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Potential confinement property of the parabolic Anderson model

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Abstract. We consider the parabolic Anderson model, the Cauchy problem for the heat equation with random potential in $\mathbb{Z}^d$. We use i.i.d. potentials $\xi:\mathbb{Z}^d \to \mathbb{R}$ in the third universality class, namely the class of *almost bounded potentials*, in the classification of van der Hofstad, König and Mörters [*Commun. Math. Phys.* **267** (2006) 307–353]. This class consists of potentials whose logarithmic moment generating function is regularly varying with parameter $\gamma = 1$, but do not belong to the class of so-called double-exponentially distributed potentials studied by Gärtner and Molchanov [*Probab. Theory Related Fields* **111** (1998) 17–55].

In [*Commun. Math. Phys.* **267** (2006) 307–353] the asymptotics of the expected total mass was identified in terms of a variational problem that is closely connected to the well-known logarithmic Sobolev inequality and whose solution, unique up to spatial shifts, is a perfect parabola. In the present paper we show that those potentials whose shape (after appropriate vertical shifting and spatial rescaling) is away from that parabola contribute only negligibly to the total mass. The topology used is the strong L^1-topology on compacts for the exponentials of the potential. In the course of the proof, we show that any sequence of approximate minimisers of the above variational formula approaches some spatial shift of the minimiser, the parabola.

Résumé. Nous considérons le modèle Anderson parabolique donné par le problème de Cauchy pour l'équation de la chaleur avec un potentiel aléatoire dans $\mathbb{Z}^d$. Nous utilisons des potentiels i.i.d. $\xi:\mathbb{Z}^d \to \mathbb{R}$ qui sont dans la troisième classe d'universalité de la classification de van der Hofstad, König and Mörters [*Commun. Math. Phys.* **267** (2006) 307–353]. Cette classe, les potentiels *presque bornés*, contient les potentiels dont le logarithme de la transformée de Laplace est de variation régulière avec paramètre $\gamma = 1$, mais qui n'appartiennent pas à la classe des potentiels "double-exponentially distributed" étudiée par Gärtner et Molchanov dans [*Probab. Theory Related Fields* **111** (1998) 17–55].

Dans [*Commun. Math. Phys.* **267** (2006) 307–353], le comportement asymptotique de l'espérance de la masse totale est décrit par un problème variationnel qui est lié à l'inégalité de Sobolev logarithmique. La solution de ce problème, qui est unique à des translations spatiales près, est une parabole. Dans cet article, nous montrons que la contribution à la masse totale des potentiels qui ont (après un changement d'échelle) une forme différente est négligeable. Nous utilisons la topologie L^1 sur les compacts pour l'exponentielle du potentiel. Au cours de la preuve, nous montrons que n'importe quelle suite des solutions approximatives du problème variationnel converge vers une translation spatiale de la solution qui est une parabole.

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1. Introduction and results

1.1. *The parabolic Anderson model*

We consider the continuous solution $v:[0,\infty)\times\mathbb{Z}^d\to[0,\infty)$ to the Cauchy problem for the heat equation with random coefficients and localised initial datum,

$$\frac{\partial}{\partial t}v(t,z)=\Delta^{\mathrm{d}}v(t,z)+\xi(z)v(t,z),\quad \text{for } (t,z)\in(0,\infty)\times\mathbb{Z}^d, \tag{1.1}$$

$$v(0,z)=\mathbb{1}_0(z),\quad \text{for } z\in\mathbb{Z}^d. \tag{1.2}$$

Here $\xi=(\xi(z):z\in\mathbb{Z}^d)$ is an i.i.d. random potential with values in $[-\infty,\infty)$, and Δ^{d} is the discrete Laplacian,

$$\Delta^{\mathrm{d}}f(z)=\sum_{y\sim z}\bigl[f(y)-f(z)\bigr],\quad \text{for } z\in\mathbb{Z}^d,\ f:\mathbb{Z}^d\to\mathbb{R}.$$

The parabolic problem (1.1) is called the *parabolic Anderson model*. The operator $\Delta^{\mathrm{d}}+\xi$ appearing on the right is called the *Anderson Hamiltonian*; its spectral properties are well-studied in mathematical physics. Equation (1.1) describes a random mass transport through a random field of sinks and sources, corresponding to lattice points z with $\xi(z)<0$, respectively, >0. There is an interpretation in terms of the expected number of particles at time t in the site x for a branching process with random space-dependent branching rates. We refer the reader to [10,14] and [5] for more background and to [8] for a survey on mathematical results.

The long-time behaviour of the parabolic Anderson problem is well-studied in the mathematics and mathematical physics literature because it is an important example of a model exhibiting an *intermittency effect*. This means, loosely speaking, that most of the total mass of the solution,

$$U(t)=\sum_{z\in\mathbb{Z}^d}v(t,z),\quad \text{for } t>0, \tag{1.3}$$

is concentrated on a small number of remote islands, called the *intermittent islands*. A manifestation of intermittency in terms of the moments of $U(t)$ is as follows. For $0<p<q$, the main contribution to the qth moment of $U(t)$ comes from islands that contribute only negligibly to the pth moments. Therefore, intermittency can be defined by the requirement,

$$\limsup_{t\to\infty}\frac{\langle U(t)^p\rangle^{1/p}}{\langle U(t)^q\rangle^{1/q}}=0,\quad \text{for } 0<p<q, \tag{1.4}$$

where $\langle\cdot\rangle$ denotes expectation with respect to ξ. Whenever ξ is truly random, the parabolic Anderson model is intermittent in this sense, see [10], Theorem 3.2.

We work under the assumption that all positive exponential moments of $\xi(0)$ are finite and that the upper tails of $\xi(0)$ possess some mild regularity property. One of the main results of [12] is that four different universality classes of long-time behaviours of the parabolic Anderson model can be distinguished: the so-called double-exponential distribution and some degenerate version of it studied by Gärtner, Molchanov and König [9,11], bounded from above potentials studied by Biskup and König [2], and so-called almost bounded potentials studied by van der Hofstad, König and Mörters [12].

In the present paper, we only consider the class of *almost bounded potentials*, which we will recall in Section 1.3. It is our main purpose to determine those shapes of the random potential ξ that contribute most to the expectation of the total mass, asymptotically as $t\to\infty$. In other words, we will find a shifted, rescaled version, $\overline{\xi}_t$, of ξ and an explicit deterministic function $\widehat{\psi}:\mathbb{R}^d\to\mathbb{R}$ such that the main contribution to $\langle U(t)\rangle$ comes from the event $\{\overline{\xi}_t\approx\widehat{\psi}\}$, in a sense that will be specified below. This is what we call a *potential confinement property*; it is a specification of the intermittency phenomenon for the moments of $U(t)$.

1.2. *Remarks on the literature*

To the best of our knowledge, the only potential confinement property that has been proved for the parabolic Anderson model in the literature is in [9] for the universality class of the double-exponential distribution, including its degenerate version. That paper works in the almost-sure setting and proves that the main contribution to the total mass $U(t)$ comes from islands in which the potential looks like the maximisers of the relevant variational formula, but from no other island else. That formula is the discrete variant of the formula for χ appearing in the present paper in (1.11) below, i.e., for the discrete Laplace operator on $\mathbb{Z}^d$ instead of the continuous one on $\mathbb{R}^d$.

There is a "dual" confinement property in the parabolic Anderson model, the confinement of the path of the random walk in the Feynman–Kac formula, see (2.2) below. This property says that the maximal contribution to the expected total mass $U(t)$ comes from those random walk paths whose shape, after appropriate rescaling, resembles the minimisers of the "dual" version of the characteristic variational problem, but from no other paths else. See Lemma 3.1 for the dual representation of χ in the case handled in the present paper, see (1.11) below. This property is proved in $d=2$ by Bolthausen [3] in an important special case of the universality class of potentials that are bounded from above: they assume only the two values 0 and $-\infty$ in [3]. A similar result, also in $d=2$, was independently derived by Sznitman [16] for the spatially continuous variant for Brownian motion in a Poisson trap field. The characteristic variational problem is in that case

$$\chi = \inf\big\{\|\nabla g\|_2^2 + \rho\big|\mathrm{supp}(g)\big| \colon\ g \in H^1\big(\mathbb{R}^d\big), \|g\|_2 = 1, \mathrm{supp}(g) \text{ compact}\big\};$$

the function g^2 plays the role of the normalised rescaled occupation measures of the walk, respectively of the Brownian motion. The restriction to $d=2$ was removed by Povel [15], after suitable isoperimetric inequalities derived in the analysis literature had become known.

1.3. *Almost bounded potentials*

The class of potentials we will be working with is determined by the following. We need to introduce the logarithmic moment generating function of $\xi(0)$ given by

$$H(t) = \log\big\langle e^{t\xi(0)}\big\rangle, \quad t \in \mathbb{R}. \tag{1.5}$$

Assumption (AB). *There is a parameter* $\rho \in (0,\infty)$ *and a continuous function* $\kappa : (0,\infty) \to (0,\infty)$ *with* $\lim_{t\to\infty} \kappa(t)/t = 0$ *such that, for all* $y \ge 0$,

$$\lim_{t\to\infty} \frac{H(yt) - yH(t)}{\kappa(t)} = \rho \cdot y \log y. \tag{1.6}$$

This is class (iii) of [12], the class of *almost bounded potentials*. The convergence in (1.6) is uniform in $y \in [0, M]$ for any $M > 0$. Both H and κ are regularly varying with index $\gamma = 1$. According to [1], Theorem 3.7.3, (1.6) is satisfied for $\kappa(t) = H(t) - \int_1^t H(s)/s\,\mathrm{d}s$. If ξ satisfies (1.6), then $C\xi$ satisfies (1.6) with ρ replaced by $C\rho$, for any $C > 0$. The class of almost bounded potentials comprises both potentials that are unbounded and bounded to ∞; in the latter case we assume, without loss of generality, that $\operatorname{esssup} \xi(0) = 0$. Examples are found by putting $\log \mathrm{Prob}(\xi(0) > r) = -e^{f(r)}$ for some positive increasing smooth function f and considering the limit as $r \uparrow \infty$ in the unbounded case and $r \uparrow 0$ in the bounded case. Then $H(t) = \sup_{r\in\mathbb{R}}[tr - e^{f(r)}] = tf(t) - e^{f(r(t))}$ for some $r(t) \to \infty$ resp. $r(t) \to 0$ as $t \to \infty$. If one now assumes that the function $f'(r(\cdot))$ is slowly varying at infinity, then the potential turns out to be almost bounded. Specific examples are $f(r) = r^2$ for an unbounded potential and $f(r) = -1/r$ for a bounded one, see [12], Section 1.4.3.

Another important object is the function $\alpha : (0,\infty) \to (0,\infty)$ defined by

$$\kappa\left(\frac{t}{\alpha(t)^d}\right) = \frac{t}{\alpha(t)^{d+2}}, \quad t \gg 1. \tag{1.7}$$

We also will write α_t instead of $\alpha(t)$. Informally, $\alpha(t)$ is the order of the diameter of the intermittent islands for the moments. That is, the expected total mass $\langle U(t)\rangle$ is well-approximated by the sub-sum $\langle \sum_{|x|\le R\alpha_t} v(t,x)\rangle$ in a certain sense, after the limits $t \to \infty$ and afterwards $R \to \infty$ are taken.

Lemma 1.1. *The function* α *is well defined, up to asymptotic equivalence. Furthermore,* $\lim_{t\to\infty}\alpha(t)=\infty$, *and* α *is slowly varying. In particular,* $\lim_{t\to\infty} t\alpha_t^{-d}=\infty$. *Furthermore, for any* $M>0$,

$$H\left(\frac{t}{\alpha_t^d}\cdot y\right)-y\cdot H\left(\frac{t}{\alpha_t^d}\right)=\frac{t}{\alpha_t^{d+2}}\cdot\rho\cdot y\log y\cdot\big(1+\mathrm{o}(1)\big)\quad \textit{uniformly in } y\in[0,M]. \tag{1.8}$$

Proof. All assertions besides the last one follow directly from [12], Proposition 1.2. The last one follows from (1.7) by substituting t with $t\alpha_t^{-d}$. □

1.4. *Asymptotics for the expected total mass*

One of the main results of [12], see Theorem 1.4, is the description of the asymptotic behavior of the expected total mass of the parabolic Anderson model for almost bounded potentials:

Theorem 1.2. *Assume that the potential distribution satisfies Assumption* (AB). *Then there is a number* $\chi\in\mathbb{R}$, *depending only on the dimension d and the parameter* ρ *appearing in Assumption* (AB), *such that*

$$\lim_{t\to\infty}\frac{\alpha(t)^2}{t}\log\big(\langle U(t)\rangle \mathrm{e}^{-H(t\alpha(t)^{-d})\alpha(t)^d}\big)=-\chi. \tag{1.9}$$

The description of χ is highly interesting and shows a rich structure, some of which we want to explore in the present paper. The following objects will play a crucial role in the following. By $\mathcal{C}(\mathbb{R}^d)$ we denote the set of continuous functions $\mathbb{R}^d\to\mathbb{R}$. For $\psi\in\mathcal{C}(\mathbb{R}^d)$ define

$$\mathcal{L}(\psi)=\frac{\rho}{\mathrm{e}}\int_{\mathbb{R}^d}\mathrm{e}^{\psi(x)/\rho}\,\mathrm{d}x\quad\text{and}\quad \lambda(\psi)=\sup_{\substack{g\in H^1(\mathbb{R}^d)\\ \|g\|_2=1}}\big\{\langle\psi,g^2\rangle-\|\nabla g\|_2^2\big\}, \tag{1.10}$$

where $H^1(\mathbb{R}^d)$ is the usual Sobolev space, ∇ the usual (distributional) gradient and $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_2$ are the inner product and the norm on $L^2(\mathbb{R}^d)$. Then $\lambda(\psi)$ is the top of the spectrum of the operator $\Delta+\psi$ in $H^1(\mathbb{R}^d)$. If ψ decays at infinity sufficiently fast towards $-\infty$, then $\mathcal{L}(\psi)$ is finite and $\lambda(\psi)$ is the principal L^2-eigenvalue of $\Delta+\psi$ in $\mathbb{R}^d$. Now we can identify χ explicitly, see [12], Proposition 1.11.

Lemma 1.3. *The limit* χ *in* (1.9) *is identified as*

$$\chi=\inf_{\psi\in\mathcal{C}(\mathbb{R}^d):\mathcal{L}(\psi)<\infty}\big[\mathcal{L}(\psi)-\lambda(\psi)\big]. \tag{1.11}$$

Furthermore, the infimum is uniquely, up to spatial shifts, attained at the parabola

$$\widehat{\psi}(x)=\rho+\rho\frac{d}{2}\log\frac{\rho}{\pi}-\rho^2|x|^2,\quad x\in\mathbb{R}^d.$$

In particular, $\chi=\rho d(1-\frac{1}{2}\log\frac{\rho}{\pi})$.

1.5. *Heuristic explanation*

The content of Theorem 1.2, in combination with Lemma 1.3, can heuristically be explained in terms of a large-deviation statement as follows. Introduce the vertically shifted and rescaled version of the potential ξ,

$$\xi_t(z)=\xi(z)-\frac{\alpha(t)^d}{t}H\left(\frac{t}{\alpha(t)^d}\right),\quad z\in\mathbb{Z}^d, \tag{1.12}$$

$$\overline{\xi}_t(x)=\alpha(t)^2\xi_t\big(\lfloor\alpha(t)x\rfloor\big),\quad x\in\mathbb{R}^d. \tag{1.13}$$

Then $\overline{\xi}_t$ is a random step function $\mathbb{R}^d \to \mathbb{R}$. Using a Fourier expansion with respect to the eigenfunctions of $\Delta^{\mathrm{d}} + \xi$ in large, t-dependent boxes, one can show that the total mass $U(t)$ is asymptotically equal to $\exp\{t\lambda^{\mathrm{d}}_{t\log t}(\xi)\}$, where $\lambda^{\mathrm{d}}_{t\log t}(V)$ denotes the principal eigenvalue of the operator $\Delta^{\mathrm{d}} + V$ in the centred box with radius $t\log t$ with zero boundary condition, for any potential $V : \mathbb{Z}^d \to \mathbb{R}$. Some technical work is done to show that $\lambda^{\mathrm{d}}_{t\log t}(\xi)$ may asymptotically be replaced by the eigenvalue $\lambda^{\mathrm{d}}_{R\alpha_t}(\xi)$ in the much smaller box of radius $R\alpha_t$. More precisely, the replacement error is exponential on the scale t/α_t^2, and its rate vanishes if the limit $R \to \infty$ is eventually taken. Using (1.13) and asymptotic scaling properties of $\lambda^{\mathrm{d}}_{R\alpha_t}(\cdot)$, we see that

$$U(t)\mathrm{e}^{-H(t\alpha(t)^{-d})\alpha(t)^d} \approx \exp\bigl\{t\lambda^{\mathrm{d}}_{R\alpha_t}(\xi_t)\bigr\} \approx \exp\biggl\{\frac{t}{\alpha(t)^2}\lambda_R(\overline{\xi}_t)\biggr\},$$

where $\lambda_R(\psi)$ denotes the principal eigenvalue of $\Delta + \psi$ in the box $Q_R = [-R, R]^d$ with zero boundary condition; note that the term $-H(t\alpha(t)^{-d})\alpha(t)^d$ is absorbed in the vertically shifted potential, ξ_t.

