Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2010, Vol. 46, No. 1, 1–23
DOI: 10.1214/08-AIHP306

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

The heat equation on manifolds as a gradient flow in the Wasserstein space

Matthias Erbar

Rheinische Friedrich Wilhelms Universität Bonn, Germany. E-mail: matthias.erbar@web.de

Received 23 June 2008; revised 27 October 2008; accepted 12 November 2008

Abstract. We study the gradient flow for the relative entropy functional on probability measures over a Riemannian manifold. To this aim we present a notion of a Riemannian structure on the Wasserstein space. If the Ricci curvature is bounded below we establish existence and contractivity of the gradient flow using a discrete approximation scheme. Furthermore we show that its trajectories coincide with solutions to the heat equation.

Résumé. Nous étudions les flux gradients dans l'espace des mesures de probabilité sur une variété Riemannienne pas nécessairement compacte. Dans ce but nous munissons l'espace de Wasserstein avec une sorte de structure Riemannienne. Si la courbure de Ricci de la variété est bornée inférieurement nous démontrons qu'il existe un flux gradient contractif pour l'entropie relative. Il est construit explicitement en utilisant une approximation variationelle discrète. De plus ses trajectoires Coïncident avec les solutions à l'équation de la chaleur.

MSC: 35A15; 58J35; 60J60

Keywords: Gradient flow; Wasserstein metric; Relative entropy; Heat equation

1. Introduction and statement of the main results

Since the work of Otto (see e.g. [7]) it is known that many equations of diffusion type on $\mathbb{R}^n$ might be interpreted as gradient flows for an appropriate functional on the Wasserstein space of probability measures equipped with the L^2-Wasserstein metric. In this context the heat equation corresponds to the relative entropy, $H(\mu) = \int \log \rho \, \mathrm{d}\mu$ if $\mu = \rho \cdot vol$. This interpretation has been made rigorous since giving precise meaning to the notion of gradient flow in an abstract metric space. We refer to the book of Ambrosio, Gigli and Savaré [1] for a comprehensive treatment of gradient flows in metric spaces. As a standard example they show that there exists a unique gradient flow of the relative entropy functional on the Wasserstein space $\mathscr{P}_2(\mathbb{R}^n)$ whose trajectories coincide with solutions to the heat equation. These results have partly been extended from the Euclidean to the Riemannian setting. Savaré [11] showed that there exists a unique gradient flow for certain functionals on metric spaces whose squared distance satisfies a concavity property. As special case this includes energy functionals on the Wasserstein space over a compact Riemannian manifold. In an independent related work Ohta [10] established existence of a unique gradient flow for (geodesically-)semiconvex functionals on Wasserstein spaces over compact Alexandrov spaces. In the special case of a compact Riemannian manifold with nonnegative sectional curvature he proves coincidence of the trajectories of the gradient flow for the free energy with solutions to the Fokker–Planck equation. However if the manifold is non-compact the picture is less complete. If the Ricci curvature is bounded below Villani ([14], Chapter 23) shows that smooth sufficiently integrable solutions to a large class of diffusive PDE's constitute trajectories of the gradient flow for the associated functional. The assumption of a lower bound on the Ricci curvature is essential to guarantee that the entropy functional is displacement K-convex, i.e. K-convex along geodesics in the Wasserstein space (cf. Definition 3.4).

In this paper we present a notion of Riemannian structure on the Wasserstein space to define gradient flows adopting the approach in [1]. We will study in detail the gradient flow for the relative entropy functional under the assumption of a lower bound on the Ricci curvature. Building on the results of Villani we show that its trajectories actually coincide with solutions to the heat equation. This yields contractivity of the gradient flow in the Wasserstein distance. Furthermore we provide a time discrete approximation of the gradient flow, thus explicitly constructing its trajectories and establishing existence.

Throughout this paper let M be a smooth (of class C^3), connected and complete Riemannian manifold with Riemannian volume m. We denote by $\mathcal{P}_2(M)$ the Wasserstein space of probability measures on M equipped with the L^2-Wasserstein distance d_W (see Section 2). We shall assume that the Ricci curvature is bounded below, i.e. $Ric \geq K$ for some $K \in \mathbb{R}$ in the sense that $Ric_x(\boldsymbol{\xi}, \boldsymbol{\xi}) \geq K|\boldsymbol{\xi}|^2$ for all $x \in M, \boldsymbol{\xi} \in T_x M$. In the first part of this paper we will develop a notion of Riemannian structure on the Wasserstein space which provides the framework to give a precise definition of gradient flows in $\mathcal{P}_2(M)$ (see Definition 3.7). For the present purpose of this introduction say that a trajectory of the gradient flow for the relative entropy H is an absolutely continuous curve $(\mu_t)_{t\geq 0}$ in $\mathcal{P}_2(M)$ satisfying the evolution variational inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}d_W^2(\mu_t, \sigma) + \frac{K}{2}d_W^2(\mu_t, \sigma) \leq H(\sigma) - H(\mu_t) \quad \forall \sigma \in D(H), \tag{1}$$

where $D(H) = \{\mu \in \mathcal{P}_2(M)\colon\ H(\mu) < \infty\}$. We will see later (4.5) that the gradient flow of H satisfies this property. Our main results are the following:

Theorem 1. *There exists a unique gradient flow* $\sigma : [0, \infty) \times \mathcal{P}_2(M) \to D(H) \subset \mathcal{P}_2(M)$ *for the relative entropy* H *and it satisfies*

$$d_W(\mu_t, \nu_t) \leq \mathrm{e}^{-Kt} d_W(\mu_0, \nu_0) \quad \textit{for all } \mu_0, \nu_0 \in \mathcal{P}_2(M),\ t > 0,$$

where $\mu_t = \sigma(t, \mu_0),\ \nu_t = \sigma(t, \nu_0)$.

Theorem 2. *Let* $(\mu_t)_{t\geq 0}$ *be a continuous curve in* $\mathcal{P}_2(M)$. *Then the following are equivalent*:

(i) $(\mu_t)_{t\geq 0}$ *is a trajectory of the gradient flow for the relative entropy* H,
(ii) μ_t *is given by* $\mu_t(\mathrm{d}x) = \rho_t(x) \cdot m(\mathrm{d}x)$ *for* $t > 0$, *where* $(\rho_t)_{t>0}$ *is a solution to the heat equation*

$$\partial_t \rho_t(x) = \Delta \rho_t(x) \quad \textit{on } (0, \infty) \times M$$

satisfying the conditions

$$H(\rho_t \cdot m) < \infty \quad \forall t > 0 \quad \textit{and} \quad \int_{s_0}^{s_1}\int_M \frac{|\nabla \rho_t|^2}{\rho_t}\,\mathrm{d}m\,\mathrm{d}t < \infty \quad \forall 0 < s_0 < s_1. \tag{2}$$

The structure of this paper is as follows. In Sections 2 and 3 we will lay down the foundation to define gradient flows in $\mathcal{P}_2(M)$ without any assumption on the Ricci curvature. We shall adopt the approach of [1] which they apply in the case of $\mathbb{R}^n$. We will use the Riemannian structure of the underlying space to endow the Wasserstein space with a notion of a Riemannian differentiable structure which gives the theory of gradient flows an appealing formal analogy to the classical setting. In Section 2 we will introduce a notion of tangent bundle to $\mathcal{P}_2(M)$ which allows us to assign a tangent vector to an absolutely continuous curve $(\mu_t)_t$ in $\mathcal{P}_2(M)$. Heuristically this is the vector field governing the evolution of μ_t via the continuity equation, chosen in such a way that the kinetic energy is minimal. We will establish a subdifferential calculus for functionals on $\mathcal{P}_2(M)$ in Section 3. Our definition of gradient flows in $\mathcal{P}_2(M)$ is modeled on classical gradient flows of smooth functionals on a Riemannian manifold by demanding that the tangent to the curve belongs to the subdifferential of the functional. We transpose the arguments and constructions of [1], Chapters 8 and 10, from $\mathbb{R}^n$ to the Riemannian setting. From then on we will focus on the relative entropy functional and assume that $Ric \geq K$. Building on the results in [14] we compute explicitly its subdifferential in Section 4. This will yield coincidence of the gradient flow with the heat flow as stated in Theorem 2 and make it possible to derive contractivity of the gradient flow and thus uniqueness. To establish existence we will introduce in Section 5 the time discrete variational approximation scheme of Otto. By this mean we will construct trajectories of the gradient flow for initial measures with finite entropy. In Section 6 we will use the contractivity to extend the gradient flow semigroup to arbitrary initial measures and thus conclude the proof of Theorem 1.

2. Metric and Riemannian structure of $\mathscr{P}_2(M)$

Recall that $(M, \langle\cdot,\cdot\rangle)$ is a smooth (of class C^3) connected complete Riemannian manifold. Let m denote the Riemannian volume and d the Riemannian distance on M. The Wasserstein space over M is defined as the space of Borel probability measures on M with finite second moment:

$$\mathscr{P}_2(M) := \left\{\mu \in \mathscr{P}(M) \;\Big|\; \int_M d^2(x_0, x)\,\mathrm{d}\mu(x) < \infty \text{ for some (hence all) } x_0 \in M\right\}.$$

Given $\mu, \nu \in \mathscr{P}_2(M)$ their L^2-Wasserstein distance is defined by

$$d_W(\mu, \nu) := \inf\left\{\int_{M\times M} d^2(x, y)\,\mathrm{d}\pi(x, y) \;\Big|\; \pi \text{ is a coupling of } \mu \text{ and } \nu\right\}^{1/2}. \tag{3}$$

Here a probability measure $\pi \in \mathscr{P}(M \times M)$ is called a coupling of μ and ν if its marginals are μ and ν, i.e. $\pi(A \times M) = \mu(A)$, $\pi(M \times A) = \nu(A)$ for all Borel sets $A \subset M$. For any $\mu, \nu \in \mathscr{P}_2(M)$ there is at least one minimizer of (3) called an optimal coupling [14], Theorem 4.1. Formula (3) defines a metric on $\mathscr{P}_2(M)$ (see [14], Definition 6.1), in fact $(\mathscr{P}_2(M), d_W)$ is a polish space. We denote the subspace of probability measures absolutely continuous w.r.t. the volume measure m by $\mathscr{P}_2^{\mathrm{ac}}(M)$. The variational problem (3) is a particular instance of the Monge–Kantorovich mass transfer problem for the cost function d^2. See [14], Part I, for a comprehensive treatment of optimal transportation on Riemannian manifolds. Let us recall the following results.

Proposition 2.1. *Let $\mu \in \mathscr{P}_2^{\mathrm{ac}}(M)$ and $\nu \in \mathscr{P}_2(M)$. Then there exists a unique minimizer π in (3) and $\pi = (id, F)_\#\mu$ for a μ-a.e. uniquely determined Borel map $F : M \to M$.*

Moreover, π-a.e. pair (x, y) is connected by a unique minimizing geodesic. Hence there exists a μ-a.s. unique vector field $\boldsymbol{\Psi}_\mu^\nu$ such that for μ-a.e. $x \in M$:

(i) $F(x) = \exp_x(\boldsymbol{\Psi}_\mu^\nu(x))$,
(ii) $t \mapsto \exp_x(t\boldsymbol{\Psi}_\mu^\nu(x))$, $t \in [0, 1]$ *is a minimizing geodesic.*

Proof. In the case of a compact Riemannian manifold (or compactly supported μ and ν) these results are due to McCann (cf. [9], Theorem 9). For a proof that the optimal coupling is unique and deterministic in the noncompact case see [14], Theorem 10.41. In fact the proof also shows that π-a.e. pair (x, y) is connected by a unique minimizing geodesic (cf. also [14], (10.32)). □

The vector field $\boldsymbol{\Psi}_\mu^\nu$ will be referred to as the optimal transport vector field. Note that we have the identity: $d_W^2(\mu, \nu) = \int |\boldsymbol{\Psi}_\mu^\nu|^2\,\mathrm{d}\mu$. One easily checks that

$$t \mapsto \mu_t = \exp\big(t\boldsymbol{\Psi}_\mu^\nu\big)_\#\mu, \quad t \in [0, 1],$$

is a (constant speed) geodesic in $\mathscr{P}_2(M)$ connecting μ to ν. It turns out that this is the unique geodesic connecting μ to ν and that any two measures in $\mathscr{P}_2(M)$ can be connected by geodesic (cf. [14], Corollaries 7.22, 7.23). By uniqueness of the optimal transport vector field it is immediate that $\boldsymbol{\Psi}_\mu^{\mu_t} = t\cdot\boldsymbol{\Psi}_\mu^\nu$.

We say that a sequence μ_n of probability measures on M converges weakly to μ if for every bounded and continuous function $f \in C_b^0(M) : \int f\,\mathrm{d}\mu_n \to \int f\,\mathrm{d}\mu$. It can be shown ([14], Theorem 6.9) that convergence in the metric d_W is equivalent to weak convergence of probability measures plus convergence of second moments.

When considering curves in the metric space $\mathscr{P}_2(M)$ our usual regularity assumption will be absolute continuity.

Definition 2.2 (Locally absolutely continuous curves). *Let (X, d) be a metric space and $I \subset \mathbb{R}$ an open interval. A curve $c: I \longrightarrow X$ is said to be locally absolutely continuous of order p (denoted by $c \in AC^p_{\mathrm{loc}}(I, X)$) if there exists $u \in L^p_{\mathrm{loc}}(I)$ such that*

$$d\big(c(s), c(t)\big) \leq \int_s^t u(\tau)\,\mathrm{d}\tau \quad \forall s \leq t \in I. \tag{4}$$

It can be shown ([1], Theorem 1.1.2) that for every locally absolutely continuous curve c the metric derivative

$$\left|c'\right|(t) := \lim_{h\to 0} \frac{d(c(t+h), c(t))}{|h|}$$

exist for a.e. $t \in I$ and is the minimal u in (4).

