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Some properties of superprocesses under a stochastic flow

Kijung Lee[a], Carl Mueller[b,1,2] and Jie Xiong[c,d,2]

[a] *Department of Mathematics, Ajou University, Suwon, 443-749, Korea*
[b] *Department of Mathematics, University of Rochester, Rochester, NY 14627, USA*
[c] *Department of Mathematics, University of Tennessee, Knoxville, TN 37996-1300, USA*
[d] *Department of Mathematics, Hebei Normal University, Shijiazhuang 050016, PRC*

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Abstract. For a superprocess under a stochastic flow in one dimension, we prove that it has a density with respect to the Lebesgue measure. A stochastic partial differential equation is derived for the density. The regularity of the solution is then proved by using Krylov's L_p-theory for linear SPDE.

Résumé. Nous montrons que, sous un flot stochastique en dimension un, un superprocess a une densité par rapport à la mesure de Lebesgue. Nous déduisons une équation différentielle stochastique satisfaite par la densité. Nous montrons ensuite la régularité de la solution en utilisant la theorie de Krylov pour les EDPS linéaires dans L_p.

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1. Introduction

Superprocesses under stochastic flows have been studied by many authors since the work of Wang [10,11] and Skoulakis and Adler [8]. At first, this problem was studied as the high-density limit of a branching particle system with the motion of each particle governed by an independent Brownian motion as well as by a common Brownian motion which determines the stochastic flow. The limit was characterized by a martingale problem whose uniqueness was established using moment duality. Before we go any further, let us introduce the model in more detail.

Let $b:\mathbb{R}^d \to \mathbb{R}^d$, $\sigma_1, \sigma_2:\mathbb{R}^d \to \mathbb{R}^{d\times d}$ be measurable functions. Let $W, B_1, B_2, \ldots$ be independent d-dimensional Brownian motions. Consider a branching particle system performing independent binary branching. That is, each particle splits in two or dies with equal probability. Between branching times, the motion of the ith particle is governed by the following stochastic differential equation (SDE):

$$\mathrm{d}\eta_i(t) = b\bigl(\eta_i(t)\bigr)\,\mathrm{d}t + \sigma_1\bigl(\eta_i(t)\bigr)\,\mathrm{d}W_t + \sigma_2\bigl(\eta_i(t)\bigr)\,\mathrm{d}B_i(t).$$

Let $\mathcal{M}_F(\mathbb{R}^d)$ denote the space of finite nonnegative measures on $\mathbb{R}^d$, and recall that $C_0^2(\mathbb{R}^d)$ denotes the space of twice continuously differentiable functions of compact support. Skoulakis and Adler [8] prove that in the high-density limit with times between branching tending to 0, the limiting process X_t is the unique solution to the following martingale problem (MP):

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We can realize X_t on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ such that X_t is $\mathcal{F}_t$ adapted and the following holds: For $X_0 = \mu \in \mathcal{M}_F(\mathbb{R}^d)$, and for any $\phi \in C_0^2(\mathbb{R}^d)$,

$$M_t(\phi) \equiv \langle X_t, \phi\rangle - \langle \mu, \phi\rangle - \int_0^t \langle X_s, L\phi\rangle \,\mathrm{d}s \tag{1.1}$$

is a continuous $(\mathbb{P}, \mathcal{F}_t)$-martingale with quadratic variation process

$$\langle M(\phi)\rangle_t = \int_0^t \left(\langle X_s, \phi^2\rangle + \left|\langle X_s, \sigma_1^T \nabla\phi\rangle\right|^2\right) \mathrm{d}s, \tag{1.2}$$

where

$$L\phi = \sum_{i=1}^d b^i \partial_i \phi + \frac{1}{2}\sum_{i,j=1}^d a^{ij}\partial_{ij}^2\phi,$$

$a^{ij} = \sum_{k=1}^d \sum_{\ell=1}^2 \sigma_\ell^{ik}\sigma_\ell^{kj}$, ∂_i means the partial derivative with respect to the ith component of $x \in \mathbb{R}^d$, $\partial_{ij} = \partial_i\partial_j$, σ_1^T is the transpose of the matrix σ_1, $\nabla = (\partial_1, \ldots, \partial_d)^T$ is the gradient operator and $\langle \mu, f\rangle$ represents the integral of the function f with respect to the measure μ.

It was conjectured in [8] that the conditional log-Laplace transform of X_t should be the unique solution to a nonlinear stochastic partial differential equation (SPDE). Namely

$$\mathbb{E}_\mu\left[\mathrm{e}^{-\langle X_t, f\rangle} | W\right] = \mathrm{e}^{-\langle \mu, y_{0,t}\rangle} \tag{1.3}$$

and

$$y_{s,t}(x) = f(x) + \int_s^t \left(Ly_{r,t}(x) - y_{r,t}(x)^2\right) \mathrm{d}r + \int_s^t \nabla^T y_{r,t}(x)\sigma_1(x)\,\hat{\mathrm{d}}W(r), \tag{1.4}$$

where $\hat{\mathrm{d}}W(r)$ represents the backward Itô integral:

$$\int_s^t g(r)\,\hat{\mathrm{d}}W(r) = \lim_{|\Delta|\to 0}\sum_{i=1}^n g(r_i)\big(W(r_i) - W(r_{i-1})\big),$$

where $\Delta = \{r_0, r_1, \ldots, r_n\}$ is a partition of $[s,t]$ and $|\Delta|$ is the maximum length of the subintervals. This conjecture was confirmed by Xiong [12] under the following boundedness conditions (BC): $f \geq 0, b, \sigma_1, \sigma_2$ are bounded with bounded first and second derivatives. $\sigma_2^T\sigma_2$ is uniformly positive definite, σ_1 has third continuous bounded derivatives. We also assume that f is of compact support.

Making use of the conditional log-Laplace functional, the long-term behavior of this process was studied in [13]. Also, the model has been extended in that paper to allow infinite nonnegative measures $\mu \in \mathcal{M}_{\text{tem}}(\mathbb{R}^d)$, namely, $\int_{\mathbb{R}^d} \mathrm{e}^{-\lambda|x|}\mu(\mathrm{d}x) < \infty$ for some $\lambda > 0$. A similar model has been investigated by Wang [11] and Dawson et al. [1] when the spatial dimension is 1. Further, in that case, it is proved by Dawson et al. [2] that their process is density-valued and solves a SPDE. The regularity of the solution was left *open* in that article.

We formulate the main result of this paper which deals with the case $d = 1$. For any real number n and $p \in [2, \infty)$, we let H_p^n denote the space of Bessel potentials defined on $\mathbb{R}$ with norm

$$\|u\|_{n,p} = \left\|(I - \Delta)^{n/2}u\right\|_p.$$

(See, for instance, pp. 186–187 in [7] for an explanation of this space.) We let ∂_t (∂_x, resp.) denote the partial derivative with respect to t (x, resp.).

Theorem 1.1. *Suppose that* (*BC*) *is satisfied with* $d = 1$ *and* $\mu \in \mathcal{M}_{\text{tem}}(\mathbb{R})$. *We also assume* $p \in [2, \infty)$. *Then*:

(i) *X_t is absolutely continuous with respect to Lebesgue measure and the density $X(t,x)$ satisfies the SPDE*

$$\partial_t X = L^* X - \partial_x(\sigma_1 X)\dot{W}_t + \sqrt{X}\dot{B}_{tx} \tag{1.5}$$

in the sense that, for any f satisfying conditions in (*BC*) *and $t > 0$,*

$$\langle X(t,\cdot), f\rangle = \langle \mu, f\rangle + \int_0^t \langle X(s,\cdot), Lf\rangle \,\mathrm{d}s + \int_0^t \langle X(s,\cdot), \sigma_1 f'\rangle \,\mathrm{d}W_s + \int_0^t \int_{\mathbb{R}} \sqrt{X(s,x)} f(x) B(\mathrm{d}s\,\mathrm{d}x) \tag{1.6}$$

holds a.s., where B is a Brownian sheet and L^* is the adjoint operator of L.

