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Between Paouris concentration inequality and variance conjecture

B. Fleury

Université Pierre et Marie Curie, Equipe d'Analyse Fonctionnelle, Institut de Mathématiques de Jussieu, boite 186, 4 place Jussieu, 75252 Paris Cedex 05, France. E-mail: fleury_bruno@yahoo.fr

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Abstract. We prove an almost isometric reverse Hölder inequality for the Euclidean norm on an isotropic generalized Orlicz ball which interpolates Paouris concentration inequality and variance conjecture. We study in this direction the case of isotropic convex bodies with an unconditional basis and the case of general convex bodies.

Résumé. Nous prouvons une inégalité inverse Hölder presque isométrique pour la norme euclidienne sur une boule d'Orlicz généralisée isotrope qui interpole l'inégalité de concentration de Paouris et la conjecture de la variance. Nous étudions dans ce sens le cas des corps convexes isotropes à base inconditionnelle et celui des corps convexes généraux.

MSC: 46B07; 46B09

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1. Introduction

Let K be a convex body in $\mathbb{R}^n$ and let $X=(X_1,\dots,X_n)$ be a random vector uniformly distributed in K. We suppose that K is in an isotropic position that means:

1. $\mathrm{vol}_n(K)=1$ (where vol_n stands for the Lebesgue measure on $\mathbb{R}^n$);
2. the barycenter of K $(\mathbb{E}X)$ is 0;
3. the expectations $\mathbb{E}\langle X,\theta\rangle^2=L_K^2$ do not depend on $\theta\in S^{n-1}$.

It is known that every convex body has an affine image which is isotropic. We denote by $|x|$ the Euclidean norm of $x\in\mathbb{R}^n$.

Under the isotropic condition, Paouris [14] showed that for some absolute constants $c_1>0$ and $c_2>0$ and for any real $p\in[2,c_1\sqrt{n}]$,

$$\left(\mathbb{E}|X|^p\right)^{1/p}\le c_2\left(\mathbb{E}|X|^2\right)^{1/2}. \tag{1}$$

Besides, Bobkov and Koldobsky [4] emphasized (considering a particular case of a conjecture of Kannan, Lovász and Simonovits [11]) that the ratio $\sigma_K^2=\frac{\mathrm{Var}\,|X|^2}{nL_K^4}$ should be bounded from above by a universal constant which can be written

$$\left(\mathbb{E}|X|^4\right)^{1/4}\le\left(1+\frac{C}{n}\right)\left(\mathbb{E}|X|^2\right)^{1/2} \tag{2}$$

for some numerical constant $C > 0$. Anttila, Ball and Perissinaki proved this conjecture in [1] for the l_p^n-balls by showing in this case that

$$\mathrm{cov}\big(X_i^2, X_j^2\big) \leq 0 \tag{3}$$

for any i and j in $\{1, \dots, n\}$ with $i \neq j$. Wojtaszczyk [17] extended this property (3) [and thus (2)] for the generalized Orlicz balls, that is to say when

$$K = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n, \sum_{i=1}^n f_i\big(|x_i|\big) \leq 1 \right\},$$

where, for any $i \in \{1, \dots, n\}$, $f_i : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{\infty\}$ is convex and satisfies: $f_i(0) = 0$, $\exists t \in \mathbb{R}_*^+$, $f_i(t) \neq 0$ and $\exists s \in \mathbb{R}_*^+$, $f_i(s) \neq \infty$. Recently, Klartag [10] proved (2) for the unconditional convex bodies (the convex bodies which are symmetric with respect to the coordinate hyperplanes). But the general case remains open.

In this direction, it is natural to try to estimate the ratio $\frac{(\mathbb{E}|X|^p)^{1/p}}{(\mathbb{E}|X|^2)^{1/2}}$ for $p \in [4, c_1\sqrt{n}]$. This question is connected to the estimate of the spectral-gap for convex bodies. For any random vector Y in $\mathbb{R}^n$ with the law μ_Y, we denote $\lambda_1(Y) = \lambda_1(\mu_Y)$ the spectral-gap of μ_Y that is to say the best constant $A \geq 0$ such that for any sufficiently smooth function $f : \mathbb{R}^n \to \mathbb{R}$

$$A\,\mathrm{Var}\big[f(Y)\big] \leq \mathbb{E}\big|\nabla f(Y)\big|^2.$$

Kannan, Lovász and Simonovits conjectured in [11] that, under the isotropy assumption, we have

$$\lambda_1(X) \geq \frac{1}{cL_K^2} \tag{4}$$

for some absolute constant $c > 0$. Up to now, this conjecture was proved only for the l_p^n-balls with $p \in [1, \infty]$ [8,16]. It is well known that an estimate of spectral gap implies moment bounds for Lipschitz functions. We observe that this conjecture implies

$$\big(\mathbb{E}|X|^p\big)^{1/p} \leq \left(1 + \frac{a_1 p}{n}\right)\big(\mathbb{E}|X|^2\big)^{1/2} \tag{5}$$

for any $p \in [2, a_2\sqrt{n}]$ where $a_1 > 0$ and $a_2 > 0$ are numerical constants. The following statement is the main result of this paper.

Theorem 1. *There exist universal constants $C_1 > 0, \dots, C_6 > 0$ such that*:

1. *for any random vector X uniformly distributed on an isotropic generalized Orlicz ball and for any $p \in [2, C_1\sqrt{n}]$*,

$$\big(\mathbb{E}|X|^p\big)^{1/p} \leq \left(1 + \frac{C_2 p}{n}\right)\big(\mathbb{E}|X|^2\big)^{1/2};$$

2. *for any random vector X uniformly distributed on an isotropic unconditional convex body and for any $p \in [2, C_3\frac{\sqrt{n}}{\log(n)}]$*,

$$\big(\mathbb{E}|X|^p\big)^{1/p} \leq \left(1 + \frac{C_4 p}{n}\right)\big(\mathbb{E}|X|^2\big)^{1/2};$$

3. *for any random vector X uniformly distributed on an isotropic convex body and for any $p \in [2, C_5 n^{1/10.02}]$*,

$$\big(\mathbb{E}|X|^p\big)^{1/p} \leq \left(1 + \frac{C_6 p}{n^{1/5.01}}\right)\big(\mathbb{E}|X|^2\big)^{1/2}.$$

These reverse Hölder inequalities obviously give concentration inequalities for the Euclidean norm. The Paouris inequality (1) for the convex bodies [14] is equivalent to the concentration inequality within a Euclidean ball

$$\forall t \geq 1 \quad \mathbb{P}\big(|X| \geq Ct\big(\mathbb{E}|X|^2\big)^{1/2}\big) \leq \mathrm{e}^{-c\sqrt{n}t} \tag{6}$$

for some numerical constants $c > 0$ and $C > 0$. The inequality (2) implies the concentration inequality within a thin Euclidean shell

$$\forall t > 0 \quad \mathbb{P}\big(\big||X| - \big(\mathbb{E}|X|^2\big)^{1/2}\big| \geq t\big(\mathbb{E}|X|^2\big)^{1/2}\big) \leq C\mathrm{e}^{-c(\sqrt{n}t)^{1/2}} \tag{7}$$

for other absolute constants c and C. This is a consequence of the $\psi_{1/2}$-behaviour of the polynomial $|\cdot|^2 - \mathbb{E}|X|^2$ on K. More precisely, Bobkov [2] showed that, for any polynomials P of degree d and any $p \geq 1$, $(\mathbb{E}|P(X)|^{p/d})^{1/p} \leq c_0 p\mathbb{E}|P(X)|^{1/d}$ for some numerical constant c_0. (7) does not give the optimal dependence on n for $t \geq 1$ as in (6). But Theorem 1 implies:

Corollary 2. *There exist universal constants $c > 0$ and $C > 0$ such that, for any random vector X uniformly distributed on an isotropic generalized Orlicz ball*

$$\forall t > 0 \quad \mathbb{P}\big(\big||X| - \big(\mathbb{E}|X|^2\big)^{1/2}\big| \geq t\big(\mathbb{E}|X|^2\big)^{1/2}\big) \leq C\mathrm{e}^{-c\sqrt{n}t}.$$

In the general case, the best deviation inequalities for the Euclidean norm on K were proved by Klartag in [9]:

$$\forall t \in (0,1] \quad \mathbb{P}\big(\big||X| - \big(\mathbb{E}|X|^2\big)^{1/2}\big| \geq t\big(\mathbb{E}|X|^2\big)^{1/2}\big) \leq C'\mathrm{e}^{-c't^{3.33}n^{0.33}} \tag{8}$$

and by Paouris (6) for $t \geq C$ [14]. Emphasize that assertion 2 of Theorem 1 for $p \in [2, cn^{1/4}]$ is a consequence of (7) and that one can deduce assertion 3 from (8). Moreover, the inequality (8) implies an almost isometric moment bound for $p \in [cn^{1/10.01}, c'n^{0.33}]$, as will be shown later in Lemma 6.

