The infinite valley for a recurrent random walk in random environment

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Abstract. We consider a one-dimensional recurrent random walk in random environment (RWRE). We show that the – suitably centered – empirical distributions of the RWRE converge weakly to a certain limit law which describes the stationary distribution of a random walk in an infinite valley. The construction of the infinite valley goes back to Golosov, see Comm. Math. Phys. 92 (1984) 491–506. As a consequence, we show weak convergence for both the maximal local time and the self-intersection local time of the RWRE and also determine the exact constant in the almost sure upper limit of the maximal local time.


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1. Introduction and statement of the results

Let \( \omega = (\omega_x)_{x \in \mathbb{Z}} \) be a collection of i.i.d. random variables taking values in \((0, 1)\) and let \( P \) be the distribution of \( \omega \). For each \( \omega \in \Omega = (0, 1)^\mathbb{Z} \), we define the random walk in random environment (abbreviated RWRE) as the time-homogeneous Markov chain \((X_n)\) taking values in \(\mathbb{Z}_+\), with transition probabilities

\[
P_\omega(X_{n+1} = x + 1 | X_n = x) = \omega_x = 1 - P_\omega(X_{n+1} = x - 1 | X_n = x)\quad \text{for } x > 0, \quad \text{and } X_0 = 0.
\]

We equip \( \Omega \) with its Borel \( \sigma \)-field \( \mathcal{F} \) and \( \mathbb{Z}^\mathbb{N} \) with its Borel \( \sigma \)-field \( \mathcal{G} \). The distribution of \( (\omega, (X_n)) \) is the probability measure \( \mathbb{P} \) on \( \Omega \times \mathbb{Z}^\mathbb{N} \) defined by

\[
\mathbb{P} \left[ F \times G \right] = \int_F P_\omega[G] \, P(\omega), \quad F \in \mathcal{F}, \ G \in \mathcal{G}.
\]

Let \( \rho_i = \rho_i(\omega) := (1 - \omega_i)/\omega_i \). We will always assume that

\[
\int \log \rho_0(\omega) \, P(\omega) = 0, \tag{1.1}
\]

\[
P[\delta \leq \omega_0 \leq 1 - \delta] = 1 \quad \text{for some } \delta \in (0, 1), \tag{1.2}
\]

\[
\text{Var}(\log \rho_0) > 0. \tag{1.3}
\]

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The first assumption, as shown in [11], implies that for $P$-almost all $\omega$, the Markov chain $(X_n)$ is recurrent, the second is a technical assumption which could probably be relaxed but is used in several places, and the third assumption excludes the deterministic case. Usually, one defines in a similar way the RWRE on the integer axis, but for simplicity, we stick to the RWRE on the positive integers; see Section 4 for the results for the usual RWRE model. A key property of recurrent RWRE is its strong localization: under the assumptions above, Sinai [10] showed that $X_n/(\log n)^2$ converges in distribution. A lot more is known about this model; we refer to the survey by Zeitouni [12] for limit theorems, large deviations results, and for further references.

Let $\xi(n, x) := |\{0 \leq j \leq n: X_j = x\}|$ denote the local time of the RWRE in $x$ at time $n$ and $\xi^*(n) := \sup_{x \in \mathbb{Z}} \xi(n, x)$ the maximal local time at time $n$. It was shown in [7] and [8] that

$$\limsup_{n \to \infty} \frac{\xi^*(n)}{n} > 0, \quad \mathbb{P}\text{-a.s.} \quad (1.4)$$

(Clearly this lim sup is at most $1/2$.) In addition, a $0$–$1$ law (see [5]) says that $\limsup_{n \to \infty} \frac{\xi^*(n)}{n}$ is $\mathbb{P}$-almost surely a constant. The constant however was not known. We give its value in Theorem 1.1.

Our main result (Theorem 1.2 below) shows weak convergence for the process $(\xi(n, x), x \in \mathbb{Z})$ – after a suitable normalization – in a function space. In particular, it will imply the following theorem.

**Theorem 1.1.** Let $M := \sup\{s: s \in \text{supp}(\omega_0)\} \in (\frac{1}{2}, 1]$ and $w := \inf\{s: s \in \text{supp}(\omega_0)\} \in [0, \frac{1}{2})$. Then

$$\limsup_{n \to \infty} \frac{\xi^*(n)}{n} = \frac{(2M - 1)(1 - 2w)}{2(M - w)\min\{M, 1 - w\}}, \quad \mathbb{P}\text{-a.s.} \quad (1.5)$$

In particular, if $M = 1 - w$, we have

$$\limsup_{n \to \infty} \frac{\xi^*(n)}{n} = \frac{2M - 1}{2M}, \quad \mathbb{P}\text{-a.s.} \quad (1.6)$$

Define the potential $V = (V(x), x \in \mathbb{Z})$ by

$$V(x) := \begin{cases} \sum_{i=1}^{x} \log \rho_i, & x > 0, \\ 0, & x = 0, \\ -\sum_{i=x+1}^{0} \log \rho_i, & x < 0, \end{cases}$$

and $C(x, x+1) := \exp(-V(x))$. For each $\omega$, the Markov chain is an electrical network in the sense of [4], where $C(x, x+1)$ is the conductance of the bond $(x, x+1)$. In particular, $\mu(x) := \exp(-V(x - 1)) + \exp(-V(x)), x > 0, \mu(0) = 1$ is a (reversible) invariant measure for the Markov chain.

