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Large deviations for voter model occupation times in two dimensions

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Abstract. We study the decay rate of large deviation probabilities of occupation times, up to time t, for the voter model $\eta:\mathbb{Z}^2\times[0,\infty)\to\{0,1\}$ with simple random walk transition kernel, starting from a Bernoulli product distribution with density $\rho\in(0,1)$. In [*Probab. Theory Related Fields* **77** (1988) 401–413], Bramson, Cox and Griffeath showed that the decay rate order lies in $[\log(t),\log^2(t)]$.

In this paper, we establish the true decay rates depending on the level. We show that the decay rates are $\log^2(t)$ when the deviation from ρ is maximal (i.e., $\eta\equiv 0$ or 1), and $\log(t)$ in all other situations. This answers some conjectures in [*Probab. Theory Related Fields* **77** (1988) 401–413] and confirms nonrigorous analysis carried out in [*Phys. Rev. E* **53** (1996) 3078–3087], [*J. Phys. A* **31** (1998) 5413–5429] and [*J. Phys. A* **31** (1998) L209–L215].

Résumé. On étudie le taux de décroissance des probabilités de grandes déviations des temps d'occupation, jusqu'à l'instant t, du modèle du votant $\eta:\mathbb{Z}^2\times[0,\infty)\to\{0,1\}$ ayant le noyau de transition d'une marche aléatoire simple et partant d'une distribution produit de Bernoulli de paramètre $\rho\in(0,1)$. Dans [*Probab. Theory Related Fields* **77** (1988) 401–413], Bramson, Cox et Griffeath ont montré que l'ordre du taux de décroissance se situe dans $[\log(t),\log^2(t)]$.

Dans cet article, nous établissons les taux de décroissance exacts dépendant du niveau. On prouve que les taux de décroissance sont $\log^2(t)$ lorsque la déviation de ρ est maximale (i.e., $\eta\equiv 0$ ou 1), et $\log(t)$ dans toutes les autres situations. Ceci répond à une conjecture de [*Probab. Theory Related Fields* **77** (1988) 401–413] et confirme l'analyse non rigoureuse effectuée dans [*Phys. Rev. E* **53** (1996) 3078–3087], [*J. Phys. A* **31** (1998) 5413–5429] et [*J. Phys. A* **31** (1998) L209–L215].

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1. Introduction and main results

1.1. *The voter model*

We consider the simple voter model in $\mathbb{Z}^2$ corresponding to the simple random walk. In general dimensions this voter model is a Markov process on $\{0,1\}^{\mathbb{Z}^d}$ with operator

$$\Omega f(\eta)=\frac{1}{2d}\sum_{x\in\mathbb{Z}^d}\sum_{y\sim_n x}\big(f\big(\eta^{x,y}\big)-f(\eta)\big), \tag{1.1.1}$$

where $x \sim_n y$ means x and y are nearest neighbours on the $\mathbb{Z}^d$ lattice and $\eta^{x,y}$ is the configuration

$$\begin{cases} \eta^{x,y}(z) = \eta(z) & \text{for } z \neq x, \\ \eta^{x,y}(x) = \eta(y). \end{cases} \tag{1.1.2}$$

This process was introduced independently by Clifford and Sudbury [3] and by Holley and Liggett [11]. There the basic results concerning equilibria were shown: for recurrent random walks (i.e., $d \leq 2$) the only extremal equilibria are δ_0 and δ_1 whereas for transient random walks there exists for each $\rho \in [0,1]$ an extremal, translation invariant ergodic equilibrium of density ρ, μ_ρ (and these are the totality of extremal equilibria). In the transient case the measures μ_ρ are the limits for distributions of the process begun with initial measure ν_ρ for which $(\eta(x)\colon x \in \mathbb{Z}^d)$ are i.i.d. Bernoulli (ρ) random variables. Details for this and much more can be found in Liggett [14].

In this note our analysis will rely heavily on the duality of the voter model with coalescing random walks (as exploited in [2] and [4–6]): given distinct space time points $(x_i, t_i)_{i=1}^r$ in $\mathbb{Z}^d \times [0,\infty)$, the joint distribution of $(\eta_{t_i}(x_i))_{i=1}^r$ can be determined via coalescing random walks $(\chi_t^i\colon t \geq 0)$ defined as follows: (suppose without loss of generality that $0 \leq t_1 \leq t_2 \leq \cdots \leq t_r$) $\chi_t^i = x_i$ for $0 \leq t \leq t_r - t_i$, thereafter χ^i evolves as a simple random walk. If for $i < j$, $s \geq t_r - t_i$, $\chi_s^i = \chi_s^j$, then $\chi_{s'}^i = \chi_{s'}^j$ for all $s' \geq s$. That is the random walks are coalescing. Otherwise the random walks evolve independently. The joint law of $(\eta_{t_1}(x_1), \eta_{t_2}(x_2), \ldots, \eta_{t_r}(x_r))$ is that of $(\eta_0(\chi_{t_r}^1), \eta_0(\chi_{t_r}^2), \ldots, \eta_0(\chi_{t_r}^r))$. The clear exposition of the Harris construction of the voter model found in Durrett [8] is here recommended.

We will in this article be concerned with the behaviour, for t large, of

$$\frac{T_t}{t} = \frac{1}{t}\int_0^t \eta_s(0)\,\mathrm{d}s \tag{1.1.3}$$

for $(\eta_s\colon s \geq 0)$ a voter model begun with initial measure ν_ρ, $\rho \in (0,1)$. In the transient regime, the behaviour is equivalent to that for a voter model begun with initial distribution μ_ρ. This problem was discussed in a series of papers by Cox and Griffeath [5] and [6] and Bramson, Cox and Griffeath [2]. It follows from the duality description also, as noted in these articles, that T_t may be understood as follows.