(ii) *We take a finite time T and let $\varphi_t(x)$ be the normal density with mean* 0 *and variance t. If, in addition to the previous conditions, μ is finite and satisfies*

$$\sup_{t,x} \langle \mu, \varphi_t(x - \cdot)\rangle < \infty, \tag{1.7}$$

and μ is in $H_p^{1/2-\varepsilon-2/p}$ for some $\varepsilon \in (0, \frac{1}{2})$ and p satisfying $\frac{1}{2} - \varepsilon - \frac{1}{p} > 0$, then the density $X(t,x)$ is Hölder continuous in x with index $\frac{1}{2} - \varepsilon - \frac{1}{p}$ for (a.e.) $t \in [0, T]$ (a.s.).

Remark 1.2. *Let C_0^∞ be the space of infinitely differentiable functions of compact support. If the measure μ as a distribution is in C_0^∞, that is, there is $\psi \in C_0^\infty$ such that $\langle \mu, \phi\rangle = \int_{\mathbb{R}} \psi(x)\phi(x)\,\mathrm{d}x$ for any $\phi \in C_0^\infty$, then the finite measure μ satisfies* (1.7) *and $\mu \in H_p^{1/2-\varepsilon-2/p}$ for any $p \geq 2$ and $\varepsilon \in (0, \frac{1}{2})$. Hence, we can have any number $\alpha \in (0, \frac{1}{2})$, in particular, close to $\frac{1}{2}$ as the index for Hölder continuity.*

This paper is organized as follows: We prove in Section 2 that X_t is absolutely continuous with respect to Lebesgue measure and show that the density $X(t,x)$ satisfies (1.5). In Section 3 we show the Hölder continuity of $X(t,x)$, as the main result of this paper, by freezing the nonlinear term in (1.5) and applying Krylov's L_p-theory for linear SPDE to get the Hölder continuity.

Remark 1.3. *Suppose that we apply the usual integral equation as in* [9], *Chapter* 3, *for* (1.5) *in order to prove the Hölder continuity. Then formally we have*

$$X(t,x) = \int p_0(t,x,y) X(0,y)\,\mathrm{d}y + \int_0^t \int \sigma_1(y) X(s,y) \partial_y p_0(t-s,x,y)\,\mathrm{d}y\,\mathrm{d}W_s + \int_0^t \int \sqrt{X(s,y)} p_0(t-s,x,y) B(\mathrm{d}s\,\mathrm{d}y),$$

where p_0 is the transition function of the Markov process with generator L. However, the second term on the right-hand side of the above equation is about

$$\int_0^t (t-s)^{-1/2}\,\mathrm{d}W_s,$$

which is not convergent. Therefore, the convolution argument used by Konno and Shiga [6] *does not apply to our model.*

We note that the SPDE in [2] is (1.5) in current paper with $\dot{W}_t$ replaced by a space–time noise which is colored in space and white in time. We believe that the method of this paper can be applied to that equation to prove the regularity for its solution.

2. Absolute continuity of X_t for $d = 1$

Let $p_0(t, x, y)$ and $q_0(t, (x_1, x_2), (y_1, y_2))$ be the transition density functions of the Markov processes $\eta_1(t)$ and $(\eta_1(t), \eta_2(t))$, respectively. By Theorem 1.5 of [12], we have

$$\mathbb{E}\big[\langle X_t, f\rangle\big] = \int_{\mathbb{R}^2} f(y) p_0(t, x, y)\,\mathrm{d}y\,\mu(\mathrm{d}x) \tag{2.1}$$

and

$$\begin{aligned}\mathbb{E}\big[\langle X_t, f\rangle\langle X_t, g\rangle\big] &= \int_{\mathbb{R}^4} f(y_1)g(y_2)q_0\big(t, (x_1, x_2), (y_1, y_2)\big)\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mu(\mathrm{d}x_1)\mu(\mathrm{d}x_2) \\ &\quad + 2\int_0^t \mathrm{d}s \int_{\mathbb{R}^4} p_0(t-s, z, y) f(z_1)g(z_2)q_0\big(s, (y, y), (z_1, z_2)\big)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}y\,\mu(\mathrm{d}z).\end{aligned} \tag{2.2}$$

Theorem 2.1. *If $\mu(\mathbb{R}) < \infty$, then X_t has a density $X(t, \cdot) \in H_2^0 = L^2(\mathbb{R})$ for almost every t a.s.*

Proof. Take $f = p_0(\varepsilon, x, \cdot)$ and $g = p_0(\varepsilon', x, \cdot)$ in (2.2). Note that as $\varepsilon, \varepsilon' \to 0$,

$$\int_{\mathbb{R}^2} p_0(\varepsilon, x, z_1) p_0\big(\varepsilon', x, z_2\big) p_0(t-s, z, y) q_0\big(t, (y, y), (z_1, z_2)\big)\,\mathrm{d}z_1\,\mathrm{d}z_2 \to p_0(t-s, z, y) q_0\big(t, (y, y), (x, x)\big).$$

By Theorem 6.4.5 in Friedman [3], we have

$$p_0(\varepsilon, x, y) \le c\varphi_{c'\varepsilon}(x-y), \tag{2.3}$$

$$q_0\big(s, (y, y), (z_1, z_2)\big) \le c\varphi_{c's}(y - z_1)\varphi_{c's}(y - z_2), \tag{2.4}$$

where we recall that $\varphi_t(x)$ is the normal density with mean 0 and variance t. Note that c' is a constant which is usually greater than 1. Since it does not play an essential role, to simplify the notations, we assume $c' = 1$ throughout this section. Hence,

$$\begin{aligned}&\int_{\mathbb{R}^2} p_0(\varepsilon, x, z_1) p_0\big(\varepsilon', x, z_2\big) p_0(t-s, z, y) q_0\big(s, (y, y), (z_1, z_2)\big)\,\mathrm{d}z_1\,\mathrm{d}z_2 \\ &\quad \le c\int_{\mathbb{R}^2} \varphi_\varepsilon(x - z_1)\varphi_{\varepsilon'}(x - z_2)\varphi_{t-s}(z-y)\varphi_s(y - z_1)\varphi_s(y - z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2 \\ &\quad = c\varphi_{s+\varepsilon}(x-y)\varphi_{s+\varepsilon'}(x-y)\varphi_{t-s}(z-y).\end{aligned}$$