We now endow $\mathcal{P}_2(M)$ with a kind of Riemannian differentiable structure and construct the tangent space $T_\mu\mathcal{P}_2(M)$ to the Wasserstein space at a given measure μ. We employ the approach of Ambrosio–Gigli–Savaré ([1], Chapter 8) and transpose their construction for the Euclidean case $\mathcal{P}_2(\mathbb{R}^n)$ to our Riemannian setting. Let us denote by $L^2(\mu, TM)$ the Hilbert space of measurable vector fields $\mathbf{w}: M \longrightarrow TM$, $\mathbf{w}(x) \in T_xM$ with finite 2-norm:

$$\|\mathbf{w}\|_{L^2(\mu,TM)} = \left(\int_M \big\langle \mathbf{w}(x), \mathbf{w}(x)\big\rangle_x \,\mathrm{d}\mu(x)\right)^{1/2}.$$

Definition 2.3. *Let $\mu \in \mathcal{P}_2(M)$. We define $T_\mu\mathcal{P}_2(M) := \overline{\{\nabla\varphi \mid \varphi \in \mathcal{C}_c^\infty(M)\}}^{L^2(\mu,TM)}$.*

Here $\mathcal{C}_c^\infty(M)$ denotes the space of smooth compactly supported functions on M. $T_\mu\mathcal{P}_2(M)$ is a Hilbert space endowed with the natural L^2 scalar product which is our formal analogue to a Riemannian metric. We have the following characterization of tangent vector fields in terms of a minimizing property.

Lemma 2.4. *Let $\mu \in \mathcal{P}_2(M)$ and $\mathbf{v} \in L^2(\mu, TM)$. Then the following are equivalent*:

(i) $\mathbf{v} \in T_\mu\mathcal{P}_2(M)$,
(ii) $\|\mathbf{v} + \mathbf{w}\|_{L^2(\mu,TM)} \geq \|\mathbf{v}\|_{L^2(\mu,TM)}$ $\forall \mathbf{w} \in L^2(\mu, TM)$ *such that* $\mathrm{div}(\mathbf{w}\mu) = 0$.

If equality holds in (ii) *for some* $\mathbf{w} \in L^2(\mu, TM)$ *with* $\mathrm{div}(\mathbf{w}\mu) = 0$ *then* $\mathbf{w} = 0$.

Here $\mathrm{div}(\mathbf{w}\mu) = 0$ *is to be understood in the sense of distributions*, *i.e.*

$$\int_M \langle\nabla\varphi, \mathbf{w}\rangle\,\mathrm{d}\mu = 0 \quad \forall\varphi \in \mathcal{C}_c^\infty(M).$$

Proof. Note that $\{\mathbf{w} \in L^2(\mu, TM) | \operatorname{div}(\mathbf{w}\mu) = 0\}$ is the orthogonal complement of the closed subspace $T_\mu\mathcal{P}_2(M)$ in $L^2(\mu, TM)$. Then the assertions are straightforward. □

Note that if we denote by Π the orthogonal projection onto $T_\mu\mathcal{P}_2(M)$ then by (ii) we have that $\|\mathbf{w}\|_{L^2(\mu,TM)} \geq \|\Pi\mathbf{w}\|_{L^2(\mu,TM)}$ for all $\mathbf{w} \in L^2(\mu, TM)$.

Our definition of tangent bundle to $\mathcal{P}_2(M)$ is justified by the following proposition.

Proposition 2.5. *Let I be an open interval and $(\mu_t)_t \in AC^2_{\mathrm{loc}}(I, \mathcal{P}_2(M))$ with metric derivative $|\mu'| \in L^2_{\mathrm{loc}}(I)$. Then there exists a vector field $\mathbf{v}: I \times M \to TM$, $(t, x) \mapsto \mathbf{v}_t(x) \in T_xM$ with $\|\mathbf{v}_t\|_{L^2(\mu_t,TM)} \in L^2_{\mathrm{loc}}(I)$ such that*

$$\mathbf{v}_t \in T_{\mu_t}\mathcal{P}_2(M) \quad \textit{for a.e. } t \in I \tag{5}$$

and the continuity equation

$$\partial_t\mu_t + \mathrm{div}(\mathbf{v}_t\mu_t) = 0 \quad \textit{in } I \times M \tag{6}$$

holds in the sense of distributions, *i.e.*

$$\int_I \int_M \big(\partial_t \varphi(t,x) + \langle \mathbf{v}_t(x), \nabla\varphi(t,x)\rangle\big)\,\mathrm{d}\mu_t(x)\,\mathrm{d}t = 0 \quad \forall \varphi \in \mathcal{C}_c^\infty(I\times M).$$

The vector field $\mathbf{v}_t$ *is uniquely determined in* $L^2(\mu_t, TM)$ *by* (5) *and* (6) *for a.e.* $t \in I$ *and we have* $\|\mathbf{v}_t\|_{L^2(\mu_t,TM)} = |\mu'|(t)$.

Conversely, let $(\mu_t)_{t\in I}$ *be a curve in* $\mathcal{P}_2(M)$ *satisfying* (6) *for some vector field* $(\mathbf{v}_t)_t$ *with*

$$\int_s^r \|\mathbf{v}_t\|^2_{L^2(\mu_t,TM)}\,\mathrm{d}t < \infty \quad \forall s < r \in I.$$

Then $(\mu_t)_t$ *is locally absolutely continuous in* I *and* $|\mu'|(t) \le \|\mathbf{v}_t\|_{L^2(\mu_t,TM)}$.

We will view the vector field $\mathbf{v}_t$ characterized by the previous proposition as the tangent vector to the curve $(\mu_t)_t$ at time t. Note that among all vector fields satisfying (6) this one is also unique such that $\|\mathbf{v}_t\|_{L^2(\mu_t,TM)}$ is minimal for a.e. $t \in I$. By Lemma 2.4 this minimality is characterized by $\mathbf{v}_t \in T_{\mu_t}\mathcal{P}_2(M)$. Intuitively, if we picture μ_t as a cloud of gas, the tangent vector field is the unique velocity field governing the evolution of μ_t via the continuity equation with minimal kinetic energy.

By Proposition 2.5 we recover in our Riemannian setting the Benamou–Brenier formula (7) [3], which shows that our formal Riemannian structure on $\mathcal{P}_2(M)$ is consistent with the metric space structure. For any $\mu_0, \mu_1 \in \mathcal{P}_2(M)$, we have

$$d_W^2(\mu_0,\mu_1) = \min\left\{\int_0^1 \|\mathbf{w}_t\|^2_{L^2(\mu_t,TM)}\,\mathrm{d}t\right\}, \tag{7}$$

where the minimum is taken over all absolutely continuous curves $(\mu_t)_{t\in[0,1]}$ connecting μ_0 to μ_1 with tangent vector field $(\mathbf{w}_t)_t$. Just recall that $|\mu'|(t) \le \|\mathbf{w}_t\|_{L^2(\mu_t,TM)}$ for any admissible curve to see that $d_W^2(\mu_0,\mu_1)$ is dominated by the right-hand side. Choose a geodesic and its tangent vector field to obtain equality.

Proof of Proposition 2.5. It suffices to prove the case, where $|\mu'| \in L^2(I)$. Indeed, in the general case exhaust I by compact intervals. By uniqueness the resulting vector fields yield a well defined vector field on I. So note first that for every $\varphi \in \mathcal{C}_c^\infty(I\times M)$ the map $t \mapsto \mu_t(\varphi)$ is in $AC^2(I,\mathbb{R})$. Indeed choosing an optimal coupling $\pi_{s,t}$ of μ_s and μ_t and applying the Hölder inequality we have

$$\big|\mu_t(\varphi) - \mu_s(\varphi)\big| = \left|\int_{M\times M} \varphi(x) - \varphi(y)\,\mathrm{d}\pi_{s,t}(x,y)\right| \le \mathrm{Lip}(\varphi) d_W(\mu_s,\mu_t)$$

which implies absolute continuity. To estimate the metric derivative of $\mu_t(\varphi)$ consider the upper semicontinuous and bounded map

$$H(x,y) := \begin{cases} \big|\nabla\varphi(x)\big| & \text{if } x = y, \\ \frac{|\varphi(x)-\varphi(y)|}{d(x,y)} & \text{if } x \ne y \end{cases}$$

and set $\pi_h := \pi_{s+h,s}$. By Hölder inequality we obtain

$$\frac{|\mu_{s+h}(\varphi) - \mu_s(\varphi)|}{|h|} \le \frac{1}{|h|}\int_{M\times M} d(x,y)H(x,y)\,\mathrm{d}\pi_h \le \frac{d_W(\mu_{s+h},\mu_s)}{|h|}\left(\int_{M\times M} H^2(x,y)\,\mathrm{d}\pi_h\right)^{1/2}.$$

Let $(\mu_t)_t$ be metrically differentiable in s. As $h \to 0$ the marginals of π_h converge weakly to μ_s and therefore π_h must converge weakly to $(id,id)_\#\mu_s$ the unique optimal coupling of μ_s and μ_s (cf. [14], Theorem 5.20). Hence, we have

$$\limsup_{h\to 0} \frac{|\mu_{s+h}(\varphi) - \mu_s(\varphi)|}{|h|} \le \big|\mu'\big|(s)\left(\int_M |\nabla\varphi|^2\,\mathrm{d}\mu_s\right)^{1/2} = \big|\mu'\big|(s)\|\nabla\varphi\|_{L^2(\mu_s,TM)}. \tag{8}$$

Now set $Q := I \times M$ and define a measure on Q by $\lambda = \int_I \mu_t \,\mathrm{d}t$. Let $\varphi \in \mathcal{C}_c^\infty(Q)$. Then we have using Fatou's lemma and (8)

$$\left|\int_Q \partial_s \varphi \,\mathrm{d}\lambda\right| = \lim_{h\downarrow 0}\left|\int_Q \frac{\varphi(s,x)-\varphi(s-h,x)}{h}\,\mathrm{d}\lambda(s,x)\right| = \lim_{h\downarrow 0}\left|\int_I \frac{1}{h}\big(\mu_s\big(\varphi(s,\cdot)\big) - \mu_{s+h}\big(\varphi(s,\cdot)\big)\big)\,\mathrm{d}s\right|$$
$$\leq \int_I \left|\mu'\right|(s)\left(\int_M |\nabla\varphi|^2(s,x)\,\mathrm{d}\mu_s(x)\right)^{1/2}\mathrm{d}s \leq \left(\int_I \left|\mu'\right|^2(s)\,\mathrm{d}s\right)^{1/2}\|\nabla\varphi\|_{L^2(\lambda,TM)}.$$

Let $\mathcal{V} := \overline{\{\nabla\varphi \mid \varphi \in \mathcal{C}_c^\infty(Q)\}}^{L^2(\lambda,TM)}$. The previous estimate shows that the linear functional

$$L(\nabla\varphi) := -\int_Q \partial_s\varphi\,\mathrm{d}\lambda$$

can be uniquely extended to a bounded functional on $\mathcal{V}$. By the Riesz representation theorem we obtain a unique $\mathbf{v} \in \mathcal{V}$ with $\|\mathbf{v}\|^2_{L^2(\lambda,TM)} \leq \||\mu'|\|_{L^2(I)}$ such that

$$\int_Q \langle \mathbf{v}, \nabla\varphi\rangle\,\mathrm{d}\lambda = L(\nabla\varphi) = -\int_Q \partial_s\varphi\,\mathrm{d}\lambda \quad \forall\varphi\in\mathcal{C}_c^\infty(Q).$$

Setting $\mathbf{v}_t = \mathbf{v}(t,\cdot)$ we obtain the continuity equation (6). Repeating the argument above with $J \subset I$ the uniqueness of the Riesz representation yields that $\mathbf{v}' := \mathbf{v}|_{J\times M}$ represents the restriction of L to $\overline{\{\nabla\varphi \mid \varphi\in\mathcal{C}_c^\infty(J\times M)\}}^{L^2(\lambda,TM)}$ and in particular

$$\int_J \|\mathbf{v}_s\|^2_{L^2(\mu_s,TM)}\,\mathrm{d}s = \left\|\mathbf{v}'\right\|^2_{L^2(\lambda,TM)} \leq \int_J \left|\mu'\right|^2(s)\,\mathrm{d}s.$$

Since J is arbitrary we obtain $\|\mathbf{v}_t\|_{L^2(\mu_t,TM)} \leq |\mu'|(t)$ for a.e. $t\in I$. Equality will follow from the converse implication.

Let us now show that $\mathbf{v}_t \in T_{\mu_t}\mathcal{P}_2(M)$ for a.e. $t\in I$. Let $\varphi_n \in \mathcal{C}_c^\infty(Q)$ such that $\nabla\varphi_n \to \mathbf{v}$ in $L^2(\lambda,TM)$ as $n\to\infty$. Then there is a subsequence $(n_k)_k$ such that for a.e. $t\in I: \nabla\varphi_{n_k}(t,\cdot)\to\mathbf{v}_t$ in $L^2(\mu_t,TM)$. This yields the claim.

Let $\mathbf{w}\in L^2(\lambda,TM)$ be another vector field satisfying (5) and (6). Then $\mathrm{div}((\mathbf{w}_t - \mathbf{v}_t)\mu_t) = 0$ for a.e. t. Hence Lemma 2.4 yields $\|\mathbf{w}_t\|_{L^2(\mu_t,TM)} = \|\mathbf{v}_t\|_{L^2(\mu_t,TM)}$.

Further $\mathbf{u} := \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}$ also satisfies (6) and $\mathbf{u}_t \in T_{\mu_t}\mathcal{P}_2(M)$. By the strict convexity of the L^2-norm we infer that $\mathbf{v}_t = \mathbf{w}_t$ for a.e. $t\in I$.