The paper is organized as follows. In Section 2, we will give some preliminary observations and we will explain how to deduce Corollary 2 from Theorem 1. We will prove Theorem 1 in Section 3 for the generalized Orlicz balls by applying the negative association property got by Pilipczuk and Wojtaszczyk [15], which generalizes (3). The unconditional case will be studied in Section 4. The proof uses the main results got by Klartag in [10]. In Section 5, we will give a proof of Theorem 1 for the general convex bodies which does not use (8) and is interesting in its own right. We will use the almost radial behavior of marginals of isotropic log-concave measures studied by Klartag to get (8) [9] and we will estimate the spectral gap of measure projections.

The letters $c, c', C, C', c_1, \ldots$ stand for various positive universal constants, whose value may change from one line to the next.

2. Preliminaries

Definition 3. *Let X be a random vector on $\mathbb{R}^n$ such that for all $p \geq 0$, $\mathbb{E}|X|^p < \infty$. X will be said to satisfy the inequality $(\mathcal{V})$ with a constant $A > 0$ if for any real $p \geq 2$,*

$$\operatorname{Var}|X|^p \leq A\frac{p^2}{n}\mathbb{E}|X|^{2p}.$$

Kannan, Lovász and Simonovits' conjecture for the functions $|\cdot|^p$ implies that the random vectors uniformly distributed on an isotropic convex body K satisfy the inequality $(\mathcal{V})$ with a universal constant. Indeed, (4), the Hölder inequality and the isotropic position of K lead to

$$\operatorname{Var}|X|^p \leq cL_K^2 p^2\mathbb{E}|X|^{2p-2} \leq cL_K^2 p^2\big(\mathbb{E}|X|^{2p}\big)^{1-1/p} \leq \frac{cp^2}{n}\mathbb{E}|X|^{2p}.$$

Lemma 4. *Let X be a random vector on $\mathbb{R}^n$ such that for all $p \geq 0$, $\mathbb{E}|X|^p < \infty$ and let r and p_0 be positive reals with $p_0 \leq \sqrt{n}$. Define D_1, D_2 and D_3 in the following way*

$$D_1 = \inf\left\{d > 0, \forall p \in \left[2, \frac{p_0}{\sqrt{d}}\right], \left(\mathbb{E}|X|^p\right)^{1/p} \leq \left(1 + \frac{dp}{n}\right)\left(\mathbb{E}|X|^2\right)^{1/2}\right\},$$

$$D_2 = \inf\left\{d > 0, \forall p \in \left[2, \frac{p_0}{\sqrt{d}}\right], \operatorname{Var}|X|^p \leq \frac{dp^2}{n}\mathbb{E}|X|^{2p}\right\},$$

$$D_3 = \inf\left\{d > 0, \forall p \in \left[\frac{2}{r}, \frac{p_0}{r\sqrt{d}}\right] \cap \mathbb{N}, \operatorname{Var}|X|^{rp} \leq \frac{d(rp)^2}{n}\mathbb{E}|X|^{2rp}\right\}.$$

Then, there exist positive reals a, b depending only on r such that

$$D_3 \leq D_2 \leq aD_1 \leq bD_3.$$

In particular, if X satisfies $(\mathcal{V})$ with a constant A, we have for some universal constants $c_1 > 0$ and $c_2 > 0$,

$$\forall p \in \left[2, c_1\frac{\sqrt{n}}{\sqrt{A}}\right] \quad \left(\mathbb{E}|X|^p\right)^{1/p} \leq \left(1 + \frac{c_2 A p}{n}\right)\left(\mathbb{E}|X|^2\right)^{1/2}. \tag{9}$$

Proof. The existence of an absolute constant a such that $D_2 \leq aD_1$ is a consequence of the growth of $t \mapsto (\mathbb{E}|X|^t)^{1/t}$. $D_3 \leq D_2$ is clear. To get the third inequality, we introduce the function $\phi : t \mapsto \log(\mathbb{E}|X|^t)^{1/t}$ and we observe by the Jensen inequality

$$\phi'(t) = \frac{1}{t^2}\frac{\operatorname{Ent}|X|^t}{\mathbb{E}|X|^t} = \frac{1}{t^2}\mathbb{E}\left[\log\left(\frac{|X|^t}{\mathbb{E}|X|^t}\right)\frac{|X|^t}{\mathbb{E}|X|^t}\right] \leq \frac{1}{t^2}\log\frac{\mathbb{E}|X|^{2t}}{(\mathbb{E}|X|^t)^2} \leq \frac{1}{t^2}\frac{\operatorname{Var}|X|^t}{(\mathbb{E}|X|^t)^2}.$$

Moreover, the convexity of $s \mapsto \phi(1/s)$ means that $t \mapsto \operatorname{Ent}|X|^t/\mathbb{E}|X|^t$ is nondecreasing. Hence, for any $q \geq 2$ and for any integer p such that $\frac{q}{r} \leq p \leq \frac{q}{r} + 1$, we have

$$\phi'(q) \leq \frac{1}{q^2}\frac{\operatorname{Ent}|X|^{rp}}{\mathbb{E}|X|^{rp}} \leq \frac{1}{q^2}\frac{\operatorname{Var}|X|^{rp}}{(\mathbb{E}|X|^{rp})^2} \leq \frac{2D_3(rp)^2}{q^2 n} \leq \frac{2(r+1)^2 D_3}{n}$$

if $2D_3 r^2 p^2 \leq p_0^2$. Integrating this inequality, we get for any $q \in [2, \frac{p_0}{\sqrt{2D_3}(r+1)}]$, $\frac{(\mathbb{E}|X|^q)^{1/q}}{(\mathbb{E}|X|^2)^{1/2}} \leq \mathrm{e}^{2(r+1)^2 D_3 q/n} \leq 1 + \frac{4(r+1)^2 D_3 q}{n}$ since $p_0 \leq \sqrt{n}$. The lemma follows. □

Corollary 2 is a consequence of the following lemmas.

Lemma 5. *Let X be a random vector uniformly distributed on an isotropic convex body. If X satisfies the inequality $(\mathcal{V})$ with a constant $A \geq 1$ then, for any $t > 0$, we have*

$$\mathbb{P}\left(|X| \geq (1+t)\left(\mathbb{E}|X|^2\right)^{1/2}\right) \leq C\mathrm{e}^{-c(\sqrt{n}/\sqrt{A})t} \tag{10}$$

for some absolute constants $c > 0$ and $C > 0$.