Let $\widetilde{V} = (\widetilde{V}(x), x \in \mathbb{Z})$ be a collection of random variables distributed as $V$ conditioned to stay non-negative for $x > 0$ and strictly positive for $x < 0$. Due to (1.3), such a distribution is well-defined, see, for example, Bertoin [1] or Golosov [6]. Moreover, it has been shown that

$$\sum_{x \in \mathbb{Z}} \exp(-\widetilde{V}(x)) < \infty, \quad (1.7)$$

see [6], p. 494. For each realization of $(\widetilde{V}(x), x \in \mathbb{Z})$ consider the corresponding Markov chain on $\mathbb{Z}$, which is an electrical network with conductances $\tilde{C}(x, x+1) := \exp(-\widetilde{V}(x))$. Intuitively, this Markov chain is a random walk in the “infinite valley” given by $\widetilde{V}$. As usual, $\tilde{\mu}(x) := \exp(-\widetilde{V}(x - 1)) + \exp(-\widetilde{V}(x)), x \in \mathbb{Z}$ is a reversible measure for this Markov chain. But, due to (1.7), we can normalize $\tilde{\mu}$ to get a reversible probability measure $\nu$, defined by

$$\nu(x) := \frac{\exp(-\widetilde{V}(x - 1)) + \exp(-\widetilde{V}(x))}{2 \sum_{x \in \mathbb{Z}} \exp(-\widetilde{V}(x))}, \quad x \in \mathbb{Z}. \quad (1.8)$$

Note that in contrast to the original RWRE which is null-recurrent, the random walk in the “infinite valley” given by $\widetilde{V}$ is positive recurrent.
Let $\ell^1$ be the space of real-valued sequences $\ell := \{\ell(x), x \in \mathbb{Z}\}$ satisfying $\|\ell\| := \sum_{x \in \mathbb{Z}} |\ell(x)| < \infty$. Let

$$c_n := \min \left\{ x \geq 0 : V(x) - \min_{0 \leq y \leq c_n} V(y) \geq \log n + (\log n)^{1/2} \right\}$$

and

$$b_n := \min \left\{ x \geq 0 : V(x) = \min_{0 \leq y \leq c_n} V(y) \right\},$$

see Figure 1. We will consider $\{\xi(n, x), x \in \mathbb{Z}\}$ as a random element of $\ell^1$ (of course, $\xi(n, y) = 0$ for $y < 0$). Here is our main result.

**Theorem 1.2.** Consider $\{\xi(n, x), x \in \mathbb{Z}\}$ as a random element of $\ell^1$ under the probability $P$. Then,

$$\frac{\xi(n, b_n + x)}{n}, x \in \mathbb{Z} \xrightarrow{\text{law}} \nu, \quad n \to \infty,$$

(1.9)

where $\xrightarrow{\text{law}}$ denotes convergence in distribution. In other words, the distributions of $\{\frac{\xi(n, b_n + x)}{n}, x \in \mathbb{Z}\}$ converge weakly to the distribution of $\nu$ (as probability measures on $\ell^1$).

The following corollary is immediate.

**Corollary 1.1.** For each continuous functional $f : \ell^1 \to \mathbb{R}$ which is shift-invariant, we have

$$f \left( \left\{ \frac{\xi(n, x)}{n}, x \in \mathbb{Z} \right\} \right) \xrightarrow{\text{law}} f \left( \{v(x), x \in \mathbb{Z}\} \right).$$

(1.10)

**Example 1 (Self-intersection local time).** Let

$$f(\{\ell(x), x \in \mathbb{Z}\}) = \sum_{x \in \mathbb{Z}} \ell(x)^2.$$
Corollary 1.1 yields that
\[ \frac{1}{n^2} \sum_{x \in \mathbb{Z}} \xi(n, x)^2 \xrightarrow{\text{law}} \sum_{x \in \mathbb{Z}} v(x)^2. \]  
(1.11)

This confirms Conjecture 7.4 in [9], at least for the RWRE on the positive integers; for the RWRE on the integer axis, see Section 4.

Example 2 (Maximal local time). Let
\[ f(\{\ell(x), x \in \mathbb{Z}\}) = \sup_{x \in \mathbb{Z}} \ell(x). \]

Corollary 1.1 yields that
\[ \frac{\xi^*(n)}{n} \xrightarrow{\text{law}} \sup_{x \in \mathbb{Z}} v(x). \]  
(1.12)

2. Proof of Theorem 1.2

The key lemma is

Lemma 2.1. For any \( K \in \mathbb{N} \),
\[ \left\{ \frac{\xi(n, b_n + x)}{n}, -K \leq x \leq K \right\} \xrightarrow{\text{law}} \left\{ v(x), -K \leq x \leq K \right\}. \]  
(2.1)

Given Lemma 2.1, the proof of Theorem 1.2 is straightforward. It suffices to show that for every function \( f : \ell^1 \to \mathbb{R} \), \( f \) bounded and uniformly continuous, we have
\[ \mathbb{E} \left[ f \left( \left\{ \frac{\xi(n, b_n + x)}{n}, x \in \mathbb{Z} \right\} \right) \right] \to \mathbb{E} \left[ f \left( \left\{ v(x), x \in \mathbb{Z} \right\} \right) \right], \quad n \to \infty. \]  
(2.2)

For \( \ell \in \ell^1 \), define \( \ell_K \) by \( \ell_K(x) = \ell(x) I_{[-K \leq x \leq K]} \) \( (x \in \mathbb{Z}) \). For \( f : \ell^1 \to \mathbb{R} \), define \( f_K \) by \( f_K(\ell) = f(\ell_K) \) \( (\ell \in \ell^1) \). Due to Lemma 2.1,
\[ \mathbb{E} \left[ f_K \left( \left\{ \frac{\xi(n, b_n + x)}{n}, x \in \mathbb{Z} \right\} \right) \right] \to \mathbb{E} \left[ f_K \left( \left\{ v(x), x \in \mathbb{Z} \right\} \right) \right], \quad n \to \infty. \]  
(2.3)

Fix \( \varepsilon > 0 \). Since \( f \) is uniformly continuous, there is \( \tilde{\delta} \) such that we have for \( \ell \in \ell^1 \)
\[ \|\ell_K - \ell\|_1 \leq \tilde{\delta} \quad \Rightarrow \quad |f_K(\ell) - f(\ell)| \leq \varepsilon. \]  
(2.4)