A Harris system for the voter model $(\eta_t\colon t \geq 0)$ is a collection of independent rate $\frac{1}{2d}$ Poisson processes $N^{x,y}$ for every ordered pair x, y with $y \sim_n x$. From this system $\eta_\cdot$ evolves by stipulating that for $x \in \mathbb{Z}^d$, $\eta_t(x)$ changes value (or flips) only at times t in $N^{x,y}$ for some $y \sim_n x$. At such a time t we put $\eta_t(x) = \eta_t(y)$. If for $t \in N^{x,y}$, $\eta_{t-}(x) = \eta_t(y)$ then there is no change in value for $\eta_\cdot(x)$ at time t. Given this system we can define for each $x \in \mathbb{Z}^d$ and $t \geq 0$ dual simple random walks, $(Z_s^{x,t}\colon 0 \leq s \leq t)$ with $Z_0^{x,t} = x$, as follows:

$$Z_s^{x,t} \neq Z_{s-}^{x,t} \quad \Longleftrightarrow \quad t - s \in N^{Z_{s-}^{x,t}, w} \tag{1.1.4}$$

for some $w \sim_n Z_{s-}^{x,t}$. In which case $Z_\cdot^{x,t}$ jumps from $Z_{s-}^{x,t}$ to w at time s.

The importance of these random walks lies in the following properties:

(1) $\eta_t(x) = \eta_0(Z_t^{x,t})$ and
(2) the random walks $\{Z_\cdot^{x,t}\}_{x \in \mathbb{Z}^d}$ are independent until they meet.

Furthermore it may be seen that if $0 \leq t \leq t'$ then the random walks $Z_\cdot^{x,t}$ and $Z_\cdot^{x,t'}$ are coalescing in the sense that

(3) if for some $s \in [0,t]$, $Z_s^{x,t} = Z_{t'-t+s}^{x,t'}$, then $Z_u^{x,t} = Z_{t'-t+u}^{x,t'}$ for all $s \leq u \leq t$.

Thus we have a system of coalescing random walks $(\chi_v^s, v \in [0,s]) = Z_v^{0,s}$ on $\mathbb{Z}^d$, so that $\chi_0^s = 0$ and by property (3) above if for $0 \leq v \leq s \leq s'$, $\chi_v^s = \chi_{s'-s+v}^{s'}$, then $\chi_u^s = \chi_{s'-s+u}^{s'}$ for all $v \leq u \leq s$. We call the collection of random walks $\{\chi_\cdot^s\}_{s \geq 0}$ the coalescing random walks associated with $\eta_\cdot(0)$.

Let $O_x = \lambda_t(\{s \in [0,t]\colon \chi_s^s = x\})$ with λ_t the Lebesgue measure on $[0,t]$, then

$$T_t = \sum_{x \in \{\chi_s^s : s \in [0,t]\}} O_x \eta_0(x). \tag{1.1.5}$$

For η_0 distributed as product measure ν_ρ the duality representation immediately yields

$$\begin{aligned}\operatorname{Var}\left(\frac{T_t}{t}\right) &= \frac{1}{t^2}\int_0^t\int_0^t \operatorname{Cov}\big(\eta_s(0),\eta_{s'}(0)\big)\,\mathrm{d}s'\,\mathrm{d}s \\ &= \frac{1}{t^2}\int_0^t\int_0^t P\big(\chi_s^s=\chi_{s'}^{s'}\big)\rho(1-\rho)\,\mathrm{d}s'\,\mathrm{d}s. \qquad (1.1.6)\end{aligned}$$

For $s>s'$, $P(\chi_s^s=\chi_{s'}^{s'})$ is easily seen to be the probability that a random walk issuing from the origin hits the origin during the interval $[s-s', s+s']$. If one chooses s, s' uniformly on $[0,t]$ this probability is easily seen to tend to zero as $t\to\infty$ for transient random walks. However for recurrent random walks it may tend to zero as $t\to\infty$ (for $d=2$) or it may tend to a non zero limit ($d=1$). From this we obtain: for η_0 distributed by ν_ρ,

$$\frac{T_t}{t}\longrightarrow\rho \quad \text{in probability} \quad \text{if and only if} \quad d\ge 2. \qquad (1.1.7)$$

In fact, the convergence in (1.1.7) holds a.s. (see Cox and Griffeath [5]).

1.2. *Asymptotic behavior of occupation time*

Bramson, Cox and Griffeath [2] obtained large deviation bounds: for each $\alpha\in(\rho,1]$ there exist positive finite constants $C_1=C_1(d)$, $C_2=C_2(d,\alpha)$ such that, for t sufficiently large,

$$\begin{cases}\mathrm{e}^{-C_1\log^2(t)}\le\mathbb{P}_{\nu_\rho}(T_t\ge\alpha t)\le\mathrm{e}^{-C_2\log(t)} & \text{if } d=2,\\ \mathrm{e}^{-C_1 b_t}\le\mathbb{P}_{\nu_\rho}(T_t\ge\alpha t)\le\mathrm{e}^{-C_2 b_t} & \text{if } d\ge 3,\end{cases} \qquad (1.2.1)$$

with

$$b_t=\begin{cases}\sqrt{t} & \text{if } d=3,\\ \dfrac{t}{\log t} & \text{if } d=4,\\ t & \text{if } d\ge 5.\end{cases} \qquad (1.2.2)$$

By symmetry arguments, the large deviation regime is the same for the deviations $T_t/t\le\alpha$ with $\alpha\in[0,\rho)$.