Now we take expectations with respect to the potential and find that the expected total mass is given in terms of an exponential moment of $\lambda_R(\overline{\xi}_t)$ on the scale $t\alpha_t^{-2}$. The following lemma is one key property of the shifted and rescaled potential $\overline{\xi}_t$ and gives to the functional $\mathcal{L}$ defined in (1.10) the meaning of a large-deviation rate function. We introduce the set $\mathcal{F}(Q_R)$ of all measurable functions $\psi : Q_R \to \mathbb{R}$ that are bounded from above.

Lemma 1.4 (LDP for $\overline{\xi}_t$). *Fix $R > 0$. Then the restriction of $(\overline{\xi}_t)_{t>0}$ to Q_R satisfies a large-deviation principle with speed $t\alpha_t^{-2}$ and rate function*

$$\mathcal{L}_R : \mathcal{F}(Q_R) \to \mathbb{R}, \qquad \mathcal{L}_R(\psi) = \frac{\rho}{\mathrm{e}} \int_{Q_R} \mathrm{e}^{\psi(x)/\rho}\,\mathrm{d}x, \tag{1.14}$$

with respect to the topology that is induced by test integrals against all nonnegative continuous functions $Q_R \to [0, \infty)$.

Sketch of proof. We identify the limiting cumulant generating function,

$$\Lambda_R(f) = \lim_{t\to\infty} \frac{\alpha(t)^2}{t} \log\biggl\langle \exp\biggl\{\frac{t}{\alpha(t)^2}\int_{Q_R} \overline{\xi}_t(x) f(x)\,\mathrm{d}x\biggr\}\biggr\rangle,$$

for any continuous nonnegative $f : Q_R \to [0, \infty)$. Indeed, we shall show that $\Lambda_R(f)$ exists and is equal to $\mathcal{H}_R(f) = \rho\int_{Q_R} f(x)\log f(x)\,\mathrm{d}x$. Then the well-known Gärtner–Ellis theorem ([6], Section 4.5.3), yields the result, since $\mathcal{H}_R$ is the Legendre transform of $\mathcal{L}_R$, see also Lemma 3.1 below.

An explicit calculation using (1.12), Assumption (AB) and (1.7) shows that

$$\begin{aligned}\frac{\alpha(t)^2}{t} \log\biggl\langle \exp\biggl\{\frac{t}{\alpha(t)^2}\int_{Q_R} \overline{\xi}_t(x) f(x)\,\mathrm{d}x\biggr\}\biggr\rangle &= \rho\bigl(1 + \mathrm{o}(1)\bigr)\int_{Q_R} \mathrm{d}x\, f(x) \log\biggl(\int_{\lfloor x\alpha_t\rfloor/\alpha(t) + Q_{1/\alpha(t)}} f(y)\,\mathrm{d}y\,\alpha(t)^d\biggr) \\ &= \rho\bigl(1 + \mathrm{o}(1)\bigr)\int_{Q_R} f(x)\log f(x)\,\mathrm{d}x.\end{aligned}$$

Obviously, this implies that $\Lambda_R(f)$ exists and equals $\mathcal{H}_R(f)$. $\square$

We kept this proof short since we are not going to use Lemma 1.4 in our proofs. Loosely speaking, this principle says that

$$\lim_{t\to\infty} \frac{\alpha(t)^2}{t} \log \mathrm{Prob}(\overline{\xi}_t \approx \psi \text{ in } Q_R) = -\mathcal{L}_R(\psi), \tag{1.15}$$

for sufficiently regular functions ψ. Using this principle in combination with Varadhan's lemma ([6], Section 4.3) and making R very large, we arrive at

$$\begin{aligned}\big\langle U(t)\big\rangle \mathrm{e}^{-H(t\alpha(t)^{-d})\alpha(t)^d} &\approx \left\langle \exp\left\{\frac{t}{\alpha(t)^2}\lambda_R(\overline{\xi}_t)\right\}\right\rangle \approx \exp\left\{\frac{t}{\alpha(t)^2}\sup_{\psi\in\mathcal{F}(Q_R)}\big[\lambda_R(\psi)-\mathcal{L}_R(\psi)\big]\right\} \\ &\approx \exp\left\{-\chi\frac{t}{\alpha(t)^2}\right\}.\end{aligned}$$

This ends the heuristic derivation of Theorem 1.2. Hence, we see that there is a competition between two forces for large R: the potential tries to keep the value of the eigenvalue $\lambda_R(\overline{\xi}_t)$ as high as possible, but has to pay an amount of $\mathcal{L}_R(\overline{\xi}_t)$ for doing that. The best contribution comes from potentials $\overline{\xi}_t$ that make an optimal compromise, i.e., optimize the difference of the two contributions. This is precisely what is expressed in (1.11).

1.6. *Our result*: *potential confinement*

The purpose of the present paper is to give rigorous substance to the heuristics of Section 1.5. We prove that there is a one-to-one correspondence between approximate minimisers ψ of the variational formula in (1.11) and the contribution to the expected total mass coming from the event $\{\overline{\xi}_t \approx \psi\}$. More precisely, we prove that the contribution to the expected total mass that comes from potential shapes outside a neighborhood of any shift of the parabola $\widehat{\psi}$ is asymptotically negligible with respect to the full expectation.

Let us first introduce the topology of potentials we are working with. We write $Q_R = [-R, R]^d$ for the centred cube of sidelength $2R$. Introduce the distance

$$\operatorname{dist}(f_1, f_2) = \sum_{r=1}^{\infty} 2^{-r}\phi\left(\int_{Q_r}\big|f_1(x)-f_2(x)\big|\,\mathrm{d}x\right), \qquad f_1, f_2 \in L^1\big(\mathbb{R}^d\big), \tag{1.16}$$

where $\phi(s) = \frac{s}{1+s}$ for $s > 0$. The metric dist induces the topology of L^1-convergence on every compact subset of $\mathbb{R}^d$. For describing general potential realisations, we enlarge the space of continuous functions to a much larger function set, the set $\mathcal{F}$ of all measurable functions $\psi:\mathbb{R}^d \to \mathbb{R}$ that are bounded from above. Now we can formulate our main result, a law of large numbers for $\overline{\xi}_t$ defined in (1.12) and (1.13) towards the set of minimizers of the formula in (1.11).

Theorem 1.5 (Potential confinement). *Suppose that Assumption* (AB) *holds. Then*

$$\lim_{t\to\infty}\frac{\langle U(t)\mathbb{1}_{\widehat{\Gamma}_{t,\varepsilon}}(\overline{\xi}_t)\rangle}{\langle U(t)\rangle} = 0, \tag{1.17}$$

where

$$\widehat{\Gamma}_{t,\varepsilon} = \bigcap_{M\in(0,\infty)}\bigcap_{x\in Q_{t\log t}}\big\{\psi\in\mathcal{F}:\ \operatorname{dist}\big(\mathrm{e}^{(\psi(x+\cdot)\wedge M)/\rho}, \mathrm{e}^{\widehat{\psi}(\cdot)/\rho}\big) > \varepsilon\big\}. \tag{1.18}$$

Theorem 1.5 says that the totality of all potential realisations ξ such that every shift of $\mathrm{e}^{(\overline{\xi}_t\wedge M)/\rho}$ is, for any $M > 0$, away from the Gaussian density $\mathrm{e}^{\widehat{\psi}/\rho}$ by some positive amount gives a negligible contribution to the expected total mass. It is sufficient to consider only shifts by amounts $\le t\log t$ since the mass coming from farther away contributes negligibly at time t anyway. It will turn out in the proof that the quotient on the left-hand side of (1.17) decays exponentially on the scale $t\alpha(t)^{-2}$. The appearance of the parameter M is necessary since distances between $\mathrm{e}^{\overline{\xi}_t\wedge M/\rho}$ and $\mathrm{e}^{\overline{\xi}_t/\rho}$ cannot be controlled on that exponential scale.

A similar confinement property can also be formulated for the moments of $U(t)$ instead of $U(t)$ itself; the proof will be not much different, but some additional technical steps will have to be added.

With much more work, it should be possible to derive a large-deviations principle for $\overline{\xi}_t$ under the measure with density $U(t)/\langle U(t)\rangle$ with an explicit rate function; then Theorem 1.5 would be a corollary of this principle.

An interesting open task is to formulate and prove a "dual" confinement property or even a large-deviations principle, for the appropriately rescaled random walk local times in the Feynman–Kac representation in (2.2) below. The characteristic variational problem is introduced in Lemma 3.1 below, and its minimiser, $\widehat{g}^2$, will play the same role for the local times as the minimiser $\widehat{\psi}$ in (1.11) for the potential. The main point will be to determine an appropriate topology for both the probabilistic and the functional analytic arguments. Presumably, our Lemma 3.2 will be crucial in that proof.

1.7. *Comments on the proof*

The proof of Theorem 1.5 has a functional analytic side and a probabilistic side. On one hand, we show that any sequence of functions that asymptotically minimise $\mathcal{L} - \lambda$ in (1.11) converge, after an appropriate spatial translation, to the minimiser $\widehat{\psi}$ in the topology used in Theorem 1.5, and on the other hand we derive effective estimates for the expectation of the total mass on the event that $\overline{\xi}_t$ is bounded away from $\widehat{\psi}$ in the same sense. The main point is that these two properties have to be proved in the same topology, which is a nontrivial issue. Note that the topology we work with is much stronger than the one in which we have a large-deviation principle, see Lemma 1.4. In the literature, other topologies are considered in which the variational formula in (1.11) has a related approximation property (see the remarks at the beginning of Section 3); however these topologies turned out to be not suitable for our probabilistic approach.

The analysis part of the proof will be handled in Section 3 by more or less standard methods from analysis. The probabilistic part is treated in Section 2. The large-deviations principle of Lemma 1.4 can serve as a guidance only since the topology used in that principle is too weak. Our proof indeed follows another route, which we informally describe now.

Similarly to the heuristics of Section 1.5, we have

$$\mathrm{e}^{-H(t\alpha(t)^{-d})\alpha(t)^d}\big\langle U(t)\mathbb{1}_{\widehat{\Gamma}_{t,\varepsilon}}(\overline{\xi}_t)\big\rangle \approx \left\langle \exp\left\{\frac{t}{\alpha(t)^2}\lambda_R(\overline{\xi}_t)\right\}\mathbb{1}_{\Gamma_{R,\varepsilon}}(\overline{\xi}_t)\right\rangle,$$

where $\Gamma_{R,\varepsilon}$ is some finite-box approximation of $\widehat{\Gamma}_{t,\varepsilon}$. Now we add und subtract the term $t\alpha(t)^{-2}\rho\log(\frac{\mathrm{e}}{\rho}\mathcal{L}_R(\overline{\xi}_t))$ in the exponent. The difference term is estimated against the variational formula

$$-\chi_R(\varepsilon) = \sup_{\psi\in\Gamma_{R,\varepsilon}}\left(\lambda_R(\psi) - \rho\log\left(\frac{\mathrm{e}}{\rho}\mathcal{L}_R(\psi)\right)\right),$$

such that we have

$$\mathrm{e}^{-H(t\alpha(t)^{-d})\alpha(t)^d}\big\langle U(t)\mathbb{1}_{\widehat{\Gamma}_{t,\varepsilon}}(\overline{\xi}_t)\big\rangle \le \mathrm{e}^{-t\alpha(t)^{-2}\chi_R(\varepsilon)}\left\langle \exp\left\{\frac{t}{\alpha(t)^2}\rho\log\left(\frac{\mathrm{e}}{\rho}\mathcal{L}_R(\overline{\xi}_t)\right)\right\}\right\rangle.$$

With the help of the principle in Lemma 1.4 and Varadhan's lemma one can convince oneself that the exponential rate (on the scale $t\alpha(t)^{-2}$) of the last expectation should be equal to

$$\sup_{\psi}\left(\rho\log\left(\frac{\mathrm{e}}{\rho}\mathcal{L}_R(\psi)\right) - \mathcal{L}_R(\psi)\right) = \sup_{l\in(0,\infty)}\left(\rho\log\frac{\mathrm{e}l}{\rho} - l\right).$$

(However, the proof of that fact cannot be done with the help of Lemma 1.4, since the functional $\psi \mapsto \rho\log(\frac{\rho}{\mathrm{e}}\mathcal{L}_R(\psi))$ is not bounded and continuous in the topology used in that lemma.) The right-hand side is easily seen to be zero with unique minimiser $\mathcal{L}_R(\psi) = l = \rho$. Hence, the only task that is left to do is to prove that $\liminf_{R\to\infty}\chi_R(\varepsilon) > \chi$. This is indeed true; it relies on the representation

$$-\chi = \sup_{\psi}\left(\lambda_R(\psi) - \rho\log\left(\frac{\mathrm{e}}{\rho}\mathcal{L}_R(\psi)\right)\right);$$

see [12]. From this estimate, Theorem 1.5 follows since the denominator of (1.17) has the strictly larger exponential rate $-\chi$ by Theorem 1.2.

2. Proof of Theorem 1.5

In this section, we prove Theorem 1.5. Recall that we suppose that Assumption (AB) holds, and recall the parameter $\rho \in (0,\infty)$ from that assumption. Comparing to Theorem 1.2, it is easy to see that the following proposition immediately implies Theorem 1.5.

Proposition 2.1. *For any $\varepsilon > 0$,*

$$\limsup_{t\to\infty} \frac{\alpha_t^2}{t} \log\bigl(\mathrm{e}^{-\alpha_t^d H(t/\alpha_t^d)}\bigl\langle U(t)\mathbb{1}_{\widehat{\Gamma}_{t,\varepsilon}}(\overline{\xi}_t)\bigr\rangle\bigr) < -\chi. \tag{2.1}$$

Indeed, Theorem 1.2 says that the denominator of (1.17), after inserting the factor $\mathrm{e}^{-\alpha_t^d H(t/\alpha_t^d)}$ both in numerator and denominator, has the exponential rate $-\chi$, while the rate of the numerator is strictly smaller, according to Proposition 2.1 (both on the scale $t\alpha_t^{-2}$). Hence, Proposition 2.1 implies that the quotient in (1.17) even decays exponentially on the scale $t\alpha_t^{-2}$.