It remains to show the converse implication. So let $(\mu_t)_{t\in I}$ be a curve in $\mathcal{P}_2(M)$ satisfying (6) for a square integrable vector field $(\mathbf{v}_t)_t$. In estimating the difference $d_W(\mu_s,\mu_t)$ a recent result of Bernard ([4], Theorem 5.8) is very useful as pointed out to the author by Savaré. It states that the solution $(\mu_t)_t$ to the continuity equation for the vector field $(\mathbf{v}_t)_t$ can be represented as a superposition of random solutions to the equation

$$\dot\gamma(t) = \mathbf{v}_t\big(\gamma(t)\big). \tag{9}$$

Precisely there exists a probability measure Π on the space of continuous curves $\mathcal{C}^0(I,M)$ such that

$$(e_t)_\#\Pi = \mu_t \quad \forall t\in I,$$

where e_t is the evaluation map. Moreover, Π-a.e. $\gamma\in\mathcal{C}^0(I,M)$ is differentiable at a.e. $t\in I$ and satisfies (9). Since $(e_s,e_t)_\#\Pi$ is a coupling of μ_s and μ_t we can now estimate using Jensen's inequality:

$$d_W^2(\mu_s,\mu_t) \leq \int d^2\big(\gamma(s),\gamma(t)\big)\,\mathrm{d}\Pi(\gamma) \leq \int\left(\int_s^t |\dot\gamma|(r)\,\mathrm{d}r\right)^2\mathrm{d}\Pi(\gamma)$$
$$\leq (t-s)\int\int_s^t \left|\mathbf{v}_r\big(\gamma(r)\big)\right|^2\mathrm{d}r\,\mathrm{d}\Pi(\gamma) = (t-s)\int_s^t\int_M |\mathbf{v}_r|^2\,\mathrm{d}\mu_r\,\mathrm{d}r.$$

This implies that $(\mu_t)_t$ is absolutely continuous and $|\mu'|(t) \le \|\mathbf{v}_t\|_{L^2(\mu_t,TM)}$ for a.e. $t \in I$. $\square$

To conclude this section we shed some more light on the relation between tangent vector fields and optimal transport maps. We observe that optimal transport vector fields $\boldsymbol{\Psi}_\mu^\nu$ (see Proposition 2.1) are tangent to $\mathcal{P}_2(M)$ at the initial measure. Moreover, we recover the tangent to a curve in $\mathcal{P}_2(M)$ as the limit of secants made up of transport maps.

Lemma 2.6. *Let $\mu \in \mathcal{P}_2^{ac}(M)$, $\nu \in \mathcal{P}_2(M)$ and let $\boldsymbol{\Psi}_\mu^\nu$ be the optimal transport vector field. Then $\boldsymbol{\Psi}_\mu^\nu \in T_\mu\mathcal{P}_2(M)$.*

Proof. Approximate μ and ν by compactly supported measures μ_n, ν_n. Proposition 13.2 of [14] shows that this can be done in such a way, namely by compactly exhausting the set of geodesics along which mass is being transported, that the optimal transport vector fields $\boldsymbol{\Psi}_{\mu_n}^{\nu_n}$ and $\boldsymbol{\Psi}_\mu^\nu$ coincide a.e. on the support of μ_n. Possibly after setting $\boldsymbol{\Psi}_{\mu_n}^{\nu_n} = 0$ outside $\operatorname{supp}(\mu_n)$ we clearly have $\boldsymbol{\Psi}_{\mu_n}^{\nu_n} \to \boldsymbol{\Psi}_\mu^\nu$ in $L^2(\mu, TM)$. In the compactly supported case $\boldsymbol{\Psi}_{\mu_n}^{\nu_n}$ is given as $\nabla\psi$, where ψ is a function satisfying the so-called $\frac{d^2}{2}$-convexity property which is Lipschitz on $\operatorname{supp}(\mu_n)$ ([14], Theorems 10.41 and 10.26). By setting $\psi = 0$ outside $\operatorname{supp}(\mu_n)$ and regularizing (e.g. in local charts) we can find $\psi_\varepsilon \in C_c^\infty(M)$ such that $\nabla\psi_\varepsilon \to \nabla\psi$ m-a.e. Since μ is absolutely continuous the dominated convergence theorem yields convergence in $L^2(\mu, TM)$. This shows $\boldsymbol{\Psi}_\mu^\nu \in T_\mu\mathcal{P}_2(M)$. $\square$

Lemma 2.7. *Let $(\mu_t)_t \in AC^2_{\mathrm{loc}}(I, \mathcal{P}_2^{\mathrm{ac}}(M))$ and $(\mathbf{v}_t)_t$ the tangent vector field characterized by Proposition 2.5. Then we have for a.e. $t \in I$:*

$$\lim_{h\to 0} \frac{1}{h}\boldsymbol{\Psi}_{\mu_t}^{\mu_{t+h}} = \mathbf{v}_t \quad \text{in } L^2(\mu_t, TM). \tag{10}$$

Proof. Choose a countable subset $\mathcal{D} \subset C_c^\infty(M)$ which is dense w.r.t. the norm

$$\|\varphi\|_{C^1} := \sup_{x\in M}\big|\varphi(x)\big| + \sup_{x\in M}\big\|\nabla\varphi(x)\big\|_x.$$

Fix $\varphi \in \mathcal{D}$ and let $\eta \in C_c^\infty(I)$. By the continuity equation (6) and a change of variables we have

$$\begin{aligned} 0 &= \int_I\int_M \partial_t\eta(t)\cdot\varphi + \langle\nabla\varphi, \mathbf{v}_t\rangle\eta(t)\,\mathrm{d}\mu_t\,\mathrm{d}t \\ &= \int_I\Big[-\lim_{h\to 0}\frac{\mu_{t+h}(\varphi) - \mu_t(\varphi)}{h} + \int_M\langle\nabla\varphi, \mathbf{v}_t\rangle\,\mathrm{d}\mu_t\Big]\eta(t)\,\mathrm{d}t. \end{aligned}$$

Since η is arbitrary we must have that for a.e. $t \in I$:

$$\lim_{h\to 0}\frac{\mu_{t+h}(\varphi) - \mu_t(\varphi)}{h} = \int_M\langle\nabla\varphi, \mathbf{v}_t\rangle\,\mathrm{d}\mu_t. \tag{11}$$

Since $\mathcal{D}$ is countable and the metric derivative exists a.e. we can find a negligible set of times $\mathcal{N}$ such that for every $t \in I\setminus\mathcal{N}$ we have that (11) holds for every $\varphi \in \mathcal{D}$ and $s \mapsto \mu_s$ is metrically differentiable at t. Now fix $t \in I\setminus\mathcal{N}$ and set

$$\boldsymbol{\Psi}_h := \frac{1}{h}\boldsymbol{\Psi}_{\mu_t}^{\mu_{t+h}}.$$

Let $\boldsymbol{\Psi}_0$ be a weak limit point of $\boldsymbol{\Psi}_h$ in $L^2(\mu_t, TM)$ as $h \to 0$. For $\varphi \in \mathcal{D}$ we have using Taylor expansion

$$\frac{1}{h}\big(\mu_{t+h}(\varphi) - \mu_t(\varphi)\big) = \frac{1}{h}\int_M \varphi\circ\exp(h\boldsymbol{\Psi}_h) - \varphi\,\mathrm{d}\mu_t = \int_M\langle\nabla\varphi, \boldsymbol{\Psi}_h\rangle\,\mathrm{d}\mu_t + \mathrm{O}(h).$$

Combining this with (11) we obtain

$$\int_M\langle\nabla\varphi, \boldsymbol{\Psi}_0\rangle\,\mathrm{d}\mu_t = \int_M\langle\nabla\varphi, \mathbf{v}_t\rangle\,\mathrm{d}\mu_t.$$

By the density of $\mathcal{D}$ we conclude that in the sense of distributions

$$\operatorname{div}\big((\boldsymbol{\Psi}_0 - \mathbf{v}_t) \cdot \mu_t\big) = 0. \tag{12}$$

Since the L^2-norm is weakly lower semicontinuous we have that

$$\begin{aligned} \|\boldsymbol{\Psi}_0\|_{L^2(\mu_t,TM)} &\le \liminf_{h\to 0} \|\boldsymbol{\Psi}_h\|_{L^2(\mu_t,TM)} = \lim_{h\to 0} \frac{1}{h} d_W(\mu_{t+h}, \mu_t) \\ &= |\mu'|(t) = \|\mathbf{v}_t\|_{L^2(\mu_t,TM)}. \end{aligned} \tag{13}$$

From (12) and (13) we infer with Lemma 2.4 that $\boldsymbol{\Psi}_0 = \mathbf{v}_t$. Since weak convergence and convergence of the norm imply strong convergence this yields the claim. □

3. Subdifferentials and gradient flows

Let $\Phi : \mathcal{P}_2(M) \to (-\infty, +\infty]$ be a functional on the Wasserstein space. Denote by

$$D(\Phi) := \big\{\mu \in \mathcal{P}_2(M) \mid \Phi(\mu) < \infty\big\}$$

the proper domain of Φ. We will establish a notion of (sub-)differential of Φ modeled on the classical one in linear spaces. Recall that for a functional $F : X \to \mathbb{R} \cup \{\infty\}$ on a Hilbert space X the subdifferential at a point $x \in D(F)$ is defined by

$$v \in \partial F(x) : \quad \Leftrightarrow \quad F(y) - F(x) \ge \langle v, y - x\rangle + \mathrm{o}\big(\|y - x\|\big).$$

To transpose this definition to the Wasserstein space we proceed as follows: if μ is the reference point we intend the scalar product to be taken in $L^2(\mu, TM)$ and the displacement vector $y - x$ corresponds to the optimal transport vector field $\boldsymbol{\Psi}_\mu^\nu$, which is defined if $\mu \in \mathcal{P}_2^{\mathrm{ac}}(M)$. Hence, we shall assume that

$$D(\Phi) \subset \mathcal{P}_2^{\mathrm{ac}}(M).$$

Note that this is the case for the relative entropy functional.

Definition 3.1 (Subdifferential). *Let $\mu \in D(\Phi)$ and $\mathbf{w} \in L^2(\mu, TM)$. We say that $\mathbf{w}$ belongs to the subdifferential $\partial\Phi(\mu)$ if*

$$\Phi(\nu) - \Phi(\mu) \ge \int_M \big\langle \mathbf{w}, \boldsymbol{\Psi}_\mu^\nu\big\rangle \,\mathrm{d}\mu + \mathrm{o}\big(d_W(\mu, \nu)\big) \quad \forall \nu \in \mathcal{P}_2(M).$$

A vector field $\mathbf{w} \in \partial\Phi(\mu)$ is said to be a strong subdifferential if it also satisfies

$$\Phi\big(\exp(\boldsymbol{\Psi})_\# \mu\big) - \Phi(\mu) \ge \int_M \langle \mathbf{w}, \boldsymbol{\Psi}\rangle \,\mathrm{d}\mu + \mathrm{o}\big(\|\boldsymbol{\Psi}\|_{L^2(\mu,TM)}\big) \quad \forall \boldsymbol{\Psi} \in L^2(\mu, TM).$$

Note that the vector $\mathbf{w}$ in the definition of subdifferential only acts on tangent vectors by Lemma 2.6. We conclude that the orthogonal projection $\Pi\mathbf{w}$ on $T_\mu\mathcal{P}_2(M)$ is in $\partial\Phi(\mu)$ whenever $\mathbf{w}$ is. The following lemma shows that subdifferentials in $T_\mu\mathcal{P}_2(M)$ are strong subdifferentials. It is a direct adaption of [2], Proposition 4.2, to the Riemannian setting.

Lemma 3.2. *Let $\mu \in D(\Phi)$ and $\mathbf{w} \in \partial\Phi(\mu) \cap T_\mu\mathcal{P}_2(M)$. Then $\mathbf{w}$ is a strong subdifferential.*

Proof. We argue by contradiction. Suppose $\mathbf{w}$ is not a strong subdifferential. Then there is $\delta > 0$ and a sequence $(\mathbf{u}_n)_{n\in\mathbb{N}} \subset L^2(\mu, TM)$ such that

$$\varepsilon_n := \|\mathbf{u}_n\|_{L^2(\mu,TM)} \to 0 \quad \text{and} \quad \Phi(\mu_n) - \Phi(\mu) - \int_M \langle \mathbf{w}, \mathbf{u}_n\rangle \,\mathrm{d}\mu \le -\delta\varepsilon_n, \tag{14}$$

where $\mu_n := \exp(\mathbf{u}_n)_\# \mu$. Setting $\boldsymbol{\Psi}_n := \boldsymbol{\Psi}_\mu^{\mu_n}$ we have

$$\|\boldsymbol{\Psi}_n\|_{L^2(\mu,TM)} = d_W(\mu,\mu_n) \le \|\mathbf{u}_n\|_{L^2(\mu,TM)} = \varepsilon_n \to 0. \tag{15}$$

Hence by the definition of subdifferential there is N such that

$$\Phi(\mu_n) - \Phi(\mu) \ge \int_M \langle \mathbf{w}, \boldsymbol{\Psi}_n \rangle \,\mathrm{d}\mu - \frac{\delta}{2}\varepsilon_n \quad \forall n > N. \tag{16}$$

Combining (14) and (16) yields:

$$\int_M \langle \mathbf{w}, \boldsymbol{\Psi}_n - \mathbf{u}_n \rangle \,\mathrm{d}\mu \le -\frac{\delta}{2}\varepsilon_n \quad \forall n > N. \tag{17}$$

Note that by (14) and (15) $(\boldsymbol{\Psi}_n/\varepsilon_n)_n$ and $(\mathbf{u}_n/\varepsilon_n)_n$ are bounded sequences in $L^2(\mu, TM)$. Hence up to extracting a subsequence we can assume that

$$\frac{\boldsymbol{\Psi}_n}{\varepsilon_n} \rightharpoonup \boldsymbol{\Psi}, \qquad \frac{\mathbf{u}_n}{\varepsilon_n} \rightharpoonup \mathbf{u} \quad \text{weakly in } L^2(\mu, TM).$$

Dividing by ε_n in (17) and passing to the limit as $n \to \infty$ yields

$$\int_M \langle \mathbf{w}, \boldsymbol{\Psi} - \mathbf{u} \rangle \,\mathrm{d}\mu \le -\frac{\delta}{2}. \tag{18}$$

Now let $\varphi \in \mathcal{C}_c^\infty(M)$. As $\|\mathbf{u}_n\|, \|\boldsymbol{\Psi}_n\| \to 0$ we obtain by Taylor expansion that for some constant C and all n large enough:

$$\begin{aligned} 0 &= \int_M \varphi\big(\exp_x\big(\boldsymbol{\Psi}_n(x)\big)\big) - \varphi\big(\exp_x\big(\mathbf{u}_n(x)\big)\big) \,\mathrm{d}\mu \\ &\le \int_M \langle \nabla\varphi, \boldsymbol{\Psi}_n - \mathbf{u}_n \rangle \,\mathrm{d}\mu + C\big(\|\boldsymbol{\Psi}_n\|^2 + \|\mathbf{u}_n\|^2\big). \end{aligned} \tag{19}$$

Now dividing by ε_n and passing to the limit in (19) yields:

$$\int_M \langle \nabla\varphi, \boldsymbol{\Psi} - \mathbf{u} \rangle \,\mathrm{d}\mu \ge 0 \quad \forall \varphi \in \mathcal{C}_c^\infty(M).$$

Since $\mathbf{w}$ belongs to $T_\mu\mathcal{P}_2(M)$ in which gradients of smooth functions are dense we also have

$$\int_M \langle \mathbf{w}, \boldsymbol{\Psi} - \mathbf{u} \rangle \,\mathrm{d}\mu \ge 0$$

in contradiction to (18). Hence $\mathbf{w}$ must be a strong subdifferential. □

As a metric substitute for the norm of the gradient of a functional we define the metric slope.