Hence
\[ \mathbb{E} \left[ \left| f_K \left( \left\{ \frac{\xi(n, b_n + x)}{n}, x \in \mathbb{Z} \right\} \right) - f \left( \left\{ \frac{\xi(n, b_n + x)}{n}, x \in \mathbb{Z} \right\} \right) \right| \right] \leq \varepsilon \]  
(2.5)

provided that
\[ \mathbb{E} \left[ \sum_{x : |x| > K} \frac{\xi(n, b_n + x)}{n} \right] \leq \tilde{\delta}. \]  
(2.6)

Due to Lemma 2.1, \( \mathbb{E}\left[ \sum_{x : |x| > K} \frac{\xi(n, b_n + x)}{n} \right] \) converges to \( \mathbb{E}\left[ \sum_{x : |x| > K} v(x) \right] \). But, for \( K \) large enough, \( \mathbb{E}\left[ \sum_{x : |x| > K} v(x) \right] \leq \tilde{\delta} \) since \( \mathbb{E}[v(\cdot)] \) is a probability measure on \( \mathbb{Z} \). Together with (2.6) and (2.3), this implies (2.2).
Proof of Lemma 2.1. We proceed in several steps:

(i) Define

\[ T(y) := \text{inf}\{n \geq 1: X_n = y\}, \tag{2.7} \]

the first hitting time of \( y \). For \( \varepsilon > 0 \), let \( A_n \) denote the event \( \{T(b_n) \leq \varepsilon n\} \). Then, \( \mathbb{P}[A_n] \to 1 \) for \( n \to \infty \).

Proof: see Golosov [6], Lemma 1.

(ii) Let \( B_n \) denote the event \( \{\text{the RWRE exits}[0, c_n] \text{ before time } n\} = \{T(c_n) < n\} \). Then, for \( P\)-a.a. \( \omega \), \( P_\omega[B_n | X_0 = b_n] \to 0 \) for \( n \to \infty \).

Proof: Due to [6], Lemma 7, we have for all \( m \)

\[ P_\omega[T(c_n) < m | X_0 = b_n] \leq \text{const} \cdot m e^{-\log n - (\log n)^{1/2}}. \]

Taking \( m = n \), we conclude that \( P_\omega[B_n | X_0 = b_n] \to 0 \).

(iii) Due to (i) and (ii), we can consider, instead of \( P_\omega \), a finite Markov chain \( \tilde{P}_\omega = \tilde{P}_\omega^{(n)} \) started from \( b_n \), in the valley \([0, c_n]\). More precisely, the Markov chain \( \tilde{P}_\omega \) is the original Markov chain \( P_\omega \) with reflection at \( c_n \), i.e. for \( 0 < x < c_n \), \( \tilde{P}_\omega[X_{n+1} = x + 1 | X_n = x] = \omega_x = 1 - \tilde{P}_\omega[X_{n+1} = x - 1 | X_n = x] = \alpha bn, x = \mu(\omega), \) see also [12], formula (2.1.4). We now decompose the paths of the Markov chain \( \tilde{P}_\omega \) into excursions from \( b_n \) to \( b_n \). Let \( x \in [-b_n, c_n - b_n], x \neq 0 \) and denote by \( Y_{b_n,x} \) the number of visits to \( b_n + x \) before returning to \( b_n \). The distribution of \( Y_{b_n,x} \) is “almost geometric”: we have

\[ \mu_\omega(x) := \begin{cases} \frac{1}{Z_\omega} (e^{-V(x)} + e^{-V(x-1)}), & 0 < x < c_n, \\ \frac{1}{Z_\omega} e^{-V(0)}, & x = 0, \\ \frac{1}{Z_\omega} e^{-V(c_n-1)}, & x = c_n, \end{cases} \]

where \( Z_\omega = 2 \sum_{x=0}^{c_n-1} e^{-V(x)} \).

(iv) Recall (2.7). We note for further reference that for \( 0 < b < y < i \),

\[ P_\omega[T(b) < T(i) | X_0 = y] = \sum_{j=y}^{i-1} e^{V(j)} \left( \sum_{j=b}^{i-1} e^{V(j)} \right)^{-1}. \tag{2.8} \]

This follows from direct computation, using \( C_{x,x+1} = e^{-V(x)} \), see also [12], formula (2.1.4). We now decompose the paths of the Markov chain \( \tilde{P}_\omega \) into excursions from \( b_n \) to \( b_n \). Let \( x \in [-b_n, c_n - b_n], x \neq 0 \) and denote by \( Y_{b_n,x} \) the number of visits to \( b_n + x \) before returning to \( b_n \). The distribution of \( Y_{b_n,x} \) is “almost geometric”: we have

\[ \tilde{P}_\omega[Y_{b_n,x} = m] = \begin{cases} \alpha (1 - \beta)^{m-1} \beta, & m = 1, 2, 3, \ldots, \\ 1 - \alpha, & m = 0, \end{cases} \]

where \( \alpha = \alpha_{b_n,x} = \tilde{P}_\omega[T(b_n + x) < T(b_n) | X_0 = b_n], \beta = \beta_{b_n,x} = \tilde{P}_\omega[T(b_n) < T(b_n + x) | X_0 = b_n + x] \). In particular,

\[ \tilde{P}_\omega[Y_{b_n,x}] = \frac{\alpha}{\beta} = \frac{\mu_\omega(b_n + x)}{\mu_\omega(b_n)}, \tag{2.9} \]

where \( \mu_\omega \) is the reversible measure for the Markov chain, see above. Note that (2.9) also applies for \( x = 0 \), with \( Y_{b_n,0} = 1 \). Now, with \( \text{Var}_\omega(Y_{b_n,x}) \) denoting the variance of \( Y_{b_n,x} \) w.r.t. \( \tilde{P}_\omega \),

\[ \text{Var}_\omega(Y_{b_n,x}) = \frac{\alpha(2 - \beta - \alpha)}{\beta^2} \leq \frac{2 \mu_\omega(b_n + x)}{\mu_\omega(b_n)} \leq \frac{4}{\beta}. \tag{2.10} \]