Proof. By the concentration inequality (6), it is sufficient to prove (10) for $t \leq c$ where c is a numerical constant. Moreover, we can suppose $t \geq \frac{\sqrt{A}}{\sqrt{n}}$. Taking $p = \frac{c_1\sqrt{n}}{\sqrt{A}}$ in (9), we get $(\mathbb{E}|X|^{c_1\sqrt{n}/\sqrt{A}})^{\sqrt{A}/(c_1\sqrt{n})} \leq (1 + \frac{c_3\sqrt{A}}{\sqrt{n}})(\mathbb{E}|X|^2)^{1/2}$. Then, Markov's inequality gives for any $t \in [\frac{\sqrt{A}}{\sqrt{n}}, c]$

$$\mathbb{P}\left[|X| \geq (1+c_4 t)\left(\mathbb{E}|X|^2\right)^{1/2}\right] \leq \mathbb{P}\left[|X| \geq (1+t)\left(\mathbb{E}|X|^{c_1\sqrt{n}/\sqrt{A}}\right)^{\sqrt{A}/(c_1\sqrt{n})}\right] \leq (1+t)^{-c_1\sqrt{n}/\sqrt{A}}.$$

The lemma is thus proved. □

The following lemma was proved by Klartag in the first version of [10] by exploiting the log-concavity of $t \mapsto \mathbb{P}(|X| \le e^t)$ for the unconditonal convex bodies got by Cordero–Erausquin, Fradelizi and Maurey in [5]. We reproduce below its proof for the convenience of the reader.

Lemma (Klartag [10]). *There exist absolute constants $c > 0$ and $C > 0$ such that, for any random vector X uniformly distributed on an isotropic unconditional convex body, we have*

$$\forall t \in (0,1] \quad \mathbb{P}\big(|X| \le (1-t)\big(\mathbb{E}|X|^2\big)^{1/2}\big) \le C e^{-c\sqrt{n}t}.$$

Proof. According to Klartag [10], X satisfies (2), that is to say, $\operatorname{Var}|X|^2 \le \frac{C}{n}(\mathbb{E}|X|^2)^2$. By Markov's inequality, we get thus

$$\mathbb{P}\bigg(|X| \le \bigg(1 + \frac{c}{\sqrt{n}}\bigg)\big(\mathbb{E}|X|^2\big)^{1/2}\bigg) \ge \frac{3}{4} \quad \text{and} \quad \mathbb{P}\bigg(|X| \le \bigg(1 - \frac{c}{\sqrt{n}}\bigg)\big(\mathbb{E}|X|^2\big)^{1/2}\bigg) \le \frac{1}{4}$$

for some numerical constant $c > 0$. Since $t \mapsto \mathbb{P}(|X| \le e^t)$ is log-concave [5], for any positive reals a and b and for any reals $u \ge 1$ and $s \ge 1$ such that $\frac{1}{u} + \frac{1}{s} = 1$, we have

$$\mathbb{P}\big(|X| \le ab\big(\mathbb{E}|X|^2\big)^{1/2}\big) \ge \big[\mathbb{P}\big(|X| \le a^u\big(\mathbb{E}|X|^2\big)^{1/2}\big)\big]^{1/u}\big[\mathbb{P}\big(|X| \le b^s\big(\mathbb{E}|X|^2\big)^{1/2}\big)\big]^{1/s}.$$

Taking $a = (1 - \frac{c}{\sqrt{n}})(1 + \frac{c}{\sqrt{n}})^{-1/s}$, $b = (1 + \frac{c}{\sqrt{n}})^{1/s}$ and thus $ab = 1 - \frac{c}{\sqrt{n}}$ and $a^u = (\frac{1-c/\sqrt{n}}{1+c/\sqrt{n}})^u(1 + \frac{c}{\sqrt{n}}) \ge e^{-c'u/\sqrt{n}} \ge 1 - \frac{c'u}{\sqrt{n}}$, we obtain for any $u \ge 1$:

$$\mathbb{P}\bigg(|X| \le \bigg(1 - \frac{c'u}{\sqrt{n}}\bigg)\big(\mathbb{E}|X|^2\big)^{1/2}\bigg) \le \bigg(\frac{1}{3}\bigg)^u \le 3\bigg(\frac{1}{3}\bigg)^u.$$

Since this inequality is obvious for $u \le 1$, the lemma is proved. □

The following lemma shows that (7) implies assertion 2 of Theorem 1 for $p \in [2, cn^{1/4}]$ by taking $\alpha = 1/2$ and $\beta = 1/2$. For $\alpha = 0.33/3.33 \approx 1/10$ and $\beta = 3.33$, it gives assertion 3 of Theorem 1 by (8).

Lemma 6. *Let a, b, α and β be positive reals such that $\alpha \le \frac{1}{2}$ and $\alpha\beta \le \frac{1}{2}$. Let X be a random vector in $\mathbb{R}^n$ which satisfies the concentration inequality*

$$\forall t > 0 \quad \mathbb{P}\big(\big||X| - \big(\mathbb{E}|X|^2\big)^{1/2}\big| \ge t\big(\mathbb{E}|X|^2\big)^{1/2}\big) \le a e^{-b(n^\alpha t)^\beta}\mathbf{1}_{t \le 1} + a e^{-b\sqrt{n}t}\mathbf{1}_{t>1}.$$

Then, we have:

1. *for all $p \in [2, c_1 n^{\alpha \min(\beta,1)}]$, $(\mathbb{E}|X|^p)^{1/p} \le (1 + \frac{C_1 p}{n^{2\alpha}})(\mathbb{E}|X|^2)^{1/2}$,*
2. *if $\beta > 1$, for $p \in [c_1 n^\alpha, c_2 n^{\alpha\beta}]$, $(\mathbb{E}|X|^p)^{1/p} \le (1 + C_2(\frac{p}{n^{\alpha\beta}})^{1/(\beta-1)})(\mathbb{E}|X|^2)^{1/2}$,*

where c_1, c_2, C_1, C_2 are positive constants depending only on a, b and β.

Proof. We will denote by $c_1, C_1, \ldots$ positive constants depending only on a, b, α and β. Let $Y = \frac{|X|^2}{\mathbb{E}|X|^2} - 1$. The concentration assumption means for Y that

$$\forall t > 0 \quad \mathbb{P}\big(|Y| \ge t\big) \le C_1 e^{-c_1(n^\alpha t)^\beta}\mathbf{1}_{t \le 1} + C_1 e^{-c_1\sqrt{n}\sqrt{t}}\mathbf{1}_{t>1}.$$

This inequality yields, via the integration by parts $\mathbb{E}|Y|^k = k\int_0^\infty t^{k-1}\mathbb{P}(|Y| \ge t)\,dt$,

$$\forall k \in \big[1, c_2 n^{\alpha\beta}\big] \quad \big(\mathbb{E}|Y|^k\big)^{1/k} \le C_2\frac{k^{1/\beta}}{n^\alpha} + C_2\frac{k^2}{n} \le C_3\frac{k^{1/\beta}}{n^\alpha}.$$

Therefore, since $\mathbb{E}Y=0$, we get for any integer $q\in[1,c_2n^{\alpha\beta}]$,

$$\frac{\mathbb{E}|X|^{2q}}{(\mathbb{E}|X|^2)^q}=\mathbb{E}(1+Y)^q=1+\sum_{k=2}^{q}\binom{q}{k}\mathbb{E}Y^k\le 1+\sum_{k=2}^{q}\left(\frac{C_4qk^{1/\beta-1}}{n^\alpha}\right)^k\le 1+2\max_{2\le k\le q}\left(\frac{C_5qk^{1/\beta-1}}{n^\alpha}\right)^k.$$

Consider the $g:k\mapsto(\frac{C_5qk^{1/\beta-1}}{n^\alpha})^k$ and set $k_0=\mathrm{e}^{-1}(\frac{C_5q}{n^\alpha})^{1/(1-1/\beta)}$.

• If $\beta<1$, g is decreasing on $(0,k_0]$ and increasing on $[k_0,\infty)$. Thus, $\max_{2\le k\le q}g(k)=\max(g(2),g(q))=g(2)\le C_6\frac{q^2}{n^{2\alpha}}$ for $q\le c_3n^{\alpha\beta}$ and the result is proved.