As

$$\begin{aligned}&\lim_{\varepsilon,\varepsilon'\to 0}\int_0^T \mathrm{d}t \int \mathrm{d}x \int_0^t \mathrm{d}s \int_{\mathbb{R}^2} \varphi_{s+\varepsilon}(x-y)\varphi_{s+\varepsilon'}(x-y)\varphi_{t-s}(z-y)\,\mathrm{d}y\,\mu(\mathrm{d}z) \\ &\quad = \lim_{\varepsilon,\varepsilon'\to 0}\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s\,\varphi_{2s+\varepsilon+\varepsilon'}(0)\mu(\mathbb{R}) = \int_0^T \mathrm{d}t \int_0^t \mathrm{d}s\,\varphi_{2s}(0)\mu(\mathbb{R}) \\ &\quad = \int_0^T \mathrm{d}t \int \mathrm{d}x \int_0^t \mathrm{d}s \int_{\mathbb{R}^2} \varphi_{t-s}(z-y)\varphi_s(x-y)\varphi_s(x-y)\,\mathrm{d}y\,\mu(\mathrm{d}z),\end{aligned}$$

by the dominated convergence theorem, we see that as $\varepsilon, \varepsilon' \to 0$,

$$\begin{aligned}&\int_0^T \mathrm{d}t \int \mathrm{d}x \int_0^t \mathrm{d}s \int_{\mathbb{R}^4} p_0(t-s, z, y) p_0(\varepsilon, x, z_1) p_0(\varepsilon', x, z_2) q_0\big(s, (y, y), (z_1, z_2)\big)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}y\,\mu(\mathrm{d}z) \\ &\quad \to \int_0^T \mathrm{d}t \int \mathrm{d}x \int_0^t \mathrm{d}s \int_{\mathbb{R}^2} p_0(t-s, z, y) q_0\big(t, (y, y), (x, x)\big)\,\mathrm{d}y\,\mu(\mathrm{d}z).\end{aligned}$$

Similarly, we have

$$\int_0^T \mathrm{d}t \int \mathrm{d}x \int_{\mathbb{R}^4} p_0(\varepsilon, x, y_1) p_0(\varepsilon', x, y_2) q_0\big(t, (x_1, x_2), (y_1, y_2)\big) \,\mathrm{d}y_1 \,\mathrm{d}y_2 \,\mu(\mathrm{d}x_1)\mu(\mathrm{d}x_2)$$
$$\to \int_0^T \mathrm{d}t \int \mathrm{d}x \int_{\mathbb{R}^2} q_0\big(t, (x_1, x_2), (x, x)\big)\mu(\mathrm{d}x_1)\mu(\mathrm{d}x_2).$$

Hence

$$\int_0^T \mathrm{d}t \int \mathrm{d}x \,\mathbb{E}\big[\langle X_t, p(\varepsilon, x, \cdot)\rangle\langle X_t, p(\varepsilon', x, \cdot)\rangle\big]$$
$$\to \int_0^T \mathrm{d}t \int \mathrm{d}x \int_{\mathbb{R}^2} q_0\big(t, (x_1, x_2), (x, x)\big)\mu(\mathrm{d}x_1)\mu(\mathrm{d}x_2)$$
$$+ \int_0^T \mathrm{d}t \int \mathrm{d}x \int_0^t \mathrm{d}s \int_{\mathbb{R}^2} p_0(t-s, x, y) q_0\big(t, (y, y), (x, x)\big) \,\mathrm{d}y \,\mu(\mathrm{d}x).$$

From this, we can show that $\{\langle X_t, p_0(\frac{1}{n}, x, \cdot)\rangle\colon n = 1, 2, \ldots\}$ is a Cauchy sequence in $L^2(\Omega \times [0, T] \times \mathbb{R})$. This implies the existence of the density $X(t, x)$ of X_t in $L^2(\Omega \times [0, T] \times \mathbb{R})$. □

To consider the case for μ being σ-finite, we use the following lemma about conditional martingale problem (CMP), which is proved in [13].

Lemma 2.2. (i) *If X_t is the solution to MP, then there exists a Brownian motion W such that X_t is the solution to CMP with this W. That is, for any $\phi \in C_0^2(\mathbb{R})$,*

$$N_t(\phi) \equiv \langle X_t, \phi\rangle - \langle \mu, \phi\rangle - \int_0^t \langle X_s, L\phi\rangle \,\mathrm{d}s - \int_0^t \big\langle X_s, \sigma_1^T \nabla\phi\big\rangle \,\mathrm{d}W_s \tag{2.5}$$

is a continuous $(\mathbb{P}, \mathcal{G}_t)$-martingale with quadratic variation process

$$\big\langle N(\phi)\big\rangle_t = \int_0^t \big\langle X_s, \phi^2\big\rangle \,\mathrm{d}s, \tag{2.6}$$

where $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}_\infty^W$.

(ii) *If X_t is a solution to CMP with a Brownian motion W, then it is a solution to MP.*

Since we wish to consider a sequence of solutions to MP on the same probability space, we need the following technical lemmas. First we cite Theorem 3.1, p. 13, of [4]. They define a "standard measurable space," but for our purposes it is enough to note that a Polish space is a standard measurable space.

Lemma 2.3. *Let $(\Omega, \mathcal{F})$ be a standard measurable space and P be a probability on $(\Omega, \mathcal{F})$. Let $\mathcal{G}$ be a sub σ-field of $\mathcal{F}$. Then a regular conditional probability $\{p(\omega, A)\}$ given $\mathcal{G}$ exists uniquely.*

The next lemma yields a sequence of regular conditional probabilities with conditional independence.

Lemma 2.4. *Let W be a random variable, and let (X_i, W_i): $i = 1, 2, \ldots$ be a sequence of random vectors with components X_i, W_i taking values in Polish spaces $\mathcal{X}, \mathcal{W}$, respectively. Suppose that for each i, W_i and W are equal in law. Then we can realize W and the vectors (X_i, W_i) on a common probability space such that the following holds: For all i, $W_i = W$. Furthermore, given W, the random variables $\{X_i\}$ are conditionally independent.*

Proof. The random vector (X_i, W_i) induces a probability P_i on the Polish space $\mathcal{X} \times \mathcal{W}$, and of course, the random variable $W = W_i$ does not depend on i. Using Lemma 2.3, we can construct the regular conditional probability $\mu^i_{(w,x)}(A)$ given $W_i = w$, on measurable sets $A \subset \mathcal{X} \times \mathcal{W}$, where $(x, w) \in \mathcal{X} \times \mathcal{W}$. Note that $\mu^i_w(\cdot) = \mu^i_{(x,w)}(\cdot)$ does not depend on x. Also, we can define a measure $\nu^i_w(B) = \mu^i_w(B \times \mathcal{W})$ on measurable sets $B \subset \mathcal{X}$. For each $w \in \mathcal{W}$, we construct the product measure ν_w on $\mathcal{X}^\infty := \bigotimes_{i=1}^\infty \mathcal{X}_i$ with $\mathcal{X}_i = \mathcal{X}$,

$$\nu_w = \bigotimes_{i=1}^\infty \nu^i_w.$$

Then we construct a probability P on $\mathcal{X}^\infty \otimes \mathcal{W}$ by

$$P(A) = \int_{\mathcal{W}} \nu_w(A_w) P^W(\mathrm{d}w), \tag{2.7}$$

where A_w denotes the section $A_w = \{x \in \mathcal{X}^\infty\colon (x, w) \in A\}$. The random variables $X_1, X_2, \dots$ and W are defined on $\mathcal{X}^\infty \otimes \mathcal{W}$ in the usual way, as are the σ-fields generated by these random variables. The reader can check that ν_w is a regular conditional probability with respect to P, given the σ-field generated by W. Now from (2.7) we immediately see that under P, the X_i are conditionally independent given W, and the proof of Lemma 2.4 is complete. □

Next we consider an infinite measure μ.