For \( x > 1 \),

\[ \beta = (1 - \omega_{b_n,x}) \tilde{P}_\omega[T(b_n) < T(b_n + x) | X_0 = b_n + x - 1]. \]
Taking into account (2.8) yields
\[ \beta = (1-\omega(b_n+x)) \left( \sum_{j=b_n}^{b_n+x-1} e^{V(j)-V(b_n+x-1)} \right)^{-1}, \] (2.11)

and (2.11) applies also to \( x = 1 \). In particular, recalling (1.2), there is a constant \( a = a(K, \delta) > 0 \) such that for \( 1 \leq x \leq K \), uniformly in \( n \),
\[ \text{Var}_\omega(Y_{b_n,x}) \leq a(K, \delta). \] (2.12)

In the same way, one obtains (2.12) for \(-K \leq x \leq 0\). We note for further reference that due to (2.10) and (2.11), there is a constant \( \tilde{a} = \tilde{a}(\delta) > 0 \) such that for \( x \in [-b_n, c_n - b_n] \),
\[ \text{Var}_\omega(Y_{b_n,x}) \leq \tilde{a}(\delta)c_n (\log n + (\log n)^{1/2}). \] (2.13)

(v) Denote by \( k_n \) the number of excursions from \( b_n \) to \( b_n \) before time \( n \). It follows from (2.9) that the average length \( \gamma_n \) of an excursion from \( b_n \) to \( b_n \) under \( \tilde{P}_\omega \) is given by
\[ \gamma_n = \sum_{y=0}^{c_n} \frac{\mu_\omega(y)}{\mu_\omega(b_n)} \geq 2. \]

Fix \( \varepsilon \in (0,1) \). We show that for \( P \)-a.a. \( \omega \),
\[ \tilde{P}_\omega \left[ \left| \frac{k_n}{n} - \frac{1}{\gamma_n} \right| \geq \varepsilon \right] \to 0, \quad n \to \infty. \] (2.14)

Proof: Let \( Y_{b_n,x}^{(1)}, Y_{b_n,x}^{(2)}, Y_{b_n,x}^{(3)}, \ldots \) be i.i.d. copies of \( Y_{b_n,x} \), and denote by \( E_n^{(i)} = \sum_{y=-b_n}^{c_n-b_n} Y_{b_n,x}^{(i)} \) the length of the \( i \)th excursion from \( b_n \) to \( b_n \). Then, \( \{k_n \geq m\} \subseteq \{\sum_{i=1}^{m} E_n^{(i)} \leq n\} \) and \( \{k_n \leq m\} \subseteq \{\sum_{i=1}^{m} E_n^{(i)} \geq n\} \). Hence,
\[ \tilde{P}_\omega \left[ \left| \frac{k_n}{n} - \frac{1}{\gamma_n} \right| \geq \varepsilon \right] = \tilde{P}_\omega \left[ \sum_{i=1}^{\lfloor n(1/\gamma_n + \varepsilon) \rfloor} E_n^{(i)} \leq n \right] + \tilde{P}_\omega \left[ \sum_{i=1}^{\lfloor n(1/\gamma_n - \varepsilon) \rfloor} E_n^{(i)} \geq n \right]. \] (2.15)

To handle the first term in (2.15), recall \( E_n^{(1)}, E_n^{(2)}, \ldots \) are i.i.d. with expectation \( \gamma_n \) under \( \tilde{P}_\omega \) and apply Chebyshev’s inequality:
\[ \tilde{P}_\omega \left[ \frac{1}{n(1/\gamma_n + \varepsilon)} \sum_{i=1}^{\lfloor n(1/\gamma_n + \varepsilon) \rfloor} (E_n^{(i)} - \gamma_n) \leq \frac{\varepsilon \gamma_n}{1 + \gamma_n \varepsilon} \right] \leq \frac{\text{Var}_\omega(E_n^{(1)}) (1 + \gamma_n \varepsilon)^2}{n(1/\gamma_n + \varepsilon)^2 \gamma_n^4}. \] (2.16)

Now, use the inequality \( \text{Var}(\sum_{j=1}^{N} X_j) \leq N \sum_{j=1}^{N} \text{Var}(X_j) \) and (2.13) to get from (2.16) that
\[ \tilde{P}_\omega \left[ \sum_{i=1}^{\lfloor n(1/\gamma_n + \varepsilon) \rfloor} E_n^{(i)} \leq n \right] \leq \frac{c_n^4 \tilde{a}(\delta)^2 (\log n)^3}{n} \frac{(1 + \gamma_n \varepsilon)^2}{\varepsilon^2 \gamma_n^4} \leq \frac{c_n^4 \tilde{a}(\delta)^2 (\log n)^3}{n} \frac{2}{\varepsilon^2}. \]

Due to Chung’s law of the iterated logarithm, \( \liminf_{n \to \infty} (\log \log n)^{1/2} (\max_{0 \leq x \leq n} V(x) - \min_{0 \leq x \leq n} V(x)) = \text{a.s.} \) a strictly positive constant, and this implies \( c_n^4 / n \to 0 \) for \( P \)-a.a. \( \omega \). We conclude that the first term in (2.15) goes to 0 for a.a. \( \omega \). The second term in (2.15) is treated in the same way.