1.3. *Results*

Given the bounds of [2] cited in the previous section, in so far as the exponential order of large deviations is concerned, the only outstanding case is the two-dimensional one. Throughout the rest of the paper, we assume that $d=2$. The following two results constitute a full resolution of the question of exponential order for the large deviations of T_t.

Theorem 1.3.1. *There exist positive finite constants C_1, C_2 such that, for t sufficiently large,*

$$\mathrm{e}^{-C_1\log^2(t)}\le\mathbb{P}_{\nu_\rho}(T_t=t)\le\mathrm{e}^{-C_2\log^2(t)}. \qquad (1.3.1)$$

Theorem 1.3.2. *For each $\alpha\in(\rho,1)$, there exist positive finite constants $C_1=C_1(\alpha)$, $C_2=C_2(\alpha)$ such that, for t sufficiently large,*

$$\mathrm{e}^{-C_1\log(t)}\le\mathbb{P}_{\nu_\rho}(T_t\ge\alpha t)\le\mathrm{e}^{-C_2\log(t)}. \qquad (1.3.2)$$

By (1.2.1), it only remains to prove the upper bound in Theorem 1.3.1 and lower bound in Theorem 1.3.2. If $g(t)$ and $h(t)$ are real functions, we write $g(t)\asymp h(t)$ as $t\to\infty$ when

$$0<\liminf_{t\to\infty} g(t)/h(t)\le\limsup_{t\to\infty} g(t)/h(t)<\infty. \qquad (1.3.3)$$

2. Discussion

The study of T_t was initiated by Cox and Griffeath [5] who noted that the question of its large deviations belonged naturally with related issues arising in the Ising and percolation models, but that in contrast (and due to the tractable duality) with these, progress in identifying the effect at low dimensions was possible. Nevertheless questions remain.

The behavior of T_t in low dimensions has motivated studies in the Physics community. Due to the recurrence of simple random walks, as $t \to \infty$, the simple voter model forms larger and larger clusters when $d \le 2$ (a more detailed analysis of clustering can be found in [6]). Therefore, a consensus of opinion is approached as $t \to \infty$. In words, that means that the system coarsens. A natural question is to study, for such a coarsening system, the asymptotic behavior of the persistence probability $\mathbb{P}(T_t = t)$, i.e., the probability that a given site will never change its state as time goes to infinity. To be in accordance with the physicist terminology, consider the voter model $\zeta : \mathbb{Z}^d \times [0, \infty) \to \{-1, 1\}$ (as a spin system) with opinions -1 and $+1$. Define the *mean magnetization* at time t by

$$M(t) = \frac{1}{t} \int_0^t \zeta(0, s)\, \mathrm{d}s, \qquad M(t) \in [-1, 1]. \tag{2.0.4}$$

In the case considered the initial distribution was symmetric w.r.t. -1 and 1 and so $\mathbb{E}(M(t)) = 0$. Then for all $x > 0$, the distribution of the mean magnetization, $P(t, x) = \mathbb{P}(M(t) \ge x)$, and $R(t, x) = P(M(s) \ge x, \forall s \le t)$ represent the *deviation* of $M(t)$ from its mean and the probability of *persistent large deviations*, respectively. Then, assuming that $\zeta(0, 0) = 1$,

$$P(t, 1) = R(t, 1) = \mathbb{P}\big(\zeta(0, s) = 1, \forall 0 \le s \le T\big) \tag{2.0.5}$$

is the so-called *persistent probability* and corresponds to the object of study of Theorem 1.3.1. Ben-Naim, Frachebourg and Krapivsky [1] showed convincingly via numerical methods that there exists some $C > 0$ such that

$$P(t, 1) \asymp \mathrm{e}^{-C \log^2(t)}, \quad t \gg 1. \tag{2.0.6}$$

Howard and Godrèche [12] confirm nonrigorously this result both by using path-integral methods and Monte Carlo simulations. After a sharper analysis, Dornic and Godrèche [7] concluded that

$$P(t, x) \asymp \mathrm{e}^{-I(x) \log(t)} \quad \text{and} \quad R(t, x) \asymp \mathrm{e}^{-J(x) \log^2(t)}, \quad t \gg 1 \tag{2.0.7}$$

with $\lim_{x \to 1} I(x) = \infty$ and $\lim_{x \to 1} J(x) = C$ for some constant $C > 0$. This is in accordance with Theorems 1.3.1 and 1.3.2.

3. Proofs

3.1. *Proof of Theorem* 1.3.1

Let $\chi = (\chi^t)_{t \ge 0} = (\chi^t_s, s \in [0, t])_{t \ge 0}$ be the coalescing random walks associated with $\eta_\cdot(0)$ for a voter model $(\eta_t\colon t \ge 0)$. Denote by P and E, respectively, probability and expectation associated with χ. The dual relationship between voter model and coalescing random walks lead to the following lemma (see Bramson, Cox and Griffeath [2], Section 1 for details).

Lemma 3.1.1. *For all $t > 0$*

$$\mathbb{P}_{\nu_\rho}(T_t = t) = E\big(\rho^{\#\chi^t}\big), \tag{3.1.1}$$

where $\#\chi^t$ denote the number of distinct sites in the collection $\{\chi^s_s\colon 0 \le s \le t\}$.

Then, the proof of Theorem 1.3.1 reduces to the following proposition.

Proposition 3.1.2. *There exist $K_1, K_2 > 0$ so that*

$$P\big(\#\chi^t \le K_1 \log^2(t)\big) \le \mathrm{e}^{-K_2 \log^2(t)} \tag{3.1.2}$$

for all $t > 0$ sufficiently large.