• If $\beta>1$, g is increasing on $(0,k_0]$ and decreasing on $[k_0,\infty)$. When $k_0\le 2$, that is to say, $q\le c_4n^\alpha$, $\max_{2\le k\le q}g(k)=g(2)\le C_7\frac{q^2}{n^{2\alpha}}$. When $k_0\in[2,q]$, that is to say, $q\in[c_4n^\alpha,c_5n^{\alpha\beta}]$, $\max_{2\le k\le q}g(k)=g(k_0)=\exp((1-1/\beta)\mathrm{e}^{-1}(\frac{C_5q}{n^\alpha})^{\beta/(\beta-1)})$ and hence

$$\frac{(\mathbb{E}|X|^{2q})^{1/2q}}{(\mathbb{E}|X|^2)^{1/2}}\le 3^{1/2q}\exp\left(C_8\left(\frac{q}{n^{\alpha\beta}}\right)^{1/(\beta-1)}\right)\le 1+C_2\left(\frac{p}{n^{\alpha\beta}}\right)^{1/(\beta-1)}.\qquad\square$$

3. Case of generalized Orlicz balls

Recall that K is a generalized Orlicz ball if there exist convex increasing functions $f_i:[0,\infty)\to[0,\infty]$, $i\in\{1,\dots,n\}$ which satisfy $f_i(0)=0$, $\exists t\in\mathbb{R}_*^+$, $f_i(t)\ne 0$ and $\exists s\in\mathbb{R}_*^+$, $f_i(s)\ne\infty$, such that

$$K=\left\{x=(x_1,\dots,x_n)\in\mathbb{R}^n,\sum_{i=1}^n f_i\bigl(|x_i|\bigr)\le 1\right\}.$$

According to Lemma 4, Theorem 1 for the generalized Orlicz balls is a consequence of the following result:

Theorem 7. *If X is a random vector uniformly distributed on an isotropic generalized Orlicz ball, then X satisfies the inequality $(\mathcal{V})$ with a universal constant C, that is to say,*

$$\forall p\ge 2\quad \operatorname{Var}|X|^p\le\frac{Cp^2}{n}\mathbb{E}|X|^{2p}.$$

The proof uses mainly the following theorem:

Theorem (Pilipczuk–Wojtaszczyk [15]). *Let X be a random vector uniformly distributed on a generalized Orlicz ball. For any coordinate-wise increasing bounded functions f, g and any disjoint subsets $\{i_1,\dots,i_k\}$ and $\{j_1,\dots,j_l\}$ of $\{1,\dots,n\}$, we have*

$$\operatorname{cov}\bigl(f\bigl(|X_{i_1}|,\dots,|X_{i_k}|\bigr),g\bigl(|X_{j_1}|,\dots,|X_{j_l}|\bigr)\bigr)\le 0. \tag{11}$$

Notation 8. *For any n-tuples $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ and $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$ with $|\alpha|=\sum_{i=1}^n\alpha_i=q$, we denote by x^α the real number $\prod_{i=1}^n x_i^{\alpha_i}$ and by $\binom{q}{\alpha}$ the multinomial coefficient $\frac{q!}{\prod_{i=1}^n\alpha_i!}$ in such a way that Newton's formula is*

$$\left(\sum_{i=1}^n x_i\right)^q=\sum_{|\alpha|=q}\binom{q}{\alpha}x^\alpha.$$

Otherwise for any $y=(y_1,\dots,y_n)\in\mathbb{R}^n$ and for any $i\in\{1,\dots,n\}$ we denote the orthogonal projection of y on $e_i^\perp$ by $\check{y}_i=(y_1,\dots,y_{i-1},0,y_{i+1},\dots,y_n)$ the orthogonal projection of y on $e_i^\perp$.

Proof of Theorem 7. The Pilipczuk–Wojtaszczyk theorem shows that, for any $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}^n$ with disjoint supports, we have

$$\operatorname{cov}\big(X^{2\alpha}, X^{2\beta}\big) \le 0. \tag{12}$$

In the case where the supports of α and β are not disjoint, we use the obvious upper bound $\operatorname{cov}(X^{2\alpha}, X^{2\beta}) \le \mathbb{E}X^{2(\alpha+\beta)}$. Since the variables X_i are ψ_1, that is to say, for any $r \ge 2$ $(\mathbb{E}|X_i|^r)^{1/r} \le r(\mathbb{E}|X_i|^2)^{1/2}$ ([13], Appendix III), we have besides

$$\mathbb{E}X_i^{2(\alpha_i+\beta_i)} \le \big(\mathbb{E}X_i^{4\alpha_i}\big)^{1/2}\big(\mathbb{E}X_i^{4\beta_i}\big)^{1/2} \le (4\alpha_i L_K)^{2\alpha_i}(4\beta_i L_K)^{2\beta_i} = \alpha_i^{2\alpha_i}\beta_i^{2\beta_i}\big(16L_K^2\big)^{(\alpha_i+\beta_i)}.$$

When i belongs to the supports of α and β, $(\alpha_i+\beta_i)e_i$ and $\check{\alpha}_i+\check{\beta}_i$ have disjoint supports and (12) gives $\operatorname{cov}(X_i^{2(\alpha_i+\beta_i)}, X^{2(\check{\alpha}_i+\check{\beta}_i)}) \le 0$. Hence

$$\mathbb{E}X^{2(\alpha+\beta)} \le \mathbb{E}X_i^{2(\alpha_i+\beta_i)}\mathbb{E}X^{2(\check{\alpha}_i+\check{\beta}_i)} \le \alpha_i^{2\alpha_i}\beta_i^{2\beta_i}\big(16L_K^2\big)^{(\alpha_i+\beta_i)}\mathbb{E}X^{2(\check{\alpha}_i+\check{\beta}_i)}. \tag{13}$$

Consequently, we get by Newton's formula for any integer $q > 0$,

$$\begin{aligned}
\operatorname{Var}|X|^{2q} &= \operatorname{Var}\Big[\textstyle\sum_{|\alpha|=q} \binom{q}{\alpha} X^{2\alpha}\Big] \le \sum_{\substack{|\alpha|=q,|\beta|=q\\ \operatorname{supp}(\alpha)\cap\operatorname{supp}(\beta)\neq\varnothing}} \binom{q}{\alpha}\binom{q}{\beta}\operatorname{cov}\big(X^{2\alpha}, X^{2\beta}\big)\\
&\le \sum_{\bigcup_{i=1}^n\{(\alpha,\beta),|\alpha|=q,|\beta|=q,\alpha_i\neq 0,\beta_i\neq 0\}} \binom{q}{\alpha}\binom{q}{\beta}\operatorname{cov}\big(X^{2\alpha}, X^{2\beta}\big)\\
&\le \sum_{i=1}^n\sum_{\alpha_i=1}^q\sum_{\beta_i=1}^q \binom{q}{\alpha_i}\binom{q}{\beta_i}\alpha_i^{2\alpha_i}\beta_i^{2\beta_i}\big(16L_K^2\big)^{(\alpha_i+\beta_i)} \sum_{\substack{|\check{\alpha}_i|=q-\alpha_i\\ |\check{\beta}_i|=q-\beta_i}} \binom{q-\alpha_i}{\check{\alpha}_i}\binom{q-\beta_i}{\check{\beta}_i}\mathbb{E}X^{2(\check{\alpha}_i+\check{\beta}_i)}\\
&= \sum_{i=1}^n\sum_{1\le k\le q}\sum_{1\le h\le q}\binom{q}{k}\binom{q}{h}h^{2h}k^{2k}\big(16L_K^2\big)^{(h+k)}\mathbb{E}|\check{X}_i|^{4q-2(k+h)}.
\end{aligned}$$

The Hölder inequality gives for any i, $h \ge 1$ and $k \ge 1$,

$$\mathbb{E}|\check{X}_i|^{4q-2(k+h)} \le \mathbb{E}|X|^{4q-2(k+h)} \le \big(\mathbb{E}|X|^{4q}\big)^{1-(k+h)/(2q)} \le \frac{\mathbb{E}|X|^{4q}}{(nL_K^2)^{k+h}}.$$

Hence, we obtain

$$\operatorname{Var}|X|^{2q} \le n\bigg(\textstyle\sum_{k=1}^q \binom{q}{k}\Big(\frac{16k^2}{n}\Big)^k\bigg)^2\mathbb{E}|X|^{4q}.$$