Theorem 2.5. *If $\mu \in \mathcal{M}_{\mathrm{tem}}(\mathbb{R})$, then X_t has a density $X(t, x)$.*

Proof. If μ is σ-finite, we can take $\mu = \sum_{n=1}^\infty \mu^n$ with μ^n finite. We may and will assume that μ^n is supported on $\{x \in \mathbb{R}\colon n \le |x| \le n + 1\}$. Given initial values $X^n_0 = \mu^n$, it is not hard to show that the solutions X^n_t to CMP and the corresponding noises W^n take values in a Polish space. Furthermore the W^n are equal in distribution. Thus Lemma 2.4 shows that we may consider the X^n as driven by a single noise W, and we may assume that the X^n are conditionally independent given W. We can also check that

$$X_t = \sum_{n=1}^\infty X^n_t$$

is the solution to CMP with initial μ. The key is the proof of the continuity of X in $\mathcal{M}_{\mathrm{tem}}(\mathbb{R})$, which we give now.

Let $\lambda > 0$ satisfy $\int_{\mathbb{R}} \mathrm{e}^{-\lambda|x|}\mu(\mathrm{d}x) < \infty$ and take any $\phi \in C^2_0(\mathbb{R})$. Since ϕ, ϕ', ϕ'' are compactly supported, we can always choose a finite constant N so that ϕ, ϕ', ϕ'' are bounded by $N\mathrm{e}^{-\lambda|x|}$. We apply the Burkholder–Davis–Gundy inequality with (1.1) and (1.2) with X and μ replaced by X^n and μ^n. Then there exists a constant N_0 such that

$$\mathbb{E}\Big[\sup_{0\le t\le T} \big|\langle X^n_t, \phi\rangle\big|^2\Big] \le N_0\bigg(\big|\langle\mu^n, \phi\rangle\big|^2 + \mathbb{E}\bigg[\int_0^T \big(\big|\langle X^n_t, L\phi\rangle\big|^2 + \langle X^n_t, \phi^2\rangle + \big|\langle X^n_t, \sigma_1\phi'\rangle\big|^2\big)\,\mathrm{d}t\bigg]\bigg),$$

where the constant N_0 depends only on T and the operator L has the simple expression, $L\phi = b\phi' + \frac{1}{2}(\sigma_1^2 + \sigma_2^2)\phi''$ in our case of $d = 1$. Using all the assumptions on b, σ_1, σ_2 in (BC) and the choice of N above, we further obtain

$$\mathbb{E}\Big[\sup_{0\le t\le T} \big|\langle X^n_t, \phi\rangle\big|^2\Big] \le N_1\Big(\big|\langle\mu^n, \mathrm{e}^{-\lambda|\cdot|}\rangle\big|^2 + \sup_{0\le t\le T}\mathbb{E}\big|\langle X^n_t, \mathrm{e}^{-2\lambda|\cdot|}\rangle\big| + \sup_{0\le t\le T}\mathbb{E}\big[\big|\langle X^n_t, \mathrm{e}^{-\lambda|\cdot|}\rangle\big|^2\big]\Big) \tag{2.8}$$

for some constant N_1.

To proceed further, we need the following inequality:

$$\int \mathrm{e}^{-\lambda|x|}\varphi_{c't}(x - y)\,\mathrm{d}x \le N_2\mathrm{e}^{-\lambda|y|}$$

which we can see by

$$\begin{aligned}\int \mathrm{e}^{-\lambda|x|+\lambda|y|}\varphi_{c't}(x-y)\,\mathrm{d}x &= \int \mathrm{e}^{-\lambda|x+y|+\lambda|y|}\varphi_{c't}(x)\,\mathrm{d}x \le \int \mathrm{e}^{\lambda|x|}\varphi_{c't}(x)\,\mathrm{d}x \\ &= 2\int_0^\infty \mathrm{e}^{\lambda x}\frac{1}{\sqrt{2\pi c't}}\mathrm{e}^{-x^2/(2c't)}\,\mathrm{d}x = 2\int_0^\infty \mathrm{e}^{-(z-\lambda\sqrt{c't})^2/2}\,\mathrm{d}z\cdot \mathrm{e}^{\lambda^2c't/2} \\ &\le 2\mathrm{e}^{\lambda^2c'T/2}.\end{aligned}$$

By (2.1) and (2.3), we have

$$\begin{aligned}\mathbb{E}\left|\left\langle X_t^n,\mathrm{e}^{-2\lambda|\cdot|}\right\rangle\right| = \mathbb{E}\left\langle X_t^n,\mathrm{e}^{-2\lambda|\cdot|}\right\rangle &\le c\int_{\mathbb{R}^2}\mathrm{e}^{-2\lambda|y|}\varphi_{c't}(x-y)\,\mathrm{d}y\,\mu^n(\mathrm{d}x) \\ &\le N_3\int_{\mathbb{R}}\mathrm{e}^{-2\lambda|x|}\mu^n(\mathrm{d}x) \le N_3\mathrm{e}^{-\lambda n}\int_{\mathbb{R}}\mathrm{e}^{-\lambda|x|}\mu^n(\mathrm{d}x)\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^\infty\Bigl(\sup_{0\le t\le T}\mathbb{E}\left|\left\langle X_t^n,\mathrm{e}^{-2\lambda|\cdot|}\right\rangle\right|\Bigr)^{1/2} &\le N_3^{1/2}\sum_{n=1}^\infty\bigl(\mathrm{e}^{-\lambda n}\bigr)^{1/2}\biggl(\int_{\mathbb{R}}\mathrm{e}^{-\lambda|x|}\mu^n(\mathrm{d}x)\biggr)^{1/2} \\ &\le N_3^{1/2}\biggl(\sum_{n=1}^\infty\mathrm{e}^{-\lambda n}\biggr)^{1/2}\biggl(\sum_{n=1}^\infty\int_{\mathbb{R}}\mathrm{e}^{-\lambda|x|}\mu^n(\mathrm{d}x)\biggr)^{1/2} \\ &= N_3^{1/2}\biggl(\frac{1}{\mathrm{e}^\lambda-1}\biggr)^{1/2}\biggl(\int_{\mathbb{R}}\mathrm{e}^{-\lambda|x|}\mu(\mathrm{d}x)\biggr)^{1/2} < \infty. \end{aligned} \tag{2.9}$$

Meanwhile, by (2.2)–(2.4), we get

$$\begin{aligned}\mathbb{E}\bigl[\left|\left\langle X_t^n,\mathrm{e}^{-\lambda|\cdot|}\right\rangle\right|^2\bigr] &\le N_4\int_{\mathbb{R}^4}\mathrm{e}^{-\lambda|y_1|}\mathrm{e}^{-\lambda|y_2|}\varphi_{c't}(y_1-x_1)\varphi_{c't}(y_2-x_2)\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mu^n(\mathrm{d}x_1)\mu^n(\mathrm{d}x_2) \\ &\quad + N_5\int_0^t\mathrm{d}s\int_{\mathbb{R}^4}\mathrm{e}^{-\lambda|z_1|}\mathrm{e}^{-\lambda|z_2|}\varphi_{c'(t-s)}(z-y)\varphi_{c's}(z_1-y)\varphi_{c's}(z_2-y)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}y\,\mu^n(\mathrm{d}z) \\ &\le N_6\biggl(\int\mathrm{e}^{-\lambda|x|}\mu^n(\mathrm{d}x)\biggr)^2 + N_7\int_0^t\mathrm{d}s\int_{\mathbb{R}^2}\varphi_{c'(t-s)}(z-y)\mathrm{e}^{-2\lambda|y|}\,\mathrm{d}y\,\mu^n(\mathrm{d}z) \\ &\le N_6\biggl(\int\mathrm{e}^{-\lambda|x|}\mu^n(\mathrm{d}x)\biggr)^2 + N_8\int_0^t\mathrm{d}s\int\mathrm{e}^{-2\lambda|z|}\mu^n(\mathrm{d}z) \\ &\le N_6\biggl(\int\mathrm{e}^{-\lambda|x|}\mu^n(\mathrm{d}x)\biggr)^2 + N_9\mathrm{e}^{-\lambda n}\int\mathrm{e}^{-\lambda|z|}\mu^n(\mathrm{d}z) \\ &\le N_{10}\biggl(\int\mathrm{e}^{-\lambda|x|}\mu^n(\mathrm{d}x)+\mathrm{e}^{-\lambda n}\biggr)^2\end{aligned}$$