Indeed, combining Lemma 3.1.1 and Proposition 3.1.2, we get

$$\begin{aligned}\mathbb{P}_{\nu_\rho}(T_t = t) &= E\big(\rho^{\#\chi^t}\mathbb{1}\{\#\chi^t \le K_1 \log^2(t)\}\big) + E\big(\rho^{\#\chi^t}\mathbb{1}\{\#\chi^t > K_1 \log^2(t)\}\big) \\ &\le P\big(\#\chi^t \le K_1 \log^2(t)\big) + \rho^{K_1 \log^2(t)} \\ &\le \mathrm{e}^{-C_2 \log^2(t)},\end{aligned} \tag{3.1.3}$$

where in the last inequality we choose K_1 small enough and t sufficiently large. This completes the proof of Theorem 1.3.1. The next section is devoted to the proof of Proposition 3.1.2.

3.2. *Proof of Proposition* 3.1.2

The overall strategy is to show that on an interval $[3t/4, t]$ with probability of the order $1 - \mathrm{e}^{-C_1 \log(t)}$ for some universal $C_1 > 0$, the stream of coalescing random walks produces $C_1 \log(t)$ distinct random walks which hit the annulus $B(0, \sqrt{2t}) \setminus B(0, \sqrt{t})$, where $B(0, t) = \{x \in \mathbb{Z}^2\colon |x| \le t\}$ $(t \ge 0)$, before time $t/2$ and do not leave in dual time $[t/2, t]$. If we call this event A_t, then it can be shown that $A_t, A_{t/2}, A_{t/4}, \ldots$ are independent, each producing with probability $1 - \mathrm{e}^{-C_1 \log(t)}$, of the order $\log(t)$ distinct random walks. This will be enough to show Proposition 3.1.2.

In order to prove Proposition 3.1.2, we need a number of preparatory results concerning ordinary and coalescing random walks. Let $X = (X(u)\colon u \ge 0)$ be a simple random walk on $\mathbb{Z}^2$ with continuous time transition probability kernel $p_u(\cdot)$. Denote by P^x its probability law starting from $x \in \mathbb{Z}^2$ and for all $y \in \mathbb{Z}^2$, $t > 0$, let

$$\tau_y = \inf\{u > 0\colon X(u) = y\} \quad \text{and} \quad \sigma_t = \inf\{u > 0\colon |X(u)| \ge t\}. \tag{3.2.1}$$

We refer to Lawler [13] for hitting probabilities for the two-dimensional simple random walk:

Lemma 3.2.1. *Uniformly for $x \in \mathbb{Z}^2 \setminus \{0\}$, $|x| \le \sqrt{t}$,*

$$P^x(\tau_0 < \sigma_{\sqrt{t}}) \asymp \frac{\log(\sqrt{t}) - \log(|x|)}{\log(\sqrt{t})} \quad \text{as } t \to \infty \tag{3.2.2}$$

and

$$P^x(\tau_0 < t) \asymp \frac{\log(\sqrt{t}) - \log(|x|) + 1}{\log(\sqrt{t})} \quad \text{as } t \to \infty. \tag{3.2.3}$$

Proof. The proof can be found in Lawler [13], Proposition 1.6.7 in the case of discrete time random walks. The transfer to continuous time is easy. □

We now consider two independent simple random walks $\{X(u)\colon u \ge 0\}$ and $\{Y(u)\colon u \ge s\}$, both starting from 0 in the sense that $X(0) = 0 = Y(s)$. We are interested in the probability that

$$\{\exists s \le u \le t\colon X(u) = Y(u)\} := A^{X,Y}(s, t). \tag{3.2.4}$$

Lemma 3.2.2. *There exists positive constants K_3, K_4 so that for $s \in (t/\log(t), t/2)$ and t large,*

$$K_3 \frac{\log(t) - \log(s)}{\log(t)} \le P\big(A^{X,Y}(s, t)\big) \le K_4 \frac{\log(t) - \log(s)}{\log(t)}. \tag{3.2.5}$$

Proof. We first show the lower bound $P(A^{X,Y}(s,t))$. We condition on the value of $X(s)$. Thus, $P(A^{X,Y}(s,t)|X(s)=x)$ is equal to $P^x(\tau_0 < 2(t-s))$, since $(Y(s+u)-X(u))_{u\geq 0}$ is a speed two random walk. Then given the constraints on s we have

$$P\big(A^{X,Y}(s,t)|X(s)=x\big) \geq P^x(\tau_0 < t). \tag{3.2.6}$$

So

$$\begin{aligned} P\big(A^{X,Y}(s,t)\big) &\geq \sum_{|x|\leq\sqrt{s}/2} P\big(X(s)=x\big)P^x(\tau_0<t) \\ &\geq C\sum_{|x|\leq\sqrt{s}/2} P\big(X(s)=x\big)\frac{\log(t)-\log(s)}{\log(t)}, \end{aligned} \tag{3.2.7}$$

by Lemma 3.2.1, for universal strictly positive C. This in turn is

$$\geq CC'\frac{\log(t)-\log(s)}{\log(t)}, \tag{3.2.8}$$

by the central limit for $X(s)$. For the opposite inequality we obtain, arguing similarly, that

$$P\left(A^{X,Y}(s,t)\cap\left\{\big|Y(s)\big|\geq\frac{\sqrt{s}}{2}\right\}\right)\leq C''\frac{\log(t)-\log(s)}{\log(t)}. \tag{3.2.9}$$

So it suffices to bound appropriately

$$\begin{aligned} P\left(A^{X,Y}(s,t)\cap\left\{\big|Y(s)\big|<\frac{\sqrt{s}}{2}\right\}\right) &= \sum_{i=1}^{\lceil\log_2(\sqrt{s})\rceil} P\big(A^{X,Y}(s,t)\cap\big\{\big|Y(s)\big|\in\big[\sqrt{s}2^{-i-1},\sqrt{s}2^{-i}\big)\big\}\big) \\ &\quad + P\big(Y(s)=0\big). \end{aligned} \tag{3.2.10}$$