Definition 3.3 (Metric slope). *For $\mu \in D(\Phi)$ we define*:

$$|\partial\Phi|(\mu) := \limsup_{\nu\to\mu} \frac{(\Phi(\mu) - \Phi(\nu))^+}{d_W(\mu,\nu)}. \tag{20}$$

Note that if $\mathbf{w} \in \partial\Phi(\mu)$ it is immediate from the definitions that $|\partial\Phi|(\mu) \le \|\mathbf{w}\|_{L^2(\mu,TM)}$. In the sequel we will exploit that the functional we consider enjoys a certain convexity property along geodesics in $\mathcal{P}_2(M)$. More precisely:

Definition 3.4. *Let $K \in \mathbb{R}$. The functional Φ is called (displacement) K-convex, if for each pair $\mu_0, \mu_1 \in D(\Phi)$ there exists a geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathcal{P}_2(M)$ such that for each $t \in [0, 1]$:*

$$\Phi(\mu_t) \leq t\Phi(\mu_1) + (1-t)\Phi(\mu_0) - t(1-t)\frac{K}{2}d_W^2(\mu_0, \mu_1). \tag{21}$$

Note that in our case there is exactly one geodesic connecting μ_0 to μ_1 by Proposition 2.1 since we assume that $D(\Phi) \subset \mathcal{P}_2^{\mathrm{ac}}(M)$. This notion of convexity for functionals on the Wasserstein space has been introduced by McCann in [8]. For K-convex functionals the subdifferential can be characterized by a variational inequality.

Lemma 3.5. *Let Φ be K-convex and let $\mu \in D(\Phi)$, $\mathbf{w} \in L^2(\mu, TM)$. Then $\mathbf{w} \in \partial\Phi(\mu)$ iff*

$$\Phi(\nu) - \Phi(\mu) \geq \int_M \langle \mathbf{w}, \boldsymbol{\Psi}_\mu^\nu \rangle \,\mathrm{d}\mu + \frac{K}{2}d_W^2(\mu, \nu) \quad \forall \nu \in \mathcal{P}_2(M). \tag{22}$$

The metric slope can be represented as

$$|\partial\Phi|(\mu) = \sup_{\nu\neq\mu}\left(\frac{\Phi(\mu) - \Phi(\nu)}{d_W(\nu, \mu)} + \frac{K}{2}d_W(\nu, \mu)\right)^+. \tag{23}$$

Proof. The if implication is obvious. So let $\mathbf{w} \in \partial\Phi(\mu)$ and $\nu \in D(\Phi)$ (otherwise, there is nothing to prove). Let $(\mu_t)_{t\in[0,1]}$ be the geodesic between μ and ν. K-convexity (21) yields:

$$\frac{\Phi(\mu_t) - \Phi(\mu)}{t} \leq \Phi(\nu) - \Phi(\mu) - (1-t)\frac{K}{2}d_W^2(\mu, \nu). \tag{24}$$

By the definition of subdifferential we also have:

$$\liminf_{t\downarrow 0}\frac{\Phi(\mu_t) - \Phi(\mu)}{t} \geq \liminf_{t\downarrow 0}\frac{1}{t}\int_M \langle \mathbf{w}, \boldsymbol{\Psi}_\mu^{\mu_t}\rangle \,\mathrm{d}\mu = \int_M \langle \mathbf{w}, \boldsymbol{\Psi}_\mu^\nu\rangle \,\mathrm{d}\mu,$$

where we have used the fact that $\boldsymbol{\Psi}_\mu^{\mu_t} = t\cdot\boldsymbol{\Psi}_\mu^\nu$ and $d_W(\mu, \mu_t) = t\cdot d_W(\mu, \nu)$. To obtain the nontrivial inequality for the metric slope we use again (24) and

$$\liminf_{t\downarrow 0}\frac{\Phi(\mu_t) - \Phi(\mu)}{t} = \liminf_{t\downarrow 0}\frac{\Phi(\mu_t) - \Phi(\mu)}{d_W(\mu, \mu_t)}d_W(\mu, \nu) \geq -|\partial\Phi|(\mu)\cdot d_W(\mu, \nu).$$

□

Proposition 3.6 (Chain rule). *Let Φ be K-convex. Let (a, b) be an open interval and $(\mu_t)_t \in AC^2((a, b), D(\Phi))$ with tangent vector field $(\mathbf{v}_t)_t$. Assume that*

$$\int_b^a |\partial\Phi|(\mu_t)\cdot\left|\mu'\right|(t)\,\mathrm{d}t < \infty. \tag{25}$$

Then the map $t \mapsto \Phi(\mu_t)$ is absolutely continuous in (a, b) and we have for a.e. t:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mu_t) = \int_M \langle \boldsymbol{\xi}, \mathbf{v}_t\rangle \,\mathrm{d}\mu_t \quad \forall \boldsymbol{\xi} \in \partial\Phi(\mu_t). \tag{26}$$

Proof. For absolute continuity of the map $t \mapsto \Phi(\mu_t)$ see the proof of [1], Corollary 2.4.10, which relies on (23). In view of (25) and Lemma 2.7 we have for a.e. t

$$|\partial\Phi|(\mu_t) < \infty, \qquad \frac{1}{h}\boldsymbol{\Psi}_{\mu_t}^{\mu_{t+h}} \overset{h\to 0}{\to} \mathbf{v}_t \quad \text{in } L^2(\mu_t), \qquad s \mapsto \Phi(\mu_s) \text{ is differentiable at } t.$$

Let us compute the derivative of $\Phi(\mu_s)$ at such a time t: for $\boldsymbol{\xi} \in \partial\Phi(\mu_t)$ we can estimate

$$\begin{aligned}\Phi(\mu_{t+h}) - \Phi(\mu_t) &\geq \int_M \langle \boldsymbol{\xi}, \boldsymbol{\Psi}_{\mu_t}^{\mu_{t+h}} \rangle \,\mathrm{d}\mu_t + \frac{K}{2} d_W^2(\mu_t, \mu_{t+h}) \\ &= h \int_M \langle \boldsymbol{\xi}, \mathbf{v}_t \rangle \,\mathrm{d}\mu_t + h \int_M \left\langle \boldsymbol{\xi}, \frac{1}{h} \boldsymbol{\Psi}_{\mu_t}^{\mu_{t+h}} - \mathbf{v}_t \right\rangle \mathrm{d}\mu_t + \frac{K}{2} d_W^2(\mu_t, \mu_{t+h}) \\ &= h \int_M \langle \boldsymbol{\xi}, \mathbf{v}_t \rangle \,\mathrm{d}\mu_t + \mathrm{o}(h).\end{aligned}$$

Dividing by h and taking left and right limits yields the claim

$$\frac{\mathrm{d}^+}{\mathrm{d}t} \Phi(\mu_t) \geq \int_M \langle \boldsymbol{\xi}, \mathbf{v}_t \rangle \,\mathrm{d}\mu_t \geq \frac{\mathrm{d}^-}{\mathrm{d}t} \Phi(\mu_t).$$

□

With the notions of tangent and subdifferential at hand we can now define gradient flows in the Wasserstein space $\mathcal{P}_2(M)$ in analogy to gradient flows of smooth functionals on Riemannian manifolds.

Definition 3.7 (Gradient flow). *Let $(\mu_t)_{t\geq 0}$ be a continuous curve in $\mathcal{P}_2(M)$ which belongs to $AC^2_{\mathrm{loc}}((0,\infty), \mathcal{P}_2(M))$. Let $(\mathbf{v}_t)_t$ be the tangent vector field characterized by Proposition 2.5. We say that $(\mu_t)_{t\geq 0}$ is a trajectory of the gradient flow for Φ if it satisfies the gradient flow equation*

$$-\mathbf{v}_t \in \partial\Phi(\mu_t) \quad \textit{for a.e. } t > 0. \tag{27}$$

Observe that the gradient flow equation implies that $|\mu'|(t) \cdot |\partial\Phi|(\mu_t) \leq \|\mathbf{v}_t\|^2_{L^2(\mu_t, TM)}$. The right-hand side is in $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ so Proposition 3.6 yields that the map $t \mapsto \Phi(\mu_t)$ is locally absolutely continuous in $(0,\infty)$ and for a.e. $t > 0$:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mu_t) = \|\mathbf{v}_t\|^2_{L^2(\mu_t, TM)}. \tag{28}$$

In particular $\Phi(\mu_t) < \infty$ for all $t > 0$. Integrating (28) we obtain the so called energy identity:

$$\Phi(\mu_b) - \Phi(\mu_a) = -\int_a^b \|\mathbf{v}_t\|^2_{L^2(\mu_t, TM)} \,\mathrm{d}t \quad \forall 0 < a < b. \tag{29}$$

4. Identification of the gradient flow for the relative entropy

In this section we shall turn to the relative entropy functional and compute its subdifferential. This will yield the proof of Theorem 2.

Recall that the relative entropy $H : \mathcal{P}_2(M) \to [-\infty, \infty]$ is defined by

$$H(\mu) := \int_M \rho \log\rho \,\mathrm{d}m, \tag{30}$$

provided that μ is absolutely continuous with density ρ and $\rho(\log\rho)_+$ is integrable. Otherwise set $H(\mu) := \infty$. In particular $D(H) \subset \mathcal{P}_2^{\mathrm{ac}}(M)$. Convexity properties of this functional are intimately related to the curvature of M. Sturm and others (see [12], Theorem 1.3) showed that H is displacement K-convex if and only if

$$Ric_x(\boldsymbol{\xi}, \boldsymbol{\xi}) \geq K \cdot \|\boldsymbol{\xi}\|^2 \quad \forall x \in M,\ \boldsymbol{\xi} \in T_x M. \tag{31}$$

So let us assume from now on that the Ricci curvature of M is bounded below by some $K \in \mathbb{R}$ in the sense of (31) which we will denote in short by $Ric \geq K$. Under this assumption the relative entropy functional H takes values in

$(-\infty, \infty]$, precisely we prove an estimate which will be useful later on. Fix a point $o \in M$ and denote the second moment by

$$m_2(\mu) := \int_M d^2(o,x)\,\mathrm{d}\mu(x). \tag{32}$$

Lemma 4.1. *Let $Ric \geq K$. For any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ depending only on ε such that for any $\mu \in \mathcal{P}_2(M)$:*

$$H(\mu) \geq -C_\varepsilon - \varepsilon \cdot m_2(\mu) > -\infty. \tag{33}$$

Proof. Let $\varepsilon > 0$. We can assume that $\mu = \rho \cdot m \in \mathcal{P}_2^{\mathrm{ac}}(M)$, otherwise there is nothing to prove. Let $c > 0$ to be chosen later and observe that $r|\log r| \leq \sqrt{r}$ for all $r \in [0,1]$. Using this in the second step we can bound $H(\mu)_-$ as

$$\begin{aligned}\int_M (\rho\log\rho)_-\,\mathrm{d}m &= \int_{\{\rho \leq \mathrm{e}^{-cd(o,\cdot)}\}} \rho|\log\rho|\,\mathrm{d}m + \int_{\{\mathrm{e}^{-cd(o,\cdot)}<\rho\leq 1\}} \rho|\log\rho|\,\mathrm{d}m \\ &\leq \int_M \mathrm{e}^{-c/2d(o,x)}\,\mathrm{d}m(x) + c\int_M d(o,x)\rho(x)\,\mathrm{d}m(x) \\ &\leq \int_M \mathrm{e}^{-c/2d(o,x)}\,\mathrm{d}m(x) + \varepsilon\cdot m_2(\mu) + \frac{c^2}{4\varepsilon},\end{aligned}$$

where we used the elementary inequality $cy \leq \varepsilon y^2 + c^2/(4\varepsilon)$. By the Bishop volume comparison theorem ([5], Theorem 3.9), the volume of geodesic balls $B_r(o)$ in M grows at most exponentially, i.e. $m(B_r(o)) \leq a(\mathrm{e}^{br} - 1)$ for suitable constant $a, b > 0$. Hence we conclude that for $c > 2b$

$$\int_M \mathrm{e}^{-c/2d(o,x)}\,\mathrm{d}m(x) = \int_0^1 m\big(\{\mathrm{e}^{-c/2d(o,x)} \geq t\}\big)\,\mathrm{d}t < \infty,$$

which proves the claim. □

We now compute explicitly the subdifferential of the relative entropy functional starting with the following lemma which shows that we can take "directional derivatives" of the entropy along the geodesic flow of smooth vector fields. Let us denote by $\mathcal{C}_c^\infty(M, TM)$ the space of smooth compactly supported vector fields on M.