One of the most important tools in the study of the parabolic Anderson model is the *Feynman–Kac formula*, which represents the solution of (1.1) and its total mass in terms of an exponential expectation of a functional of simple random walk $(X(s)\colon s\in[0,t])$ on $\mathbb{Z}^d$ with generator Δ^{d}. We denote by $\mathbb{P}_z$ and $\mathbb{E}_z$ probability and expectation with respect to the random walk, when started at $z\in\mathbb{Z}^d$. The walker's *local times* are denoted by $\ell_t(z) = \int_0^t \delta_z(X(s))\,\mathrm{d}s$, the amount of time the walker spends at $z\in\mathbb{Z}^d$ by time $t>0$. Note that $\int_0^t V(X(s))\,\mathrm{d}s = \langle V,\ell_t\rangle$ for functions $V:\mathbb{Z}^d\to\mathbb{R}$, where $\langle f,g\rangle = \sum_{z\in\mathbb{Z}^d} f(z)g(z)$ for any f, g. Then, also using (1.12), the Feynman–Kac formula may be formulated by saying

$$\mathrm{e}^{-\alpha_t^d H(t/\alpha_t^d)} U(t) = \mathbb{E}_0\biggl[\exp\biggl\{\int_0^t \xi_t\bigl(X(s)\bigr)\,\mathrm{d}s\biggr\}\biggr] = \mathbb{E}_0\bigl[\mathrm{e}^{\langle \ell_t,\xi_t\rangle}\bigr]. \tag{2.2}$$

We divide the proof of Proposition 2.1 into a sequence of steps. In Section 2.1 we show how we reduce the infinite state space $\mathbb{Z}^d$ to some finite large box. In Section 2.2 we replace the shifted and rescaled potential, $\overline{\xi}_t$, by a truncated version $\overline{\xi}_t\wedge M$ and show that the replacement error vanishes as $M\to\infty$. This technical step turns out to be crucial in Section 2.3 since our proof of Lemma 2.6 would fail for $\overline{\xi}_t$ in place of $\overline{\xi}_t\wedge M$. After the two preparatory steps in Sections 2.1 and 2.2, the main strategy of the proof of Proposition 2.1 is carried out in Section 2.3.

2.1. *Reduction to a large box*

Our first main step is to estimate the expectation on the left-hand side of (2.1) in terms of a finite-box version. In other words, we argue that we may replace the full state space, $\mathbb{Z}^d$, by a box with a radius of order α_t. We will also insert an appropriate scaling, which will turn the discrete box of order α_t into continuous cubes of finite-order radius. By $B_R = [-R,R]^d\cap\mathbb{Z}^d$ and $Q_R = [-R,R]^d$ we denote the discrete box and the continuous cube of radius $\lfloor R\rfloor$, resp. R. A finite-cube version of the distance dist defined in (1.16), appropriate for our purposes, is

$$\mathrm{d}_R(\psi_1,\psi_2) = \int_{Q_R} \bigl|\mathrm{e}^{\psi_1(x)/\rho} - \mathrm{e}^{\psi_2(x)/\rho}\bigr|\,\mathrm{d}x, \qquad \psi_1,\psi_2\in\mathcal{F}, \tag{2.3}$$

where we recall that $\mathcal{F}$ denotes the set of all measurable functions $\mathbb{R}^d\to\mathbb{R}$ that are bounded from above.

Lemma 2.2 (Reduction to a large box). *Fix $\varepsilon>0$. Then there is $C>0$ such that for all $R\ge 2-\log\frac{\varepsilon}{2}$ and $t\gg 1$,*

$$\begin{aligned}&\mathrm{e}^{-\alpha_t^d H(t/\alpha_t^d)}\bigl\langle U(t)\mathbb{1}_{\widehat{\Gamma}_{t,\varepsilon}}(\overline{\xi}_t)\bigr\rangle\\ &\quad\le \mathrm{e}^{H(2t)/2-\alpha_t^d H(t/\alpha_t^d)}\mathrm{e}^{-(t\log t)/2} + \mathrm{e}^{Ct/(R^2\alpha_t^2)}\bigl\langle \mathbb{E}^{t,R}\bigl[\mathrm{e}^{\langle \ell_t,\xi_t\rangle}\bigr]\mathbb{1}_{\{\forall M>0:\,\overline{\xi}_t\wedge M\in\Gamma_{R,\varepsilon/2}\}}\bigr\rangle,\end{aligned} \tag{2.4}$$

where we abbreviate $\mathbb{E}^{t,R}[\cdots] = \mathbb{E}_0[\cdots \cdot \mathbb{1}_{\{\mathrm{supp}\,\ell_t \subset B_{3R\alpha_t}\}}]$, *and we put*

$$\Gamma_{R,\varepsilon} = \bigcap_{x\in Q_{2R}} \{\psi \in \mathcal{F} : \mathrm{d}_R(\psi(x+\cdot), \widehat{\psi}(\cdot)) > \varepsilon\}, \quad \varepsilon > 0. \tag{2.5}$$

Lemma 2.2 reduces the $\overline{\xi}_t$-expectation to an expectation of the restriction to the cube Q_{3R}; the constraint that any shift is away from $\widehat{\psi}$ on any compact subset of $\mathbb{R}^d$ is replaced by the requirement that the shift by any amount $\le 2R$ is away from the Q_R-restriction of $\widehat{\psi}$ in $L^1(Q_R)$-sense.

Proof of Lemma 2.2. This is a refinement of the proofs of [2], Proposition 4.4 and [12], Lemma 3.2. Indeed, we use the Feynman–Kac formula for $U(t)$ and distinguish the contributions from those paths that leave, or do not leave, respectively, the box $B_{t\log t}$ up to time t. The first contribution can be estimated against the first term on the right of (2.4), as is seen in [12], Lemma 3.2, together with the subsequent text (see the display above (3.18) there). In order to see that the second contribution can be estimated against the second term on the right-hand side of (2.4), we have to repeat parts of the proof of [2], Proposition. 4.4; we shall replace the R there by $2R\alpha(t)$.

As a first step, we estimate, with the help of a Fourier expansion, against the principal eigenvalue. For $V : \mathbb{Z}^d \to \mathbb{R}$, let $\lambda^{\mathrm{d}}_{t\log t}(V)$ be the principal eigenvalue of $\Delta^{\mathrm{d}} + V$ in the box $B_{t\log t}$ with zero boundary condition. Then a Fourier expansion shows in a standard way that

$$\mathbb{E}_0\bigl[\mathrm{e}^{\langle \ell_t, V\rangle} \mathbb{1}_{\{\mathrm{supp}(\ell_t)\subset B_{t\log t}\}}\bigr] \le \mathrm{e}^{\mathrm{o}(t/\alpha_t^2)} \mathrm{e}^{t\lambda^{\mathrm{d}}_{t\log t}(V)},$$

where $\mathrm{o}(t\alpha_t^{-2})$ does not depend on the potential V. Now [2], Proposition 4.4, says that the eigenvalue in the box $B_{t\log t}$ may be estimated from above against a small error plus the maximal eigenvalue in certain, mutually overlapping boxes:

$$\lambda^{\mathrm{d}}_{t\log t}(V) \le \max_{k\in B_{t\log t}} \lambda^{\mathrm{d}}_{4kR\alpha(t)+B_{3R\alpha(t)}}(V) + \frac{C}{R^2\alpha(t)^2},$$

where $C > 0$ does not depend on V nor on R nor on t. (Here $\lambda^{\mathrm{d}}_B(V)$ denotes the eigenvalue of $\Delta^{\mathrm{d}} + V$ in a bounded set $B \subset \mathbb{Z}^d$ with zero boundary condition.) Hence, we may estimate

$$\bigl\langle \mathrm{e}^{t\lambda^{\mathrm{d}}_{t\log t}(\xi)} \mathbb{1}_{\widehat{\Gamma}_{t,\varepsilon}}(\overline{\xi}_t)\bigr\rangle \le \mathrm{e}^{Ct/(R^2\alpha_t^2)} \sum_{k\in B_{t\log t}} \bigl\langle \mathrm{e}^{t\lambda^{\mathrm{d}}_{4kR\alpha(t)+B_{3R\alpha(t)}}(\xi)} \mathbb{1}_{\widehat{\Gamma}_{t,\varepsilon}}(\overline{\xi}_t)\bigr\rangle. \tag{2.6}$$

Now we estimate $\mathbb{1}_{\widehat{\Gamma}_{t,\varepsilon}}(\overline{\xi}_t)$. Observe that

$$\widehat{\Gamma}_{t,\varepsilon} \subset \bigcap_{M\in(0,\infty)} \bigcap_{k\in B_{t\log t}} \bigcap_{x\in Q_{2R}} \Bigl\{\psi \in \mathcal{F} :\ \mathrm{d}_R\bigl(\psi(4kR+x+\cdot)\wedge M, \widehat{\psi}(\cdot)\bigr) > \frac{\varepsilon}{2}\Bigr\}. \tag{2.7}$$

In order to show this, we show that the complement of the right side is contained in the complement of the left side. Pick $k \in B_{t\log t}$ and $x \in Q_{2R}$ and $\psi \in \mathcal{F}$ such that

$$\frac{\varepsilon}{2} \ge \mathrm{d}_R\bigl(\psi(4kR+x+\cdot)\wedge M, \widehat{\psi}(\cdot)\bigr) = \int_{Q_R} \bigl|\mathrm{e}^{(\psi(4kR+x+y)\wedge M)/\rho} - \mathrm{e}^{\widehat{\psi}(y)/\rho}\bigr|\,\mathrm{d}y.$$

Then, for $\widetilde{x} = 4kR + x$, we have (recalling that $\phi(s) = \frac{s}{1+s}$ is increasing in s), for any $M > 0$,

$$\begin{aligned}\mathrm{dist}\bigl(\mathrm{e}^{(\psi(\widetilde{x}+\cdot)\wedge M)/\rho}, \mathrm{e}^{\widehat{\psi}(\cdot)/\rho}\bigr) &\le \sum_{r=1}^{R} 2^{-r}\phi\biggl(\int_{Q_r} \bigl|\mathrm{e}^{(\psi(\widetilde{x}+y)\wedge M)/\rho} - \mathrm{e}^{\widehat{\psi}(y)/\rho}\bigr|\,\mathrm{d}y\biggr) + \sum_{r>R} 2^{-r} \\ &\le \phi\biggl(\int_{Q_R} \bigl|\mathrm{e}^{(\psi(\widetilde{x}+y)\wedge M)/\rho} - \mathrm{e}^{\widehat{\psi}(y)/\rho}\bigr|\,\mathrm{d}y\biggr) + 2^{-R} \le \frac{\varepsilon}{2} + 2^{-R} < \varepsilon,\end{aligned}$$

by our assumption that $R > 2 - \log \frac{\varepsilon}{2}$. Hence, ψ lies in $\widehat{\Gamma}^{\mathrm{c}}_{t,\varepsilon}$, which shows that (2.7) holds.

Now we use (2.7) on the right-hand side of (2.6) and obtain that

$$\begin{aligned}
&\bigl\langle \mathrm{e}^{t\lambda^{\mathrm{d}}_{t\log t}(\xi)} \mathbb{1}_{\widehat{\Gamma}_{t,\varepsilon}}(\overline{\xi}_t)\bigr\rangle \\
&\quad\le \mathrm{e}^{Ct/(R^2\alpha_t^2)} \sum_{k\in B_{t\log t}} \Bigl\langle \mathrm{e}^{t\lambda^{\mathrm{d}}_{4kR\alpha(t)+B_{3R\alpha(t)}}(\xi)} \prod_{\widetilde{k}\in B_{t\log t}} \mathbb{1}_{\{\forall M>0\ \forall x\in Q_{2R}: \mathrm{d}_R(\overline{\xi}_t(4\widetilde{k}R+x+\cdot)\wedge M,\widehat{\psi}(\cdot))>\varepsilon/2\}}\Bigr\rangle \\
&\quad\le \mathrm{e}^{Ct/(R^2\alpha_t^2)} 3^d (t\log t)^d \bigl\langle \mathrm{e}^{t\lambda^{\mathrm{d}}_{3R\alpha(t)}(\xi)} \mathbb{1}_{\{\forall M>0:\, \overline{\xi}_t\wedge M\in\Gamma_{R,\varepsilon/2}\}}\bigr\rangle, \qquad (2.8)
\end{aligned}$$

where we have estimated the product of indicators against the kth factor, and we have used the shift-invariance of the potential. Now enlarge C in order to absorb the term $3^d(t\log t)^d$. □

2.2. *Truncating the potential*

In the next lemma, we replace the random potential by a truncated version. In the proof of Lemma 2.6 below it will turn out to be crucial that the random potential under interest is bounded from above, hence Lemma 2.3 is a necessary preparation for that.

Lemma 2.3 (Truncating the potential). *Fix $R > 0$ and $\varepsilon > 0$. Then*

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{\alpha_t^2}{t} \log\bigl\langle \mathbb{E}^{t,R}\bigl[\mathrm{e}^{\langle \ell_t,\xi_t\rangle}\bigr] \mathbb{1}_{\{\forall M>0: \overline{\xi}_t\wedge M\in\Gamma_{R,\varepsilon}\}}\bigr\rangle \\
&\quad\le \limsup_{M\to\infty}\limsup_{t\to\infty} \frac{\alpha_t^2}{t} \log\bigl\langle \mathbb{E}^{t,R}\bigl[\mathrm{e}^{\langle \ell_t,\xi_t\wedge(M/\alpha_t^2)\rangle}\bigr] \mathbb{1}_{\Gamma_{R,\varepsilon}}(\overline{\xi}_t\wedge M)\bigr\rangle. \qquad (2.9)
\end{aligned}$$

Proof. It is clear that we may estimate, for any $M > 0$,

$$\mathbb{1}_{\{\forall \widetilde{M}>0: \overline{\xi}_t\wedge\widetilde{M}\in\Gamma_{R,\varepsilon}\}} \le \mathbb{1}_{\Gamma_{R,\varepsilon}}(\overline{\xi}_t\wedge M).$$

Fix some small parameter $\eta > 0$. In the expectation on the left-hand side of (2.9) we insert the sum of the indicators on the event $\{\langle \ell_t, \xi_t - \xi_t\wedge(M/\alpha_t^2)\rangle \le \eta t/\alpha_t^2\}$ and on the opposite event. On the first event, we estimate $\langle \ell_t,\xi_t\rangle \le \eta t/\alpha_t^2 + \langle \ell_t, \xi_t\wedge(M/\alpha_t^2)\rangle$ in the exponent. The second indicator is estimated as follows:

$$\mathbb{1}_{\{\langle \ell_t,\xi_t-\xi_t\wedge(M/\alpha_t^2)\rangle>\eta t/\alpha_t^2\}} \le \mathrm{e}^{-K\eta t/\alpha_t^2} \mathrm{e}^{K\langle \ell_t,\xi_t-\xi_t\wedge(M/\alpha_t^2)\rangle},$$

where $K\in(0,\infty)$ is some large auxiliary parameter. This gives that

$$\begin{aligned}
&\bigl\langle \mathbb{E}^{t,R}\bigl[\mathrm{e}^{\langle \ell_t,\xi_t\rangle}\bigr] \mathbb{1}_{\{\forall \widetilde{M}>0: \overline{\xi}_t\wedge\widetilde{M}\in\Gamma_{R,\varepsilon}\}}\bigr\rangle \\
&\quad\le \mathrm{e}^{\eta t/\alpha_t^2} \bigl\langle \mathbb{E}^{t,R}\bigl[\mathrm{e}^{\langle \ell_t,\xi_t\wedge(M/\alpha_t^2)\rangle}\bigr] \mathbb{1}_{\Gamma_{R,\varepsilon}}(\overline{\xi}_t\wedge M)\bigr\rangle + \mathrm{e}^{-K\eta t/\alpha_t^2} \bigl\langle \mathbb{E}^{t,R}\bigl[\mathrm{e}^{\langle \ell_t,\xi_t\rangle} \mathrm{e}^{K\langle \ell_t,\xi_t-\xi_t\wedge(M/\alpha_t^2)\rangle}\bigr]\bigr\rangle. \qquad (2.10)
\end{aligned}$$

The last expectation is estimated with the help of the Cauchy–Schwarz inequality:

$$\bigl\langle \mathbb{E}^{t,R}\bigl[\mathrm{e}^{\langle \ell_t,\xi_t\rangle} \mathrm{e}^{K\langle \ell_t,\xi_t-\xi_t\wedge(M/\alpha_t^2)\rangle}\bigr]\bigr\rangle \le \bigl\langle \mathbb{E}^{t,R}\bigl[\mathrm{e}^{2\langle \ell_t,\xi_t\rangle}\bigr]\bigr\rangle^{1/2} \bigl\langle \mathbb{E}^{t,R}\bigl[\mathrm{e}^{2K\langle \ell_t,\xi_t-\xi_t\wedge(M/\alpha_t^2)\rangle}\bigr]\bigr\rangle^{1/2}. \qquad (2.11)$$

We are going to show that, for any $K\in(0,\infty)$,

$$\limsup_{M\to\infty}\limsup_{t\to\infty} \frac{\alpha_t^2}{t} \log\bigl\langle \mathbb{E}^{t,R}\bigl[\mathrm{e}^{K\langle \ell_t,\xi_t-(\xi_t\wedge(M/\alpha_t^2))\rangle}\bigr]\bigr\rangle \le 0, \qquad (2.12)$$

i.e., the exponential rate (on the scale t/α_t^2) of the second term on the right-hand side of (2.11) vanishes as $M\to\infty$. In the course of the proof, it will become obvious that the first term on the right-hand side of (2.11) has a bounded exponential rate. Hence, the assertion of the lemma follows from considering the large-t rate in (2.10) and making $M\to\infty$, $K\to\infty$ and $\eta\downarrow 0$.