and we have

$$\sum_{n=1}^\infty\Bigl(\sup_{0\le t\le T}\mathbb{E}\bigl[\left|\left\langle X_t^n,\mathrm{e}^{-\lambda|\cdot|}\right\rangle\right|^2\bigr]\Bigr)^{1/2} \le N_{10}^{1/2}\biggl(\int\mathrm{e}^{-\lambda|x|}\mu(\mathrm{d}x)+\frac{1}{\mathrm{e}^\lambda-1}\biggr). \tag{2.10}$$

Thus (2.8)–(2.10) give us

$$\mathbb{E}\left[\sum_{n=1}^{\infty}\sup_{0\le t\le T}\left|\langle X_t^n,\phi\rangle\right|\right] \le \sum_{n=1}^{\infty}\left(\mathbb{E}\left[\sup_{0\le t\le T}\left|\langle X_t^n,\phi\rangle\right|^2\right]\right)^{1/2}$$
$$\le N_{11}\left[\int e^{-\lambda|x|}\mu(dx) + \left(\frac{1}{e^{\lambda}-1}\int e^{-\lambda|x|}\mu(dx)\right)^{1/2} + \int e^{-\lambda|x|}\mu(dx) + \frac{1}{e^{\lambda}-1}\right]$$
$$< \infty$$

and, with probability 1, the following uniform convergence holds:

$$\lim_{m\to\infty}\sup_{0\le t\le T}\left|\sum_{n=1}^{m}\langle X_t^n,\phi\rangle - \langle X_t,\phi\rangle\right| = \lim_{m\to\infty}\sup_{0\le t\le T}\left|\sum_{n=m+1}^{\infty}\langle X_t^n,\phi\rangle\right| \le \lim_{m\to\infty}\sum_{n=m+1}^{\infty}\sup_{0\le t\le T}\left|\langle X_t^n,\phi\rangle\right| = 0.$$

Hence, the continuity of $\langle X^n,\phi\rangle$ gives us the continuity of $\langle X,\phi\rangle$, which implies the continuity of X in $\mathcal{M}_{\text{tem}}(\mathbb{R})$.

Let

$$X(t,x) = \sum_{n=1}^{\infty} X^n(t,x). \tag{2.11}$$

By (2.1), we have

$$\mathbb{E}\left[X^n(t,x)\right] = \int_{\mathbb{R}} p_0(t,y,x)\mu^n(dy).$$

As

$$p_0(t,x,y) \le c\varphi_t(x-y) \le c(t,\lambda,x)e^{-\lambda|y|},$$

for any $\lambda > 0$, we have

$$\mathbb{E}\left[\sum_{n=1}^{\infty} X^n(t,x)\right] = \sum_{n=1}^{\infty}\int_{\mathbb{R}} p_0(t,y,x)\mu^n(dy) = \int_{\mathbb{R}} p_0(t,y,x)\mu(dy) < \infty.$$

Hence, $X(t,x)$ is well-defined by (2.11). It is then easy to show that $X(t,x)\,dx = X_t(dx)$. □

The following theorem implies Theorem 1.1(i).

Theorem 2.6. *X_t is the unique* (*in law*) *solution to the SPDE* (1.5).

Proof. Note that $N_t(\phi)$ in (2.5) is a continuous $(\mathbb{P},\mathcal{G}_t)$-martingale with quadratic variation process

$$\langle N(\phi)\rangle_t = \int_0^t\int_{\mathbb{R}}\left(\sqrt{X(s,x)}\phi(x)\right)^2 dx\,ds.$$

By the martingale representation theorem ([5], Theorem 3.3.5), there exists an $L^2(\mathbb{R})$-cylindrical Brownian motion $\tilde{B}$ on an extension of $(\Omega,\mathcal{F},\mathcal{G}_t,\mathbb{P})$ such that

$$N_t = \int_0^t \langle\sqrt{X_s}, d\tilde{B}_s\rangle_{L^2(\mathbb{R})}.$$

There exists a standard Brownian sheet B such that

$$\tilde{B}_t(h) = \int_0^t\int_{\mathbb{R}} h(x)B(ds\,dx), \quad \forall h \in L^2(\mathbb{R}).$$

Therefore,

$$N_t(\phi) = \int_0^t \int_{\mathbb{R}} \sqrt{X(s,x)}\phi(x) B(\mathrm{d}s\, \mathrm{d}x).$$

As B is a Brownian sheet on an extension of $\mathcal{G}_t$, it is easy to show that B is independent of W. Thus, X_t is a (weak) solution to (1.5).

On the other hand, if X_t is a (weak) solution to (1.5), it must be a solution to the MP (1.1, 1.2). The uniqueness (in law) for the solution to (1.5) then follows from that of the MP (1.1, 1.2). □

3. Hölder continuity for $d = 1$

In this section we prove Theorem 1.1(ii).

We note that the functions b, σ_1, σ_2 are scalar functions on $\mathbb{R}$ since $d = 1$ and we have $L = \frac{1}{2}a\partial_{xx} + b\partial_x$, $L^* = \frac{1}{2}a\partial_{xx} + (a' - b)\partial_x + (\frac{1}{2}a'' - b')$ with $a = \sigma_1^2 + \sigma_2^2$.

We will need the following lemma, which is about the moments of X.

Lemma 3.1. *If μ is finite and satisfies* (1.7), *then*

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} X(t,x)^n \,\mathrm{d}x\, \mathrm{d}t\right] < \infty \tag{3.1}$$

for all $n \in \mathbb{N}$.