Given the condition that $s\leq t/2$,

$$\frac{\log(t)-\log(s)}{\log(t)}\geq\frac{\log(2)}{\log(t)}\gg P\big(Y(s)=0\big) \tag{3.2.11}$$

for $s\geq t/\log(t)$, so we may ignore the term $P(Y(s)=0)$. By the local central limit theorem (see e.g. Durrett [9]),

$$P\big(Y(s)\in\big[\sqrt{s}2^{-i-1},\sqrt{s}2^{-i}\big)\big)\leq K2^{-2i} \tag{3.2.12}$$

for universal K. By Lemma 3.2.1 and given the condition that $s\in(t/\log(t),t/2)$,

$$P\big(A^{X,Y}(s,t)\big|\big|Y(s)\big|\in\big[\sqrt{s}2^{-i-1},\sqrt{s}2^{-i}\big)\big)\leq\frac{1}{\log(t)}\big(\log(t)-\log(s)+(2i+3)\log(2)+2\big). \tag{3.2.13}$$

Combining (3.2.12) and (3.2.13), we get

$$\begin{aligned} &\sum_{i=1}^{\lceil\log_2(\sqrt{s})\rceil} P\big(A^{X,Y}(s,t)\cap\big\{\big|Y(s)\big|\in\big[\sqrt{s}2^{-i-1},\sqrt{s}2^{-i}\big)\big\}\big) \\ &\quad\leq K\sum_{i=1}^{\lceil\log_2(\sqrt{s})\rceil} 2^{-2i}\frac{\log(t)-\log(s)+(2i+3)\log(2)+2}{\log(t)} \leq K'\frac{\log(t)-\log(s)}{\log(t)}, \end{aligned} \tag{3.2.14}$$

for some $K'>0$ and we are done. □

Corollary 3.2.3. *Given $C > 1$ let*

$$R = \left\lceil \frac{\log(t)}{5C} \right\rceil \tag{3.2.15}$$

and let $(Y^k(t)\colon\ t \geq t_k)$, $0 \leq k \leq R$, be independent random walks starting at

$$Y^k(t_k) = 0 \quad \text{with } t_k = \frac{kCt}{\log(t)}. \tag{3.2.16}$$

Then, there exists some universal (not depending on C) strictly positive K_5 so that, for all t sufficiently large,

$$E(V) \leq \frac{K_5}{C} \quad \text{with } V = \sum_{k=1}^{R} \mathbb{1}\left\{A^{Y^0,Y^k}(t_k, t)\right\}. \tag{3.2.17}$$

Remark 3.2.4. *$E(V|Y^0)$ is a functional of the random walk path independent of the random walks Y^k, $1 \leq k \leq R$, and can and will be considered as defined for any random walk starting at the origin, see Corollary* 3.2.8.

Proof of Corollary 3.2.3. By Lemma 3.2.2, for all t sufficiently large

$$\begin{aligned} E(V) &\leq -K_5 \sum_{k=1}^{\lceil \log(t)/(5C) \rceil} \frac{\log(kC/\log(t))}{\log(t)} \leq -\frac{K_5}{C} \int_0^{\log(t)/(5C)+1} \frac{C}{\log(t)} \log\left(\frac{Cx}{\log(t)}\right) \mathrm{d}x \\ &\leq -\frac{K_5}{C} \int_0^1 \log(x)\, \mathrm{d}x = \frac{K_5}{C}. \end{aligned} \tag{3.2.18}$$

□

We now collect a few nice properties of our random walks: let $(X(u)\colon\ u \geq 0)$ be a simple random walk starting at $X(0) = 0$. For $t \geq 0$, recall that $B(0,t) = \{x \in \mathbb{Z}^2\colon\ |x| \leq t\}$.

Lemma 3.2.5. *For all $t \geq 0$ and for whatever finite choice of $C \geq 1$,*

$$P^0\left(X(u) \in B\left(0, t^{1/3}\right) \text{ for some } u \in \left(t_1 - 1, t\right)\right) \longrightarrow 0 \quad \text{as } t \to \infty, \tag{3.2.19}$$

for $t_1 = Ct/\log(t)$.

Remark 3.2.6. *We will explain the choice of $t_1 - 1$ later (see Remark* 3.2.10).

Proof of Lemma 3.2.5. First, remark that

$$P\left(X(t_1) \geq \frac{\sqrt{t}}{\log(t)}\right) \uparrow 1 \quad \text{as } t \to \infty. \tag{3.2.20}$$

For any random process $(Z(t)\colon\ t \geq 0)$ on $\mathbb{Z}^2$ denote

$$S_t(Z) = \inf\left\{s\colon\ \left|Z(s)\right| \leq t\right\}. \tag{3.2.21}$$

Therefore, in order to prove (3.2.19), it suffices to prove that for all $|x| \geq \sqrt{t}/\log(t)$

$$P^x\left(S_{t^{1/3}}(X) < t\right) \to 0 \quad \text{as } t \to \infty. \tag{3.2.22}$$

But this follows from random walks embedding into Brownian motions and the fact that (3.2.22) is fulfilled when a two-dimensional Brownian motion is considered instead of X. □

The following is simply a consequence of the invariance principle.

Lemma 3.2.7. *As* $t \to \infty$,

$$P^0\left(\left|X(u)\right| \in \left(\sqrt{t}, \sqrt{2t}\right) \forall \frac{t}{4} \le u \le t\right) \to P\left(\left|B(u)\right| \in \left(1, \sqrt{2}\right) \forall \frac{1}{4} \le u \le 1\right) = \alpha > 0, \tag{3.2.23}$$

where B *denotes a standard two-dimensional Brownian motion*.