To conclude, it is sufficient to use Lemma 4 and to observe that if $q \le \frac{1}{20}\sqrt{n}$, we have

$$\begin{aligned}
\sum_{k=1}^q\binom{q}{k}\Big(\frac{16k^2}{n}\Big)^k &\le \sum_{k=1}^q\Big(\frac{50kq}{n}\Big)^k = \frac{50q}{n}\sum_{k=0}^{q-1}\Big(\frac{50q}{n}\Big)^k(k+1)^{k+1}\\
&\le \frac{50q}{n}\sum_{k=0}^{q-1}\Big(\frac{200kq}{n}\Big)^k \le \frac{50q}{n}\sum_{k=0}^{q-1}\Big(\frac{200q^2}{n}\Big)^k \le \frac{100q}{n}.
\end{aligned}$$

□

4. Case of convex bodies with an unconditional basis

Recall that a convex body K in $\mathbb{R}^n$ is unconditional if K is symmetric with respect to the coordinate hyperplanes, that is to say

$$\forall(\varepsilon_1,\dots,\varepsilon_n)\in\{-1,1\}^n, \forall x=(x_1,\dots,x_n)\in K \quad (\varepsilon_1 x_1,\dots,\varepsilon_n x_n)\in K.$$

We repeat the arguments used by Klartag [10] to prove the variance conjecture for the unconditional convex bodies. The main tool of the proof is the following result based on analysis of the Neumann Laplacian on convex domains.

Theorem (Klartag [10]). *Let X be a random vector uniformly distributed on an unconditional convex body K of $\mathbb{R}^n$. For any $i\in\{1,\dots,n\}$ and for any $x\in\mathbb{R}^n$ denote $B_i^+(x)$ and $B_i^-(x)$ the points such that $[B_i^-(x),B_i^+(x)]=K\cap(x+\mathbb{R}e_i)$ (with $\langle B_i^+(x),e_i\rangle\geq 0$ and $\langle B_i^-(x),e_i\rangle\leq 0$). Let $f:\mathbb{R}^n\to\mathbb{R}$ be an unconditional function of class C^1 (that is to say, such that, for any $(x_1,\dots,x_n)\in\mathbb{R}^n$ and for any $(\varepsilon_1,\dots,\varepsilon_n)\in\{-1,1\}^n$, $f(\varepsilon_1x_1,\dots,\varepsilon_nx_n)=f(x_1,\dots,x_n)$). Then,*

$$\operatorname{Var}\bigl[f(X)\bigr]\leq\mathbb{E}\Biggl[\sum_{i=1}^n\bigl(f(X)-f\bigl(B_i^+(X)\bigr)\bigr)^2\Biggr]. \tag{14}$$

If $h:\mathbb{R}\to\mathbb{R}$ is an even function of class C^1 and $r\in\mathbb{R}^+$, by integration by parts and by Cauchy–Schwarz inequality, we have

$$\int_{-r}^r\bigl(h(t)-h(r)\bigr)^2\,\mathrm{d}t\leq 4\int_{-r}^r t^2\bigl(h'(t)\bigr)^2\,\mathrm{d}t.$$

For the functions $h_i:t\mapsto f(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n)$ (where f satisfies the assumptions of the previous theorem) and $r=\langle B_i^+(x),e_i\rangle$ for $i\in\{1,n\}$, this inequality gives after an integration on $K\cap e_i^\perp$:

$$\begin{aligned}
&\int_K\bigl[f(x)-f\bigl(B_i^+(x)\bigr)\bigr]^2\,\mathrm{d}x\\
&\quad=\int_{K\cap e_i^\perp}\mathrm{d}y\int_{-\langle B_i^+(y),e_i\rangle}^{\langle B_i^+(y),e_i\rangle}\bigl[f(y+te_i)-f\bigl(y+\langle B_i^+(y),e_i\rangle e_i\bigr)\bigr]^2\,\mathrm{d}t\\
&\quad\leq 4\int_{K\cap e_i^\perp}\mathrm{d}y\int_{-\langle B_i^+(y),e_i\rangle}^{\langle B_i^+(y),e_i\rangle}t^2\bigl(\partial_i f(y+te_i)\bigr)^2\,\mathrm{d}t=4\int_K x_i^2\bigl(\partial_i f(x)\bigr)^2\,\mathrm{d}x.
\end{aligned}$$

Hence (14) implies

$$\operatorname{Var}\bigl[f(X)\bigr]\leq 4\sum_{i=1}^n\mathbb{E}\bigl[X_i^2\bigl(\partial_i f(X)\bigr)^2\bigr]. \tag{15}$$

Remark. *If X is uniformly distributed on an isotropic generalized Orlicz ball, the inequalities*

$$\operatorname{cov}\Biggl(\prod_{i=1}^n|X_i|^{\alpha_i},\prod_{i=1}^n|X_i|^{\beta_i}\Biggr)\leq 0 \quad \textit{if}\ \operatorname{supp}(\alpha)\cap\operatorname{supp}(\beta)=\varnothing, \tag{16}$$

$$\operatorname{cov}\Biggl(\prod_{i=1}^n|X_i|^{\alpha_i},\prod_{i=1}^n|X_i|^{\beta_i}\Biggr)\leq\mathbb{E}\prod_{i=1}^n|X_i|^{\alpha_i+\beta_i} \quad \textit{if}\ \operatorname{supp}(\alpha)\cap\operatorname{supp}(\beta)\neq\varnothing \tag{17}$$

for any $\alpha\in(\mathbb{R}^+)^n$ and $\beta\in(\mathbb{R}^+)^n$, imply (15) for any function f such that $f(x)=\sum_\alpha a_\alpha\prod_{i=1}^n|x_i|^{\alpha_i}$ with $a_\alpha\geq 0$ for any α (expanding the variance is sufficient). Hence, (15) applied to the function $|\cdot|^p$ gives the same estimate of

$\mathrm{Var}\,|X|^p$ *as one given by* (16) *and* (17). *On the other hand,* (13) *is false in the unconditional case. More precisely, there does not exist some universal constant* $C > 0$ *such that, for any* $i \in \{1, \dots, n\}$, *any* $\gamma \in \mathbb{N}^n$ *with* $\gamma_i = 2$ *and any random vector* X *uniformly distributed on an unconditional convex body,* $\mathbb{E}\prod_{k=1}^n |X_k|^{\gamma_k} \le C\mathbb{E}|X_i|^2\mathbb{E}\prod_{k\neq i}|X_k|^{\gamma_k}$. *Let us repeat the counterexample to the square negative correlation property given by Wojtaszczyk in* [17] *for unconditional bodies. Let* $X = (Y, Z)$ *be a random vector on* $\mathbb{R}^{n-2} \times \mathbb{R}^2 = \mathbb{R}^n$ *uniformly distributed on the unconditional convex body* $L = \{x = (y, z) \in \mathbb{R}^{n-2} \times \mathbb{R}^2, |y|_1 + |z|_\infty \le 1\}$. *Then, for any* $(\delta_1, \delta_2) \in \mathbb{N}^2$,

$$\begin{aligned}\int_{L\cap\mathbb{R}^n_+} z_1^{\delta_1} z_2^{\delta_2}\,\mathrm{d}y\,\mathrm{d}z &= \int_{y\in\mathbb{R}^{n-2}_+:|y|_1\le 1}\mathrm{d}y\int_{[0,1-|y|_1]^2} z_1^{\delta_1} z_2^{\delta_2}\,\mathrm{d}z = \frac{\int_{y\in\mathbb{R}^{n-2}_+:|y|_1\le 1}\mathrm{d}y\,(1-|y|_1)^{\delta_1+\delta_2+2}}{(\delta_1+1)(\delta_2+1)}\\ &= \frac{1}{(n-3)!(\delta_1+1)(\delta_2+1)}\int_0^1 t^{n-3}(1-t)^{\delta_1+\delta_2+2}\,\mathrm{d}t\\ &= \frac{(\delta_1+\delta_2+2)!}{(\delta_1+\delta_2+n)!(\delta_1+1)(\delta_2+1)}.\end{aligned}$$