Lemma 4.2. *Let $\boldsymbol{\xi} \in \mathcal{C}_c^\infty(M, TM)$ and $\mu = \rho \cdot m \in D(H)$. We set $T_t := \exp(t\boldsymbol{\xi})$ for $t \in \mathbb{R}$ and $\mu_t := T_{t\#}\mu$. Then we have*:

$$\lim_{t\to 0} \frac{H(\mu_t) - H(\mu)}{t} = -\int_M \rho \cdot \operatorname{div}\boldsymbol{\xi}\,\mathrm{d}m. \tag{34}$$

Proof. For t sufficiently small T_t is a diffeomorphism. We denote by $\mathcal{J}_t := \det(\mathrm{d}T_t)$ its Jacobian determinant. Since $\boldsymbol{\xi}$ is compactly supported and $\mathcal{J}_0 = 1$ there is $c > 0$ such that $1/c \leq \mathcal{J}_t(x) \leq c$ for $x \in M, t \in [-\varepsilon, \varepsilon]$ and ε small enough. By the change of variables formula μ_t is again absolutely continuous with density $\rho_t = (\rho \cdot \mathcal{J}_t^{-1}) \circ T_t^{-1}$ and

$$H(\mu_t) = \int_M F(\rho_t)\,\mathrm{d}m = \int_M F\left(\frac{\rho}{\mathcal{J}_t}\right)\mathcal{J}_t\,\mathrm{d}m < \infty,$$

where $F(r) := r \cdot \log(r)$. Direct computation yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(F\left(\frac{\rho(x)}{\mathcal{J}_t(x)}\right)\mathcal{J}_t(x)\right) = -\rho(x)\frac{\dot{\mathcal{J}}_t(x)}{\mathcal{J}_t(x)}.$$

Note that $\mathcal{J}_0 = 1$, $\dot{\mathcal{J}}_0 = \operatorname{div} \boldsymbol{\xi}$ (see [5], Section 3.4). Since $\dot{\mathcal{J}}_t \mathcal{J}_t^{-1}$ is bounded for $t \in [-\varepsilon, \varepsilon]$ we can differentiate under the integral and obtain

$$\lim_{t\to 0} \frac{H(\mu_t) - H(\mu)}{t} = \int_M \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F\left(\frac{\rho}{\mathcal{J}_t}\right) \mathcal{J}_t \,\mathrm{d}m = -\int_M \rho \cdot \operatorname{div} \boldsymbol{\xi} \,\mathrm{d}m. \qquad \square$$

The next proposition determines the subdifferential of the relative entropy functional. We shall denote by $W^{1,1}(M)$ the space of integrable functions on M whose distributional gradient is given by an integrable vector field.

Proposition 4.3. *Let $Ric \geq K$ and $\mu = \rho \cdot m \in D(H)$. Then the following are equivalent*:

(i) $|\partial H|(\mu) < \infty$,
(ii) $\rho \in W^{1,1}(M)$, $\nabla \rho = \mathbf{w} \cdot \rho$ *for some* $\mathbf{w} \in L^2(\mu, TM)$.

In this case $\mathbf{w}$ is the unique strong subdifferential at μ and $|\partial H|(\mu) = \|\mathbf{w}\|_{L^2(\mu,TM)}$. Further $\mathbf{w} \in T_\mu \mathcal{P}_2(M)$.

Proof. Let $|\partial H|(\mu) < \infty$. By definition ρ is integrable. We have to show that the distributional gradient of ρ is represented by a integrable vector field. By Lemma 4.2 we have that

$$\left| -\int_M \rho \cdot \operatorname{div} \boldsymbol{\xi} \,\mathrm{d}m \right| = \lim_{t\to 0} \left| \frac{H(\mu_t) - H(\mu)}{d_W(\mu, \mu_t)} \right| \frac{d_W(\mu, \mu_t)}{|t|} \leq |\partial H|(\mu) \cdot \|\boldsymbol{\xi}\|_{L^2(\mu,TM)}$$

holds for any $\boldsymbol{\xi} \in \mathcal{C}_c^\infty(M, TM)$. Here we estimated $d_W(\mu, \mu_t)$ with the coupling $(id, \exp(t\boldsymbol{\xi}))_\# \mu$. By the Riesz representation theorem there is $\mathbf{w} \in L^2(\mu, TM)$ with $\|\mathbf{w}\|_{L^2(\mu,TM)} \leq |\partial H|(\mu)$ such that

$$-\int_M \rho \cdot \operatorname{div} \boldsymbol{\xi} \,\mathrm{d}m = \int_M \langle \mathbf{w}, \boldsymbol{\xi} \rangle \,\mathrm{d}\mu = \int_M \langle \mathbf{w}, \boldsymbol{\xi} \cdot \rho \rangle \,\mathrm{d}m \quad \forall \boldsymbol{\xi} \in \mathcal{C}_c^\infty(M, TM).$$

This shows that the distributional gradient of ρ is given by $\mathbf{w} \cdot \rho$ which is indeed integrable since μ is a probability measure: $\|\mathbf{w} \cdot \rho\|_{L^1(m,TM)} = \|\mathbf{w}\|_{L^1(\mu,TM)} \leq \|\mathbf{w}\|_{L^2(\mu,TM)} < \infty$.

Given (ii) Theorem 23.13 in [14] shows that $\mathbf{w} = \nabla\rho/\rho$ is indeed a subdifferential at μ and hence $|\partial H|(\mu) \leq \|\mathbf{w}\|_{L^2(\mu,TM)}$. Since the orthogonal projection $\Pi \mathbf{w}$ of $\mathbf{w}$ on $T_\mu \mathcal{P}_2(M)$ is also in $\partial H(\mu)$ we have

$$|\partial H|(\mu) \leq \|\Pi \mathbf{w}\|_{L^2(\mu,TM)} \leq \|\mathbf{w}\|_{L^2(\mu,TM)} \leq |\partial H|(\mu).$$

Hence we must have $\mathbf{w} \in T_\mu \mathcal{P}_2(M)$ and $\mathbf{w}$ is a strong subdifferential by Lemma 3.2. It remains to show uniqueness. So let $\mathbf{v}$ be another strong subdifferential at μ and $\boldsymbol{\xi} \in \mathcal{C}_c^\infty(M, TM)$. With the notation of Lemma 4.2 we have

$$H(\mu_t) - H(\mu) \geq \int_M \langle \mathbf{v}, t\boldsymbol{\xi} \rangle \,\mathrm{d}\mu + \mathrm{o}(t).$$

Dividing by t and taking left and right limits we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}^{+} H(\mu_t) \geq \int_M \langle \mathbf{v}, \boldsymbol{\xi} \rangle \,\mathrm{d}\mu \geq \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}^{-} H(\mu_t).$$

The same inequalities hold for $\mathbf{w}$ and Lemma 4.2 shows that the left and right derivatives coincide, hence it follows that

$$\int_M \langle \mathbf{v} - \mathbf{w}, \boldsymbol{\xi} \rangle \,\mathrm{d}\mu = 0 \quad \forall \boldsymbol{\xi} \in \mathcal{C}_c^\infty(M, TM).$$

Thus we must have $(\mathbf{v} - \mathbf{w}) \cdot \rho = 0$ m-a.e. on M and hence $\mathbf{v} = \mathbf{w}$ μ-a.e. on M. $\square$

The strategy to determine the subdifferential is inspired by similar arguments given in [1], Section 10.4, in the special case of $M = \mathbb{R}^n$. The main part of Proposition 4.3 showing that $\nabla\rho/\rho$ is a subdifferential at $\rho \cdot m$ is due to

Villani [14]. The proof therein mainly relies on the K-convexity of the functional and covers a wide class of integral functionals.

Now that we know explicitly the subdifferential we are prepared to prove our main Theorem 2 establishing the equivalence of the gradient flow for the relative entropy with the heat flow.

Proof of Theorem 2. (i) $\Rightarrow$ (ii): Let $(\mu_t)_t$ be a trajectory of the gradient flow with tangent vector field $(\mathbf{v}_t)_t$ characterized by Proposition 2.5. Recall from (28) that $H(\mu_t) < \infty$ for all $t > 0$ and hence every μ_t has a density ρ_t. By the definition of gradient flow we have $-\mathbf{v}_t \in \partial H(\mu_t)$ for a.e. $t > 0$ and in particular $|\partial H|(\mu_t) < \infty$. By the definition of the tangent vector field $\mathbf{v}_t \in T_{\mu_t}\mathcal{P}_2(M)$ for a.e. $t > 0$. Hence Lemma 3.2 shows that $\mathbf{v}_t$ is a strong subdifferential. Now Proposition 4.3 characterizing the unique strong subdifferential yields that $\rho_t \in W^{1,1}(M)$ and $-\mathbf{v}_t = \nabla\rho_t/\rho_t$ for a.e. $t > 0$. Hence the continuity equation (6) becomes

$$\int_I \int_M \partial_t \varphi(t,x)\rho_t(x) - \big\langle \nabla\rho_t(x), \nabla\varphi(t,x)\big\rangle \mathrm{d}m(x)\,\mathrm{d}t = 0 \quad \forall \varphi \in \mathcal{C}_c^\infty\big((0,\infty)\times M\big),$$

i.e. $(\rho_t)_{t>0}$ is a weak solution to the heat equation. It is well known that this already implies that $(\rho_t)_{t>0}$ has a smooth version solving the heat equation.

(ii) $\Rightarrow$ (i): The heat equation for $(\rho_t)_t$ can be recast as a continuity equation for $(\mu_t)_t$:

$$\partial_t\mu_t + \operatorname{div}\left(-\frac{\nabla\rho_t}{\rho_t}\cdot\mu_t\right) = 0.$$

From the integrability of $-\mathbf{v}_t := \frac{\nabla\rho_t}{\rho_t}$ and Proposition 2.5 we infer that $(\mu_t)_t$ is locally absolutely continuous in $(0,\infty)$. Moreover, the integrability assumption (2) ensures that $\|\nabla\rho_t/\rho_t\|_{L^2(\mu_t,TM)} < \infty$ for a.e. $t > 0$. Hence Proposition 4.3 again shows that $-\mathbf{v}_t$ is the unique strong subdifferential at μ_t and $\mathbf{v}_t \in T_{\mu_t}\mathcal{P}_2(M)$ for a.e. $t > 0$. Hence $(\mathbf{v}_t)_t$ must be the tangent vector field characterized by Proposition 2.5 and $-\mathbf{v}_t \in \partial H(\mu_t)$ for a.e. $t > 0$. Thus $(\mu_t)_{t\geq 0}$ is a trajectory of the gradient flow for H. $\square$

The strategy to prove contractivity of the gradient flow in the Wasserstein distance is to control the infinitesimal behavior of $d_W(\mu_t,\nu_t)$ along two trajectories μ_t and ν_t. The crucial ingredient is (35) which can be read as a formula of first variation in the Wasserstein space $\mathcal{P}_2(M)$.

Proposition 4.4 (Contractivity of the gradient flow). *Assume Ric $\geq K$. Let $(\mu_t)_{t\geq 0}$ and $(\nu_t)_{t\geq 0}$ be two trajectories of the gradient flow for H. Then*

$$d_W(\mu_t,\nu_t) \leq \mathrm{e}^{-Kt} d_W(\mu_0,\nu_0).$$

In particular for a given initial value μ_0 there is at most one trajectory of the gradient flow.

Proof. Denote $\mathbf{v}_t, \mathbf{w}_t$ the tangent vector fields of $(\mu_t)_t$ and $(\nu_t)_t$. By Theorem 2 and Proposition 4.3 we can assume that $\mathbf{v}_t = -\nabla\rho_t/\rho_t$, where $(\rho_t)_{t>0}$ solves the heat equation. Certainly ρ_t is strictly positive for every $t > 0$. Indeed, by the maximum principle for the heat equation we have $\rho_t(x) \geq \int p(t-\varepsilon,x,y)\rho_\varepsilon(y)\,\mathrm{d}m(y)$, where $p(s,x,y)$ is the heat kernel on M which is strictly positive (cf. [6]). Hence $\mathbf{v}_t$ is smooth and thus locally Lipschitz and so is $\mathbf{w}_t$ for the same reasons. Now we can apply [14], Theorem 23.9, to see that the Wasserstein distance is differentiable along these curves and that we have for a.e. $t > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t} d_W^2(\mu_t,\nu_t) = -2\int_M \big\langle \mathbf{v}_t, \boldsymbol{\Psi}_{\mu_t}^{\nu_t}\big\rangle \mathrm{d}\mu_t - 2\int_M \big\langle \mathbf{w}_t, \boldsymbol{\Psi}_{\nu_t}^{\mu_t}\big\rangle \mathrm{d}\nu_t. \tag{35}$$

By Lemma 3.5, characterizing the subdifferential of a displacement convex functional, we also have for a.e. $t > 0$:

$$\int_M \big\langle -\mathbf{v}_t, \boldsymbol{\Psi}_{\mu_t}^{\nu_t}\big\rangle \mathrm{d}\mu_t \leq H(\nu_t) - H(\mu_t) - \frac{K}{2} d_W^2(\mu_t,\nu_t),$$

$$\int_M \big\langle -\mathbf{w}_t, \boldsymbol{\Psi}_{\nu_t}^{\mu_t}\big\rangle \mathrm{d}\nu_t \leq H(\mu_t) - H(\nu_t) - \frac{K}{2} d_W^2(\mu_t,\nu_t).$$

Together we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t} d_W^2(\mu_t, \nu_t) \leq -2K \cdot d_W^2(\mu_t, \nu_t).$$

Thus the assertion follows by an application of Gronwall's lemma. □

The proof of contractivity given here is due to Villani, see [14], Theorem 23.25, but note the deviating definition of gradient flow.