We are ready to choose our constant C: we choose C so that for K_5 as in Corollary 3.2.3 and α as above,

$$\frac{K_5}{C} \le \frac{\alpha^2}{10^4}. \tag{3.2.24}$$

Corollary 3.2.8. *For* $E(V|X)$ *as defined in Remark* 3.2.4 *and* t *sufficiently large*, *the probability that the path* $\{(u, X(u))\colon 0 \le u \le t\}$ *is such that either*:

(i) $E(V|X) \ge \alpha/10^2$, *or*
(ii) $|X(u)| \notin (\sqrt{t}, \sqrt{2t})$ *for some* $u \in (t/4, t]$, *or*
(iii) $|X(u)| < t^{1/3}$ *for some* $u \in [t_1 - 1, t]$,

is at most $1 - 2\alpha/3$.

Proof. By Corollary 3.2.3 and our choice of C, we have

$$P\left(E(V|X) \ge \frac{\alpha}{10^2}\right) \le \frac{10^2 K_5}{\alpha C} \le \frac{\alpha}{10^2}. \tag{3.2.25}$$

Then, combining Lemmas 3.2.5–3.2.7 and (3.2.25), we get the claim. $\square$

We consider the system of coalescing random walks $(X^i(s)\colon t_i \le s \le t, 0 \le i \le R) = (\chi_{s-t_i}^{t-t_i}\colon t_i \le s \le t, 0 \le i \le R)$. We are interested in the number of distinct random walks at time t which satisfy

$$\left|X^i(u)\right| \in \left(\sqrt{t}, \sqrt{2t}\right), \quad \forall u \in \left[\frac{t}{2}, t\right], \tag{3.2.26}$$

where X^i, $0 \le i \le R$, are coalescing random walks defined in (3.2.16). We will in turn let the random walks evolve until something "bad" happens. This will mean the violation of some given conditions: Define times:

(a) $T^{i,a} = \inf\{s \ge t_{i+1}\colon \sum_{j=i+1}^R P(X^j(v) = X^i(v) \text{ for some } v \in [t_{i+1}, s] | X^i) \ge \alpha/10^2\}$;
(b) $T^{i,b} = \inf\{s \ge t_{i+1} - 1\colon |X^i(s)| \le t^{1/3}\}$;
(c) $T^{i,c} = \inf\{s \ge t_i + t/4\colon |X^i(s)| \notin (\sqrt{t}, \sqrt{2t})\}$;
(d) $T^i = t \wedge T^{i,a} \wedge T^{i,b} \wedge T^{i,c}$,

and kill (or freeze) the random walk X^i at time T^i.

Remark 3.2.9. *Note that in* (c), *since for all* $0 \le i \le R$, $t_i \le t/4$, X^i *will satisfy* (3.2.26) *if* $T^i = t$. *Note that in* (a), *because the coalescing random walks are stopped as soon as they meet and are independent up until they meet*, *we can apply Corollary* 3.2.3.

We first consider the consequence of our definition of T^i: we define the random variables $C_{i,j}$, $0 \le i < j \le R$, by

$$C_{i,j} = P\left(X^j(v) = X^i(v) \text{ for some } v \in \left[t_j, T^i\right] \middle| X^i, T^i\right). \tag{3.2.27}$$

We have for any $j \in \{i+1, i+2, \dots, R\}$ that

$$\begin{aligned} C_{i,j} &= P\left(X^j(v) = X^i(v) \text{ for some } v \in \left[t_i, T^i\right) \middle| X^i, T^i\right) \\ &\quad + P\left(X^j\left(T^i\right) = X^i\left(T^i\right), X^j(v) \ne X^i(v), \forall v < T^i \middle| X^i, T^i\right). \end{aligned} \tag{3.2.28}$$

By the definition of $T^{i,b}$, $|X^i(T^i)| \geq t^{1/3} - 1$ unless $T^i < t_{i+1}$, in which case $\{(X^j(s), s)\colon s \geq t_j\}$ cannot hit $\{(X^i(u), u)\colon t_i \leq u \leq T^i\}$. Therefore, using a simple bound for $p_s(\cdot)$ (use e.g. continuous version of Lawler [13], Theorem 1.2.1, inequality (1.10)), there exists some universal $K > 0$ so that

$$\begin{aligned} P\bigl(X^j\bigl(T^i\bigr) = X^i\bigl(T^i\bigr), X^j(v), \neq X^i(v)\, \forall v < T^i \bigm| X^i, T^i\bigr) &\leq \sup_{|x| \geq t^{1/3}-1} \sup_{u \geq 0} p_u(x) \\ &\leq \frac{K}{(t^{1/3}-1)^2}. \end{aligned} \tag{3.2.29}$$

Remark 3.2.10. *It is above all here we see the validity of the of the definition of $T^{i,b}$, since this assures that for any $t_{i+1} \leq s \leq T^i$, $|X^i(s)| \geq t^{1/3} - 1$. Obviously the 1 is arbitrary and could be replace by any $\lambda > 0$.*

Combining (3.2.28) and (3.2.29) and summing over $i \leq j \leq R$ with $i < R$, we obtain (recalling (a))

$$\begin{aligned} \sum_{j=i+1}^{R} C_{i,j} &\leq \frac{\alpha}{10^2} + \frac{RK}{(t^{1/3}-1)^2} \\ &< \frac{\alpha}{99}, \end{aligned} \tag{3.2.30}$$

for t sufficiently large.