We now prove (2.12). We first sum on all subsets S of $B = B_{3R\alpha_t}$ in which the potential ξ_t is larger than M/α_t^2 and distinguish large and small such sets. This distinction is made with the help of a small auxiliary parameter $\tau \in (0,\infty)$:

$$\begin{aligned}\big\langle \mathbb{E}^{t,R}\big[\mathrm{e}^{K\langle \ell_t, \xi_t-(\xi_t\wedge M/\alpha_t^2)\rangle}\big]\big\rangle &\le \Big\langle \mathbb{E}^{t,R}\Big[\exp\Big\{K\sum_{z\in B}\ell_t(z)\cdot\xi_t(z)\cdot\mathbb{1}_{\{\xi_t(z)>M/\alpha_t^2\}}\Big\}\Big]\Big\rangle \\ &\le \sum_{S\subset B\,:\,|S|\ge\tau\alpha_t^d}\big\langle \mathbb{E}^{t,R}\big[\mathrm{e}^{K\sum_{z\in S}\ell_t(z)\xi_t(z)}\big]\mathbb{1}_{\{S=\{z\in B:\xi_t(z)>M/\alpha_t^2\}\}}\big\rangle \\ &\quad+\sum_{S\subset B\,:\,|S|<\tau\alpha_t^d}\big\langle \mathbb{E}^{t,R}\big[\mathrm{e}^{K\sum_{z\in S}\ell_t(z)\cdot\xi_t(z)}\big]\big\rangle. \end{aligned}\tag{2.13}$$

In the following, we show that the exponential rate of the first term tends to $-\infty$ as $M\to\infty$ for any $\tau>0$, and the rate of the second vanishes as $\tau\downarrow 0$.

We first consider a summand of the first sum, where $|S|\ge\tau\alpha_t^d$. By using the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}&\big\langle \mathbb{E}^{t,R}\big[\mathrm{e}^{K\sum_{z\in S}\ell_t(z)\cdot\xi_t(z)}\big]\mathbb{1}_{\{S=\{z\in B\,:\,\xi_t(z)>M/\alpha_t^2\}\}}\big\rangle \\ &\quad\le \big\langle \mathbb{E}^{t,R}\big[\mathrm{e}^{2K\sum_{z\in S}\ell_t(z)\cdot\xi_t(z)}\big]\big\rangle^{1/2}\cdot \mathrm{Prob}\big(S=\big\{z\in B\colon\ \xi_t(z)>M/\alpha_t^2\big\}\big)^{1/2}.\end{aligned}\tag{2.14}$$

We first estimate the probability. We use the definition of $\xi_t(z)$ in (1.12), the independence of the $\xi(z)$ for different z, the Markov inequality and the definition of $H(\cdot)$ in (1.5), to obtain

$$\begin{aligned}\mathrm{Prob}\big(S=\big\{z\colon\ \xi_t(z)>M/\alpha_t^2\big\}\big)^{1/2} &\le \mathrm{Prob}\bigg(\xi(0)>\frac{H(t/\alpha_t^d)}{t/\alpha_t^d}+\frac{M}{\alpha_t^2}\bigg)^{|S|/2} \\ &= \mathrm{Prob}\big(\mathrm{e}^{\xi(0)t/\alpha_t^d}>\mathrm{e}^{H(t/\alpha_t^d)}\mathrm{e}^{Mt/\alpha_t^{d+2}}\big)^{|S|/2} \\ &\le \big(\mathrm{e}^{-Mt/\alpha_t^{d+2}}\mathrm{e}^{-H(t/\alpha_t^d)}\big\langle \mathrm{e}^{\xi(0)t/\alpha_t^d}\big\rangle\big)^{|S|/2}\le \mathrm{e}^{-t\alpha_t^{-2}M\tau/2}.\end{aligned}\tag{2.15}$$

The exponential rate of this tends to $-\infty$ as $M\to\infty$. Hence, it suffices to show that the exponential rate of the expectation on the right-hand side of (2.14) is finite on the scale $t\alpha_t^{-2}$. This is surprisingly difficult and cannot be handled with rough arguments. We do this by extending some results of [12], Section 3.2, the only additional issue being that the set B is replaced by some subset S of B. This is some technical issue since ξ_t can assume also negative values, such that we have to repeat some of the steps from [12]. We use the definition of ξ_t in (1.12), apply Fubini's theorem, execute the expectation with respect to ξ, recall the definition of $H(\cdot)$ in (1.5) and use the abbreviation $h_t(z)=H(2K\ell_t(z))-2K\ell_t(z)\frac{H(t/\alpha_t^d)}{t/\alpha_t^d}$. This gives

$$\big\langle \mathbb{E}^{t,R}\big[\mathrm{e}^{2K\sum_{z\in S}\ell_t(z)\cdot\xi_t(z)}\big]\big\rangle=\mathbb{E}^{t,R}\big[\mathrm{e}^{\sum_{z\in S}h_t(z)}\big].\tag{2.16}$$

We split the sum on $z\in S$ into the subsums where $\ell_t(z)\le t/\alpha_t^d$ and the remainder. For $\ell_t(z)\le t/\alpha_t^d$ we may apply the asymptotics for H from Lemma 1.1. This gives, as $t\to\infty$, also using the relation between α_t and $\kappa(t)$ in (1.7),

$$\begin{aligned}\sum_{z\in S:\ell_t(z)\le t/\alpha_t^d}h_t(z) &\le \big(\rho+\mathrm{o}(1)\big)\kappa\big(t/\alpha_t^d\big)\sum_{z\in S:\ell_t(z)\le t/\alpha_t^d}2K\ell_t(z)\frac{\alpha_t^d}{t}\log\bigg(2K\ell_t(z)\frac{\alpha_t^d}{t}\bigg) \\ &\le 2\rho 2K\log(2K)\frac{t}{\alpha_t^{d+2}}|S|\le C\frac{|S|}{|B|}\frac{t}{\alpha_t^2},\end{aligned}\tag{2.17}$$

where $C\in(0,\infty)$ depends on ρ and K (and R) only.

Now we handle the subsum on $z\in S$ satisfying $\ell_t(z)>t/\alpha_t^d$. For this purpose, we need the following estimate for differences of H-terms, which follows from [1], Theorem 3.8.6(a). For any $\delta\in(0,\frac12]$, there are $A,t_0\in(1,\infty)$ such that

$$\frac{H(ty) - yH(t)}{\kappa(t)} \le Ay^{1+\delta}, \quad y \in [1, \infty), t \in [t_0, \infty). \tag{2.18}$$

We pick a small $\delta > 0$ and apply (2.18) with δ replaced by $\delta^2/3$, for $y = 2K\ell_t(z)\alpha_t^d/t$ and with t/α_t^d instead of t, to get, for z satisfying $\ell_t(z) > t/\alpha_t^d$,

$$\begin{aligned} h_t(z) &= H\big(2K\ell_t(z)\big) - 2K\ell_t(z)\frac{\alpha_t^d}{t}H\big(t/\alpha_t^d\big) \le A\kappa\big(t/\alpha_t^d\big)\bigg(2K\ell_t(z)\frac{\alpha_t^d}{t}\bigg)^{1+\delta^2/3} \\ &= \widetilde{C}\frac{t}{\alpha_t^{d+2}}\bigg(\frac{\alpha_t^d}{t}\ell_t(z)\bigg)^{1+\delta^2/3}, \end{aligned} \tag{2.19}$$

where we have used the definition of α_t in (1.7), and $\widetilde{C} \in (0, \infty)$ depends on A and K only. Using this and (2.17) in (2.16), we can estimate

$$\mathbb{E}^{t,R}\big[\mathrm{e}^{\sum_{z\in S} h_t(z)}\big] \le \mathrm{e}^{(C|S|t)/(|B|\alpha_t^2)}\mathbb{E}^{t,R}\bigg[\exp\bigg\{\widetilde{C}\frac{t}{\alpha_t^2}\alpha_t^{d\delta^2/3}\sum_{z\in S:\ell_t(z)>t/\alpha_t^d}\bigg(\frac{1}{t}\ell_t(z)\bigg)^{1+\delta^2/3}\bigg\}\bigg]. \tag{2.20}$$

Now in the same way as in [12], Section 3.2, we see that, for any probability measure μ on B, for any $0 < a \le b < 1$ and $0 < c$,

$$\sum_{z\in B:\mu(z)>\alpha_t^{-d}} \mu(z)^{1+a} \le \alpha_t^{d[(1-b)(b-a)+c(1+a-b)]}\bigg(\sum_{z\in B}\mu(z)^{1+b+c}\bigg)^{1+a-b}. \tag{2.21}$$

This is proved as follows, using Jensen's inequality (we write $\sum_z$ instead of $\sum_{z\in B:\mu(z)>\alpha_t^{-d}}$):

$$\begin{aligned} \sum_z \mu(z)^{1+a} &= \bigg(\sum_z \mu(z)^b\bigg)\bigg(\sum_z \frac{\mu(z)^b}{\sum_z \mu(z)^b}\mu(z)^{1+a-b}\bigg) \le \bigg(\sum_z \mu(z)^b\bigg)\bigg(\sum_z \frac{\mu(z)^{1+b}}{\sum_z \mu(z)^b}\bigg)^{1+a-b} \\ &\le \bigg(\sum_z \mu(z)^b\big(\mu(z)\alpha_t^d\big)^{1-b}\bigg)^{b-a}\bigg(\sum_z \mu(z)^{1+b}\big(\mu(z)\alpha_t^d\big)^c\bigg)^{1+a-b} \\ &\le \alpha_t^{d[(1-b)(b-a)+c(1+a-b)]}\bigg(\sum_{z\in B}\mu(z)^{1+b+c}\bigg)^{1+a-b}. \end{aligned}$$

Applying (2.21) for $\mu = \frac{1}{t}\ell_t$, $a = \delta^2/3$, $b = \delta^2/3 + \frac{\delta}{1+\delta}$ and $c = \delta - \delta^2/3 - \frac{\delta}{1+\delta}$, we obtain

$$\frac{t}{\alpha_t^2}\alpha_t^{d\delta^2/3}\sum_{z\in B:\ell_t(z)>t/\alpha_t^d}\bigg(\frac{1}{t}\ell_t(z)\bigg)^{1+\delta^2/3} \le \alpha_t^{-(d+(2-d)(1+\delta))/(1+\delta)}\|\ell_t\|_{1+\delta}, \tag{2.22}$$

where $\|\cdot\|_{1+\delta}$ denotes the $(1+\delta)$-norm on $\ell^{1+\delta}(\mathbb{Z}^d)$. Now, picking $\delta > 0$ so small that $\delta(d-2) < 2$ ([12], Proposition 2.1), states that the large-t exponential rate of the right-hand side of (2.20) on the scale t/α_t^2 vanishes as $\widetilde{C} \downarrow 0$. However, the proof shows that this rate is finite for any $\widetilde{C} \in (0, \infty)$. Using this fact in (2.16), and substituting this in (2.14) we see because of (2.15) that the exponential rate of the first sum on the right-hand side of (2.13) on the scale t/α_t^2 tends to $-\infty$ as $M \to \infty$, for any $\tau > 0$.

Now we address the second sum on the right-hand side of (2.13). We show that its large-t exponential rate on the scale t/α_t^2 vanishes as $\tau \downarrow 0$. We consider $S \subset B = B_{3R\alpha_t^d}$ with $|S| < \tau\alpha_t^d$. We start from (2.20), which is valid for any $S \subset B$. We use Hölder's inequality with new parameters $\frac{1}{p} + \frac{1}{q} = 1$ for the last sum to obtain

$$\sum_{z \in S: \ell_t(z) > t/\alpha_t^d} \left(\frac{1}{t}\ell_t(z)\right)^{1+\delta^2/3} \le |S|^{1/q} \Big(\sum_{z \in B: \ell_t(z) > t/\alpha_t^d} \left(\frac{1}{t}\ell_t(z)\right)^{p(1+\delta^2/3)} \Big)^{1/p}. \tag{2.23}$$

Now we apply (2.21) for $\mu = \frac{1}{t}\ell_t$, $a = p + p\delta^2/3 - 1$, some $b \in (a, 1)$ and $c = \frac{p-(1+b)(1+a-b)}{1+a-b}$, where we assume that $\delta > 0$ is small enough and $p > 1$ close enough to one such that all the assumptions $0 < a \le b < 1$ and $c > 0$ are satisfied. This gives, using (2.23),

$$\begin{aligned} \frac{t}{\alpha_t^2}\alpha_t^{d\delta^2/3} \sum_{z \in S: \ell_t(z) > t/\alpha_t^d} \left(\frac{1}{t}\ell_t(z)\right)^{1+\delta^2/3} &\le \frac{t}{\alpha_t^2}\alpha_t^{d\delta^2/3}\tau^{1/q}\alpha_t^{d/q}\alpha_t^{d[(1-b)(b-a)+p-(1+b)(1+a-b)]/p} \left\| \frac{1}{t}\ell_t \right\|_{1+b+c} \\ &= \tau^{1/q}\alpha_t^{-(d+(2-d)(1+\widetilde{\delta}))/(1+\widetilde{\delta})} \|\ell_t\|_{1+\widetilde{\delta}}, \end{aligned}$$

where $\widetilde{\delta} = b + c$. Picking $\delta > 0$ small enough and $p > 1$ close enough to one, we also have that $\widetilde{\delta}(d-2) < 2$, and we may again apply [12], Proposition 2.1 and see that the exponential rate of the second sum on the right-hand side of (2.13) vanishes as $\tau \downarrow 0$. This ends the proof. □

2.3. *Main part of the proof of Proposition* 2.1

Using Lemmas 2.2 and 2.3, it is clear that Proposition 2.1 now follows from the following assertion.