Remark 4.5. *Note that if we apply* [14], *Theorem* 23.9, *for a trajectory* $(\mu_t)_t$ *of the gradient flow and a constant curve* $\nu_t = \sigma$ (*i.e.* $\mathbf{w}_t = 0$) *and invoke the* K*-convexity of* H *we obtain that*

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} d_W^2(\mu_t, \sigma) = -\int_M \langle \mathbf{v}_t, \boldsymbol{\Psi}_{\mu_t}^{\sigma} \rangle \mathrm{d}\mu_t \leq H(\sigma) - H(\mu_t) - \frac{K}{2} d_W^2(\mu_t, \sigma), \tag{36}$$

which is the evolution variational inequality (1).

5. The discrete approximation scheme

To establish the existence of gradient flows for the relative entropy functional we will adapt the discrete variational approximation scheme introduced by Otto in [7] to our Riemannian setting. Using this scheme we construct trajectories for given initial value with finite entropy. The complete gradient flow with arbitrary initial values in $\mathcal{P}_2(M)$ is then constructed in the next section via an extension argument based on the contractivity in d_W.

Throughout this section we shall assume $Ric \geq K$. We will proceed as follows: Fix a time step $\tau > 0$ and an initial value $\mu_0 \in D(H)$. Recursively define a sequence $(\mu_n^\tau)_{n\in\mathbb{N}}$ of local minimizers by

$$\mu_0^\tau := \mu_0, \qquad \mu_n^\tau := \arg\min_\nu \left(H(\nu) + \frac{d_W^2(\mu_{n-1}^\tau, \nu)}{2\tau} \right). \tag{37}$$

We then define a discrete trajectory as the piecewise constant interpolant $(\overline{\mu}_t^\tau)_{t\geq 0}$ by

$$\overline{\mu}_0^\tau := \mu_0, \qquad \overline{\mu}_t^\tau := \mu_n^\tau \quad \text{if } t \in \big((n-1)\tau, n\tau\big]. \tag{38}$$

We will show that $\overline{\mu}_t^\tau \to \mu_t$ weakly as $\tau \to 0$ for every $t > 0$, such that $(\mu_t)_{t\geq 0}$ is a trajectory of the gradient flow for H according to Definition 3.7.

Proposition 5.1. *For every* $\mu_0 \in D(H)$ *and every* $\tau > 0$ *there exists a unique solution to the variational scheme* (37).

Proof. It is enough to show that for every $\mu \in \mathcal{P}_2(M)$ the functional

$$\Phi(\tau, \mu; \cdot) := H(\cdot) + \frac{d_W^2(\mu, \cdot)}{2\tau} \tag{39}$$

has a unique minimizer. Uniqueness follows from the fact that $\Phi(\tau, \mu; \cdot)$ is strictly convex in the usual sense, i.e. $\Phi(\tau, \mu; s\nu + (1-s)\nu') < s\Phi(\tau, \mu; \nu) + (1-s)\Phi(\tau, \mu; \nu')$ for $\nu \neq \nu'$. The direct method of the calculus of variations easily yields existence once we have established the following claims. Recall from (32) that m_2 denotes the second moment.

Claim 5.2. *Let* $(\nu_n)_n$ *be a minimizing sequence for the functional* (39). *Then*

$$\sup_n \big\{ m_2(\nu_n) \big\} < \infty.$$

In particular the sequence is tight and hence admits a weak limit point $\nu \in \mathcal{P}_2(M)$.

Claim 5.3. *Let* $\nu_n \to \nu$ *weakly such that* $\sup_n\{m_2(\nu_n)\} < \infty$. *Then*

$$\Phi(\tau, \mu; \nu) \le \liminf \Phi(\tau, \mu; \nu_n).$$

Proof of Claim 5.2. Using the estimate $m_2(\nu_n) \le 2 \cdot m_2(\mu) + 2 \cdot d_W^2(\mu, \nu_n)$ (which follows immediately from the inequality $d(o,x)^2 \le 2d(o,y)^2 + 2d(x,y)^2$ and the definition of d_W) and combining it with (33) we obtain:

$$\Phi(\tau, \mu; \nu_n) \ge -\frac{m_2(\mu)}{\tau} - C_\varepsilon + \left(\frac{1}{2\tau} - \varepsilon\right) m_2(\nu_n). \tag{40}$$

Choosing $\varepsilon < \frac{1}{2\tau}$ we see that $(m_2(\nu_n))_n$ must be bounded. In particular the function $d(o,\cdot)$ is uniformly integrable w.r.t. $(\nu_n)_n$ which ensures tightness ([1], Remark 5.1.5), and thus relative compactness by Prokhorov's theorem. Any weak limit point ν satisfies $m_2(\nu) \le \liminf m_2(\nu_n) < \infty$ ([1], Lemma 5.1.7) and must hence belong to $\mathcal{P}_2(M)$. □

Proof of Claim 5.3. Lower semicontinuity of $d_W^2(\mu, \cdot)$ w.r.t. weak convergence is well known (cf. [14], Remark 6.12 or [1], Proposition 7.1.3). To conclude we note that H is also lower semicontinuous w.r.t. weak convergence on sets of bounded second moment. We state this result separately in the next proposition for future reference. □

Proposition 5.4. *Let* $Ric \ge K$. *Let* $\nu_n \to \nu$ *weakly such that* $\sup_n\{m_2(\nu_n)\} < \infty$. *Then*

$$H(\nu) \le \liminf_{n\to\infty} H(\nu_n).$$

In particular H *is lower semicontinuous w.r.t. convergence in* d_W.

Proof. We reduce to the case of a finite reference measure instead of m. Define a probability measure $\gamma := \mathrm{e}^{-V} \cdot m$, where $V(x) := c \cdot d(o,x) + a$ for suitable constants a, c (recall from the proof of Lemma 4.1 that $c \cdot d(o,\cdot)$ is integrable). Then write for $\nu = \rho \cdot \gamma = \rho \mathrm{e}^{-V} \cdot m$:

$$H(\nu) = \int_M \rho \log \rho \, \mathrm{d}\gamma - \int_M V \, \mathrm{d}\nu =: H(\nu|\gamma) - \int_M V \, \mathrm{d}\nu.$$

Lower semicontinuity of the relative entropy functional $H(\cdot|\gamma)$ with finite reference measure γ w.r.t. weak convergence is proven in [13], Lemma 4.1. Alternatively it follows immediately from the following representation formula ([2], Lemma 3.18, here $M = \mathbb{R}^n$ but the very same arguments hold in our case)

$$H(\nu|\gamma) = \sup\left\{\int_M \phi \, \mathrm{d}\nu - \int_M \mathrm{e}^\phi - 1 \, \mathrm{d}\gamma \,\middle|\, \phi \in C_b^0(M)\right\}.$$

Since the second moments of ν_n are bounded V is uniformly integrable w.r.t. $(\nu_n)_n$ and we have $\nu_n(V) \to \nu(V)$ as $n \to \infty$, which proves the claim. Since convergence in d_W implies weak convergence and convergence of second moments we obtain in particular that H is l.s.c. on $(\mathcal{P}_2(M), d_W)$. □

Proposition 5.5. *For every* $\mu_0 \in D(H)$ *there exists a unique trajectory of the gradient flow* $(\mu_t)_{t\ge 0}$ *with initial value* μ_0. *For every* $t \ge 0$ *we have* $\overline{\mu}_t^\tau \to \mu_t$ *weakly as* $\tau \to 0$.

Proof. *Step 1: discrete equation.* We start by making precise why we call $\overline{\mu}^\tau$ a discrete solution. Let us introduce the piecewise constant velocity vector field

$$\overline{V}_t^\tau := -\frac{1}{\tau} \boldsymbol{\Psi}_{\mu_n^\tau}^{\mu_{n-1}^\tau}, \quad t \in \big((n-1)\tau, n\tau\big].$$

Then $\overline{\mu}^\tau$ satisfies the "discrete gradient flow equation":

$$-\overline{V}_t^\tau \in \partial H\big(\overline{\mu}_t^\tau\big) \quad \text{for every } t > 0. \tag{41}$$

In view of our characterization of the subdifferential (Proposition 4.3) this follows immediately from the following claim:

Claim 5.6. *Let $\nu = \rho \cdot m$ be a minimizer of $\Phi(\tau, \mu; \cdot)$. Then*

$$-\int_M \rho \cdot \operatorname{div} \boldsymbol{\xi} \, \mathrm{d}m = \int_M \left\langle \frac{1}{\tau} \boldsymbol{\Psi}_\nu^\mu, \boldsymbol{\xi} \right\rangle \mathrm{d}\nu \quad \forall \boldsymbol{\xi} \in C_c^\infty(M, TM). \tag{42}$$

This implies that $\rho \in W^{1,1}(M)$ and $\frac{1}{\tau}\boldsymbol{\Psi}_\nu^\mu = \nabla\rho/\rho$.

For later reference we rephrase (41) using (42) in the form

$$\int_M \operatorname{div} \boldsymbol{\xi} \, \mathrm{d}\overline{\mu}_t^\tau = \int_M \langle \overline{V}_t^\tau, \boldsymbol{\xi} \rangle \mathrm{d}\overline{\mu}_t^\tau \quad \forall \boldsymbol{\xi} \in C_c^\infty(M, TM). \tag{43}$$

Proof of Claim 5.6. For $\delta > 0$ we set $\nu_\delta := T_{\delta\#}\nu$, where $T_\delta(x) := \exp_x(\delta \boldsymbol{\xi}(x))$. Let π be the optimal coupling of μ and ν. Since ν is a minimizer, we have

$$\begin{aligned} 0 &\le \frac{1}{\delta}\left[H(\nu_\delta) - H(\nu) + \frac{d_W^2(\mu, \nu_\delta)}{2\tau} - \frac{d_W^2(\mu, \nu)}{2\tau} \right] \\ &\le \frac{1}{\delta}\big[H(\nu_\delta) - H(\nu)\big] + \frac{1}{2\tau\delta}\left[\int_M d^2\big(x, T_\delta(y)\big) - d^2(x, y) \, \mathrm{d}\pi(x, y) \right], \end{aligned} \tag{44}$$

where we used the coupling $(id, T_\delta)_\#\pi$ to estimate $d_W(\mu, \nu_\delta)$. We want to pass to the limit as $\delta \downarrow 0$. Recall that for π-a.e. (x, y), $\boldsymbol{\Psi}_\nu^\mu(y)$ is the initial velocity of a minimizing geodesic connecting y to x. Hence we deduce from the first variation formula (see e.g. [5], 2.3) that

$$\limsup_{\delta \downarrow 0} \frac{1}{\delta}\Big(d^2\big(x, T_\delta(y)\big) - d^2(x, y)\Big) \le -2\big\langle \boldsymbol{\Psi}_\nu^\mu(y), \boldsymbol{\xi}(y)\big\rangle.$$

Furthermore by the triangle inequality we can bound the difference quotient above as

$$\frac{1}{\delta}\big|d^2\big(x, T_\delta(y)\big) - d^2(x, y)\big| \le \frac{1}{\delta} d\big(y, T_\delta(y)\big) \cdot \Big(d\big(y, d\big(T_\delta(y)\big)\big) + 2d(x, y)\Big) \le \delta C^2 + 2C \cdot d(x, y),$$

where $C = \sup\{|\boldsymbol{\xi}(x)|, \ x \in M\}$. Since $d(x, y)$ is integrable w.r.t. π we can apply Fatou's lemma to pass to the limsup in the second term of (44). Combining with Lemma 4.2 we obtain

$$0 \le -\int_M \rho \cdot \operatorname{div} \boldsymbol{\xi} \, \mathrm{d}m - \int_M \left\langle \frac{1}{\tau}\boldsymbol{\Psi}_\nu^\mu, \boldsymbol{\xi} \right\rangle \mathrm{d}\nu. \tag{45}$$

Since $\boldsymbol{\xi}$ was arbitrary we must have equality in (45) which yields the claim. □

Step 2: a priori estimates. Fix a time horizon $T > 0$. We show that there exists a constant $C > 0$ depending only on T and μ_0 such that for all $m \le n$, τ with $n\tau \le T$:

$$m_2\big(\mu_n^\tau\big) \le C, \tag{46}$$

$$-C \le H\big(\mu_n^\tau\big) \le C, \tag{47}$$

$$\sum_{k=1}^n \frac{d_W^2(\mu_{k-1}^\tau, \mu_k^\tau)}{2\tau} \le C, \tag{48}$$

$$d_W\big(\mu_n^\tau, \mu_m^\tau\big) \le \sqrt{\tau(n-m)}C. \tag{49}$$

Indeed, by definition of the approximation scheme we have

$$\frac{d_W^2(\mu_{k-1}^\tau,\mu_k^\tau)}{2\tau}+H\big(\mu_k^\tau\big)\le H\big(\mu_{k-1}^\tau\big),\qquad \sum_{k=1}^n\frac{d_W^2(\mu_{k-1}^\tau,\mu_k^\tau)}{2\tau}\le H(\mu_0)-H\big(\mu_n^\tau\big),\tag{50}$$

which yields the second inequality in (47). Arguing as in Claim 5.2, using Cauchy–Schwarz and finally applying Lemma 4.1 we can estimate the second moments as:

$$\begin{aligned}m_2\big(\mu_n^\tau\big)&\le 2d_W^2\big(\mu_n^\tau,\mu_0\big)+2m_2(\mu_0)\le 2n\sum_{k=1}^n d_W^2\big(\mu_{k-1}^\tau,\mu_k^\tau\big)+2m_2(\mu_0)\\&\le 4\tau n\big[H(\mu_0)-H\big(\mu_n^\tau\big)\big]+2m_2(\mu_0)\le 4T\big[H(\mu_0)+C_\varepsilon+\varepsilon\cdot m_2\big(\mu_n^\tau\big)\big]+2m_2(\mu_0).\end{aligned}$$