Definition 3.2.11. *We say $1 \leq j \leq R$ is* good *if*

$$\sum_{i=0}^{j-1} C_{i,j} \leq \frac{2\alpha}{99}. \tag{3.2.31}$$

Lemma 3.2.12. *At least $R/2$ of the j are good.*

Proof. By (3.2.30), we have

$$\sum_{i=0}^{R-1} \sum_{j=i+1}^{R} C_{i,j} \leq \frac{R\alpha}{99}. \tag{3.2.32}$$

Thus,

$$\sum_{j=1}^{R} \sum_{i=0}^{j-1} C_{i,j} \leq \frac{R\alpha}{99}, \tag{3.2.33}$$

from which we obtain the result. □

Definition 3.2.13. *We say a random walk $\{X^j(t_j + u)\colon u \geq 0\}$ is* successful *if*

(i) *the stopping time T^j is equal to t;*
(ii) *X^j does not hit a previous stopped random walk, i.e., for all $i < j$ and $s \in [t_j, T^i]$, $X^i(s) \neq X^j(s)$.*

We consider now a somewhat unnatural filtration $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_R$. Each of whose σ-fields will be based on the Poisson processes generating the coalescing random walks. They are defined in the following way: $\mathcal{F}_0$ is trivial; $\mathcal{F}_1$ is the σ-field generated by $(X^0(u), 0 \leq u \leq T^0)$; $\mathcal{F}_r$ with $2 \leq r \leq R$ is the σ-field generated by $\mathcal{F}_{r-1}$ and the random walk X^{r-1} stopped at $T^{r-1} \vee S^{r-1}$, where S^{r-1} is the first time $(X^{r-1}(u), u)$ hits a previous (stopped) random walk. One way to see $\mathcal{F}_r$ is as the σ-field generated by the Harris system viewed along the paths of the X^i, $i \leq r-1$, that is to say with information on $N^{x,y}$ for all y on interval I for $X^i(s) = x$ on I. It is clearly seen that on the σ-field $\mathcal{F}_j$, the law of $(X^j(s), s)$ is simply a space–time random walk which evolves until it hits a point (y, s) such that $X^i(s) = y$ for some $i < j$ and $s \le T^i$.

Corollary 3.2.14. *If t is sufficiently large, for at least $R/2$ random walks X^j, $1 \le j \le R$,*

$$P\big(X^j \textit{ is successful}|\mathcal{F}_j\big) \ge \frac{\alpha}{2}. \tag{3.2.34}$$

Proof. By the definition of "being good" and Lemma 3.2.12, for at least $R/2$ random walks X^j, $1 \le j \le R$, we have $\sum_{i=0}^{j-1} C_{ij} \le 2\alpha/99$. Therefore, for those j,

$$\begin{aligned} P\big(X^j \text{ hits a previous stopped random walk}|\mathcal{F}_j\big) &\le \sum_{i=0}^{j-1} P\big(X^j \text{ hits } X^i \text{ stopped}|\mathcal{F}_j\big) \\ &= \sum_{i=0}^{j-1} C_{i,j} \le \frac{2\alpha}{99}. \end{aligned} \tag{3.2.35}$$

By Corollary 3.2.8, it follows that if j is good

$$P\big(X^j \text{ is successful}|\mathcal{F}_j\big) \ge \frac{2\alpha}{3} - \frac{2\alpha}{99} \ge \frac{\alpha}{2} > 0. \tag{3.2.36}$$

□

As a consequence, we have the following result.

Corollary 3.2.15. *There exists $K_6 > 0$ not depending on t so that*

$$P\big(\textit{at least } K_6 \log(t) \textit{ random walks } \big(X^j\colon\ 1 \le j \le R\big) \textit{ are successful}\big) \ge 1 - \mathrm{e}^{-K_6 \log(t)}. \tag{3.2.37}$$

In consequence, for the system χ^t, except for an event of probability at most $\exp[-K_6 \log(t)]$, *there exist at least $3t/4 \le s_1 < s_2 < \cdots < s_{\lfloor K_6 \log(t)\rfloor} \le t$, such that*

(i) $\chi_u^{s_j} \ne \chi_{s_k - s_j + u}^{s_k}$ *for all $1 \le j < k \le K_6 \log(t)$ and $0 \le u \le s_j$;*
(ii) $|\chi_u^{s_j}| \in (\sqrt{t}, \sqrt{2t})$ *for all $1 \le j \le K_6 \log(t)$ and $s_j - t/2 \le u \le s_j$.*

Proof. By Corollary 3.2.14, at least $R/2$ of the $1 \le j \le R$ satisfy (3.2.34). For notational convenience only, we assume that (3.2.34) holds for $1 \le j \le R/2$. Let $Z_j = \mathbb{1}\{X^j \text{ is successful}\}$. Therefore,

$$P\big(Z_j = 1|Z_1, Z_2, \ldots, Z_{j-1}\big) \ge \frac{\alpha}{2} \quad \forall 1 \le j \le \frac{R}{2}. \tag{3.2.38}$$

It follows that

$$P\big(\text{at least } \alpha R/8 \text{ random walks } \big(X^j\colon\ 1 \le j \le R\big) \text{ are successful}\big) \ge P\left(\sum_{i=1}^{R/2} Z_i \ge \frac{\alpha R}{8}\right). \tag{3.2.39}$$

We suppose that $(U_j\colon\ 1 \le j \le R/2)$ is an i.i.d. sequence with uniform distribution $\mathcal{U}([0,1])$ such that independently of the Harris system

$$Y_j = Z_j \mathbb{1}\big\{U_j \le \alpha \big/ \big(2P(Z_j = 1|Z_1, Z_2, \ldots, Z_{j-1})\big)\big\}. \tag{3.2.40}$$