Hence

$$\frac{\mathbb{E}|Z_1|^{\delta_1}|Z_2|^{\delta_2}}{\mathbb{E}|Z_1|^{\delta_1}\mathbb{E}|Z_2|^{\delta_2}} = \prod_{k=1}^{\delta_1}\left[\frac{1+\delta_2/(2+k)}{1+\delta_2/(n+k)}\right].$$

For $i = n-1$ *and* $\gamma = (0, (2, \gamma_n)) \in \mathbb{N}^{n-2} \times \mathbb{N}^2$, *that gives*

$$\mathbb{E}\prod_{k=1}^n |X_k|^{\gamma_k} \ge c\min(n, \gamma_n)\mathbb{E}|X_i|^2\mathbb{E}\prod_{k\neq i}|X_k|^{\gamma_k}.$$

According to Lemma 4, Theorem 1 for the unconditional convex bodies is a consequence of following result:

Theorem 9. *There exist universal constants* $c_1 > 0$ *and* $c_2 > 0$ *such that for any integer* n, *any real* $p \in [2, c_1\frac{\sqrt{n}}{\log(n)}]$ *and any random vector* X *uniformly distributed on an isotropic unconditional convex body* K *of* $\mathbb{R}^n$, *we have*

$$\mathrm{Var}\,|X|^p \le c_2\frac{p^2}{n}\mathbb{E}|X|^{2p}.$$

Proof. Applying (15) to the function $|\cdot|^p$ and using the Cauchy–Schwarz inequality, we get

$$\mathrm{Var}\,|X|^p \le 4p^2\mathbb{E}|X|_4^4|X|^{2p-4} \le 4p^2\big(\mathbb{E}|X|_4^8\big)^{1/2}\big(\mathbb{E}|X|^{4p-8}\big)^{1/2}, \tag{18}$$

where $|X|_4 = (\sum_{i=1}^n X_i^4)^{1/4}$. By Borell's lemma ([13], Appendix III), we have $(\mathbb{E}|X|_4^8)^{1/2} \le c_1 n L_K^4$. Besides, since the spectral gap of X is bounded from below by $c_2/(\log(n))^2$ [10], the Poincaré inequality for X applied to the function $|\cdot|^q$ shows that, for $q \in [1, c_3\sqrt{n}/\log(n)]$, $\mathbb{E}|X|^{2q} \le c_4(\mathbb{E}|X|^q)^2$. Therefore, for $p \in [2, c_5\sqrt{n}/\log(n)]$,

$$\big(\mathbb{E}|X|^{4p-8}\big)^{1/2} \le c_6\mathbb{E}|X|^{2p-4} \le c_6\big(\mathbb{E}|X|^{2p}\big)^{1-4/2p} \le \frac{c_6}{n^2L_K^4}\mathbb{E}|X|^{2p}.$$

The inequality (18) gives the result. □

5. General case

In this section, X belongs to the class of random vectors in $\mathbb{R}^n$ which have a log-concave law, that is to say, for all nonempty compact subsets A and B of $\mathbb{R}^n$ and $t \in [0, 1]$

$$\mathbb{P}\big(X \in \big(tA + (1-t)B\big)\big) \ge \mathbb{P}(X \in A)^t\mathbb{P}(X \in B)^{1-t},$$

X will be said isotropic if for any $\theta \in S^{n-1}$, we have

$$\mathbb{E}\langle X, \theta\rangle = 0 \quad \text{and} \quad \mathbb{E}\big|\langle X, \theta\rangle\big|^2 = 1.$$

By the Brunn–Minkowski inequality (see, for instance, [7]), if X is uniformly distributed in an isotropic convex body, $\frac{1}{L_K}X$ is log-concave and is isotropic in the previous meaning.

The proof of assertion 3 of Theorem 1 uses the approach of Klartag [9] to get (8) and the one built independently in [6]. We can summarize the arguments in the following way:

1. We reduce the estimate of the ratio $(\mathbb{E}|X|^p)^{1/p}/(\mathbb{E}|X|^2)^{1/2}$ to the estimate of this ratio for projections of X on subspaces.
2. We show that, if G_n is a standard Gaussian vector in $\mathbb{R}^n$, the inequality $(\mathcal{V})$ is satisfied by most of the projections of $X + G_n$ on subspaces with an adapted dimension. We use the main tool of Klartag which gives almost radial projections of $X + G_n$.
3. We explain how to deduce the result for X from the estimate for $X + G_n$.

Recall that $G_{n,k}$ stands for the Grassmannian of all k-dimensional subspaces in $\mathbb{R}^n$ and for any subspace $F \in G_{n,k}$, P_F stands for the orthogonal projection from $\mathbb{R}^n$ on F. Denote by $\mu_{n,k}$ the unique rotationally-invariant probability measure on $G_{n,k}$ and by ν_n the unique Haar probability measure on the special orthogonal group $SO(n)$ which is invariant under both left and right translations. $\mu_{n,k}$ and ν_n are linked by the following equality: for any measurable subset Ω of $G_{n,k}$ and for a fixed subspace $F_0 \in G_{n,k}$, we have

$$\nu_n\big(u \in SO(n), u(F_0) \in \Omega\big) = \mu_{n,k}(\Omega).$$

Furthermore recall that the geodesic distance d on the connected Riemannian manifold $SO(n)$ is equivalent to the distance defined by the Hilbert–Schmidt norm $\|\cdot\|_{\mathrm{HS}}$. More precisely for any u_1 and u_2 in $SO(n)$,

$$\|u_1 - u_2\|_{\mathrm{HS}} \le d(u_1, u_2) \le \frac{\pi}{2}\|u_1 - u_2\|_{\mathrm{HS}}. \tag{19}$$

Emphasize that ν_n satisfies the following log-Sobolev inequality. For any Lipschitz function $f : SO(n) \to \mathbb{R}$, we have

$$\mathrm{Ent}_{\nu_n}\big[f^2\big] := \int f^2 \log\big(f^2\big)\,\mathrm{d}\nu_n - \left(\int f^2\,\mathrm{d}\nu_n\right)\log\left(\int f^2\,\mathrm{d}\nu_n\right) \le \frac{C}{n}\int |\nabla f|^2\,\mathrm{d}\nu_n, \tag{20}$$

where $|\nabla f(u)| = \limsup_{d(v,u)\to 0} \frac{|f(v)-f(u)|}{d(v,u)}$ and $C > 0$ is an absolute constant. Applying (20) to the function $|f|^{p/2}$, we observe that, for any $p \ge 1$,

$$\frac{\mathrm{d}}{\mathrm{d}p}\left[\log\left(\int |f|^p\,\mathrm{d}\nu_n\right)^{1/p}\right] = \frac{1}{p^2}\frac{\mathrm{Ent}_{\nu_n}[f^p]}{\int |f|^p\,\mathrm{d}\nu_n} \le \frac{C}{n}\frac{\int |f|^{p-2}\,\mathrm{d}\nu_n}{\int |f|^p\,\mathrm{d}\nu_n}\|f\|_{\mathrm{Lip}}^2 \le \frac{C}{n}\frac{\|f\|_{\mathrm{Lip}}^2}{(\int |f|^p\,\mathrm{d}\nu_n)^{2/p}} \le \frac{C}{\mathrm{d}(f)},$$

where $d(f) = n\big(\frac{\int |f|\,\mathrm{d}\nu_n}{\|f\|_{\mathrm{Lip}}}\big)^2$. Consequently, for any $p \in [1, d(f)]$, we have

$$\left(\int_{SO(n)} |f|^p\,\mathrm{d}\nu_n\right)^{1/p} \le \left(1 + \frac{C' p}{d(f)}\right)\int_{SO(n)} |f|\,\mathrm{d}\nu_n \tag{21}$$

for a new numerical constant $C' > 0$.