Choosing $\varepsilon<\frac{1}{4T}$ we obtain (46). This in turn implies the first inequality in (47) by Lemma 4.1. Clearly (47) and (50) yield (48). Finally, to establish (49) we use again Cauchy–Schwarz, apply (50), (47) and obtain

$$\begin{aligned}d_W\big(\mu_n^\tau,\mu_m^\tau\big)&\le 2\tau\sum_{k=m+1}^n\frac{d_W(\mu_{k-1}^\tau,\mu_k^\tau)}{2\tau}\le\sqrt{2(n-m)\tau}\left(\sum_{k=1}^n\frac{d_W^2(\mu_{k-1}^\tau,\mu_k^\tau)}{2\tau}\right)^{1/2}\\&\le\sqrt{2(n-m)\tau}\big[H(\mu_0)-H\big(\mu_n^\tau\big)\big]^{1/2}.\end{aligned}$$

Step 3: limit trajectory. We show that there is a curve $(\mu_t)_{t\ge0}$ such that up to extraction of a subsequence $\overline{\mu}_t^{\tau_n}$ converges to μ_t for all t as $\tau_n\to0$. By connecting $\mu_{n-1}^\tau,\mu_n^\tau$ with a geodesic parametrized in $((n-1)\tau,n\tau]$ we obtain a curve $(\hat\mu_t^\tau)_{t\in[0,T]}$ satisfying for all $s,t\in[0,T]$:

$$d_W\big(\overline{\mu}_t^\tau,\hat\mu_t^\tau\big)\le C\sqrt{\tau}\quad\text{and}\quad d_W\big(\hat\mu_s^\tau,\hat\mu_t^\tau\big)\le C\sqrt{t-s}.\tag{51}$$

We want to apply the Arzelà–Ascoli theorem to the family $(\hat\mu^\tau)_{\tau>0}\subset C^0([0,T],\mathscr{P}(M))$, where $\mathscr{P}(M)$ is the space of probability measures on M equipped with the topology of weak convergence. This topology can be metrized for example by the Wasserstein distance $\tilde d_W$ corresponding to the bounded distance $\tilde d:=d(1+d)^{-1}$ on M which induces the same topology as d (see [14], Corollary 6.11). Since d_W is obviously stronger than $\tilde d_W$, (51) implies that the family $(\hat\mu^\tau)_{\tau>0}$ is uniformly equicontinuous in the weak topology. From (51) with $s=0$ we infer that

$$m_2(\hat\mu^\tau)\le 2\cdot d_W^2\big(\hat\mu^\tau,\mu_0\big)+2\cdot m_2(\mu_0)\le 2C^2t+2\cdot m_2(\mu_0).$$

This implies that the second moments remain bounded and hence $(\hat\mu^\tau)_\tau$ takes values in a relatively compact w.r.t. weak convergence. Hence the Arzelà–Ascoli theorem yields relative compactness of $(\hat\mu^\tau)_{\tau>0}$ in $C^0([0,T],\mathscr{P}(M))$ for every $T>0$. So we can extract a subsequence $\tau_n\to0$ such that

$$\overline{\mu}_t^{\tau_n}\to\mu_t\quad\text{weakly }\forall t\ge0.$$

By lower semicontinuity of the second moment and H (cf. Proposition 5.4) and the bounds (46), (47) we conclude that μ_t belongs to $\mathscr{P}_2(M)$ and that $H(\mu_t)<\infty$. From (51) for $s=0$ and weak lower semicontinuity of d_W ([1], 7.1.3) we deduce $\mu_t\to\mu_0$ in $\mathscr{P}_2(M)$ as $t\to0$.

Step 4: limit velocity. We show that the family of discrete velocities $(\overline{V}^{\tau_n})_n$ admits limit points in a weak sense which takes into account that the vector fields $\overline{V}^{\tau_n}$ belong to different L^2-spaces. Precisely we prove

Claim 5.7. *There is a time dependent vector field* $\mathbf{v}:(0,\infty)\times M\to TM$ *with*

$$\int_0^T\int_M\|\mathbf{v}_t\|^2\,\mathrm{d}\mu_t\,\mathrm{d}t<\infty\quad\forall T>0,\tag{52}$$

such that up to extraction of a further subsequence

$$\int_0^\infty \int_M \langle \boldsymbol{\xi}_t, \overline{V}_t^{\tau_n} \rangle \, \mathrm{d}\overline{\mu}_t^{\tau_n} \, \mathrm{d}t \overset{n\to\infty}{\longrightarrow} \int_0^\infty \int_M \langle \boldsymbol{\xi}_t, \mathbf{v}_t \rangle \, \mathrm{d}\mu_t \, \mathrm{d}t \quad \forall \boldsymbol{\xi} \in C_c^\infty\big((0,\infty) \times M, TM\big). \tag{53}$$

Proof. We will first prove the claim for a finite time horizon. Fix $T > 0$ and set $Q_T := (0, T) \times M$. Define measures on $(0, \infty) \times M$ by

$$\overline{\mu}^{\tau_n} := \int_0^\infty \overline{\mu}_t^{\tau_n} \, \mathrm{d}t, \qquad \mu := \int_0^\infty \mu_t \, \mathrm{d}t.$$

First observe that by (48)

$$S := \sup_n \int_{Q_T} \big\| \overline{V}^{\tau_n} \big\|^2 \, \mathrm{d}\overline{\mu}^{\tau_n} = \sup_n \sum_{k=1}^N \frac{d_W^2(\mu_{k-1}^{\tau_n}, \mu_k^{\tau_n})}{\tau_n} \le 2C < \infty.$$

Choose a countable set $\{\boldsymbol{\xi}_j\}_{j\in\mathbb{N}}$ dense in $C_c^\infty(Q_T, TM)$ w.r.t. the norm $\|\boldsymbol{\xi}\|_\infty := \sup_{Q_T} \|\boldsymbol{\xi}(t,x)\|$. By a diagonal argument we find a subsequence again denoted by τ_n such that

$$L(\boldsymbol{\xi}_j) := \lim_{n\to\infty} \int_{Q_T} \langle \boldsymbol{\xi}_j, \overline{V}^{\tau_n} \rangle \, \mathrm{d}\overline{\mu}^{\tau_n}$$

converges for all $j \in \mathbb{N}$. If we denote by $\mathcal{V} := \langle \boldsymbol{\xi}_j, j \in \mathbb{N} \rangle$ the linear hull and extend by linearity we obtain a linear functional $L : \mathcal{V} \to \mathbb{R}$. Since weak convergence of $\overline{\mu}_t^{\tau_n} \to \mu_t$ for all $t > 0$ implies weak convergence $\overline{\mu}^{\tau_n} \to \mu$ we find that L is continuous w.r.t. the $L^2(\mu, TM)$-norm. Indeed, we have for all $\boldsymbol{\xi} \in \mathcal{V}$:

$$\big|L(\boldsymbol{\xi})\big| = \lim_{n\to\infty} \left| \int_{Q_T} \langle \boldsymbol{\xi}, \overline{V}^{\tau_n} \rangle \, \mathrm{d}\overline{\mu}^{\tau_n} \right| \le S \cdot \liminf_{n\to\infty} \cdot \left(\int_{Q_T} \|\boldsymbol{\xi}\|^2 \, \mathrm{d}\overline{\mu}^{\tau_n} \right)^{1/2} = S \cdot \|\boldsymbol{\xi}\|_{L^2(\mu, TM)}.$$

From the Riesz theorem we obtain a vector field $\mathbf{v} \in L^2(\mu|_{Q_T}, TM)$ such that

$$\lim_n \int_{Q_T} \langle \boldsymbol{\xi}, \overline{V}^{\tau_n} \rangle \, \mathrm{d}\overline{\mu}^{\tau_n} = L(\boldsymbol{\xi}) = \int_{Q_T} \langle \boldsymbol{\xi}, \mathbf{v} \rangle \, \mathrm{d}\mu \quad \forall \boldsymbol{\xi} \in \mathcal{V}.$$

From the density of $\{\boldsymbol{\xi}_j\}_j$ w.r.t. uniform convergence we conclude that

$$\int_{Q_T} \langle \boldsymbol{\xi}, \overline{V}^{\tau_n} \rangle \, \mathrm{d}\overline{\mu}^{\tau_n} \overset{n\to\infty}{\longrightarrow} \int_{Q_T} \langle \boldsymbol{\xi}, \mathbf{v} \rangle \, \mathrm{d}\mu \quad \forall \boldsymbol{\xi} \in C_c^\infty(Q_T, TM).$$

If $T' > T$ we find that the vector field obtained for T' must coincide μ-a.e. on Q_T with the one obtained for T. Hence successively repeating the previous construction for $T \to \infty$ we obtain a vector field $\mathbf{v} : (0, \infty) \times M \to TM$ and a subsequence satisfying (52) and (53). □

Step 5: μ and v satisfy the continuity equation. Let $\varphi \in C_c^\infty((0, \infty) \times M)$ be a test function for the continuity equation (6). Using Taylor expansion we find that

$$\begin{aligned} \int_M \varphi_t \, \mathrm{d}\overline{\mu}_t^{\tau_n} - \int_M \varphi \, \mathrm{d}\overline{\mu}_{t-\tau_n}^{\tau_n} &= \int_M \varphi(t,x) - \varphi\big(t, \exp\big(-\tau_n \overline{V}_t^{\tau_n}(x)\big)\big) \, \mathrm{d}\overline{\mu}_t^{\tau_n}(x) \\ &= \tau_n \cdot \int_M \langle \nabla \varphi_t, \overline{V}_t^{\tau_n} \rangle \, \mathrm{d}\overline{\mu}_t^{\tau_n} + \varepsilon(\tau_n, \varphi, t), \end{aligned} \tag{54}$$

where the error term is bounded as $|\varepsilon(\tau_n, \varphi, t)| \le C_\varphi \tau_n^2 \|\overline{V}_t^{\tau_n}\|^2_{L^2(\overline{\mu}_t^{\tau_n})}$ for a constant C_φ depending only on the second derivatives of φ. Note that $\operatorname{supp} \varphi \subset Q_T = (0, T) \times M$ for some $T > 0$. Using the weak convergence of $\overline{\mu}^{\tau_n}$, (54) and

(53) we have that:

$$\begin{aligned}\int_0^\infty \int_M \partial_t\varphi \, \mathrm{d}\mu_t \, \mathrm{d}t &= \lim_{n\to\infty} \int_{Q_T} \partial_t\varphi \, \mathrm{d}\overline{\mu}^{\tau_n}\\ &= \lim_{n\to\infty} \frac{1}{\tau_n} \int_{Q_T} \big[\varphi(t+\tau_n,x) - \varphi(t,x)\big] \mathrm{d}\overline{\mu}^{\tau_n}(t,x)\\ &= -\lim_{n\to\infty} \frac{1}{\tau_n} \int_{(0,T)} \left[\int_M \varphi(t,\cdot)\,\mathrm{d}\overline{\mu}_t^{\tau_n} - \int_M \varphi(t,\cdot)\,\mathrm{d}\overline{\mu}_{t-\tau_n}^{\tau_n}\right] \mathrm{d}t\\ &= -\lim_{n\to\infty} \left[\int_{Q_T} \big\langle \nabla\varphi, \overline{V}^{\tau_n}\big\rangle \mathrm{d}\overline{\mu}^{\tau} + \frac{1}{\tau_n}\int_{(0,T)} \varepsilon(\tau_n,\varphi,t)\,\mathrm{d}t\right]\\ &= -\int_0^\infty \int_M \langle \nabla\varphi, \mathbf{v}_t\rangle \,\mathrm{d}\mu_t \,\mathrm{d}t.\end{aligned}$$

In the final equality we used the fact that due to (48) the second summand in the final limit is bounded by

$$C_\varphi \tau_n \int_{[0,T]} \big\|\overline{V}_t^{\tau_n}\big\|^2_{L^2(\overline{\mu}_t^{\tau_n})} \mathrm{d}t = C_\varphi \tau_n^2 \sum_{k=1}^N \frac{1}{\tau_n^2} \cdot d_W^2\big(\mu_{k-1}^{\tau_n}, \mu_k^{\tau_n}\big) \le 2C_\varphi C \tau_n$$

and thus tends to 0 as $n \to \infty$. As $\varphi \in \mathcal{C}_c^\infty((0,\infty)\times M)$ was arbitrary we conclude that the continuity equation holds in the sense of distributions.