Proof. We use the moment dual to prove (3.1). Let n_t be a pure death Markov process with $n_0 = n$ which jumps from n to $n-1$ at a rate $\frac{1}{2}n(n-1)$. Let $0 = \tau_0 < \tau_1 < \cdots < \tau_{n-1}$ be the jump times. Let $f_0 = \delta_x^{\otimes n}$ and for $t < \tau_1$, $f_t(y) = p_0^n(t, (x, \ldots, x), y)$, $\forall y \in \mathbb{R}^n$, where p_0^n is the transition function of the n-dimensional diffusion $(\eta_1(t), \ldots, \eta_n(t))$. For $f \in C(\mathbb{R}^n)$, let $G_{ij} f \in C(\mathbb{R}^{n-1})$ be given by

$$G_{ij} f(y_1, \ldots, y_{n-2}, y_{n-1}) = f(y_1, \ldots, y_{n-1}, \ldots, y_{n-1}, \ldots, y_{n-2}),$$

where y_{n-1} is at ith and jth position. Let

$$f_{\tau_1} = \Gamma_1 f_{\tau_1 -},$$

where Γ_1 is a random element chosen from $\{G_{ij}\colon 1 \le i < j \le n\}$; each element has equal probability. We continue this procedure to get the process f_t. Replace f_0 by the smooth function $f_0^\varepsilon = \varphi_\varepsilon^{\otimes n}$. Denote the process constructed above with f_0^ε in place of f_0 by f_t^ε. As in Theorem 11 in Xiong and Zhou [14], we have

$$\mathbb{E}\bigl[\langle X_t^{\otimes n}, f_0^\varepsilon\rangle\bigr] = \mathbb{E}\left[\langle \mu^{\otimes n_t}, f_t^\varepsilon\rangle \exp\left(\frac{1}{2}\int_0^t n_s(n_s - 1)\,\mathrm{d}s\right)\right].$$

Taking limits and using Fatou's lemma, we have

$$\begin{aligned}\mathbb{E}\bigl[X(t,x)^n\bigr] &\le \liminf_{\varepsilon\to 0} \mathbb{E}\left[\langle \mu^{\otimes n_t}, f_t^\varepsilon\rangle \exp\left(\frac{1}{2}\int_0^t n_s(n_s-1)\,\mathrm{d}s\right)\right] \\ &\le \exp\left(\frac{1}{2}n(n-1)t\right) \liminf_{\varepsilon\to 0} \sum_{i=1}^n \mathbb{E}\bigl[\langle \mu^{\otimes n_t}, f_t^\varepsilon\rangle 1_{\tau_{i-1} \le t < \tau_i}\bigr].\end{aligned}$$

Now we estimate the sum above. We will consider the term with $i = 3$ first. Denote the left-hand side of (1.7) by c_1 and the bound of $\sqrt{t}\varphi_t(x)$ by c_2. Denote by

$$\tilde{y}^{k\ell} = (y_1, \ldots, y_{n-2}, \ldots, y_{n-2}, \ldots, y_{n-1}),$$

where y_{n-2} is at the kth and the ℓth positions. Then

$$E\big[1_{\tau_2\le t<\tau_3} f_t^\varepsilon(x_1,\dots,x_{n-2})\big]$$
$$\le c\int_{\mathbb{R}^{n-2}} E\left[1_{\tau_2\le t<\tau_3}\prod_{i=1}^{n-2}\varphi_{t-\tau_2}(x_i-y_i)\Gamma_2 f^\varepsilon_{\tau_2-}(y)\right]\mathrm{d}y$$
$$= c\int_{\mathbb{R}^{n-2}} E\left[1_{\tau_2\le t<\tau_3}\prod_{i=1}^{n-2}\varphi_{t-\tau_2}(x_i-y_i)\sum_{1\le k<\ell\le n-2}\frac{2}{(n-2)(n-3)}f^\varepsilon_{\tau_2-}\big(\tilde{y}^{k\ell}\big)\right]\mathrm{d}y.$$

As

$$E\big[1_{\tau_2\le t<\tau_3} f^\varepsilon_{\tau_2-}\big(\tilde{y}^{k\ell}\big)\big]$$
$$\le c\int_{\mathbb{R}^{n-1}} E\left[1_{\tau_2\le t<\tau_3}\prod_{j=1}^{n-1}\varphi_{\tau_2-\tau_1}\big(\tilde{y}^{k\ell}_j-z_j\big)f^\varepsilon_{\tau_1}(z)\right]\mathrm{d}z$$
$$= c\int_{\mathbb{R}^{n-1}} E\left[1_{\tau_2\le t<\tau_3}\prod_{j=1}^{n-1}\varphi_{\tau_2-\tau_1}\big(\tilde{y}^{k\ell}_j-z_j\big)\sum_{1\le k'<\ell'\le n-1}f^\varepsilon_{\tau_1-}\big(\tilde{z}^{k'\ell'}\big)\frac{2}{(n-1)(n-2)}\right]\mathrm{d}z$$
$$= c\int_{\mathbb{R}^{n-1}} E\left[1_{\tau_2\le t<\tau_3}\prod_{j=1}^{n-1}\varphi_{\tau_2-\tau_1}\big(\tilde{y}^{k\ell}_j-z_j\big)\varphi_{\tau_1+\varepsilon}(z_1-x)\cdots\varphi_{\tau_1+\varepsilon}(z_{n-2}-x)\varphi_{\tau_1+\varepsilon}(z_{n-1}-x)^2\right]\mathrm{d}z,$$

we have

$$E\big[1_{\tau_2\le t<\tau_3}\big\langle\mu^{\otimes n-2}, f_t^\varepsilon\big\rangle\big]$$
$$\le c^2\int_{\mathbb{R}^{n-2}}\prod_{i=1}^{n-2}\int_{\mathbb{R}} E\Bigg[1_{\tau_2\le t<\tau_3}\varphi_{t-\tau_2}(x_i-y_i)\mu(\mathrm{d}x_i)\sum_{1\le k<\ell\le n-2}\frac{2}{(n-2)(n-3)}$$
$$\times\int_{\mathbb{R}^{n-1}}\prod_{j=1}^{n-1}\varphi_{\tau_2-\tau_1}\big(\tilde{y}^{k\ell}_j-z_j\big)\varphi_{\tau_1+\varepsilon}(z_1-x)\cdots\varphi_{\tau_1+\varepsilon}(z_{n-2}-x)\varphi_{\tau_1+\varepsilon}(z_{n-1}-x)^2\Bigg]\mathrm{d}z\,\mathrm{d}y$$
$$\le c^2c_1^{n-3}\int_{\mathbb{R}^{n-2}}\int_{\mathbb{R}} E\Bigg[1_{\tau_2\le t<\tau_3}\varphi_{t-\tau_2}(x_{n-2}-y_{n-2})\mu(\mathrm{d}x_{n-2})\sum_{1\le k<\ell\le n-2}\frac{2}{(n-2)(n-3)}$$
$$\times\int_{\mathbb{R}^{n-1}}\prod_{j=1}^{n-1}\varphi_{\tau_2-\tau_1}\big(\tilde{y}^{k\ell}_j-z_j\big)\varphi_{\tau_1+\varepsilon}(z_1-x)\cdots\varphi_{\tau_1+\varepsilon}(z_{n-2}-x)\varphi_{\tau_1+\varepsilon}(z_{n-1}-x)^2\Bigg]\mathrm{d}z\,\mathrm{d}y$$
$$\le c^2c_1^{n-3}\int_{\mathbb{R}}\int_{\mathbb{R}} E\Bigg[1_{\tau_2\le t<\tau_3}\varphi_{t-\tau_2}(x_{n-2}-y_{n-2})\mu(\mathrm{d}x_{n-2})\sum_{1\le k<\ell\le n-2}\frac{2}{(n-2)(n-3)}$$
$$\times\int_{\mathbb{R}^{n-1}}\frac{c_2}{\sqrt{\tau_2-\tau_1}}\varphi_{\tau_2-\tau_1}(y_{n-2}-z_k)\varphi_{\tau_1+\varepsilon}(z_1-x)\cdots\varphi_{\tau_1+\varepsilon}(z_{n-2}-x)$$
$$\times\varphi_{\tau_1+\varepsilon}(z_{n-1}-x)\frac{c_2}{\sqrt{\tau_1+\varepsilon}}\Bigg]\mathrm{d}z\,\mathrm{d}y_{n-2}$$
$$\le E\Bigg[1_{\tau_2\le t<\tau_3}\frac{c^2c_1^{n-3}c_2^2}{\sqrt{\tau_1(\tau_2-\tau_1)}}\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi_{t-\tau_2}(x_{n-2}-y_{n-2})\mu(\mathrm{d}x_{n-2})$$

$$\times \sum_{1\le k<\ell\le n-2} \frac{2}{(n-2)(n-3)} \varphi_{\tau_2+\varepsilon}(y_{n-2}-x)\,\mathrm{d}y_{n-2}\Bigg]$$
$$\le E\bigg[1_{\tau_2\le t<\tau_3} \frac{c^2 c_1^{n-3} c_2^2}{\sqrt{\tau_1(\tau_2-\tau_1)}} \int_{\mathbb{R}} \varphi_{t+\varepsilon}(x_{n-2}-x)\mu(\mathrm{d}x_{n-2})\bigg].$$