In [6], the study of moments bounds for the Euclidean norm on a convex body begins by reducing the problem to the study of the mean width of its L_p-centroid bodies. When $k = 1$, the following lemma is similar to this reduction. In this case, p^* is the parameter introduced by Paouris in [14].

Lemma 10. *Let X be an isotropic random vector in $\mathbb{R}^n$ distributed according to a log-concave law. Then, for any integer $k \in [1, n]$ and for any $p \in [2, c_1 \max((kn)^{1/3}, n^{1/2})]$, we have*:

$$\frac{(\mathbb{E}|X|^p)^{1/p}}{(\mathbb{E}|X|^2)^{1/2}} \le \left(1 + \frac{c_2 p^2}{n}\min\left(\frac{p}{k}, 1\right)\right)\int_{G_{n,k}} \frac{(\mathbb{E}|P_F X|^p)^{1/p}}{\sqrt{k}}\mu_{n,k}(\mathrm{d}F), \tag{22}$$

where $c_1 > 0$ and $c_2 > 0$ are absolute constants.

Proof. Fix an integer $k \in [1, n]$, a real $p \geq 2$ and a subspace F_0 of $G_{n,k}$. There exists a real number $a_{n,k,p}$ such that for all point $x \in \mathbb{R}^n$,

$$|x|^p = a_{n,k,p} \int_{G_{n,k}} |P_F x|^p \mu_{n,k}(\mathrm{d}F) = a_{n,k,p} \int_{SO(n)} \left|P_{F_0} u(x)\right|^p \nu_n(\mathrm{d}u).$$

Hence, denoting by G_i a standard Gaussian vector on $\mathbb{R}^i$, we have for $q \in \{2, p\}$,

$$\frac{\mathbb{E}|X|^q}{\mathbb{E}|G_n|^q} = \frac{\int_{G_{n,k}} \mathbb{E}|P_F X|^q \mu_{n,k}(\mathrm{d}F)}{\mathbb{E}|G_k|^q}. \tag{23}$$

Remark that $\mathbb{E}|G_i|^p/(\mathbb{E}|G_i|^2)^{p/2} = \Gamma(\frac{i+p}{2})\Gamma(\frac{i}{2})^{p/2-1}/\Gamma(\frac{i+2}{2})^{p/2}$ and that $(\log \circ \Gamma)'$ is concave (the Euler's formula shows that $\log \circ \Gamma$ is the sum of functions which have a negative third derivative). This implies that $i \mapsto \mathbb{E}|G_i|^p/(\mathbb{E}|G_i|^2)^{p/2}$ is decreasing. We get

$$\begin{aligned} \frac{(\mathbb{E}|X|^p)^{1/p}}{(\mathbb{E}|X|^2)^{1/2}} &= \frac{(\mathbb{E}|G_n|^p)^{1/p}}{(\mathbb{E}|G_n|^2)^{1/2}} \frac{(\mathbb{E}|G_k|^2)^{1/2}}{(\mathbb{E}|G_k|^p)^{1/p}} \frac{(\int_{G_{n,k}} \mathbb{E}|P_F X|^p \mu_{n,k}(\mathrm{d}F))^{1/p}}{(\int_{G_{n,k}} \mathbb{E}|P_F X|^2 \mu_{n,k}(\mathrm{d}F))^{1/2}} \\ &\leq \frac{(\int_{G_{n,k}} \mathbb{E}|P_F X|^p \mu_{n,k}(\mathrm{d}F))^{1/p}}{\sqrt{k}}, \end{aligned} \tag{24}$$

since $P_F X$ is isotropic. Consider the function $h_p : \mathbb{M}_n(\mathbb{R}) \to \mathbb{R}$ such that $h_p(u) = (\mathbb{E}|P_{F_0} u(X)|^p)^{1/p}$. As a consequence of Borell's lemma ([13], Appendix III), h_p satisfies the Khintchine-type inequality $h_p \leq Cph_2$ for some absolute constant $C > 0$. Thus, for any u_1 and u_2 in $SO(n)$, one has

$$\left|h_p(u_1) - h_p(u_2)\right| \leq h_p(u_1 - u_2) \leq Cph_2(u_1 - u_2) = Cp\left\|p_F(u_1 - u_2)\right\|_{\mathrm{HS}} \leq Cp\|u_1 - u_2\|_{\mathrm{HS}}.$$

Hence, the inequality (19) gives for any $u \in SO(n)$,

$$\|h_p\|_{\mathrm{Lip}} \leq Cp. \tag{25}$$

Since, by Stirling's formula, we have $a_1 \max(\sqrt{i}, \sqrt{p}) \leq (\mathbb{E}|G_i|^p)^{1/p} \leq a_2 \max(\sqrt{i}, \sqrt{p})$ for some numerical constants a_1 and a_2 and since $(\mathbb{E}|X|^p)^{1/p} \geq (\mathbb{E}|X|^2)^{1/2} = \sqrt{n}$, (23) gives for $p \leq n$,

$$\left(\int_{SO(n)} h_p^p(u)\nu_n(\mathrm{d}u)\right)^{1/p} = \frac{(\mathbb{E}|G_k|^p)^{1/p}(\mathbb{E}|X|^p)^{1/p}}{(\mathbb{E}|G_n|^p)^{1/p}} \geq a_3 \max\left(\sqrt{k}, \sqrt{p}\right). \tag{26}$$

Set $p^* = \max\{p_0 \in [2, n]\colon \forall q \in [2, p_0], q \leq d(h_q)\}$. By the inequality (21), we have for any $p \in [2, p^*]$,

$$\left(\int_{SO(n)} h_p^p \,\mathrm{d}\nu_n\right)^{1/p} \leq \left(1 + \frac{C'p}{d(h_p)}\right) \int_{SO(n)} h_p \,\mathrm{d}\nu_n. \tag{27}$$

In particular, $(\int_{SO(n)} h_p^p \,\mathrm{d}\nu_n)^{1/p} \leq C'' \int_{SO(n)} h_p \,\mathrm{d}\nu_n$. Thus, by using (25) and (26), we get

$$d(h_p) \geq c_1 n \frac{\max(k, p)}{p^2}.$$

Thanks to the inequalities (24) and (27), we get the assertion of Lemma 10 for any $p \in [2, p^*]$. To conclude, it is sufficient to observe that, if $p_0 \in [p^*, p^* + 1]$ is such that $d(h_{p_0}) \leq p_0$, then by the inequalities (25)–(27) and the fact that $p \mapsto h_p$ is non-decreasing, we get

$$p^* + 1 \geq p_0 \geq d(h_{p_0}) \geq c_2 n \frac{(\int_{SO(n)} h_{p^*} \,\mathrm{d}\nu_n)^2}{p_0^2} \geq c_3 n \frac{p^{*2}}{p_0^2} \frac{\max(k, p^*)}{p^{*2}} \geq \frac{c_3}{2} n \frac{\max(k, p^*)}{p^{*2}}.$$

Hence $p^* \geq c_4 \max(n^{1/2}, (kn)^{1/3})$. Lemma 10 is proved. □

Lemma 11. *Let U and V be two isotropic independent random vectors in $\mathbb{R}^k$. Suppose V is symmetric. If $U+V$ satisfies the inequality $(\mathcal{V})$ with a constant $A>0$ then U satisfies $(\mathcal{V})$ with a constant $c_0 A$ where c_0 is a universal constant.*

Proof. Let $p \geq 2$ be an integer. Since V is symmetric, for any nonnegative integers a, b c we have: $\mathbb{E}|U|^{2a}|V|^{2b}\langle U, V\rangle^c \geq 0$. Hence, by using the inequality $(|t|^2+|s|^2)^p \geq 2^p|t|^p|s|^p$ for any reals t and s, we get

$$\mathbb{E}|U+V|^{2p} = \mathbb{E}\big(|U|^2+|V|^2+2\langle U,V\rangle\big)^p = \sum_{a+b+c=p} \frac{p!}{a!b!c!}\mathbb{E}|U|^{2a}|V|^{2b}\big(2\langle U,V\rangle\big)^c$$

$$\geq \sum_{a+b=p} \frac{p!}{a!b!}\mathbb{E}|U|^{2a}|V|^{2b} = \mathbb{E}\big(|U|^2+|V|^2\big)^p \geq 2^p\mathbb{E}|U|^p\mathbb{E}|V|^p \geq \big(2\sqrt{k}\big)^p\mathbb{E}|U|^p.$$

