

# A characterization of harmonic measures on laminations by hyperbolic Riemann surfaces

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**Abstract.**  $\mathcal{L}$  denotes a (compact, nonsingular) lamination by hyperbolic Riemann surfaces. We prove that a probability measure on  $\mathcal{L}$  is harmonic if and only if it is the projection of a measure on the unit tangent bundle  $T^1\mathcal{L}$  of  $\mathcal{L}$  which is invariant under both the geodesic and the horocycle flows.

**Résumé.**  $\mathcal{L}$  denote une lamination (compacte, nonsingulière) par surfaces de Riemann hyperboliques. On montre qu'une mesure sur  $\mathcal{L}$  est harmonique si et seulement si elle est la projection d'une mesure sur le fibré tangent unitaire  $T^1\mathcal{L}$  qui est invariante sous les flots géodésique et horocyclique.

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## Introduction

This note studies measures associated to compact nonsingular laminations by hyperbolic Riemann surfaces. It builds on a previous article by the second author, entitled *Measures on Hyperbolic Surface Laminations* (see [14]), and improves on the main results found therein.

In this work,  $\mathcal{L}$  denotes a lamination (or foliated space), which is compact and whose leaves are hyperbolic Riemann surfaces. Each leaf has its Poincaré metric – the metric of constant curvature  $-1$  compatible with the conformal structure. We consider two different kinds of measures associated to  $\mathcal{L}$ : measures invariant under the heat diffusion along the leaves of  $\mathcal{L}$  which are called *harmonic* and measures on the unit tangent bundle  $T^1\mathcal{L}$  of  $\mathcal{L}$  that are invariant under the laminated geodesic and horocycle flows. Our main result reads as follows:

**Main Theorem.** *Let  $\mu$  be a probability measure on  $\mathcal{L}$ . Then  $\mu$  is harmonic if and only if there is a measure  $\nu$  on  $T^1\mathcal{L}$  which is invariant under both the geodesic and stable horocycle flow and that projects onto  $\mu$  (under the canonical projection  $T^1\mathcal{L} \rightarrow \mathcal{L}$ ). Furthermore, such a  $\nu$  is unique.*

This had been conjectured by Christian Bonatti.

Of course, in the statement of this theorem we can change the *stable* horocycle flow for the *unstable* horocycle flow, but asking that the measure  $\mu$  be invariant under the three flows we are considering might be too much: In fact, there is a canonical bijection between measures invariant under the three flows and the simplest kind of harmonic measures; namely, those coming from measures on transversals which are invariant under the holonomy pseudogroup. This is easy to prove and can be found in [14].

Our Main Theorem represents an improvement on the main results found in [14] (labeled Theorem 2.6 and Theorem 2.8), which state the following:

**Theorem.** 1. Any harmonic probability measure on  $\mathcal{L}$  is the projection of a measure invariant under the stable horocycle flow on  $T^1\mathcal{L}$ .

2. Any probability measure on  $T^1\mathcal{L}$  invariant under both the geodesic and the stable horocycle flow projects onto a harmonic measure on  $\mathcal{L}$ .

Remark that the first statement is a partial converse of the second one; what we do in this note is to prove the converse of 2. The techniques we use are completely different from those used in [14].

In Section 1 we review and slightly modify the construction of a measure on  $T^1\mathcal{L}$  invariant under the horocycle flow that projects onto a given harmonic measure on  $\mathcal{L}$ . We wish to prove that the measure thus constructed is also invariant under the geodesic flow. This can be reduced to a probabilistic statement on the regularity of the distribution of the radial component of the Brownian motion on the hyperbolic plane.

The key regularity statement is provided by Theorem 2 which is proved in Section 2. The main idea of the proof is to show that for large times the radial component of the hyperbolic planar Brownian motion behaves as one-dimensional Brownian motion with constant positive drift. The closeness estimates are based on Girsanov’s theorem.

The uniqueness of the measure  $\nu$  is proved in Section 3. We consider a flow box in which we disintegrate a harmonic measure  $\mu$  using Rokhlin’s theorem, and prove that there is at most one way to lift the harmonic measure in this flow box to a measure invariant under the affine group. Only local considerations are involved.

In Section 4 we state some simple applications of our Main Theorem. We state previous results regarding the uniqueness of harmonic measures which can now be translated to unique ergodicity for the action of the affine group in some examples.