Further, due to Chebyshev’s inequality, (2.9) and (2.12), we have for \(-K \leq x \leq K\)
\[ \tilde{P}_\omega \left[ \left| \frac{1}{k_n} \sum_{i=1}^{k_n} Y_{b_n,x}^{(i)} - \frac{\mu_\omega(b_n + x)}{\mu_\omega(b_n)} \right| \geq \varepsilon \right] \to 0, \quad n \to \infty. \] (2.17)
(vi) Define, for $-K \leq x \leq K$, $\rho_{n,\omega}$ by
\[
\rho_{n,\omega}(x) = \frac{\xi(n + T(b_n), b_n + x) - \xi(T(b_n), b_n + x)}{n}, \quad -K \leq x \leq K.
\] (2.18)

For $x = 0$, we have $n\rho_{n,\omega}(x) = k_n$.

We now estimate $\rho_{n,\omega}(x)$ for $-K \leq x \leq K$, $x \neq 0$:
\[
k_n\sum_{i=1}^{k_n} Y_{b_n,x}^{(i)} \leq \rho_{n,\omega}(x) \leq \frac{k_n + 1}{k_n + 1} \sum_{i=1}^{k_n+1} Y_{b_n,x}^{(i)}.
\] (2.19)

We conclude from (2.14) and (2.17) that for $-K \leq x \leq K$,
\[
\tilde{P}_\omega\left[ \left| \rho_{n,\omega}(x) - \frac{1}{\gamma_n} \frac{\mu_\omega(b_n + x)}{\mu_\omega(b_n)} \right| \geq \epsilon \right] \to 0, \quad n \to \infty.
\] (2.20)

(vii) We show that $\{ \frac{1}{\gamma_n} \frac{\mu_\omega(b_n + x)}{\mu_\omega(b_n)} , -K \leq x \leq K \}$ converges in distribution to $\{ \nu(x) , -K \leq x \leq K \}$.

Proof: Due to [6], Lemma 4, the finite-dimensional distributions of $\{ V(b_n + x) - V(b_n) \}_{x \in \mathbb{Z}}$ converge to those of $\{ \tilde{V}(x) \}_{x \in \mathbb{Z}}$. Therefore, setting $\gamma(N)_n := \frac{1}{\gamma_n} \frac{\mu_\omega(b_n + x)}{\mu_\omega(b_n)}$, we have for each $N$ that
\[
\begin{align*}
\gamma(N)_n \leq \gamma_n \quad \text{and} \quad P\left[ \gamma(N)_n \gamma_n \geq 1 - \epsilon \right] \to P\left[ \sum_{x=-\infty}^{\infty} (e^{-\tilde{V}(x)} + e^{-\tilde{V}(x-1)}) \geq 1 - \epsilon \right]
\end{align*}
\] (2.21)

for $n \to \infty$. But, due to (1.7),
\[
\left\{ \sum_{x=-N}^{N} (e^{-\tilde{V}(x)} + e^{-\tilde{V}(x-1)}) \geq (1 - \epsilon) \sum_{x=-\infty}^{\infty} (e^{-\tilde{V}(x)} + e^{-\tilde{V}(x-1)}) \right\}
\]

increases to the whole space and we conclude that the last term in (2.21) goes to 1 for $N \to \infty$.

Putting together (i)–(vii), we arrive at (2.1).

\section{3. Proof of Theorem 1.1}

We know now that the laws of $\frac{\xi^*(n)}{n}$ converge to the law of $\nu^*$ (see (1.12)), where
\[
\nu^* := \sup_{x \in \mathbb{Z}} \nu(x) = \sup_{x \in \mathbb{Z}} \frac{\exp(-\tilde{V}(x-1)) + \exp(-\tilde{V}(x))}{2 \sum_{x \in \mathbb{Z}} \exp(-\tilde{V}(x))}.
\] (3.1)

We will show that
\[
\limsup_{n \to \infty} \frac{\xi^*(n)}{n} = c, \quad \mathbb{P}\text{-a.s.},
\] (3.2)
where
\[ c = \sup\{ z : z \in \text{supp } \nu^* \}. \tag{3.3} \]

In fact, we get
\[ \limsup_{n \to \infty} \frac{\xi^*(n)}{n} \geq c, \quad \mathbb{P}\text{-a.s.} \tag{3.4} \]

from the following general lemma, whose proof is easy (and therefore omitted).

**Lemma 3.1.** Let \( (Y_n), n = 1, 2, \ldots \) be a sequence of random variables such that \( (Y_n) \) converges in distribution to a random variable \( Y \) and \( \limsup_{n \to \infty} Y_n \) is constant almost surely. Then,
\[ \limsup_{n \to \infty} Y_n \geq \sup\{ z : z \in \text{supp } Y \}. \tag{3.5} \]

To show that
\[ \limsup_{n \to \infty} \frac{\xi^*(n)}{n} \leq c, \quad \mathbb{P}\text{-a.s.}, \tag{3.6} \]

we will use a coupling argument with an environment \( \omega \) whose potential \( V \) will be shown to achieve the supremum in (3.3). Define the environment \( \omega \) as follows.
\[ \omega_x := \begin{cases} \{w, & x > 0, \\ M, & x \leq 0. \end{cases} \]

For the corresponding potential \( V \), \( \exp(-V(x)) \) is given as follows:
\[ \exp(-V(x)) = \begin{cases} \left( \frac{w}{1-w} \right)^x, & x > 0, \\ 1, & x = 0, \\ \left( \frac{M}{1-M} \right)^x, & x < 0. \end{cases} \]

For any fixed environment \( \omega \in [w, M]^\mathbb{Z} \), \( Q_\omega \) defines a Markov chain on the integers with transition probabilities \( Q_\omega[X_{n+1} = x + 1 | X_n = x] = \omega_x = 1 - Q_\omega[X_{n+1} = x - 1 | X_n = x], x \in \mathbb{Z}, \) and \( X_0 = 0 \). Then we have the following lemma.