Therefore, $\big(Y_j\colon\ 1 \le j \le R/2\big)$ a sequence of i.i.d. random variables on $\{0, 1\}$ so that

$$P(Y_j = 1) = \frac{\alpha}{2} \quad \text{and} \quad Y_j \le Z_j \quad \forall 1 \le j \le R/2. \tag{3.2.41}$$

Therefore,

$$P\left(\sum_{i=1}^{R/2} Z_i \geq \frac{\alpha R}{8}\right) \geq P\left(\sum_{i=1}^{R/2} Y_i \geq \frac{\alpha R}{8}\right). \tag{3.2.42}$$

But, by large deviations bound for Binomial process (see e.g. den Hollander [10], Chapter 1) and (3.2.15), we have

$$\begin{aligned} P\left(\sum_{i=1}^{R/2} Y_i \geq \frac{\alpha R}{8}\right) &\geq 1 - \mathrm{e}^{-K\alpha/4R} \\ &\geq 1 - \mathrm{e}^{-K\alpha/(20C)\log(t)} \end{aligned} \tag{3.2.43}$$

for some universal $K > 0$ (not depending on t). Combining (3.2.39) and (3.2.42) and (3.2.43), and reducing constants if necessary, we arrive at (3.2.37). □

Proof of Proposition 3.1.2. Let K_1 be a small positive constant to be more fully specified later. Consider for all $0 \leq i \leq K_1 \log(t)$ the events $A_i(t) =$

$$\begin{aligned} &\big\{\text{there exist at least } 3 \times 2^{-i-2}t \leq s_1 < s_2 < \cdots < s_{\lfloor K_1 \log(2^{-i}t)\rfloor} \leq 2^{-i}t, \text{ such that} \\ &\quad \text{(i) } \chi_u^{s_j} \neq \chi_{s_k - s_j + u}^{s_k} \quad \text{for all } 1 \leq j < k \leq K_1 \log\big(2^{-i}t\big) \quad \text{and} \quad 0 \leq u \leq s_j; \\ &\quad \text{(ii) } \big|\chi_u^{s_j}\big| \in \big(\sqrt{2^{-i}t}, \sqrt{2^{-i+1}t}\big) \quad \text{for all } 1 \leq j \leq K_1 \log\big(2^{-i}t\big) \quad \text{and} \quad s_j - 2^{-i-1}t \leq u \leq s_j\big\}. \end{aligned} \tag{3.2.44}$$

Thus, under this definition, Corollary 3.2.15 says that

$$P\big(A_i(t)\big) \geq 1 - \exp\big[-K_1 \log\big(2^{-i}t\big)\big] \geq 1 - \exp\left[-\frac{K_1}{2}\log(t)\right] \tag{3.2.45}$$

if K_1 is small enough. Therefore, we have that (after reducing K_1):

(i) events $A_i(t)$ are independent for $1 \leq i \leq K_1 \log(t)$;
(ii) $P(A_i(t)) \geq 1 - \exp[-K_1 \log(t)]$.

If $\sum_{i \leq K_1 \log(t)} \mathbb{1}_{A_i^c} < K_1 \log(t/2)$, then $\#\chi^t \geq K_1 \log^2(t)$. Therefore, there exists $K_2 > 0$ so that

$$\begin{aligned} P\big(\#\chi^t \leq K_1 \log^2(t)\big) &\leq P\left(\sum_{i \leq K_1 \log(t)} \mathbb{1}_{A_i^c} \geq K_1 \log\left(\frac{t}{2}\right)\right) \\ &\leq 2^{K_1 \log(t)} \exp\left[-\frac{K_1^2}{2}\log^2(t)\right] \leq \mathrm{e}^{-K_2 \log^2(t)}. \end{aligned} \tag{3.2.46}$$

□

3.3. *Proof of Theorem* 1.3.2

Denote by $\#\chi^{[r,s]}$, $0 \leq r \leq s$, the number of distinct sites in the collection $\{\chi_u^u\colon r \leq u \leq s\}$. We refer to Bramson, Cox and Griffeath [2], Section 2:

Lemma 3.3.1. *There exists some positive finite constant K so that for all $t > 1$*

$$E\big(\#\chi^{[t/2,t]}\big) \leq K \log(t). \tag{3.3.1}$$

We are now ready to prove Theorem 1.3.2.

Proof of Theorem 1.3.2. For all $\alpha \in (\rho, 1)$ and $t \geq 0$, by Jensen's inequality, we have

$$\begin{aligned}\mathbb{P}_{\nu_\rho}(T_t \geq \alpha t) &\geq \mathbb{P}_{\nu_\rho}\left(\int_{(1-\alpha)t}^{t} \mathbb{1}\big\{\eta_s(0) = 1\big\}\,\mathrm{d}s = \alpha t\right) \\ &\geq \rho^{E(\#\chi^{[(1-\alpha)t,t]})}. \end{aligned} \tag{3.3.2}$$

Split time interval $((1-\alpha)t, t]$ so that

$$\big((1-\alpha)t, t\big] \subset \bigcup_{k=0}^{\lfloor -\log_2(1-\alpha)\rfloor} \big(t2^{-k-1}, t2^{-k}\big], \tag{3.3.3}$$

then apply Lemma 3.3.1 to each $\chi^{[t2^{-k-1}, t2^{-k}]}$, $k = 0, \ldots, \lfloor -\log_2(1-\alpha)\rfloor$ to obtain

$$E\big(\#\chi^{[(1-\alpha)t,t]}\big) \leq K_1 \log(t) \tag{3.3.4}$$

for K_1 a finite positive constant large enough. Then, combining (3.3.2) and (3.3.4), we get

$$\mathbb{P}_{\nu_\rho}(T_t \geq \alpha t) \geq \mathrm{e}^{-C_1 \log(t)} \tag{3.3.5}$$

for C_1 large enough. □

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