Therefore we have

$$\int_{\mathbb{R}} \mathbb{E}\big[\langle \mu^{\otimes n_t}, f_t^{\varepsilon}\rangle 1_{\tau_2\le t<\tau_3}\big]\,\mathrm{d}x \le c^2 c_1^{n-2} c_2^2 \mathbb{E}\bigg[\frac{1}{\sqrt{\tau_1(\tau_2-\tau_1)}}\bigg] \times \mu(\mathbb{R}) < \infty,$$

where we used the assumption that μ is finite. The other terms can be proved similarly. □

By interpolation it follows that $\|\sqrt{X}\|_{\mathbb{L}_p(T)} < \infty$ for any $p \ge 2$. We note that Lemma 3.1 alone can not ensure anything about Hölder continuity of X (cf. Sobolev embedding theorem).

Let us explain our idea on Hölder continuity of X. By freezing the nonlinear term of SPDE (1.5), we consider the following auxiliary linear SPDE:

$$\begin{cases} \partial_t Y = L^* Y + \sqrt{X}\dot{B}_{tx}, \\ Y_0 = \mu, \end{cases} \tag{3.2}$$

with $\mu \in H_p^{1/2-\varepsilon-2/p}$. Then $Z := X - Y$ satisfies another linear SPDE:

$$\begin{cases} \partial_t Z = L^* Z - \big(\partial_x(\sigma_1 Z) + \partial_x(\sigma_1 Y)\big)\dot{W}_t, \\ Z_0 = 0. \end{cases} \tag{3.3}$$

Hence, we can estimate X via Y and Z by using linear SPDE theory if the coefficients of (3.2) and (3.3) are good for doing so. It turns out that (BC) serves this purpose very well.

We define few spaces for convenience of presentation. We denote

$$[f]_0 = \sup_{x\in\mathbb{R}} \big|f(x)\big|, \qquad [f]_\gamma = \sup_{x\ne y} \frac{|f(x)-f(y)|}{|x-y|^\gamma}$$

for $\gamma \in (0, 1]$. Using this notation, we define

$$\|f\|_{C^{0,\gamma}} = [f]_0 + [f]_\gamma, \qquad \|f\|_{C^{1,\gamma}} = [f]_0 + \big[f'\big]_0 + \big[f'\big]_\gamma,$$
$$\|f\|_{C^1} = [f]_0 + \big[f'\big]_0, \qquad \|f\|_{C^2} = [f]_0 + \big[f'\big]_0 + \big[f''\big]_0$$

and the following Banach spaces:

$$C^{0,\gamma} = \big\{f:\ \|f\|_{C^{0,\gamma}} < \infty\big\}, \qquad C^{1,\gamma} = \big\{f:\ f' \text{ exists and } \|f\|_{C^{1,\gamma}} < \infty\big\},$$
$$C^1 = \big\{f:\ f' \text{ exists and } \|f\|_{C^1} < \infty\big\}, \qquad C^2 = \big\{f:\ f'' \text{ exists and } \|f\|_{C^2} < \infty\big\}.$$

Remark 3.2. *Spaces $C^{0,\gamma}, C^{1,\gamma}$ are the usual Hölder spaces if $\gamma \in (0, 1)$. It is easy to see that $\|f\|_{C^{0,\gamma}} \le 3\|f\|_{C^{0,1}}$, $\|f\|_{C^{1,\gamma}} \le 3\|f\|_{C^{1,1}}$ and $\|f\|_{C^{0,1}} \le \|f\|_{C^1}$, $\|f\|_{C^{1,1}} \le \|f\|_{C^2}$ when f' or f'' exists.*

Next, we define the sense of solutions for a SPDE given. First, we recall the basic definitions of some function spaces defined in [7]. For $n \in \mathbb{R}$ and $p \in [2, \infty)$, in addition to the definition of H_p^n given before Theorem 1.1, let $H_p^n(l_2)$ be the space of ℓ_2-valued functions $g = \{g_k\}$ with norm

$$\|g\|_{n,p} = \big\|\big|(I-\Delta)^{n/2} g\big|_{l_2}\big\|_p.$$

Then we define

$$\mathbb{H}_p^n(T) = L_p\big(\Omega \times [0,T], \mathcal{P}, H_p^n\big), \qquad \mathbb{H}_p^n(T, l_2) = L_p\big(\Omega \times [0,T], \mathcal{P}, H_p^n(l_2)\big),$$

where $\mathcal{P}$ is the predictable σ-field of $\Omega \times [0,T]$. We denote $\mathbb{L}_p(T) = \mathbb{H}_p^0(T)$. Let $\{w_t^k\colon k = 1, 2, \ldots\}$ be a family of independent one-dimensional Brownian motions.

Definition 3.3. *We say $u \in \mathcal{H}_p^n(T)$ if $\partial_{xx} u \in \mathbb{H}_p^{n-2}(T)$ and $u(0,\cdot) \in L_p(\Omega, H_p^{n-2/p})$ and there exists $(f,g) \in \mathbb{H}_p^{n-2}(T) \times \mathbb{H}_p^{n-1}(T, l_2)$ such that $\forall \phi \in C_0^\infty$, (a.s.)*

$$\big\langle u(t,\cdot), \phi\big\rangle = \big\langle u_0(\cdot), \phi\big\rangle + \int_0^t \big\langle f(s,\cdot), \phi\big\rangle \mathrm{d}s + \sum_{k=0}^\infty \int_0^t \big\langle g^k(s,\cdot), \phi\big\rangle \mathrm{d}w_s^k$$

holds for all $t \le T$. We denote

$$\|u\|_{\mathcal{H}_p^n(T)} = \|\partial_{xx} u\|_{\mathbb{H}_p^{n-2}(T)} + \|f\|_{\mathbb{H}_p^{n-2}(T)} + \|g\|_{\mathbb{H}_p^{n-1}(T,l_2)} + \big(\mathbb{E}\big[\|u_0\|_{n-2/p,p}^p\big]\big)^{1/p}.$$

The reader can find motivation and detailed remarks about this definition in [7]. We understand solutions of SPDEs (3.2), (3.3) in the sense of Definition 3.3 with observation that a stochastic integral against a Brownian sheet can be expressed as a sum of countable stochastic integrals each of which is against (independent) one-dimensional Brownian motions. (See, for example, the first part of Section 8.3 of [7].)

Now, we fix $\varepsilon \in (0, \frac{1}{2})$.

Proof of Theorem 1.1(ii).