**Lemma 3.2.** Let the environment \( \omega \) be defined as above. Assume \( M \leq 1 - w \). For all \( \omega \in [w, M]^\mathbb{Z} \), all \( x \in \mathbb{Z} \) and all \( n \in \mathbb{N} \), the distribution of \( \xi(n, x) \) with respect to \( Q_\omega \) is stochastically dominated by the distribution of \( \xi(n, 0) \) with respect to \( Q_{\mathbb{Z}} \). In particular,
\[ \sup\{ z : z \in \text{supp } \text{sup } \nu(x) \} = \mathbb{V}(0) := \frac{\exp(-\mathbb{V}(1)) + \exp(-\mathbb{V}(0))}{2 \sum_{x \in \mathbb{Z}} \exp(-\mathbb{V}(x))}. \tag{3.7} \]

**Proof of Lemma 3.2.** First step. We show that with \( T(0) := \min\{n \geq 1 : X_n = 0\} \), the distribution of \( T(0) \) with respect to \( Q_{\mathbb{Z}} \) is stochastically dominated by the distribution of \( T(0) \) with respect to \( Q_\omega \). We do this by coupling two Markov chains: the Markov chain \( (\overline{X}_n) \) moves according to \( Q_{\mathbb{Z}} \), with \( \overline{X}_0 = 0 \), the Markov chain \( (X_n) \) moves according to \( Q_\omega \), with \( X_0 = 0 \), in such a way that \((\overline{X}_n)\) returns to 0 before (or at the same time as) \((X_n)\). Let \( (\overline{X}_n) \) move according to \( Q_{\mathbb{Z}} \). If \( \overline{X}_n \leq 0 \) and \( \overline{X}_{n+1} < \overline{X}_n \) and \( X_n \leq 0 \), then also \( X_{n+1} < X_n \); this is possible since \( Q_{\mathbb{Z}}[\overline{X}_{n+1} < \overline{X}_n] = 1 - M \leq Q_\omega[X_{n+1} < X_n] \). If \( \overline{X}_n > 0 \) and \( \overline{X}_{n+1} > \overline{X}_n \) and \( X_n > 0 \), then also \( X_{n+1} > X_n \); this is possible since \( Q_{\mathbb{Z}}[\overline{X}_{n+1} > \overline{X}_n] = w \leq Q_\omega[X_{n+1} > X_n] \). If \( \overline{X}_n > 0 \) and \( \overline{X}_{n+1} = \overline{X}_n \) and \( X_n < 0 \), then \( X_{n+1} < X_n \); this is possible since \( Q_{\mathbb{Z}}[\overline{X}_{n+1} = \overline{X}_n] = w \leq 1 - M \leq Q_\omega[X_{n+1} < X_n] \). Now, \((\overline{X}_n)\) visits 0 before (or at the same time as) as \((X_n)\) does and this proves the statement.
Second step. We show that for each \( n \), the distribution of \( \xi(n, 0) \) with respect to \( Q_\omega \) is stochastically dominated by the distribution of \( \xi(n, 0) \) with respect to \( Q_{\overline{\omega}} \). Let \( T_1, T_2, \ldots \) be i.i.d. copies of \( T(0) \). Then, \( Q_\omega[\xi(n, 0) \geq k] = P_\omega[\sum_{j=1}^{k} T_j \leq n] \leq Q_{\overline{\omega}}[\sum_{j=1}^{k} T_j \leq n] = Q_{\overline{\omega}}[\xi(n, 0) \geq k] \), where the inequality follows from the first step.

Third step. For \( x \in \mathbb{Z} \), let \( \theta^x \) be the shift on \( \Omega \), i.e. \( (\theta^x \omega)(y) = \omega(x + y), y \in \mathbb{Z} \). For \( x \in \mathbb{Z} \), the distribution of \( \xi(n, x) \) with respect to \( Q_\omega \) is dominated by the distribution of \( \xi(n, 0) \) with respect to \( Q_{\theta^x \omega} \) (more precisely, with \( T(x) := \inf\{n: X_n = x\} \) denoting the first hitting time of \( x \), the distribution of \( \xi((n - T(x)), 0)I_{T(x) \leq n} \) with respect to \( Q_{\theta^x \omega} \)). Now, apply the second step with \( \theta^x \omega \) instead of \( \omega \).

To show (3.7), define, for \( \omega \in [w, M]^\mathbb{Z} \), the corresponding potential \( V \) as in Section 1. Note that if \( V \) is in the support of \( \overline{V} \), the Markov chain defined by \( Q_\omega \) is positive recurrent with invariant probability measure
\[
v(x) = \frac{\exp(-V(x-1)) + \exp(-V(x))}{2\sum_{x \in \mathbb{Z}} \exp(-V(x))},
\]
and we have \( v(x) = \lim_{n \to \infty} \frac{1}{n} \xi(n, x) \), hence (3.7) follows from the stochastic domination.