Step 6: $(\mu_t)_t$ is a trajectory of the gradient flow. From step 5 and Proposition 2.5 we infer that $(\mu_t)_{t\ge 0}$ is locally absolutely continuous in $(0,\infty)$. Step 3 already showed $\mu_t \to \mu_0$ in $\mathscr{P}_2(M)$ and $H(\mu_t) < \infty$ for all $t > 0$. Hence μ_t has a density ρ_t. We will now pass to the limit in the discrete equation (43). Fix $\boldsymbol{\xi} \in C_c^\infty(M, TM)$ and consider test functions of the form $\eta(t)\cdot\boldsymbol{\xi}(x)$ for $\eta \in C_c^\infty((0,\infty))$. From the weak convergence of $\overline{\mu}_t^{\tau_n}$ for every $t > 0$ and dominated convergence we obtain:

$$\int_0^\infty \eta(t) \int_M \operatorname{div}\boldsymbol{\xi}\, \mathrm{d}\overline{\mu}_t^{\tau_n}\, \mathrm{d}t \overset{n\to\infty}{\longrightarrow} \int_0^\infty \eta(t) \int_M \operatorname{div}\boldsymbol{\xi}\, \mathrm{d}\mu_t\, \mathrm{d}t. \tag{55}$$

On the other hand we have from the weak convergence of velocities (53):

$$\int_0^\infty \int_M \big\langle \overline{V}_t^{\tau_n}, \eta(t)\cdot\boldsymbol{\xi}\big\rangle \mathrm{d}\overline{\mu}_t^{\tau_n}\, \mathrm{d}t \overset{n\to\infty}{\longrightarrow} \int_0^\infty \int_M \langle \mathbf{v}_t, \eta(t)\cdot\boldsymbol{\xi}\rangle\, \mathrm{d}\mu_t\, \mathrm{d}t. \tag{56}$$

As the sequences on the left in (55) and (56) coincide by (43) we find that:

$$\int_0^\infty \eta(t) \int_M \operatorname{div}\boldsymbol{\xi}\, \mathrm{d}\mu_t\, \mathrm{d}t = \int_0^\infty \eta(t) \int_M \langle \mathbf{v}_t, \boldsymbol{\xi}\rangle\, \mathrm{d}\mu_t\, \mathrm{d}t,$$

and hence that for a.e. $t > 0$:

$$\int_M \operatorname{div}\boldsymbol{\xi}\, \mathrm{d}\mu_t = \int_M \langle \mathbf{v}_t, \boldsymbol{\xi}\rangle\, \mathrm{d}\mu_t. \tag{57}$$

Since $C_c^\infty(M, TM)$ is separable w.r.t. convergence in the C^1-norm we can find a negligible set of times $\mathscr{N}$ such that (57) holds for all $t \in (0,\infty)\setminus\mathscr{N}$ and all $\boldsymbol{\xi} \in C_c^\infty(M, TM)$. This implies that $\rho_t \in W^{1,1}(M)$ and $\mathbf{v}_t = -\nabla\rho_t/\rho_t$. Hence the characterization of the subdifferential (Proposition 4.3) shows that

$$-\mathbf{v}_t \in \partial H(\mu_t) \quad \text{for a.e. } t > 0$$

and that $\mathbf{v}_t \in T_{\mu_t}\mathscr{P}_2(M)$. Hence $(\mathbf{v}_t)_t$ is indeed the tangent vector field characterized by Proposition 2.5 and $(\mu_t)_t$ satisfies the gradient flow equation. Finally the uniqueness of gradient flows (Proposition 4.4) shows that μ and $\mathbf{v}$ do not depend on the chosen subsequence τ_n and hence we have full convergence as $\tau \to 0$. □

6. Conclusion of the proof

Before we can finish the proof of our main theorem we need the following lemma which is of independent interest. It shows that the gradient flow has a regularizing effect.

Lemma 6.1. *Let $Ric \geq K$ and $(\mu_t)_{t\geq 0}$ be a trajectory of the gradient flow for H. Then for all $t > 0$ and any $\nu \in D(H)$ we have (with the convention $0/0 = 1$ if $K = 0$):*

$$H(\mu_t) \leq H(\nu) + \frac{K}{2(\mathrm{e}^{Kt} - 1)} d_W^2(\mu_0, \nu). \tag{58}$$

Proof. Let $\mathbf{v}_t$ be the tangent vector field. We know from [14], Theorem 23.9 (cf. again the proof of Proposition 4.4) and Lemma 3.5 characterizing the subdifferential of a K-convex functional that for a.e. $s > 0$

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{2}d_W^2(\mu_s, \nu) = -\int_M \langle \mathbf{v}_s, \boldsymbol{\Psi}_{\mu_s}^{\nu}\rangle \mathrm{d}\mu_s \leq H(\nu) - H(\mu_s) - \frac{K}{2}d_W^2(\mu_s, \nu),$$

which immediately yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{e}^{Ks}}{2}d_W^2(\mu_s, \nu)\right) \leq \mathrm{e}^{Ks}\big[H(\nu) - H(\mu_s)\big].$$

Integrating in the interval (ε, t) and recalling from (28) that $s \mapsto H(\mu_s)$ is nonincreasing we obtain

$$\frac{\mathrm{e}^{Kt}}{2}d_W^2(\mu_t, \nu) - \frac{1}{2}d_W^2(\mu_\varepsilon, \nu) \leq \frac{\mathrm{e}^{Kt} - 1}{K}\big[H(\nu) - H(\mu_t)\big].$$

Letting $\varepsilon \to 0$ obviously yields the claim as the first term on the left is positive. □

Proof of Theorem 1. Uniqueness and contractivity of the gradient flow for H we have already proven in Proposition 4.4. Combining this with Proposition 5.5 we have obtained a K-contractive semigroup $\sigma : [0, \infty) \times D(H) \to \mathcal{P}_2(M)$, such that $\mu_t := \sigma(t, \mu_0)$ is a trajectory of the gradient flow for every initial value $\mu_0 \in D(H)$. By the contractivity estimate

$$d_W\big(\sigma(t, \mu_0), \sigma(t, \nu_0)\big) \leq \mathrm{e}^{-Kt} d_W(\mu_0, \nu_0)$$

there is a unique continuous extension of the semigroup σ to the closure $\overline{D(H)}$, which coincides with $\mathcal{P}_2(M)$. Indeed, observe that $D(H)$ contains all measures of the form

$$\mu_{x,r} = \mathbf{1}_{B_r(x)}/m\big(B_r(x)\big) \cdot m$$

and their convex combinations. Hence all convex combinations of Dirac measures $\sum \alpha_i \delta_{x_i}$ belong to $\overline{D(H)}$. It is well know that the set of such measures is dense in $\mathcal{P}_2(M)$ ([14], proof of 6.18) and thus we have $\overline{D(H)} = \mathcal{P}_2(M)$. Now we have to show that the trajectories of the extended semigroup are still trajectories of the gradient flow. So let $\mu_0 \in \mathcal{P}_2(M)$ and choose $\mu_0^n \in D(H)$ for $n \in \mathbb{N}$, such that $\mu_0^n \to \mu_0$ in $\mathcal{P}_2(M)$. Then $\mu_t := \sigma(t, \mu_0) = \lim \sigma(t, \mu_0^n)$. Let $(\mathbf{v}_t^n)_t$ be the tangent vector fields for the curves $(\mu_t^n)_t$. By the definition of gradient flow we have for a.e. $t > 0$:

$$-\mathbf{v}_t^n \in \partial H\big(\mu_t^n\big). \tag{59}$$

From Proposition 4.3 characterizing the subdifferential we infer that $\rho_t^n \in W^{1,1}(M)$ and $\nabla \rho_t^n/\rho_t^n = -\mathbf{v}_t^n$, where ρ_t^n is the density of μ_t^n. In other words, this means that

$$\int_M \operatorname{div} \boldsymbol{\xi}\, \mathrm{d}\mu_t^n = \int_M \langle \mathbf{v}_t^n, \boldsymbol{\xi}\rangle \mathrm{d}\mu_t^n \quad \forall \boldsymbol{\xi} \in \mathcal{C}_c^\infty(M, TM). \tag{60}$$

Via (60) we will pass to the limit in (59). Fix some $\nu \in D(H)$. By (58) and lower semicontinuity of H (cf. Proposition 5.4) we have that for all $t > 0$:

$$H(\mu_t) \le \liminf_{n\to\infty} H\big(\mu_t^n\big) \le H(\nu) + \frac{K}{2(\mathrm{e}^{Kt}-1)} d_W^2(\mu_0, \nu) < \infty.$$

Hence μ_t is absolutely continuous and has some density ρ_t. Fix $T > \delta > 0$. Using the energy identity (29) and the estimates (33), (58) we obtain:

$$\begin{aligned}\int_\delta^T \int_M \|\mathbf{v}_t^n\|^2 \,\mathrm{d}\mu_t^n \,\mathrm{d}t &= H\big(\mu_\delta^n\big) - H\big(\mu_T^n\big) \\ &\le H(\nu) + \frac{K}{2(\mathrm{e}^{K\delta}-1)} d_W^2\big(\mu_0^n, \nu\big) + C_1 + m_2\big(\mu_T^n\big) \\ &\le C(\delta, T, \mu_0, \mu_T),\end{aligned}$$

where $C(\delta, T, \mu_0, \mu_T)$ is a constant depending only on δ, T, μ_0 and μ_T and independent of n since μ_0^n, μ_T^n converge in $\mathscr{P}_2(M)$. Starting from this bound we can argue as in step 4 of Proposition 5.5 (with $\overline{\mu}_t^{\tau_n}, \overline{V}_t^{\tau_n}$ replaced by $\mu_t^n, \mathbf{v}_t^n$) to find a time dependent vector field $\mathbf{v} : (0, \infty) \times M \to TM$ with

$$\int_\delta^T \|\mathbf{v}_t\|^2_{L^2(\mu_t, TM)} \,\mathrm{d}t < \infty \quad \forall \delta, T > 0$$

such that up to extraction of a subsequence we have for all $\boldsymbol{\xi} \in C_c^\infty((0,\infty) \times M, TM)$:

$$\int_0^\infty \int_M \big\langle \boldsymbol{\xi}_t, \mathbf{v}_t^n \big\rangle \,\mathrm{d}\mu_t^n \,\mathrm{d}t \stackrel{n\to\infty}{\longrightarrow} \int_0^\infty \int_M \langle \boldsymbol{\xi}_t, \mathbf{v}_t \rangle \,\mathrm{d}\mu_t \,\mathrm{d}t.$$

Hence we can pass to the limit in the continuity equation (6), i.e. for all $\varphi \in C_c^\infty((0,\infty) \times M)$:

$$0 = \int_0^\infty \int_M \partial_t \varphi + \big\langle \mathbf{v}_t^n, \nabla\varphi \big\rangle \,\mathrm{d}\mu_t^n \,\mathrm{d}t \stackrel{n\to\infty}{\longrightarrow} \int_0^\infty \int_M \partial_t \varphi + \langle \mathbf{v}_t, \nabla\varphi \rangle \,\mathrm{d}\mu_t \,\mathrm{d}t.$$

So $(\mu_t)_t, (\mathbf{v}_t)_t$ satisfy the continuity equation. Thus by Proposition 2.5 $(\mu_t)_t$ is locally absolutely continuous in $(0, \infty)$. Arguing as in step 6 of Proposition 5.5 we can pass to the limit in (60) and obtain that for a.e. $t > 0$ and every $\boldsymbol{\xi} \in C_c^\infty(M, TM)$:

$$\int_M \operatorname{div} \boldsymbol{\xi} \,\mathrm{d}\mu_t = \int_M \langle \mathbf{v}_t, \boldsymbol{\xi} \rangle \,\mathrm{d}\mu_t.$$

This implies that $\rho_t \in W^{1,1}(M)$ and $\mathbf{v}_t = -\nabla\rho_t/\rho_t$. Hence the characterization of the subdifferential (Proposition 4.3) shows that

$$-\mathbf{v}_t \in \partial H(\mu_t) \quad \text{for a.e. } t > 0$$

and that $\mathbf{v}_t \in T_{\mu_t}\mathscr{P}_2(M)$. Hence $(\mathbf{v}_t)_t$ is indeed the tangent vector field characterized by Proposition 2.5 and $(\mu_t)_t$ satisfies the gradient flow equation. □

References

[1] L. Ambrosio, N. Gigli and G. Savaré. *Gradient Flows in Metric Spaces and in the Space of Probability Measures*. Birkhäuser, Basel, 2005. MR2129498

[2] L. Ambrosio and G. Savaré. *Gradient Flows of Probability Measures*. In *Handbook of Differential Equations: Evolution Equations* 1–136. Elsevier, 2006.

[3] J.-D. Benamou and Y. Brenier. A computational fluid mechanics solution to the Monge–Kantorovich mass transfer problem. *Numer. Math.* **84** (2000) 375–393. MR1738163

[4] P. Bernard. Young measures, superposition and transport. *Indiana Univ. Math. J.* **57** (2008) 247–276. MR2400257

[5] I. Chavel. *Riemannian Geometry – a Modern Introduction*. *Cambridge Tracts in Mathematics* **108**. Cambridge Univ. Press, Cambridge, 1993. MR1271141

[6] J. Dodziuk. Maximum principles for parabolic inequalities and the heat flow on open manifolds. *Indiana Univ. Math. J.* **32** (1983) 703–716. MR0711862

[7] R. Jordan, D. Kinderlehrer and F. Otto. The variational formulation of the Fokker–Planck equation. *SIAM J. Math. Anal.* **29** (1998) 1–17. MR1617171

[8] R.J. McCann. A convexity principle for interacting gases. *Adv. Math.* **128** (1997) 153–179. MR1451422

[9] R.J. McCann. Polar factorization of maps on Riemannian manifolds. *Geom. Funct. Anal.* **11** (2001) 589–608. MR1844080

[10] S.-I. Ohta. Gradient flows on Wasserstein spaces over compact Alexandrov spaces. *Amer. J. Math.* 2008. To appear.

[11] G. Savaré. Gradient flows and diffusion semigroups in metric spaces under lower curvature bounds. *C. R. Acad. Sci.* **345** (2007) 151–154. MR2344814

[12] K.-T. Sturm. Convex functionals of probability measures and nonlinear diffusions on manifolds. *J. Math. Pures Appl.* **84** (2005) 149–168. MR2118836

[13] K.-T. Sturm. On the geometry of metric measure spaces. *Acta Math.* **196** (2006) 65–131. MR2237206

[14] C. Villani. *Optimal Transport, Old and New*. *Grundlehren der Mathematischen Wissenschaften* **338**. Springer, Berlin, Heidelberg, 2009.