Proposition 2.4. *For any* $\varepsilon > 0$,

$$\limsup_{R\to\infty}\limsup_{M\to\infty}\limsup_{t\to\infty} \frac{\alpha_t^2}{t} \log\big\langle \mathbb{E}^{t,R}\big[e^{\langle \ell_t, \xi_t \wedge (M/\alpha_t^2)\rangle}\big]\mathbb{1}_{\Gamma_{R,\varepsilon}}(\overline{\xi}_t \wedge M)\big\rangle < -\chi. \tag{2.24}$$

Let us now prove Proposition 2.4. For any potential $V : B_R \to \mathbb{R}$, we denote by $\lambda_R^{\mathrm{d}}(V)$ the principal eigenvalue of $\Delta^{\mathrm{d}} + V$ in the box B_R with zero boundary condition. Introduce a rescaled version of this eigenvalue by putting, for $\psi \in \mathcal{F}(Q_R)$,

$$\lambda_R^{(t)}(\psi) = \alpha_t^2 \lambda_{R\alpha(t)}^{\mathrm{d}}\left(\frac{1}{\alpha_t^2}\psi^{\mathrm{d}}\right), \quad \text{where } \psi^{\mathrm{d}}(z) = \alpha_t^d \int_{z/\alpha_t + [0,\alpha_t^{-1})^d} \psi(y)\,\mathrm{d}y \text{ for } z \in \mathbb{Z}^d. \tag{2.25}$$

Observe from (1.12) and (1.13) that $\frac{1}{\alpha_t^2}\overline{\xi}_t^{\mathrm{d}} = \xi_t$. Recall the definition of $\mathbb{E}^{t,R}$ from Lemma 2.2. Hence, using a Fourier expansion, one has, for any $R, M > 0$, as $t \to \infty$,

$$\mathbb{E}^{t,R}\big[e^{\langle \ell_t, \xi_t \wedge (M/\alpha_t^2)\rangle}\big] = e^{o(t/\alpha_t^2)} \exp\big\{t\lambda_{3R\alpha_t}^{\mathrm{d}}\big(\xi_t \wedge \big(M/\alpha_t^2\big)\big)\big\} = e^{o(t/\alpha_t^2)} \exp\left\{\frac{t}{\alpha_t^2}\lambda_{3R}^{(t)}(\overline{\xi}_t \wedge M)\right\}. \tag{2.26}$$

Now we multiply (2.26) with $\mathbb{1}_{\Gamma_{R,\varepsilon}}(\overline{\xi}_t \wedge M)$ and take expectation with respect to ξ. We subtract and add the term $t\alpha_t^{-2}\rho\log(\frac{e}{\rho}\mathcal{L}_{3R}(\overline{\xi}_t \wedge M))$ in the exponent. The next step is to pick some small parameter $\beta \in (0, \infty)$ and to distinguish the events $\{\overline{\xi}_t \wedge M \in D_{\beta,R}\}$ and its complement, where

$$D_{\beta,R} = \big\{\psi \in \mathcal{F}(Q_{3R}) : \big|\mathcal{L}_{3R}(\psi) - \rho\big| \le \beta\big\}. \tag{2.27}$$

On the event $\{\overline{\xi}_t \wedge M \in \Gamma_{R,\varepsilon}\}$, we estimate the first two terms in the exponent differently on $\{\overline{\xi}_t \wedge M \in D_{\beta,R}\}$ and on the complement:

$$\lambda_{3R}^{(t)}(\overline{\xi}_t \wedge M) - \rho\log\left(\frac{e}{\rho}\mathcal{L}_{3R}(\overline{\xi}_t \wedge M)\right) \le \begin{cases} -\chi_R(\beta, \varepsilon, t) & \text{on } \{\overline{\xi}_t \wedge M \in D_{\beta,R}\}, \\ -\chi_R(t) & \text{on } \{\overline{\xi}_t \wedge M \notin D_{\beta,R}\}, \end{cases} \tag{2.28}$$

where the variational formulas are defined by

$$-\chi_R(\beta,\varepsilon,t)=\sup\left\{\lambda_{3R}^{(t)}(\psi)-\rho\log\left(\frac{\mathrm{e}}{\rho}\mathcal{L}_{3R}(\psi)\right):\ \psi\in\Gamma_{R,\varepsilon}\cap D_{\beta,R}\right\},\tag{2.29}$$

and we put $-\chi_R(t)=-\chi_R(\infty,0,t)=\sup_{\psi\in\mathcal{F}(Q_{3R})}[\lambda_{3R}^{(t)}(\psi)-\rho\log(\frac{\mathrm{e}}{\rho}\mathcal{L}_{3R}(\psi))]$.

Making the above explicit and abbreviating

$$F_{t,R}(\psi)=\exp\left\{\frac{t}{\alpha_t^2}\rho\log\left(\frac{\mathrm{e}}{\rho}\mathcal{L}_{3R}(\psi)\right)\right\},\tag{2.30}$$

we obtain

$$\begin{aligned}
&\big\langle\mathbb{E}^{t,R}\big[\mathrm{e}^{\langle\ell_t,\xi_t\wedge(M/\alpha_t^2)\rangle}\big]\mathbb{1}_{\Gamma_{R,\varepsilon}}(\overline{\xi}_t\wedge M)\big\rangle\\
&\quad\le\mathrm{e}^{\mathrm{o}(t/\alpha_t^2)}\left\langle\exp\left\{\frac{t}{\alpha_t^2}\left[\lambda_{3R}^{(t)}(\overline{\xi}\wedge M)-\rho\log\left(\frac{\mathrm{e}}{\rho}\mathcal{L}_{3R}(\overline{\xi}_t\wedge M)\right)\right]\right\}F_{t,R}(\overline{\xi}_t\wedge M)\mathbb{1}_{\Gamma_{R,\varepsilon}}(\overline{\xi}_t\wedge M)\right\rangle\\
&\quad\le\mathrm{e}^{\mathrm{o}(t/\alpha_t^2)}\big\langle\big[\mathbb{1}_{D_{\beta,R}}(\overline{\xi}_t\wedge M)\mathrm{e}^{-t\alpha_t^{-2}\chi_R(\beta,\varepsilon,t)}+\mathbb{1}_{D_{\beta,R}^{\mathrm{c}}}(\overline{\xi}_t\wedge M)\mathrm{e}^{-t\alpha_t^{-2}\chi_R(t)}\big]F_{t,R}(\overline{\xi}_t\wedge M)\mathbb{1}_{\Gamma_{R,\varepsilon}}(\overline{\xi}_t\wedge M)\big\rangle\\
&\quad\le\mathrm{e}^{\mathrm{o}(t/\alpha_t^2)}\big[\mathrm{e}^{-t\alpha_t^{-2}\chi_R(\beta,\varepsilon,t)}\big\langle F_{t,R}(\overline{\xi}_t\wedge M)\big\rangle+\mathrm{e}^{-t\alpha_t^{-2}\chi_R(t)}\big\langle F_{t,R}(\overline{\xi}_t\wedge M)\mathbb{1}_{D_{\beta,R}^{\mathrm{c}}}(\overline{\xi}_t\wedge M)\big\rangle\big].
\end{aligned}\tag{2.31}$$

Now we need the following asymptotics for the approximate variational formulas:

Lemma 2.5.

(i) $\liminf_{R\to\infty}\liminf_{t\to\infty}\chi_R(t)\ge\chi.$

(ii) *For any* $\varepsilon>0$, *and any* $\beta>0$ *small enough*,

$$\liminf_{R\to\infty}\liminf_{t\to\infty}\chi_R(\beta,\varepsilon,t)>\chi.$$

The proof of Lemma 2.5 is deferred to the end of Section 3.

The large-t exponential rate of the expectation of $F_{t,R}(\overline{\xi}_t\wedge M)$ is nonpositive for any $M>0$, as is seen from an application of part (i) of the following lemma for $K=\rho$.

Lemma 2.6. (i) *Fix* $R>0$ *and* $M>0$. *Then, for any* $K>0$,

$$\limsup_{t\to\infty}\frac{\alpha_t^2}{t}\log\big\langle F_{t,R}(\overline{\xi}_t\wedge M)^{K/\rho}\big\rangle\le K\log\frac{K}{\rho}.\tag{2.32}$$

(ii) *For any* $\beta>0$,

$$\limsup_{R\to\infty}\limsup_{M\to\infty}\limsup_{t\to\infty}\frac{\alpha_t^2}{t}\log\big\langle F_{t,R}(\overline{\xi}_t\wedge M)\mathbb{1}_{D_{\beta,R}^{\mathrm{c}}}(\overline{\xi}_t\wedge M)\big\rangle<0.\tag{2.33}$$

It is elementary to see that an application of Lemmas 2.5 and 2.6 to the terms on the right-hand side of (2.31) implies that (2.24) holds for any $\varepsilon>0$. This ends the proof of Proposition 2.4.

It remains to prove Lemma 2.6. Let us mention that our proof crucially depends on the appearance of the cut-off potential $\overline{\xi}_t\wedge M$ instead of $\overline{\xi}_t$, even though the cut-off parameter M does not appear in the asymptotics. This is the place where Lemma 2.3 turns out to be important.

Proof of Lemma 2.6. First we prove (i). Recall the definition of $F_{t,R}(\psi)$ from (2.30). We recall that $\overline{\xi}_t(x) = \alpha_t^2 \xi_t(\lfloor x\alpha_t \rfloor)$ and rewrite

$$\big\langle F_{t,R}(\overline{\xi}_t \wedge M)^{K/\rho}\big\rangle = \bigg\langle \bigg(\alpha_t^{-d} \sum_{z\in B} \exp\bigg\{\frac{1}{\rho}\big(\alpha_t^2 \xi_t(z) \wedge M\big)\bigg\}\bigg)^{D_t}\bigg\rangle, \tag{2.34}$$

where we abbreviated $D_t = Kt/\alpha_t^2$ and $B = B_{3R\alpha_t}$. (For the ease of notation, we assume that D_t and $3R\alpha_t$ are integers.)

Now we calculate the right-hand side with the help of elementary combinatorics. We denote by $\mathcal{M}_1^{(D_t)}(B) = \{\mu \in (\frac{1}{D_t}\mathbb{N}_0)^B \colon \sum_{b\in B} \mu(b) = 1\}$ the set of probability vectors μ on B such that $D_t\mu$ has solely integer coefficients. Then we have

$$\begin{aligned}
\text{l.h.s. of (2.34)} &= \alpha_t^{-dD_t} \sum_{z_1,z_2,\dots,z_{D_t}\in B} \bigg\langle \prod_{b\in B} \exp\bigg\{\bigg(\frac{\alpha_t^2}{\rho}\xi_t(b) \wedge \frac{M}{\rho}\bigg)\#\{i\colon z_i = b\}\bigg\}\bigg\rangle \\
&= \alpha_t^{-dD_t} \sum_{\mu\in\mathcal{M}_1^{(D_t)}(B)} \#\big\{z\in B^{D_t}\colon \#\{i\colon z_i = b\} = D_t\mu(b), \forall b\big\} \prod_{b\in B} \big\langle e^{t\mu(b)(\xi_t(0)\wedge(M/\alpha_t^2))K/\rho}\big\rangle \\
&= \alpha_t^{-dD_t} \sum_{\mu\in\mathcal{M}_1^{(D_t)}(B)} \frac{D_t!}{\prod_{b\in B}(D_t\mu(b))!} \prod_{b\in B} \big\langle e^{t\mu(b)(\xi_t(0)\wedge(M/\alpha_t^2))K/\rho}\big\rangle.
\end{aligned} \tag{2.35}$$

Use Stirling's formula and recall that $D_t = Kt/\alpha_t^2$ to deduce that, uniformly in $\mu \in \mathcal{M}_1^{(D_t)}(B)$,

$$\alpha_t^{-dD_t} \frac{D_t!}{\prod_{b\in B}(D_t\mu(b))!} = e^{o(t/\alpha_t^2)} \exp\bigg\{-K\frac{t}{\alpha_t^2} \sum_{b\in B} \mu(b) \log\big(\mu(b)\alpha_t^d\big)\bigg\}. \tag{2.36}$$

Now we analyse the last product on the right-hand side of (2.35). We use the formula $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > s)\,ds$ for nonnegative random variables X, introduce an auxiliary variable $N > 0$ (which will be chosen later) and apply the Markov inequality with the map $s \mapsto s^N$. Then we make the change of measure via $s = \exp\{\frac{t}{\alpha_t^{d+2}} r\}$, which implies $\frac{ds}{dr} = \frac{t}{\alpha_t^{d+2}} \exp\{\frac{t}{\alpha_t^{d+2}} r\}$. We use the abbreviation $a = \frac{K}{\rho}\mu(b)\alpha_t^d$. Hence, we have, for any $Q \geq 0$,

$$\begin{aligned}
\big\langle e^{t\mu(b)(\xi_t(0)\wedge M/\alpha_t^2)K/\rho}\big\rangle &\leq e^{tQ/\alpha_t^{d+2}} + \int_{\exp\{tQ/\alpha_t^{d+2}\}}^{\exp\{tMa/\alpha_t^{d+2}\}} \mathrm{Prob}\big[e^{(t\xi_t(0)a)/\alpha_t^d} > s\big]\,ds \\
&\leq e^{tQ/\alpha_t^{d+2}} + \int_{\exp\{tQ/\alpha_t^{d+2}\}}^{\exp\{tMa/\alpha_t^{d+2}\}} s^{-N} \bigg\langle \exp\bigg\{\frac{t}{\alpha_t^d}\xi_t(0)aN\bigg\}\bigg\rangle\,ds \\
&= e^{tQ/\alpha_t^{d+2}} + \frac{t}{\alpha_t^{d+2}} \int_Q^{Ma} \exp\bigg\{r(1-N)\frac{t}{\alpha_t^{d+2}}\bigg\} \bigg\langle \exp\bigg\{\frac{t}{\alpha_t^d}\xi_t(0)aN\bigg\}\bigg\rangle\,dr.
\end{aligned} \tag{2.37}$$

Now we use the definition $\xi_t(z) = \xi(z) - H(t/\alpha_t^d)\alpha_t^d/t$ of the shifted potential (see (1.12)) and recall that $H(s) = \log\langle e^{s\xi(0)}\rangle$ and $t/\alpha_t^{d+2} = \kappa(t/\alpha_t^d)$, to proceed with

$$\begin{aligned}
\text{l.h.s. of (2.37)} &\leq e^{tQ/\alpha_t^{d+2}} + \frac{t}{\alpha_t^{d+2}} \int_Q^{Ma} \exp\bigg\{-(N-1)r\frac{t}{\alpha_t^{d+2}}\bigg\} \exp\bigg\{-aNH\bigg(\frac{t}{\alpha_t^d}\bigg)\bigg\} \bigg\langle \exp\bigg\{\frac{t}{\alpha_t^d}\xi(0)aN\bigg\}\bigg\rangle\,dr \\
&= e^{tQ/\alpha_t^{d+2}} + \frac{t}{\alpha_t^{d+2}} \int_Q^{Ma} \exp\bigg\{-\frac{t}{\alpha_t^{d+2}}\bigg((N-1)r + \frac{aNH(t/\alpha_t^d) - H(aNt/\alpha_t^d)}{\kappa(t/\alpha_t^d)}\bigg)\bigg\}\,dr.
\end{aligned}$$

Now we have to distinguish the case of bounded aN, where we can use precise asymptotics in (1.6) for the last quotient, and the case of arbitrarily large aN, where we can only bound the last quotient. Introduce a new parameter

$L > 0$, which will later be chosen large enough. First we handle those a satisfying $a \le L$, and we now pick $N = \mathrm{e}^{r/a\rho}/a\mathrm{e}$. Note that aN lies then in the interval $[\frac{1}{\mathrm{e}}, \frac{\mathrm{e}^{M/\rho}}{\mathrm{e}}]$. Hence, we may use the asymptotics in (1.6). This gives, picking $Q = 0$,

$$\begin{aligned}\text{l.h.s. of (2.37)} &\le 1 + \mathrm{e}^{o(t/\alpha_t^{d+2})} \int_0^{Ma} \exp\biggl\{-\frac{t}{\alpha_t^{d+2}}\Bigl((N-1)r - \rho a N \log(aN)\Bigr)\biggr\}\,\mathrm{d}r \\ &= 1 + \mathrm{e}^{o(t/\alpha_t^{d+2})} \int_0^{Ma} \exp\biggl\{-\frac{t}{\alpha_t^{d+2}}\Bigl(-r + \frac{\rho}{\mathrm{e}}\mathrm{e}^{r/(a\rho)}\Bigr)\biggr\}\,\mathrm{d}r.\end{aligned}$$

The term $-r + \frac{\rho}{\mathrm{e}}\mathrm{e}^{r/(a\rho)}$ is minimal for $r = \rho a \log(a\mathrm{e})$ with value $-\rho a \log(a)$. Hence, we may estimate for $a \le L$ as follows.

$$\begin{aligned}\text{l.h.s. of (2.37)} &\le 2 + \mathrm{e}^{o(t/\alpha_t^{d+2})} Ma \exp\biggl\{\frac{t}{\alpha_t^{d+2}}\rho a \log(a)\biggr\} \\ &\le \mathrm{e}^{o(t/\alpha_t^{d+2})} \exp\biggl\{\frac{t}{\alpha_t^{d+2}}\rho a \log(a)\biggr\}.\end{aligned} \tag{2.38}$$

Now we turn to a satisfying $a > L$. This time we pick $N = r^{1/\delta}(A(1+\delta))^{-1/\delta}a^{-(1+\delta)/\delta}$, where we have picked some small $\delta > 0$. With A as in (2.18) we have, for every t large enough,

$$\frac{aNH(t/\alpha_t^d) - H(aNt/\alpha_t^d)}{\kappa(t/\alpha_t^d)} \ge -A \cdot (aN)^{1+\delta}, \quad aN \ge 1.$$

This time we pick $Q = A(1+\delta)L$ and note that $aN \ge 1$ on the integration interval $[A(1+\delta)L, Ma]$. Hence, for $a > L$, we may estimate

$$\text{l.h.s. of (2.37)} \le \exp\biggl\{\frac{A(1+\delta)L \cdot t}{\alpha_t^{d+2}}\biggr\} + \mathrm{e}^{o(t/\alpha_t^{d+2})} \int_{A(1+\delta)L}^{Ma} \exp\biggl\{\frac{t}{\alpha_t^{d+2}}\bigl(r - Nr + A(Na)^{1+\delta}\bigr)\biggr\}\,\mathrm{d}r.$$

Note that we may estimate $-Nr + A(Na)^{1+\delta} \le 0$ in the exponent. Furthermore we extend the integration area to the interval $[0, Ma]$. Hence,

$$\begin{aligned}\text{l.h.s. of (2.37)} &\le \exp\biggl\{\frac{A(1+\delta)L \cdot t}{\alpha_t^{d+2}}\biggr\} + \mathrm{e}^{o(t/\alpha_t^{d+2})} \int_0^{Ma} \exp\biggl\{\frac{t}{\alpha_t^{d+2}}r\biggr\}\,\mathrm{d}r \\ &\le \mathrm{e}^{o(t/\alpha_t^{d+2})} \exp\biggl\{\frac{t}{\alpha_t^{d+2}}Ma\biggr\},\end{aligned} \tag{2.39}$$

where the last step is valid for $M > A(1+\delta)$.