Intuitively, \( \overline{V} \) is the steepest possible (infinite) valley which maximises the occupation time of 0 (if \( M \leq 1 - w \)). Let \( \varepsilon > 0 \). Since \((X_n)\) is a positive recurrent Markov chain with respect to \( Q_{\overline{\omega}} \), we have
\[
\lim_{n \to \infty} \frac{\xi(n, 0)}{n} = \overline{V}(0), \quad Q_{\overline{\omega}}\text{-a.s.,}
\]
and
\[
Q_{\overline{\omega}}\left[ \frac{\xi(n, 0)}{n} \geq \overline{V}(0) + \varepsilon \right] \leq (2n + 1) e^{-C(\varepsilon)n},
\]
(3.9)
where \( C(\varepsilon) \) is a strictly positive constant depending only on \( \varepsilon \). We have
\[
Q_\omega\left[ \frac{\xi^*(n)}{n} \geq \overline{V}(0) + \varepsilon \right] \leq \sum_{x=-n}^{n} Q_{\theta^x \omega}\left[ \frac{\xi(n, 0)}{n} \geq \overline{V}(0) + \varepsilon \right] 
\leq (2n + 1) Q_{\overline{\omega}}\left[ \frac{\xi(n, 0)}{n} \geq \overline{V}(0) + \varepsilon \right] 
\leq (2n + 1) e^{-C(\varepsilon)n},
\]
(3.10)
where we used Lemma 3.2 for the second inequality and (3.9) for the last inequality. By the Borel–Cantelli lemma, we conclude that
\[
\limsup_{n \to \infty} \frac{\xi^*(n)}{n} \leq \overline{V}(0), \quad Q_{\omega}\text{-a.s., for all } \omega \in [w, M]^\mathbb{Z}.
\]
(3.11)
We have to show that (3.11) holds true \( \mathbb{P}\)-a.s., i.e. for our RWRE on \( \mathbb{Z}_+ \), with reflection at 0. In order to do so, we need the following modification of Lemma 3.2 for environments on \( \mathbb{Z}_+ \). Let \( K \geq 1 \) and let the environment \( \overline{\omega}^{(K)} \) be defined as follows.
\[
\overline{\omega}_x^{(K)} := \begin{cases} 
M, & 0 < x \leq K, \\
1, & x = 0, \\
\frac{w}{1-w}, & x > K.
\end{cases}
\]
For the corresponding potential \( \overline{V}^{(K)} \), \( \exp(-\overline{V}^{(K)}(x)) \) is given as follows:
\[
\exp(-\overline{V}^{(K)}(x)) = \begin{cases} 
\left( \frac{M}{1-M} \right)^x, & 0 < x \leq K, \\
\left( \frac{M}{1-M} \right)^K \left( \frac{w}{1-w} \right)^{x-K}, & x > K, \\
1, & x = 0.
\end{cases}
\]
The Markov chain on \( \mathbb{Z}_+ \) defined by \( P^{(K)}_\omega \) is positive recurrent with invariant probability measure \( \nu^{(K)} \) given by

\[
\nu^{(K)}(x) := \begin{cases} 
\frac{\exp(-\nu^{(K)}(x-1)) + \exp(-\nu^{(K)}(x))}{1 + 2 \sum_{z \geq 1} \exp(-\nu^{(K)}(z))}, & x \geq 1, \\
\frac{1}{1 + 2 \sum_{z \geq 1} \exp(-\nu^{(K)}(z))}, & x = 0.
\end{cases}
\] (3.12)

**Lemma 3.3.** Assume \( M \leq 1 - w \). For all \( K \geq 1 \), if \( \omega \in \{1 \times [w, M]\}^{\mathbb{Z}_+} \) satisfies \( \omega_0 = 1 \) and \( b_n(\omega) \to \infty \), then for all \( n \in \mathbb{N} \) such that \( b_n(\omega) \geq 2K \), and all \( x \in \mathbb{Z}, x \geq -b_n + K \), the distribution of \( \xi(n, b_n + x) \) with respect to \( P_\omega \) is stochastically dominated by the distribution of \( \xi(n, K) \) with respect to \( P^{(K)}_\omega \). The last statement follows from (3.12).

The proof of Lemma 3.3 is similar to the proof of Lemma 3.2. The last statement follows from (3.12).

Let \( \epsilon > 0 \). Choose \( K \) such that \( \nu^{(K)}(K) \leq \nu(0) + \frac{\epsilon}{2} \). Since \( \{X_n\} \) is a positive recurrent Markov chain with respect to \( P^{(K)}_\omega \), we have

\[
\lim_{n \to \infty} \frac{\xi(n, K)}{n} = \nu^{(K)}(K), \quad P^{(K)}_\omega [X_0 = K]-a.s.,
\]

and

\[
P^{(K)}_\omega \left[ \frac{\xi(n, K)}{n} \geq \nu^{(K)}(K) + \epsilon | X_0 = K \right] \leq e^{-C(\epsilon)n},
\] (3.13)

where \( C(\epsilon) \) is a strictly positive constant depending only on \( \epsilon \), and on \( K \). For \( P \)-a.a. \( \omega \), choose \( n_0(\omega) \) such that for \( n \geq n_0(\omega), b_n(\omega) \geq 2K \) and \( \frac{1}{n} \sum_{x=0}^{K} \xi(n, x) \leq \frac{\epsilon}{4} \). \( P_\omega \)-a.s. (this is possible since \( b_n(\omega) \) increases to \( \infty \), \( P \)-a.s. for \( n \to \infty \), and \( \frac{1}{n} \sum_{x=0}^{K} \xi(n, x) \to 0 \)), \( P \)-a.s. for \( n \to \infty \) since \( P_\omega \) is null-recurrent for \( P \)-a.a. \( \omega \). We then have \( P \)-a.s. for \( n \geq n_0(\omega) \)

\[
P_\omega \left[ \frac{\xi^*(n)}{n} \geq \nu(0) + \epsilon \right] \leq P_\omega \left[ \frac{\xi(n)}{n} \geq \nu^{(K)}(K) + \frac{\epsilon}{2} \right] 
\leq \sum_{x=0}^{n} P_\omega \left[ \frac{\xi(n, b_n + x)}{n} \geq \nu^{(K)}(K) + \frac{\epsilon}{4} \right]
\leq (2n + 1) P^{(K)}_\omega \left[ \frac{\xi(n, K)}{n} \geq \nu^{(K)}(K) + \frac{\epsilon}{4} | X_0 = K \right] 
\leq (2n + 1) e^{-C(\epsilon/4)n},
\] (3.14)

where we used Lemma 3.2 for the third inequality and (3.13) for the last inequality. By the Borel–Cantelli lemma, we conclude that

\[
\limsup_{n \to \infty} \frac{\xi^*(n)}{n} \leq \nu(0), \quad P_\omega \text{-a.s., for } P \text{-a.a. } \omega.
\] (3.15)

Together with (3.7), this finishes the proof in the case \( M \leq 1 - w \); the value of \( c \) is computed easily from (3.7). In the case \( M > 1 - w \), Lemma 3.2 is replaced with the following