1. We apply Theorem 8.5 of [7] to (3.2) with ε instead of κ. To do this we need the coefficients of L^* and $\sqrt{X}$ to satisfy Assumptions 8.5 and 8.6 and conditions of Theorem 8.5 for (3.2). By examining Eq. (3.2) and consulting with Lemma 8.4 of [7], we can see that if

$$\frac{1}{2}\|a\|_{C^{1,1}} + \big\|a' - b\big\|_{C^{0,1}} \le K, \quad \delta \le \frac{1}{2} a \le K, \quad \left[\frac{1}{2}a'' - b'\right]_0 \le K \tag{3.4}$$

hold for some positive constants δ, K and $\|\sqrt{X}\|_{\mathbb{L}_p(T)} < \infty$, then all the requirements above hold. In fact, (3.4) follows from (BC) along with Remark 3.2, and the boundedness of $\|\sqrt{X}\|_{\mathbb{L}_p(T)}$ follows from Lemma 3.1. Hence, by Theorem 8.5 and the fact that μ is nonrandom, we have a unique solution Y of (3.2) in $\mathcal{H}_p^{1/2-\varepsilon}(T)$ with estimate

$$\|Y\|_{\mathcal{H}_p^{1/2-\varepsilon}(T)} \le N\big(\big\|\sqrt{X}\big\|_{\mathbb{L}_p(T)} + \|\mu\|_{1/2-\varepsilon-2/p,p}\big), \tag{3.5}$$

where N depends only on $\varepsilon, p, \delta, K, T$.

2. Next, we use Theorem 5.1 in [7] for Eq. (3.3) with $n = -\frac{3}{2} - \varepsilon \in (-2, -\frac{3}{2})$. Note $\partial_x(\sigma_1 Z) = \sigma_1 \partial_x Z + \partial_x \sigma_1 Z$.

Assumptions 5.1–5.6 in [7] are required for using Theorem 5.1. These requirements are fulfilled if the following hold:

(i) $\delta' \le \frac{1}{2}a - \frac{1}{2}\sigma_1^2 = \frac{1}{2}\sigma_2^2 \le K'$
for some positive δ', K'.

(ii) a, σ_1 are Lipschitz continuous with a Lipschitz constant K'.

(iii) $a \in C^{1,\gamma_1}, \sigma_1 \in C^{0,\gamma_1}$ for some $\gamma_1 \in (\frac{1}{2}, 1)$ and $\|a\|_{C^{1,\gamma_1}} + \|\sigma_1\|_{C^{0,\gamma_1}} \le K'$.

(iv) $\partial_x(\sigma_1 Y) \in \mathbb{H}_p^{n+1}(T)\ (= \mathbb{H}_p^{-1/2-\varepsilon}(T))$.

(v) $\|a' - b\|_{C^{0,\gamma_2}} + [\frac{1}{2}a'' - b']_0 + [\sigma_1']_0 \le K'$ for some $\gamma_2 \in (\frac{1}{2}, 1)$.

In this case, conditions (i)–(iv) handle Assumptions 5.1–5.5. The condition (v) handles Assumption 5.6 in the following way: by the help of Remark 5.5, Remark 5.6 suggests sufficient conditions for Assumption 5.6 and these conditions with $n = -\frac{3}{2} - \varepsilon$ are in fact (v).

It turns out that (i)–(v) follow (BC) and Remark 3.2. On the other hand, (iv) is also satisfied. For we have the following chain of inequalities:

$$\|\partial_x(\sigma_1 Y)\|_{\mathbb{H}_p^{-1/2-\varepsilon}(T)} \le N\|\sigma_1 Y\|_{\mathbb{H}_p^{1/2-\varepsilon}(T)} \tag{3.6}$$

$$\le N\|\sigma_1\|_{C^{0,1/2-\varepsilon+1/4}}\|Y\|_{\mathbb{H}_p^{1/2-\varepsilon}(T)} \tag{3.7}$$

$$\le N\|\sigma_1\|_{C^1}\|Y\|_{\mathbb{H}_p^{1/2-\varepsilon}(T)} \tag{3.8}$$

$$\le N\|Y\|_{\mathcal{H}_p^{1/2-\varepsilon}(T)} \tag{3.9}$$

$$\le N\|\sqrt{X}\|_{\mathbb{L}_p(T)} + N\|\mu\|_{1/2-\varepsilon-2/p,p} < \infty. \tag{3.10}$$

Explanations of these inequalities are in order. The right-hand side of (3.6) follows the observation $\partial_x = \partial_x(I-\Delta)^{-1/2}(I-\Delta)^{1/2}$ and the boundedness of the operator $\partial_x(I-\Delta)^{-1/2}$. (3.7) follows Lemma 5.2(i) in [7]. Note that $\frac{1}{2}-\varepsilon+\frac{1}{4}$ is still in $(0,1)$. Up to this step, N only depends on ε, p. We have (3.8) by Remark 3.2. Next, (3.9) follows (BC) and Theorem 3.7 in [7]. N now depends only on ε, p, K, T. Finally, estimate (3.5) gives us (3.10) with $N = N(\varepsilon, p, \delta, K, T)$.

Therefore, we have a unique solution Z in $\mathcal{H}_p^{1/2-\varepsilon}(T)$ with

$$\|Z\|_{\mathcal{H}_p^{1/2-\varepsilon}(T)} \le N\|\partial_x(\sigma_1 Y)\|_{\mathbb{H}_p^{-1/2-\varepsilon}(T)} \le N\|\sqrt{X}\|_{\mathbb{L}_p(T)} + N\|\mu\|_{1/2-\varepsilon-2/p,p}, \tag{3.11}$$

where $N = N(\varepsilon, p, \delta, K, T)$.

3. Combining steps 1 and 2, we have $\hat{X} := Y + Z \in \mathcal{H}_p^{1/2-\varepsilon}(T)$ satisfying

$$\partial_t \hat{X} = L^*\hat{X} - \partial_x(\sigma_1\hat{X})\dot{W}_t + \sqrt{X}\dot{B}_{tx} \tag{3.12}$$

in the sense of Definition 3.3 with estimate

$$\|\hat{X}\|_{\mathcal{H}_p^{1/2-\varepsilon}(T)} \le N\|\sqrt{X}\|_{\mathbb{L}_p(T)} + N\|\mu\|_{1/2-\varepsilon-2/p,p}. \tag{3.13}$$

We note that $D := \hat{X} - X$ satisfies

$$\partial_t D = L^*D - \partial_x(\sigma_1 D)\dot{W}_t, \quad D(0,\cdot) \equiv 0 \tag{3.14}$$

in the sense given in Theorem 1.1(i). On the other hand, using Theorem 5.1 in [7] one more time, we note that only a trivial solution satisfies (3.14) in the sense given in Theorem 1.1(i). This observation leads us to have $D \equiv 0$, $\hat{X} = X$ and estimate (3.13) with X instead of $\hat{X}$.

By the embedding Theorem 7.1 in [7], we have

$$\left(E\int_0^T \|X(t,\cdot)\|^p_{C^{0,1/2-\varepsilon-1/p}}\,\mathrm{d}t\right)^{1/p} \le N\|X\|_{\mathcal{H}_p^{1/2-\varepsilon}(T)} \le N\|\sqrt{X}\|_{\mathbb{L}_p(T)} + N\|\mu\|_{1/2-\varepsilon-2/p,p}$$

as long as $\frac{1}{2}-\varepsilon-\frac{1}{p} > 0$. In this case we have

$$\|X(t,\cdot)\|_{C^{0,1/2-\varepsilon-1/p}} < \infty$$

for (a.e.) $t \in [0,T]$ (a.s.). Theorem 1.1(ii) follows. □

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