Now we go back to (2.35) and substitute (2.36), recall that $a = \frac{K}{\rho}\mu(b)\alpha_t^d$ and substitute (2.38) for $a \le L$ and (2.39) for $a > L$. We now write $L\rho/K$ instead of L and obtain

$$\begin{aligned}\text{l.h.s. of (2.34)} \le \mathrm{e}^{o(t/\alpha_t^2)} \sum_{\mu \in \mathcal{M}_1^{(D_t)}(B)} &\biggl(\prod_{b\in B} \exp\biggl\{-\frac{t}{\alpha_t^2}K\mu(b)\log\bigl(\mu(b)\alpha_t^d\bigr)\biggr\}\biggr) \\ &\times \biggl(\prod_{b\in B:\mu(b)\alpha_t^d \le L} \exp\biggl\{\frac{t}{\alpha_t^{d+2}}K\mu(b)\alpha_t^d \log\Bigl(\frac{K}{\rho}\mu(b)\alpha_t^d\Bigr)\biggr\}\biggr) \\ &\times \biggl(\prod_{b\in B:\mu(b)\alpha_t^d > L} \exp\biggl\{\frac{t}{\alpha_t^{d+2}}M\frac{K}{\rho}\mu(b)\alpha_t^d\biggr\}\biggr)\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{e}^{o(t/\alpha_t^2)} \sum_{\mu \in \mathcal{M}_1^{(D_t)}(B)} \exp\biggl\{\frac{t}{\alpha_t^2} K \log\Bigl(\frac{K}{\rho}\Bigr) \sum_{b \in B:\mu(b)\alpha_t^d \le L} \mu(b)\biggr\} \\
&\quad \times \exp\biggl\{\frac{t}{\alpha_t^2} \sum_{b \in B:\mu(b)\alpha_t^d > L} \Bigl(M\frac{K}{\rho}\mu(b) - K\mu(b)\log\bigl(\mu(b)\alpha_t^d\bigr)\Bigr)\biggr\}. \qquad (2.40)
\end{aligned}
$$

Estimate the last term by

$$
\frac{t}{\alpha_t^{d+2}} \sum_{b \in B:\mu(b)\alpha_t^d > L} \Bigl(M\frac{K}{\rho}\mu(b)\alpha_t^d - K\mu(b)\alpha_t^d \log\bigl(\mu(b)\alpha_t^d\bigr)\Bigr) \le \frac{t}{\alpha_t^2} \sum_{b \in B:\mu(b)\alpha_t^d > L} \mu(b)\Bigl(M\frac{K}{\rho} - K\log L\Bigr),
$$

which is nonpositive for L large enough (only depending on M, K and ρ). Now observe that $\sum_{b\in B}\mu(b) = 1$ and that the cardinality of $\mathcal{M}_1^{(D_t)}(B)$ is $\mathrm{e}^{o(t/\alpha_t^2)}$. This gives that

$$
\lim_{t\to\infty} \frac{\alpha_t^2}{t} \log\bigl\langle F_{t,R}(\overline{\xi}_t \wedge M)^{K/\rho}\bigr\rangle = \lim_{t\to\infty} \frac{\alpha_t^2}{t} \log\bigl(\text{l.h.s. of (2.34)}\bigr) \le K \log\frac{K}{\rho},
$$

and the assertion (i) is proved.

Now we prove assertion (ii). We use the exponential Chebyshev inequality and (i) as follows. We split the event $\{\overline{\xi}_t \wedge M \in D_{\beta,R}^{\mathrm{c}}\} = \{|\mathcal{L}_{3R}(\overline{\xi}_t \wedge M) - \rho| > \beta\}$ into the events $\{\mathcal{L}_{3R}(\overline{\xi}_t \wedge M) > \rho + \beta\}$ and $\{\mathcal{L}_{3R}(\overline{\xi}_t \wedge M) < \rho - \beta\}$. Let us consider only the first of these events, the other is handled in the same way. On this event, we multiply both sides of the inequality with e/ρ, take logs, multiply with $\beta t\alpha(t)^{-2}$ and take exp. This gives, recalling the definition of $F_{t,R}(\psi)$ in (2.30),

$$
\begin{aligned}
\mathbb{1}_{\{\mathcal{L}_{3R}(\overline{\xi}_t \wedge M) > \rho+\beta\}} &= \mathbb{1}_{\{F_{t,R}(\overline{\xi}_t\wedge M)^{\beta/\rho} > \exp\{t\alpha_t^{-2}\beta\log(\mathrm{e}(1+\beta/\rho))\}\}} \\
&\le F_{t,R}(\overline{\xi}_t \wedge M)^{\beta/\rho} \exp\biggl\{-\frac{t}{\alpha_t^2}\beta\log\Bigl(\mathrm{e}\Bigl(1+\frac{\beta}{\rho}\Bigr)\Bigr)\biggr\}.
\end{aligned}
$$

Hence, we can estimate, with the help of assertion (i),

$$
\begin{aligned}
\bigl\langle F_{t,R}(\overline{\xi}_t \wedge M)\mathbb{1}_{\{\mathcal{L}_{3R}(\overline{\xi}_t\wedge M) > \rho+\beta\}}\bigr\rangle &\le \bigl\langle F_{t,R}(\overline{\xi}_t \wedge M)^{\beta/\rho+1}\bigr\rangle \exp\biggl\{-\frac{t}{\alpha_t^2}\beta\log\Bigl(\mathrm{e}\Bigl(1+\frac{\beta}{\rho}\Bigr)\Bigr)\biggr\} \\
&\le \mathrm{e}^{o(t/\alpha_t^2)} \exp\biggl\{\frac{t}{\alpha_t^2}\rho\biggl[\Bigl(\frac{\beta}{\rho}+1\Bigr)\log\Bigl(\frac{\beta}{\rho}+1\Bigr) - \frac{\beta}{\rho} - \frac{\beta}{\rho}\log\Bigl(1+\frac{\beta}{\rho}\Bigr)\biggr]\biggr\} \\
&= \mathrm{e}^{o(t/\alpha_t^2)} \exp\biggl\{\frac{t}{\alpha_t^2}\rho\biggl[\log\Bigl(1+\frac{\beta}{\rho}\Bigr) - \frac{\beta}{\rho}\biggr]\biggr\}.
\end{aligned}
$$

Since the term in square brackets is negative and does not depend on R nor on M, the proof is complete. □

3. The variational formulas

In this section we identify the constant χ appearing in Theorem 1.2 in terms of a "dual" variational problem which will be of importance. Furthermore, we prove a minimisation property of that formula: every asymptotically minimising sequence converges, along a suitable subsequence, after appropriate spatial translation, towards the minimiser of the formula in $L^2(\mathbb{R}^d)$-sense. This is one of the crucial ingredients of the subsequent proof of Lemma 2.5, the last open step in the proof of Proposition 2.4.

Recall the parameter $\rho \in (0,\infty)$ from Assumption (AB). Then [12], Proposition 1.11, identifies χ as follows.

Lemma 3.1 (Dual representation of χ). *For any $g \in H^1(\mathbb{R}^d)$,*

$$\mathcal{H}(g^2) = \rho \int_{\mathbb{R}^d} g^2(x) \log g^2(x)\, \mathrm{d}x \in [-\infty, \infty) \tag{3.1}$$

is well defined. Furthermore, $\mathcal{L}$ and $\mathcal{H}$ on $L^2(\mathbb{R}^d)$ are Legendre transform of each other, more precisely,

$$\mathcal{L}(\psi) = \sup_{g \in H^1(\mathbb{R}^d)} \bigl(\langle g^2, \psi\rangle - \mathcal{H}(g^2)\bigr) \quad \text{and} \quad \mathcal{H}(g^2) = \sup_{\psi \in \mathcal{C}(\mathbb{R}^d)} \bigl(\langle g^2, \psi\rangle - \mathcal{L}(\psi)\bigr). \tag{3.2}$$

Furthermore,

$$\chi = \inf_{g \in H^1(\mathbb{R}^d): \|g\|_2 = 1} \bigl\{\|\nabla g\|_2^2 - \mathcal{H}(g^2)\bigr\}. \tag{3.3}$$

Moreover, the minimum in (3.3) *is attained, uniquely up to translation, at the Gaussian density*

$$\widehat{g}^2(x) = \left(\frac{\rho}{\pi}\right)^{d/2} \mathrm{e}^{-\rho|x|^2} = \frac{1}{\mathrm{e}} \mathrm{e}^{\widehat{\psi}(x)/\rho}, \quad x \in \mathbb{R}^d.$$

The function $\widehat{g}$ is the unique L^2-normalized positive eigenfunction of the operator $\Delta + \widehat{\psi}$ with eigenvalue $\lambda(\widehat{\psi}) = \rho - \rho d + \rho \frac{d}{2} \log \frac{\rho}{\pi}$. Furthermore, $\mathcal{L}(\widehat{\psi}) = \rho$.

The main point in the proof of Lemma 3.1 is the well-known *logarithmic Sobolev inequality*,

$$\|\nabla g\|_2^2 \ge \mathcal{H}(g^2) + \rho d\left(1 - \frac{1}{2}\log\frac{\rho}{\pi}\right), \quad g \in L^2(\mathbb{R}^d), \|g\|_2 = 1, \tag{3.4}$$

with equality if and only if g is equal to $\widehat{g}$; see, e.g., [13], Theorem 8.14.

Now we consider the infimum in (3.3) under the additional constraint that any translation of g^2 is away from the minimizer $\widehat{g}^2$ introduced in Lemma 3.1 in $L^1(\mathbb{R}^d)$-sense, i.e.,

$$\chi(\varepsilon) = \inf\bigl\{\|\nabla g\|_2^2 - \mathcal{H}(g^2)\colon g \in H^1(\mathbb{R}^d), \|g\|_2 = 1, \forall x \in \mathbb{R}^d\colon \bigl\|g^2(x + \cdot) - \widehat{g}^2(\cdot)\bigr\|_1 \ge \varepsilon\bigr\}. \tag{3.5}$$

The following lemma says that, given any L^2-normalised sequence $(g_n)_n$ of approximate minimisers of $g \mapsto \|\nabla g\|_2^2 - \mathcal{H}(g^2)$, there is some shift $x_n \in \mathbb{R}^d$ such that, along some subsequence, $g_n^2(x_n + \cdot)$ converges in $L^1(\mathbb{R}^d)$ towards the Gaussian density $\widehat{g}^2$ introduced in Lemma 3.1. Let us remark that a similar result is obtained in [4] using a different approach. It is shown there that, for any L^2-normalised $g \in H^1(\mathbb{R}^d)$,

$$\|\nabla g\|_2^2 - 2\pi \int g^2 \log g^2 \ge \chi + 2\pi \mathfrak{H}\bigl(|\mathfrak{F}g|^2 | 2^{d/2} \mathrm{e}^{-2\pi|x|^2}\bigr), \tag{3.6}$$

where $\mathfrak{F}g(x) = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} x \cdot y} g(y)\, \mathrm{d}y$ is the Fourier transform, and $\mathfrak{H}$ denotes the relative entropy between probability measures with the respective densities. Note that the latter density is equal to the Gaussian density $\widehat{g}^2$ with $\rho = 2\pi$; by χ we mean our parameter with precisely that choice of ρ. Certainly, (3.6) can easily be generalised from 2π to any value of ρ. However, the result in (3.6) is not sufficient for our purposes since we found no way to make the entropic distance between two densities used in (3.6) compatible with our large-deviations arguments for the normalised random walk occupation measures. Neither we were able to estimate the quotient on the left-hand side of (1.17) in terms of anything that involves the entropy between the square of a Fourier transform of the squareroot of a density and the Gaussian density, nor we able to apply useful probabilistic techniques to it if the distance dist introduced in (1.16) is replaced by some entropic distance in the above spirit. Hence, we do not use (3.6) in our proof.

We also would like to mention that the discrete variant of the variational formula in (3.3) (i.e., where the Laplace operator in $\mathbb{R}^d$ is replaced by its discrete version in $\mathbb{Z}^d$) has been analysed in detail in [7], Theorem 2.II and its dual variant in [9], Proposition 1.1 and Lemma 3.2. These are the formulas that appear in the analysis of the parabolic Anderson model in the universality class of the double-exponential distribution.

Lemma 3.2. *For any $\varepsilon > 0$, $\chi(\varepsilon) > \chi$.*

Proof. It is sufficient to show that, for any sequence $(g_n)_n$ in $H^1(\mathbb{R}^d)$ such that $\|g_n\|_2 = 1$ for all n and $\lim_{n\to\infty}(\|\nabla g_n\|_2^2 - \rho \int {g_n}^2 \log(g_n^2)) = \chi$, there is a suitable shift $x_n \in \mathbb{R}^d$ such that, along a suitable subsequence, $\lim_{n\to\infty} \operatorname{dist}(g_n^2(x_n + \cdot), \widehat{g}^2(\cdot)) = 0$. Let $(g_n)_n$ be such a sequence. Hence, for some $K > 0$,

$$\|\nabla g_n\|_2^2 - \rho \int {g_n}^2 \log({g_n}^2) \le K, \quad n \in \mathbb{N}. \tag{3.7}$$

Now we show that $(\|\nabla g_n\|_2)_n$ is bounded: In the case $d \ge 3$ we use Jensen's inequality and the Sobolev inequality ([13], Theorem 8.3), to estimate

$$\begin{aligned}\|\nabla g_n\|_2^2 &\le K + \rho \int g_n^2 \log(g_n^2) = K + \rho \frac{d-2}{2} \int g_n^2 \log\big(g_n^{4/(d-2)}\big)\\ &\le K + \rho \frac{d-2}{2} \log\bigg(\int g_n^{2d/(d-2)}\bigg) \le K + \rho \frac{d-2}{2} \log\big(C \|\nabla g_n\|_2^{2d/(d-2)}\big),\end{aligned} \tag{3.8}$$

where $C > 0$ is a Sobolev constant that satisfies $\int f^{2d/(d-2)} \le C \|\nabla f\|_2^{2d/(d-2)}$ for any $f \in L^{2d/(d-2)}(\mathbb{R}^d)$. Hence $(\|\nabla g_n\|_2)_n$ is bounded and therefore $(\int g_n^2 \log g_n^2)_n$ as well. In a similar way, we see the boundedness of $(\|\nabla g_n\|_2)_n$ also in $d = 2$, using the Sobolev inequality of [13], Theorem 8.5(ii). In dimension $d = 1$, we estimate, using the Sobolev inequality of [13], Theorem 8.5(i),

$$\|\nabla g_n\|_2^2 \le K + \rho \int g_n^2 \log(g_n^2) \le K + \rho \int g_n^2 \log \|g_n\|_\infty^2 \le K + \rho \log\bigg(\frac{1}{2}\|\nabla g_n\|_2^2 + \frac{1}{2}\bigg)$$

and conclude as above.