Besides, since U and V are isotropic, from the inequality $(\mathcal{V})$ and from Lemma 4, we get if $p \leq c_1\frac{\sqrt{k}}{\sqrt{A}}$,

$$\big(\mathbb{E}|U+V|^{2p}\big)^{1/2p} \leq \left(1+\frac{c_2Ap}{k}\right)\big(\mathbb{E}|U+V|^2\big)^{1/2} = \left(1+\frac{c_2Ap}{k}\right)\sqrt{2k}.$$

Hence

$$\big(\mathbb{E}|U|^p\big)^{1/p} \leq \left(1+\frac{c_2Ap}{k}\right)^2\sqrt{k} \leq \left(1+\frac{c_3Ap}{k}\right)\big(\mathbb{E}|U|^2\big)^{1/2}.$$

Consequently, the proof is complete thanks to Lemma 4. □

We will use the three results which follow. The first is a key argument in the proof of (8) (see Lemma 3.3 in [9] with $\alpha=0$, $\eta=0$, $u=\frac{4}{5.01}$ and a little alteration in the constants).

Theorem A (Klartag [9]). *Let X be a random vector in $\mathbb{R}^n$ distributed according to an isotropic log-concave density. Denote $g:\mathbb{R}^n \to \mathbb{R}^+$ the density of $Y = X + G_n$ where G_n is a standard Gaussian vector on $\mathbb{R}^n$ independent of X. For $k = \lfloor c_1 n^{1/5.01}\rfloor$, there exists a subset $\mathcal{E}$ in $G_{n,k}$ with probability $\mu_{n,k}(\mathcal{E}) \geq 1 - c_2\mathrm{e}^{-c_3k}$ such that for any subspace $F \in \mathcal{E}$, any x_1 and x_2 in F with $|x_1| = |x_2| \leq 10\sqrt{k}$, we have*

$$\left|\frac{\pi_F g(x_1)}{\pi_F g(x_2)} - 1\right| \leq \frac{1}{4},$$

where $\pi_F g$ is the density of $P_F Y$ and c_1, c_2 and c_3 are absolute constants.

Recall that, if μ_1 and μ_2 are two Borel probability measures on $\mathbb{R}^n$, $d_{\mathrm{TV}}(\mu_1,\mu_2)$ stands for the total variation distance between μ_1 and μ_2 which is defined by

$$d_{\mathrm{TV}}(\mu_1,\mu_2) = 2\sup_{A\subset\mathbb{R}^n}\big|\mu_1(A)-\mu_2(A)\big| = \int\left|\frac{\mathrm{d}\mu_1}{\mathrm{d}x}(x) - \frac{\mathrm{d}\mu_2}{\mathrm{d}x}(x)\right|\mathrm{d}x.$$

Recently, E. Milman proved the following result (see Theorem 5.5 in [12]). The proof is based on the concavity of the isoperimetric profile for the log-concave measures.

Theorem B (Milman [12]). *Let μ_1 and μ_2 be two log-concave probability measures on $\mathbb{R}^n$. If*

$$d_{\mathrm{TV}}(\mu_1,\mu_2) \leq c < 1$$

then

$$\frac{1}{a_c}\lambda_1(\mu_1) \le \lambda_1(\mu_2) \le a_c\lambda_1(\mu_1),$$

where $a_c > 0$ depends only on c.

When Z is a radial random vector on $\mathbb{R}^n$, so as to estimate the spectral-gap of Z, one must essentially estimate the spectral gaps of S^{n-1} and of the random variable $|Z|$. These estimates are well known when the law of $|Z|$ is log-concave. In this way, Bobkov showed the following result.

Theorem C (Bobkov [3]). *Let Z be a random vector on $\mathbb{R}^n$ with a radial density $\rho(|\cdot|)$. Suppose $\rho:\mathbb{R}^+ \to \mathbb{R}^+$ is log-concave. Then*

$$\lambda_1(Z) \ge \frac{c}{\mathbb{E}Z_1^2}.$$

Proof of assertion 3 of Theorem 1. Let $\mathcal{E}$ be the subset of $G_{n,k}$ given by Theorem A with $k = \lfloor c_1 n^{1/5.01}\rfloor$. Fix a subspace F in $\mathcal{E}$. Denote Z_F a random vector on F with the density $\pi_F g(|\cdot|\theta_0)$ where $\theta_0 \in S_F = S^{n-1} \cap F$ is chosen in such a way that $\int_F \pi_F g(|x|\theta_0)\,\mathrm{d}x = 1$. θ_0 exists since $\int_{S_F}\int_F \pi_F g(|x|\theta)\,\mathrm{d}x\,\sigma_F(\mathrm{d}\theta) = \int_F \pi_F g(x)\,\mathrm{d}x = 1$ (where σ_F stands for the unique rotationally-invariant Haar probability measure on S_F). Then

$$\begin{aligned} d_{\mathrm{TV}}(P_F Y, Z_F) &\le \frac{1}{4} + \int_{|x|\ge 10\sqrt{k}} \bigl(\pi_F g(x) + \pi_F g\bigl(|x|\theta_0\bigr)\bigr)\,\mathrm{d}x \\ &\le \frac{1}{2} + 2\int_{|x|\ge 10\sqrt{k}} \pi_F g(x)\,\mathrm{d}x \le \frac{27}{50} \end{aligned} \tag{28}$$

by Markov's inequality. Remark that this inequality implies $\mathbb{E}|Z_F|^2 \le C_1\mathbb{E}|P_F Y|^2 = 2C_1 k$ for some absolute constant $C_1 > 0$. Consequently, according to (28) and Theorems B and C, the spectral gap of $P_F Y$ is bounded from below by a universal constant. In particular, $P_F Y$ satisfies the inequality $(\mathcal{V})$ with a universal constant and, by Lemma 11, it is the same for $P_F X$. Therefore, thanks to Lemma 4, we have, for any $p \in [2, c_2\sqrt{k}]$,

$$\bigl(\mathbb{E}|P_F X|^p\bigr)^{1/p} \le \left(1 + \frac{C_2 p}{k}\right)\bigl(\mathbb{E}|P_F X|^2\bigr)^{1/2} = \left(1 + \frac{C_2 p}{k}\right)\sqrt{k}.$$

Besides, when $F \notin \mathcal{E}$, Borell's lemma ([13], Appendix III) shows that $(\mathbb{E}|P_F X|^p)^{1/p} \le C_3 p\sqrt{k}$ for any $p \ge 2$. Thus, we get for any $p \in [2, c_2\sqrt{k}]$,

$$\begin{aligned} \int_{G_{n,k}} \bigl(\mathbb{E}|P_F X|^p\bigr)^{1/p}\mu_{n,k}(\mathrm{d}F) &\le \left(1 + \frac{C_2 p}{k}\right)\sqrt{k} + C_3 p\sqrt{k}\mu_{n,k}\bigl(\mathcal{E}^c\bigr) \\ &\le \left(1 + \frac{C_2 p}{k}\right)\sqrt{k} + C_4 p\sqrt{k}\mathrm{e}^{-c_3 k} \le \left(1 + \frac{C_5 p}{k}\right)\sqrt{k}. \end{aligned}$$

By Lemma 10, we obtain for $p \in [1, c_4 n^{1/10.02}]$,

$$\bigl(\mathbb{E}|X|^p\bigr)^{1/p} \le \left(1 + \frac{C_6 p^3}{kn}\right)\left(1 + \frac{C_5 p}{k}\right)\bigl(\mathbb{E}|X|^2\bigr)^{1/2} \le \left(1 + \frac{C_7 p}{n^{1/5.01}}\right)\bigl(\mathbb{E}|X|^2\bigr)^{1/2}.$$

□

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