**Lemma 3.4.** Let the environment \( \omega \) be defined as above. Assume \( M > 1 - w \). For all \( \omega \in \{1 \times [w, M]\}^{\mathbb{Z}_+} \), all \( x \in \mathbb{Z} \) and all \( n \in \mathbb{N} \), the distribution of \( \xi(T(1) + n, x) - \xi(T(1), x) \) with respect to \( Q_\omega \) is stochastically dominated by the distribution of \( \xi(T(1) + n, 1) \) with respect to \( Q^{(K)}_\omega \). In particular,

\[
\sup \left\{ z : z \in \text{supp } \sup_{x \in \mathbb{Z}} \nu(x) \right\} = \nu(1) = \frac{\exp(-\nu(0)) + \exp(-\nu(1))}{2 \sum_{z \in \mathbb{Z}} \exp(-\nu(z))}.
\] (3.16)

The rest of the proof is analogous to the case \( M \leq 1 - w \).
4. Further remarks

1. Consider RWRE on the integers axis, i.e. with transition probabilities \(Q_{\omega}[X_{n+1} = x + 1|X_n = x] = \omega_x = 1 - Q_{\omega}[X_{n+1} = x - 1|X_n = x]\) for \(x \in \mathbb{Z}\). Then, Theorem 1.2 can be modified in the following way (we refer to [2] for details). First, we have to replace \(b_n\) in (1.9) with \(\hat{b}_n\) defined as follows. We call a triple \((a, b, c)\) with \(a < b < c\) a valley of \(V\) if

\[
V(b) = \min_{a \leq x \leq c} V(x), \quad V(a) = \max_{a \leq x \leq b} V(x), \quad V(c) = \max_{b \leq x \leq c} V(x).
\]

The depth of the valley is defined as \(d(a, b, c) = (V(a) - V(b)) \land (V(c) - V(b))\). Call a valley \((a, b, c)\) minimal if for all valleys \((\tilde{a}, \tilde{b}, \tilde{c})\) with \(a < \tilde{a}, \tilde{c} < c\), we have \(d(\tilde{a}, \tilde{b}, \tilde{c}) < d(a, b, c)\). Consider the smallest minimal valley \((\tilde{a}_n, \tilde{b}_n, \tilde{c}_n)\) with \(\tilde{a}_n < 0 < \tilde{c}_n\) and \(d(\tilde{a}_n, \tilde{b}_n, \tilde{c}_n) \geq \log n + (\log n)^{1/2}\), i.e. \((\tilde{a}_n, \tilde{b}_n, \tilde{c}_n)\) is a minimal valley with \(d(\tilde{a}_n, \tilde{b}_n, \tilde{c}_n) \geq \log n + (\log n)^{1/2}\) and for every other valley \((\tilde{a}_n, \tilde{b}_n, \tilde{c}_n)\) with these properties, we have \(\tilde{a}_n < \tilde{a}_n\) or \(\tilde{c}_n < \tilde{c}_n\). If there are several such valleys, take \((\tilde{a}_n, \tilde{b}_n, \tilde{c}_n)\) such that \(|b_n|\) is minimal (and \(\tilde{b}_n > 0\) if there are two possibilities). Further, one has to replace the limit measure \(\nu\) in (1.9) with \(\tilde{\nu}\) defined in as follows. Let \(\tilde{V}_{\text{left}} = (\tilde{V}_{\text{left}}(x), x \in \mathbb{Z})\) be a collection of random variables distributed as \(V\) conditioned to stay strictly positive for \(x > 0\) and non-negative for \(x < 0\). (Recall that \(\tilde{V} = (\tilde{V}(x), x \in \mathbb{Z})\) is a collection of random variables distributed as \(V\) conditioned to stay non-negative for \(x > 0\) and strictly positive for \(x \leq 0\).) Set

\[
\nu_{\text{left}}(x) := \frac{\exp(-\tilde{V}_{\text{left}}(x-1)) + \exp(-\tilde{V}_{\text{left}}(x))}{2 \sum_{y \in \mathbb{Z}} \exp(-\tilde{V}_{\text{left}}(y))}, \quad x \in \mathbb{Z}.
\]

In other words, \(\nu_{\text{left}}\) is defined in the same way as \(\nu\), replacing \(\tilde{V}\) in (1.8) with \(\tilde{V}_{\text{left}}\). Now, let \(Q\) be the distribution of \(\nu\) and \(Q_{\text{left}}\) be the distribution of \(\nu_{\text{left}}\) and let \(\tilde{\nu}\) be a random probability measure on \(\mathbb{Z}\) with distribution \(\frac{1}{2}(Q + Q_{\text{left}})\). Then, (1.9)–(1.12) hold true for RWRE on the integer axis, if we replace \(b_n\) with \(\hat{b}_n\) and \(\nu\) with \(\tilde{\nu}\).

Since the right-hand side of (1.5) is a function of \(M\) and \(w\), say \(\varphi(M, w)\), for which we have \(\varphi(M, w) = \varphi(1 - w, 1 - M)\), we see that replacing \(V\) with \(\nu_{\text{left}}\) in (3.1) yields the same constant \(c\) as in (3.3) and therefore Theorem 1.1 carries over verbatim.

2. We showed that \(\limsup_{n \to \infty} \frac{\xi_n^*(\tau)}{n} = \mathbb{P}\)-a.s. a strictly positive constant and gave its value. We have \(\liminf_{n \to \infty} \frac{\xi_n^*(\tau)}{n} = 0 = \mathbb{P}\)-a.s. which is a consequence of (1.12). It was shown in [3] that there is a strictly positive constant \(a\) such that \(\liminf_{n \to \infty} \frac{\xi_n^*(\tau)}{n \log \log \log n} = a\), \(\mathbb{P}\)-a.s. The value of \(a\) is not known.

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References