Now we construct, for any $n \in \mathbb{N}$ and any small $\delta > 0$ and any sufficiently large $R = R_\delta > 0$, some $x_n(\delta, R) \in \mathbb{R}^d$ such that

$$\int_{x_n(\delta,R)+Q_R} g_n^2(y)\,\mathrm{d}y \ge 1 - \delta. \tag{3.9}$$

We pick a smooth auxiliary function $\Phi = \Phi_R : \mathbb{R}^d \to [0, 1]$ satisfying $\operatorname{supp}(\Phi) \subset Q_R$ and $\Phi \equiv 1$ on Q_{R-1}, and we put $\Phi_x(y) = \Phi(x + y)$ for $x, y \in \mathbb{R}^d$. Consider $h_{n,x} = \Phi_x \cdot g_n$. Then we have

$$\int_{\mathbb{R}^d} \|h_{n,x}\|_2^2\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} \mathrm{d}x\, \Phi^2(x + y) g_n^2(y) = \|\Phi\|_2^2.$$

Similarly, we get

$$\begin{aligned}\int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y\, h_{n,x}^2(y) \log\big(h_{n,x}^2(y)\big) &= \int_{\mathbb{R}^d} \mathrm{d}y\, g_n^2(y) \int_{\mathbb{R}^d} \mathrm{d}x\, \Phi^2(x + y) \log\big(\Phi^2(x + y)\big) + \|\Phi\|_2^2 \int g_n^2 \log\big(g_n^2\big)\\ &= \int \Phi^2 \log\big(\Phi^2\big) + \|\Phi\|_2^2 \int g_n^2 \log\big(g_n^2\big).\end{aligned}$$

Using the product rule of differentiation, we get

$$\begin{aligned}\int_{\mathbb{R}^d} \|\nabla h_{n,x}\|_2^2\,\mathrm{d}x &= \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y\, \big[g_n^2(y)\big|\nabla\Phi(x + y)\big|^2 + \Phi^2(x + y)\big|\nabla g_n(y)\big|^2\\ &\qquad + 2g_n(y)\Phi(x + y)\nabla\Phi(x + y) \cdot \nabla g_n(y)\big]\\ &= \|\nabla\Phi\|_2^2 + \|\Phi\|_2^2 \|\nabla g_n\|_2^2 + \int_{\mathbb{R}^d} g_n(y) u_\Phi \cdot \nabla g_n(y)\,\mathrm{d}y,\end{aligned}$$

where $u_\Phi = \int_{\mathbb{R}^d} \mathrm{d}x\, \nabla(\Phi^2)(x)$. Using the Cauchy–Schwarz inequality in the last term first for the Euclidean inner product and afterwards for the integral and recalling that $\|g_n\|_2 = 1$ and that $C = \sup_{n\in\mathbb{N}} \|\nabla g_n\|_2$ is finite, we see that

$$\int_{\mathbb{R}^d} \|\nabla h_{n,x}\|_2^2 \,\mathrm{d}x \le \|\nabla\Phi\|_2^2 + \|\Phi\|_2^2 \|\nabla g_n\|_2^2 + C|u_\Phi|.$$

Now fix $\delta > 0$ and summarize the above estimates to obtain, for any $n \in \mathbb{N}$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{d}x \left[\|\nabla h_{n,x}\|_2^2 - \rho \int_{\mathbb{R}^d} \mathrm{d}y\, h_{n,x}^2(y) \log\bigl(h_{n,x}^2(y)\bigr) - (\chi+\delta)\|h_{n,x}\|_2^2\right] \\
&\quad \le \|\nabla\Phi\|_2^2 + \|\Phi\|_2^2 \left(\|\nabla g_n\|_2^2 - \rho \int g_n^2 \log\bigl(g_n^2\bigr) - \chi - \delta\right) + C|u_\Phi| - \rho \int \Phi^2 \log\bigl(\Phi^2\bigr) \\
&\quad = -\|\Phi\|_2^2\bigl(\delta - \mathrm{o}(1)\bigr) + \|\nabla\Phi\|_2^2 + C|u_\Phi| - \rho \int \Phi^2 \log\bigl(\Phi^2\bigr),
\end{aligned}$$

where $\mathrm{o}(1)$ refers to $n \to \infty$ (recall that $(g_n)_n$ is asymptotically maximal in the definition (3.3) of χ). It is possible to choose $R = R_\delta$ so large that the right-hand side is negative. Indeed, since Φ equals one in Q_{R-1} and equals zero in Q_R^{c}, all the terms $\|\nabla\Phi\|_2^2$, $|u_\Phi|$ and $\int \Phi^2 \log(\Phi^2)$ are of order R^{d-1} as $R \to \infty$, while $\|\Phi\|_2^2$ is of order R^d. Since the integral on the left-hand side is therefore also negative, there is some $x_n = x_n(\delta, R) \in \mathbb{R}^d$ such that the integrand is negative, that is,

$$\left\|\nabla \frac{h_{n,x_n}}{\|h_{n,x_n}\|_2}\right\|_2^2 - \rho \int \left(\frac{h_{n,x_n}}{\|h_{n,x_n}\|_2}\right)^2 \log\left(\frac{h_{n,x_n}}{\|h_{n,x_n}\|_2}\right)^2 \le \chi + \delta + \rho \log \|h_{n,x_n}\|_2^2,$$

using some elementary manipulations. By definition of χ in (3.3), the left-hand side is no smaller than χ, and it follows that $\delta + \rho \log \|h_{n,x_n}\|_2^2$ is nonnegative. This in turn means that $\|h_{n,x_n}\|_2^2 \ge \mathrm{e}^{-\delta/\rho} \ge 1 - \delta/\rho$. Replacing δ/ρ by δ, we have arrived at our first goal: the construction of some $x_n(\delta, R) \in \mathbb{R}^d$ such that (3.9) holds.

Now put $x_n = x_n(\frac14, R_{1/4})$. We claim that the sequence $(g_n^2(-x_n + \cdot))_{n\in\mathbb{N}}$ (conceived as probability measures on $\mathbb{R}^d$) is tight. Indeed, for any $\delta \in (0, \frac14)$ and any $n \in \mathbb{N}$, we pick R_δ and $x_n(\delta, R_\delta)$ as above. Since the masses of g_n^2 both in the box $x_n(\frac14, R_{1/4}) + Q_{R_{1/4}}$ and in the box $x_n(\delta, R_\delta) + Q_{R_\delta}$ exceed $\frac34$, the two boxes must have an nonempty intersection. Hence, the latter box is contained in the box $x_n(\frac14, R_{1/4}) + Q_{R_{1/4}+2R_\delta}$. Consequently, putting $\widetilde{R}_\delta = R_{1/4} + 2R_\delta$,

$$\int_{Q_{\widetilde{R}_\delta}} g_n^2(-x_n + y)\,\mathrm{d}y \ge \int_{x_n(\delta, R_\delta) + Q_{R_\delta}} g_n^2(y)\,\mathrm{d}y \ge 1 - \delta.$$

This shows the tightness of $(g_n^2(-x_n + \cdot))_{n\in\mathbb{N}}$.

Now we use the Banach–Alaoglu theorem [13], Theorem 2.18 and [13], Theorems 8.6, 8.7 and 2.11. Since $(\|\nabla g_n(-x_n + \cdot)\|_2)_n$ is bounded, there is a subsequence of $(g_n(-x_n + \cdot))_n$, still denoted $(g_n(-x_n + \cdot))_n$, and a $g \in H^1(\mathbb{R}^d)$ satisfying $\|g\|_2 \le 1$, such that $g_n(-x_n + \cdot)$ converges to g weakly in $L^2(\mathbb{R}^d)$ and strongly in $L^p(Q_R)$ for any $p < \frac{2d}{d-2}$ in $d \ge 3$ and for any $p < \infty$ in $d \in \{1, 2\}$ and for any $R \in (0, \infty)$ and almost everywhere, and $\nabla g_n(-x_n + \cdot)$ converges to ∇g weakly in $L^2(\mathbb{R}^d)$. Furthermore, $\|\nabla g\|_2^2 \le \liminf_{n\to\infty} \|\nabla g_n\|_2^2$. Since $(g_n^2(-x_n + \cdot))_{n\in\mathbb{N}}$ is tight and by local L^2-convergence, we also have that g is L^2-normalized.

Now we argue that $\limsup_{n\to\infty} \int g_n^2 \log(g_n^2) \le \int g^2 \log g^2$. To derive this for $d \ge 3$, we first estimate the integrals over complements of large boxes. A similar estimate as the one in (3.8) shows, for any $R > 0$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\int_{Q_R^{\mathrm{c}}} g_n^2(-x_n + y) \log\bigl(g_n^2(-x_n + y)\bigr)\,\mathrm{d}y &\le -\int_{Q_R^{\mathrm{c}}} g_n^2(-x_n + y)\,\mathrm{d}y\, \log\left(\int_{Q_R^{\mathrm{c}}} g_n^2(-x_n + y)\,\mathrm{d}y\right) \\
&\quad + C \int_{Q_R^{\mathrm{c}}} g_n^2(-x_n + y)\,\mathrm{d}y,
\end{aligned}$$

where $C > 0$ is again a Sobolev constant. By tightness, the right-hand side vanishes as $R \to \infty$. A similar argument applies for $d \le 2$.

Now we turn to the integral over the interior of a box. Observe that, for any $R > 0$, the sequence $g_n^2 \log g_n^2$ converges in probability to $g^2 \log g^2$ with respect to the normalized Lebesgue measure on Q_R, that is,

$$\lim_{n\to\infty} \int_{Q_R} \mathrm{d}y\, \mathbb{1}_{\{|g_n^2(-x_n+y)\log(g_n^2(-x_n+y)) - g^2(y)\log(g^2(y))|>\eta\}} = 0, \qquad \eta > 0,$$

as is easily deduced from the almost everywhere convergence of $g_n(-x_n + \cdot)$ to g, using Lebesgue's theorem. Furthermore, $(g_n^2 \log(g_n^2))_n$ is uniformly integrable with respect to the normalized Lebesgue measure on Q_R, which is seen, for $d \ge 3$, as follows. Note that, for any $p \in (1, \frac{d}{d-2})$ and any $\beta \in (0, 1)$, there is $c > 0$ such that

$$\bigl|x \log(x)\bigr| \le c\bigl(|x|^p + |x|^\beta\bigr), \quad x > 0, \tag{3.10}$$

and recall that $(\|g_n^2\|_{p'})_n$ and $(\|g_n\|_2)_n$ are bounded for any p' with $p < p' < \frac{d}{d-2}$. From this it is easy to deduce the uniform integrability on Q_R for $d \ge 3$. A similar argument is used for $d = 1, 2$. Hence $\lim_{n\to\infty} \int_{Q_R} g_n^2 \log(g_n^2) = \int_{Q_R} g^2 \log(g^2)$. Using the above, we even see that $\limsup_{n\to\infty} \int_{\mathbb{R}^d} g_n^2 \log(g_n^2) \le \int_{\mathbb{R}^d} g^2 \log(g^2)$.

Hence we see that g is a minimizer in the definition (3.3) of χ. Without loss of generality, we may therefore assume that g is equal to $\widehat{g}$ introduced in Lemma 3.1. Since $g_n(-x_n + \cdot)$ converges to $\widehat{g}$ on every compact subset of $\mathbb{R}^d$ in L^p for any $p \in (1, \frac{2d}{d-2})$ in $d \ge 3$ and for any $p < \infty$ in $d \le 2$, and by compactness of $(g_n^2(-x_n + \cdot)$, we have also that $g_n^2(-x_n + \cdot)$ converges towards $\widehat{g}$ in $L^2(\mathbb{R}^d)$-sense. This ends the proof. □

Now we show that the variational formula $\chi(\varepsilon)$ can be approximated by finite-box versions. Introduce

$$\chi_R(\varepsilon) = \inf\biggl\{\|\nabla g\|_2^2 - \mathcal{H}\bigl(g^2\bigr) : g \in H^1\bigl(\mathbb{R}^d\bigr), \|g\|_2 = 1, \operatorname{supp}(g) \subset Q_{3R}, \\ \forall x \in Q_{2R} : \int_{Q_R} \bigl|g^2(x+y) - \widehat{g}^2(y)\bigr|\,\mathrm{d}y \ge \varepsilon\biggr\}. \tag{3.11}$$

Then $\chi_R = \chi_R(0) = \inf\{\|\nabla g\|_2^2 - \mathcal{H}(g^2) : g \in H^1(\mathbb{R}^d), \|g\|_2 = 1, \operatorname{supp}(g) \subset Q_{3R}\}$.

Lemma 3.3 (Finite-box approximation of χ). *For any $\varepsilon \ge 0$,*

$$\liminf_{R\to\infty} \chi_R(\varepsilon) \ge \chi(\varepsilon). \tag{3.12}$$

Proof. Let $(g_R)_{R\ge1}$ be a family of L^2-normalised functions $g_R \in H^1(\mathbb{R}^d)$ satisfying $\operatorname{supp}(g_R) \subset Q_{3R}$ and $\int_{Q_R} |g_R^2(x+y) - \widehat{g}^2(y)|\,\mathrm{d}y \ge \varepsilon$ for any $x \in Q_{2R}$ such that $\|\nabla g_R\|_2^2 - \mathcal{H}(g_R^2)$ converges towards $\liminf_{R\to\infty} \chi_R(\varepsilon)$ as $R \to \infty$. Precisely as in the proof of Lemma 3.2, we see that, for some sequence $R_n \to \infty$ as $n \to \infty$, there are suitable shifts $x_n \in \mathbb{R}^d$ and some L^2-normalised $g \in H^1(\mathbb{R}^d)$ such that $g_{R_n}(x_n + \cdot)$ converges towards g in $L^2(\mathbb{R}^d)$ sense and

$$\liminf_{R\to\infty} \chi_R(\varepsilon) = \lim_{n\to\infty} \bigl(\|\nabla g_{R_n}\|_2^2 - \mathcal{H}\bigl(g_{R_n}^2\bigr)\bigr) = \|\nabla g\|_2^2 - \mathcal{H}\bigl(g^2\bigr). \tag{3.13}$$

By $L^2(\mathbb{R}^d)$-convergence of $g_{R_n}(x_n + \cdot)$ towards g, and since $\int_{Q_R} |g_R^2(x+y) - \widehat{g}^2(y)|\,\mathrm{d}y \ge \varepsilon$ for any $x \in Q_{2R}$, we know that g lies in the set of functions over which the infimum is taken in the definition (3.5) of $\chi(\varepsilon)$. Hence, the right-hand side of (3.13) is not smaller than $\chi(\varepsilon)$, which finishes the proof. □

After these preparations, we finally can prove the last building block in the proof of Proposition 2.4.

