steif. Which properties of a random sequence are dynamically sensitive? *Ann. Probab.* **31** (2003) 1–34. MR1959784
[3] A. Borodin and P. Salminen. *Handbook of Brownian Motion—Facts and Formulae*, 2nd edition. Birkhäuser, Basel, 2002. MR1912205
[4] F. Camia, L. R. G. Fontes and C.M. Newman. The scaling limit geometry of near-critical 2D percolation. *J. Statist. Phys.* **125** (2006) 1159–1175. MR2282484
[5] F. Camia, L. R. G. Fontes and C. M. Newman. Two-dimensional scaling limits via marked nonsimple loops. *Bull. Braz. Math. Soc. (N. S.)* **37** (2006) 537–559. MR2284886
[6] F. Camia, M. Joosten and R. Meester. Geometric properties of two-dimensional near-critical percolation. Revised version submitted in 2009 as Trivial, critical and near-critical scaling limits of two-dimensional percolation. *J. Statist. Phys.* To appear, 2010. Available at arXiv:0803.3785.
[7] L. R. G. Fontes, M. Isopi, C. M. Newman and K. Ravishankar. The Brownian web: Characterization and convergence. *Ann. Probab.* **32** (2004) 2857–2883. MR2094432
[8] L. R. G. Fontes, M. Isopi, C. M. Newman and K. Ravishankar. Coarsening, nucleation, and the marked Brownian web. *Ann. Inst. H. Poincaré, Probab. Statist.* **42** (2006) 37–60. MR2196970
[9] L. R. G. Fontes, C. M. Newman, K. Ravishankar and E. Schertzer. The dynamical discrete web, 2007. Available at arXiv:0704.2706.
[10] L. R. G. Fontes, C. M. Newman, K. Ravishankar and E. Schertzer. Exceptional times for the dynamical discrete web. *Stochastic Process. Appl.* **119** (2009) 2832–2858.
[11] C. Garban. Processus SLE et sensibilité aux perturbations de la percolation plane critique. Doctoral thesis, Univ. Paris Sud, Paris, 2008.
[12] C. Garban, G. Pete and O. Schramm. Private communication, 2007.
[13] C. Howitt and J. Warren. Consistent families of Brownian motions and stochastic flows of kernels. *Ann. Probab.* **37** (2009) 1237–1272.
[14] C. Howitt and J. Warren. Dynamics for the Brownian web and the erosion flow. *Stochastic Process. Appl.* **119** (2009) 2028–2051. MR2519355
[15] I. Karatzas and E. Shreve. *Brownian Motion and Stochastic Calculus*. Springer, New York, 1991. MR1121940
[16] C. M. Newman, K. Ravishankar and E. Schertzer. Scaling limit of the one-dimensional stochastic Potts model, 2009. In preparation.
[17] P. Nolin. Near-critical percolation in two dimensions. *Electron. J. Probab.* **13** (2008) 1562–1623. MR2438816
[18] P. Nolin and W. Werner. Asymmetry of near-critical percolation interfaces. *J. Amer. Math. Soc.* **22** (2009) 797–819. MR2505301
[19] E. Perkins. The exact Hausdorff measure of the level sets of Brownian motions. *Z. Wahrsch. Verw. Gebiete* **58** (1981) 373–388. MR0639146
[20] E. Schertzer, R. Sun and J.M. Swart. Special points of the Brownian net. *Electron. J. Probab.* **14** (2009) 805–864. MR2497454
[21] E. Schertzer, R. Sun and J.M. Swart. Stochastic flows in the Brownian web and net, 2009. In preparation.
[22] F. Soucaliuc, B. Tóth and W. Werner. Reflection and coalescence between independent one-dimensional Brownian paths. *Ann. Inst. H. Poincaré Probab. Statist.* **36** (2000) 509–545. MR1785393
[23] R. Sun and J. M. Swart. The Brownian net. *Ann. Probab.* **36** (2008) 1153–1208. MR2408586
[24] B. Tóth and W. Werner. The true self-repelling motion. *Probab. Theory Related Fields* **111** (1998) 375–452. MR1640799
[25] B. Tsirelson. Scaling limit, noise, stability. In *Lectures in Probability Theory and Statistics* 1–106. *Lecture Notes in Math.* **1840**. Springer, Berlin, 2004. MR2079671
[26] J. Warren. Branching processes, the Ray–Knight theorem, and sticky Brownian motion. In *Séminaire de Probabilité de Strasbourg* **31** 1–15. *Lecture Notes in Math.* **1655**. Springer, Berlin, 1997. MR1478711
[27] J. Warren. The noise made by a Poisson snake. *Electron. J. Probab.* **7** (2002), no. 21. MR1943894