Proof of Lemma 2.5. Recall the definition of $\chi_R(\beta,\varepsilon,t)$ from (2.29); recall also (2.25), (2.27) and (2.5). We prove (i) and (ii) jointly. Let $\psi_t\in\mathcal{F}(Q_{3R})$, depending on $R>0$, $\beta\in(0,\infty]$ and $\varepsilon\ge 0$, be an approximately maximizing function in the definition (2.29) of $-\chi_R(\beta,\varepsilon,t)$. More precisely, we require that

$$\lambda_{3R}^{(t)}(\psi_t)-\rho\log\biggl(\frac{\mathrm{e}}{\rho}\mathcal{L}_{3R}(\psi_t)\biggr)\le-\chi_R(\beta,\varepsilon,t)+\frac{1}{t}.$$

By the two extra conditions, we have

$$\bigl|\mathcal{L}_{3R}(\psi_t)-\rho\bigr|\le\beta\quad\text{and}\quad\mathrm{d}_R\bigl(\psi_t(x+\cdot),\widehat{\psi}(\cdot)\bigr)\ge\varepsilon\quad\text{for any } x\in Q_{2R}.$$

By the first condition, we may pick some $c\in\mathbb{R}$ (to be precise, $c=\rho\log(\rho/\mathcal{L}_{3R}(\psi_t))$) such that

$$1=\frac{1}{\mathrm{e}}\int_{Q_{3R}}\mathrm{e}^{(\psi_t(x)+c)/\rho}\,\mathrm{d}x\quad\text{and}\quad\bigl|1-\mathrm{e}^{-c/\rho}\bigr|\le\frac{\beta}{\rho}.\tag{3.14}$$

Since $\lambda_{3R}^{(t)}(\psi_t+c)=\lambda_{3R}^{(t)}(\psi_t)+c$ and $\mathcal{L}_{3R}(\psi_t+c)=\mathrm{e}^{c/\rho}\mathcal{L}_{3R}(\psi_t)$ and since $\mathcal{L}_{3R}(\psi_t+c)=\rho$ by the choice of c, we have

$$-\liminf_{t\to\infty}\chi_R(\beta,\varepsilon,t)=\limsup_{t\to\infty}\lambda_{3R}^{(t)}(\psi_t+c)-\rho.$$

Recall the Rayleigh–Ritz formula $\lambda_R^{\mathrm{d}}(V)=\max_{f\in\ell^2(B_R):\|f\|_2=1}(\langle\Delta^{\mathrm{d}}f,f\rangle+\langle V,f^2\rangle)$ for potentials $V:B_R\to\mathbb{R}$. Hence, there is an ℓ^2-normalized function $f_t\in\ell^2(\mathbb{Z}^d)$ in that is positive in $B_{3R\alpha_t}$ and zero outside and satisfies

$$\lambda_{3R}^{(t)}(\psi_t+c)=\alpha_t^2\lambda_{3R\alpha_t}^{\mathrm{d}}\biggl(\frac{1}{\alpha_t^2}\bigl[\psi_t^{\mathrm{d}}+c\bigr]\biggr)=\alpha_t^2\bigl\langle\Delta^{\mathrm{d}}f_t,f_t\bigr\rangle+\bigl\langle\psi_t^{\mathrm{d}}+c,f_t^2\bigr\rangle.\tag{3.15}$$

For any $i\in\{1,\dots,d\}$, introduce $g_t^{(i)}:\mathbb{R}^d\to[0,\infty)$ defined by

$$g_t^{(i)}(x)=\alpha_t^{d/2}\bigl[f_t\bigl(\lfloor x\alpha_t\rfloor\bigr)+\bigl(\alpha_t x_i-\lfloor\alpha_t x_i\rfloor\bigr)\bigl(f_t\bigl(\lfloor x\alpha_t\rfloor+\mathrm{e}_i\bigr)-f_t\bigl(\lfloor x\alpha_t\rfloor\bigr)\bigr)\bigr],$$

where $x=(x_i)_{i=1,\dots,d}$, and $\mathrm{e}_i\in\mathbb{R}^d$ is the ith unit vector. Abbreviate $\widetilde{x}_i=(x_j)_{j\ne i}\in\mathbb{R}^{d-1}$ and denote $g_{t,\widetilde{x}_i}^{(i)}(x_i)=g_t^{(i)}(x)$. For almost every $\widetilde{x}_i\in\mathbb{R}^{d-1}$, the map $g_{t,\widetilde{x}_i}^{(i)}$ is continuous and piecewise affine, and hence lies in $H^1(\mathbb{R})$ with support in $[-3R,3R]$. Now let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $(0,\infty)$ with $\lim_{n\to\infty}t_n=\infty$ such that the limit superior of $\lambda_{3R}^{(t)}(\psi_t+c)$ is realized along this sequence. Using Fubini's theorem and Fatou's lemma, one shows, in the same way as in the proof of [12], Proposition 5.1, that

$$\sum_{i=1}^d\int_{\mathbb{R}^{d-1}}\mathrm{d}\widetilde{x}_i\,\liminf_{n\to\infty}\int_{\mathbb{R}}\mathrm{d}x_i\,\bigl|\bigl(g_{n,\widetilde{x}_i}^{(i)}\bigr)'(x_i)\bigr|^2<\infty.$$

Furthermore, since $|x_i-\lfloor\alpha(t_n)x_i\rfloor/\alpha(t_n)|\le\alpha(t_n)^{-1}$, one also derives that

$$\lim_{n\to\infty}\bigl\|g_{t_n}^{(i)}-\alpha(t_n)^{d/2}f_{t_n}\bigl(\bigl\lfloor\alpha(t_n)\cdot\bigr\rfloor\bigr)\bigr\|_2=0.$$

Hence, one sees that, along some subsequence, for almost every $\widetilde{x}_i\in\mathbb{R}^{d-1}$, $g_{t_n,\widetilde{x}_i}^{(i)}$ converges towards some $g_{\widetilde{x}_i}^{(i)}\in H^1(\mathbb{R}^d)$. According to [13], Theorems 8.6 and 8.7, the convergence is strong in L^q for any $q<\frac{2d}{d-2}$ for $d\ge 3$ and for all $q<\infty$ for $d\in\{1,2\}$, pointwise almost everywhere and weak in L^2 for the gradients. Furthermore, as also is shown in the proof of [12], Proposition 5.1, there is some L^2-normalized $g\in H^1(\mathbb{R}^d)$ with support in Q_{3R} such that $g(x)=g_{\widetilde{x}_i}^{(i)}(x_i)$ for almost all $x\in\mathbb{R}^d$, and we have

$$\limsup_{t\to\infty}\alpha_t^2\bigl\langle\Delta^{\mathrm{d}}f_t,f_t\bigr\rangle\le-\|\nabla g\|_2^2.$$

Observe that, for any $i \in \{1, \dots, d\}$,

$$\begin{aligned}\langle \psi_t^{\mathrm{d}}, f_t^2 \rangle &= \int_{\mathbb{R}^d} \big(\psi_t(x)\big) f_t\big(\lfloor \alpha_t x \rfloor\big)^2 \alpha_t^d \,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \big(\psi_t(x)\big)\big(g_t^{(i)}(x) - \alpha_t^{d/2}\big(\alpha_t x_i - \lfloor \alpha_t x_i \rfloor\big)\big(f_t\big(\lfloor x\alpha_t \rfloor + \mathrm{e}_i\big) - f_t\big(\lfloor x \alpha_t \rfloor\big)\big)\big)^2 \,\mathrm{d}x.\end{aligned}$$

It is also clear from the proof of [12], Proposition 5.1, that the function in the brackets on the right-hand side has an L^2 distance to $g_t^{(i)}$ that vanishes as $t \to \infty$ and that $g_t^{(i)}$ converges towards g strongly in L^2. We write now g_t instead of $g_t^{(1)}/\|g_t^{(1)}\|_2$; recall that $\lim_{t\to\infty} \|g_t^{(1)}\|_2 = 1$. Hence, we have

$$\begin{aligned}-\liminf_{t\to\infty} \chi_R(\beta, \varepsilon, t) &= \limsup_{t\to\infty} \lambda_{3R}^{(t)}(\psi_t + c) - \rho \le \limsup_{n\to\infty} \big(\alpha_{t_n}^2 \langle \Delta^{\mathrm{d}} f_{t_n}, f_{t_n} \rangle + \langle \psi_t^{\mathrm{d}} + c, f_{t_n}^2 \rangle\big) - \rho \\ &\le -\|\nabla g\|_2^2 + \limsup_{n\to\infty} \langle \psi_{t_n} + c - \rho, g_{t_n}^2 \rangle.\end{aligned}$$

Now we employ the definition of $\mathcal{H}$ in (3.1) to rewrite

$$\langle \psi_{t_n} + c - \rho, g_{t_n}^2 \rangle = \mathcal{H}\big(g_{t_n}^2\big) - \rho \left\langle g_{t_n}^2, \log \frac{g_{t_n}^2}{\mathrm{e}^{(\psi_{t_n} + c - \rho)/\rho}} \right\rangle.$$

Recall that g_t is L^2-normalized and that $\mathrm{e}^{(\psi_{t_n} + c - \rho)/\rho}$ is a probability density on Q_{3R} by (3.14). Hence, the last term is equal to the entropy between the two probability measures with densities $\frac{1}{\mathrm{e}} \mathrm{e}^{(\psi_{t_n} + c)/\rho}$ resp. g_t^2. According to [6], Example 6.2.17, we can estimate this entropy against the variational distance between these measures as follows.

$$\left\langle g_{t_n}^2, \log \frac{g_{t_n}^2}{\mathrm{e}^{(\psi_{t_n} + c - \rho)/\rho}} \right\rangle \ge \frac{1}{2} \left\| g_{t_n}^2 - \frac{1}{\mathrm{e}} \mathrm{e}^{(\psi_{t_n} + c)/\rho} \right\|_{1,3R}^2,$$

where $\|\cdot\|_{1,3R}$ denotes the L^1-norm on $L^1(Q_{3R})$. In the same way as in the proof of Lemma 3.2 (see around (3.10)) one sees that $\limsup_{n\to\infty} \mathcal{H}(g_{t_n}^2) \le \mathcal{H}(g^2)$. Hence,

$$-\liminf_{t\to\infty} \chi_R(\beta, \varepsilon, t) \le \mathcal{H}\big(g^2\big) - \|\nabla g\|_2^2 - \frac{\rho}{2} \liminf_{n\to\infty} \left\| g_{t_n}^2 - \frac{1}{\mathrm{e}} \mathrm{e}^{(\psi_{t_n} + c)/\rho} \right\|_{1,3R}^2. \tag{3.16}$$

Recall that $g \in H^1(\mathbb{R}^d)$ is L^2-normalized with support in Q_{3R}. Hence, the right-hand side is not larger than $-\chi_R$, and this ends the proof of Lemma 2.5(i), since we know from Lemma 3.3 that $\lim_{R\to\infty} \chi_R = \chi$.

However, for proving (ii), we have to work harder in order to get an upper bound that is strictly smaller. Recall the definition of $\chi_R(\varepsilon)$ in (3.11). If g is bounded away from $\widehat{g}$ in the sense that $\int_{Q_R} |g^2(x+y) - \widehat{g}^2(y)| \,\mathrm{d}y \ge \frac{\varepsilon}{4\mathrm{e}}$ for any $x \in Q_{2R}$, then we can estimate the first two terms on the right-hand side of (3.16) from above against $-\chi_R(\frac{\varepsilon}{4\mathrm{e}})$, which finishes the proof of Lemma 2.5(ii), since $\liminf_{R\to\infty} \chi_R(\frac{\varepsilon}{4\mathrm{e}}) \ge \chi(\frac{\varepsilon}{4\mathrm{e}}) > \chi$ by Lemmas 3.3 and 3.2. Hence, it remains to consider the case that $\int_{Q_R} |g^2(x+y) - \widehat{g}^2(y)| \,\mathrm{d}y < \frac{\varepsilon}{4\mathrm{e}}$ for some $x \in Q_{2R}$. Then we also have $\int_{Q_R} |g_{t_n}^2(x + y) - \widehat{g}^2(y)| \,\mathrm{d}y < \frac{\varepsilon}{2\mathrm{e}}$ for all sufficiently large n, since g_{t_n} converges towards g in $L^2(Q_{3R})$. Now we estimate the last term on the right-hand side of (3.16) as follows. Recall that $\widehat{g}^2 = \frac{1}{\mathrm{e}} \mathrm{e}^{\widehat{\psi}/\rho}$ and use the reversed triangle inequality to estimate

$$\begin{aligned}\left\| g_{t_n}^2 - \frac{1}{\mathrm{e}} \mathrm{e}^{(\psi_{t_n} + c)/\rho} \right\|_{1,3R} &\ge \int_{Q_R} \left| \frac{1}{\mathrm{e}} \mathrm{e}^{[(\psi_{t_n}(x+y)) + c]/\rho} - g_{t_n}^2(x+y) \right| \mathrm{d}y \\ &\ge \mathrm{e}^{c/\rho} \int_{Q_R} \frac{1}{\mathrm{e}} \left| \mathrm{e}^{(\psi_{t_n}(x+y))/\rho} - \mathrm{e}^{\widehat{\psi}(y)/\rho} \right| \mathrm{d}y \\ &\quad - \left|\mathrm{e}^{c/\rho} - 1\right| \int_{Q_R} \frac{1}{\mathrm{e}} \mathrm{e}^{\widehat{\psi}(y)/\rho} \,\mathrm{d}y - \int_{Q_R} \left|\widehat{g}^2(y) - g_{t_n}^2(x+y)\right| \mathrm{d}y.\end{aligned} \tag{3.17}$$

Recall that $\psi_{t_n} \in \Gamma_{R,\varepsilon}$, i.e., we have in particular that $\int_{Q_R} \frac{1}{\mathrm{e}} |\mathrm{e}^{(\psi_{t_n}(x+y))/\rho} - \mathrm{e}^{\widehat{\psi}(y)/\rho}| \,\mathrm{d}y \geq \frac{\varepsilon}{\mathrm{e}}$, see (2.5). Furthermore, we use the estimate for c in (3.14) and the above mentioned one for the distance between $g_{t_n}^2$ and $\widehat{g}^2$ to see that, for $\beta > 0$ small enough, the right-hand side of (3.17) is positive and may be estimated by

$$\left\| g_{t_n}^2 - \frac{1}{\mathrm{e}} \mathrm{e}^{(\psi_{t_n}+c)/\rho} \right\|_{1,3R} \geq \mathrm{e}^{c/\rho} \frac{\varepsilon}{\mathrm{e}} - \mathrm{e}^{c/\rho} \frac{\beta}{\rho} - \frac{\varepsilon}{2\mathrm{e}}. \tag{3.18}$$

If one picks $\beta > 0$ so small that $\mathrm{e}^{c/\rho} \geq 3/4$ and $\mathrm{e}^{c/\rho} \frac{\beta}{\rho} \leq \frac{\varepsilon}{8\mathrm{e}}$, then the right-hand side of (3.18) is not smaller than $\frac{\varepsilon}{8\mathrm{e}}$. This ends the proof. □

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References

[1] N. H. Bingham, C. M. Goldie and J. L. Teugels. *Regular Variation*. Cambridge Univ. Press, 1987. MR0898871

[2] M. Biskup and W. König. Long-time tails in the parabolic Anderson model with bounded potential. *Ann. Probab.* **29** (2001) 636–682. MR1849173

[3] E. Bolthausen. Localization of a two-dimensional random walk with an attractive path interaction. *Ann. Probab.* **22** (1994) 875–918. MR1288136

[4] E. A. Carlen. Superadditivity of Fisher's information and logarithmic Sobolev inequality. *J. Funct. Anal.* **101** (1991) 194–211. MR1132315

[5] R. Carmona and S. A. Molchanov. Parabolic Anderson problem and intermittency. *Mem. Amer. Math. Soc.* **108** (1994) 1–125. MR1185878

[6] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications*, 2nd edition. Springer, New York, 1998. MR1619036

[7] J. Gärtner and F. den Hollander. Correlation structure of intermittency in the parabolic Anderson model. *Probab. Theory Related Fields* **114** (1999) 1–54. MR1697138

[8] J. Gärtner and W. König. The parabolic Anderson model. In *Interacting Stochastic Systems* 153–179. J.-D. Deuschel and A. Greven (Eds). Springer, Berlin, 2005. MR2118574

[9] J. Gärtner, W. König and S. Molchanov. Geometric characterization of intermittency in the parabolic Anderson model. *Ann. Probab.* **35** (2007) 439–499. MR2308585

[10] J. Gärtner and S. Molchanov. Parabolic problems for the Anderson model I. Intermittency and related topics. *Commun. Math. Phys.* **132** (1990) 613–655. MR1069840

[11] J. Gärtner and S. Molchanov. Parabolic problems for the Anderson model II. Second-order asymptotics and structure of high peaks. *Probab. Theory Related Fields* **111** (1998) 17–55. MR1626766

[12] R. van der Hofstad, W. König and P. Mörters. The universality classes in the parabolic Anderson model. *Commun. Math. Phys.* **267** (2006) 307–353. MR2249772

[13] E. H. Lieb and M. Loss. *Analysis*, 2nd edition. *AMS Graduate Studies* **14**. Amer. Math. Soc., Providence, RI, 2001. MR1817225

[14] S. Molchanov. Lectures on random media. In *Lectures on Probability Theory. Ecole d'Eté de Probabilités de Saint-Flour XXII-1992* 242–411. D. Bakry, R. D. Gill and S. Molchanov (Eds). *Lecture Notes in Math.* **1581**. Springer, Berlin, 1994. MR1307415

[15] T. Povel. Confinement of Brownian motion among Poissonian obstacles in R^d, $d \geq 3$. *Probab. Theory Related Fields* **114** (1999) 177–205. MR1701519

[16] A.-S. Sznitman. On the confinement property of two-dimensional Brownian motion among Poissonian obstacles. *Comm. Pure Appl. Math.* **44** (1991) 1137–1170. MR1127055