## 1. Harmonic measures as projections of measures invariant under the affine group

### 1.1. Hyperbolic surface laminations

$\mathcal{L}$  is a *lamination* if it is a separable, locally compact metrizable space that has an open covering  $\{E_i\}$  and an atlas  $\{(E_i, \varphi_i)\}$  satisfying:

1.  $\varphi_i : E_i \rightarrow D_i \times T_i$  is a homeomorphism, for some open disk  $D_i$  in  $\mathbb{R}^d$  and topological space  $T_i$ , and
2. the coordinate changes  $\varphi_j \circ \varphi_i^{-1}$  are of the form  $(z, t) \mapsto (\zeta(z, t), \tau(t))$ , where each  $\zeta$  is smooth in the  $z$  variable.

This last condition says that the sets of the form  $\varphi_i^{-1}(D_i \times \{t\})$ , called *plaques*, glue together to form  $d$ -dimensional manifolds that we call *leaves*.

$\mathcal{L}$  is a *Riemann surface lamination* if the disks  $D_i$  are open subsets of the complex plane and the maps  $\zeta$  are holomorphic in the  $z$  variable. We say that  $\mathcal{L}$  is a *hyperbolic surface lamination* if its leaves are hyperbolic Riemann surfaces.

Each leaf on a hyperbolic surface lamination  $\mathcal{L}$  has a Poincaré metric, which is the only Riemannian metric of constant curvature  $-1$  compatible with the conformal structure. According to a theorem due to Candel (see [3]), these metrics on the leaves, as well as all their derivatives, have continuous variation in the transverse direction.

In this paper,  $\mathcal{L}$  will always denote a compact hyperbolic surface lamination.

### 1.2. Harmonic measures

Each leaf  $L$  of  $\mathcal{L}$ , being a Riemannian manifold, has a Laplace–Beltrami operator  $\Delta_L$ . If  $f : \mathcal{L} \rightarrow \mathbb{R}$  is a function of class  $C^2$  in the leaf direction and  $x \in \mathcal{L}$ , we define  $\Delta f(x) = \Delta_L f|_L(x)$ , where  $L$  is the leaf passing through  $x$  and  $f|_L$  is the restriction of  $f$  to  $L$ . A probability measure  $\mu$  on  $\mathcal{L}$  is *harmonic* if  $\Delta\mu = 0$ ; i.e. if  $\int \Delta f \, d\mu = 0 \, \forall f$ .

The operator  $\Delta/2$  is the infinitesimal generator of the *laminated heat semigroup*  $\{D_t\}_{t \geq 0}$ . (This choice of the generator corresponds to the standard hyperbolic planar Brownian motion with unit diffusion.) The heat diffusion along the leaves can be described in terms of the *laminated heat kernel*,  $p : \mathcal{L} \times \mathcal{L} \times (0, +\infty) \rightarrow \mathbb{R}$  given by

$$p(x, y, t) = \begin{cases} p_L(x, y, t) & \text{if } x, y \text{ belong to the same leaf } L, \\ 0 & \text{otherwise,} \end{cases}$$

where  $p_L$  is the heat kernel on  $L$ . The laminated heat semigroup can be expressed as

$$D_t f(x) = \int_{L_x} p(x, y, t) f(y) dy,$$

where  $L_x$  is the leaf through  $x$ .

Lucy Garnett proved an Ergodic Theorem for harmonic measures, which can be found in [4] or [7].

A harmonic measure  $m$  is *ergodic* if  $\mathcal{L}$  cannot be partitioned into two measurable leaf-saturated subsets having positive  $m$  measure.

The ergodic harmonic measures can be briefly described as follows: we take a point  $x$  in  $\mathcal{L}$  and consider the Dirac delta at  $x$ , which we call  $\delta_x$ . We can diffuse this measure, obtaining for each positive time  $t$  a probability measure  $D_t \delta_x$ , whose integral on any continuous function  $f$  on  $\mathcal{L}$  is  $D_t f(x)$ . For almost all  $x$  according to any harmonic measure, the sequence of Krylov–Bogolyubov means

$$\frac{1}{n} \sum_{t=0}^{n-1} D_t \delta_x$$

has a limit which we call  $\tilde{\delta}_x$ . Ergodic measures are of the form  $\tilde{\delta}_x$ .

### 1.3. The $PSL(2, \mathbb{R})$ -action on the unit tangent bundle

If  $\mathcal{L}$  is a hyperbolic surface lamination, we call  $T^1 \mathcal{L}$  the lamination whose three dimensional leaves are the unit tangent bundles of the leaves of  $\mathcal{L}$  and that has “the same” charts as  $\mathcal{L}$ .

The *laminated geodesic flow* is the flow  $g_t$  that, restricted to the unit tangent bundle of a leaf  $L$  of  $\mathcal{L}$ , coincides with the geodesic flow in  $T^1 L$ . The *laminated stable and unstable horocycle flows*  $h^+$  and  $h^-$  are the flows that, when restricted to the unit tangent bundle of a leaf  $L$ , coincide with the stable and unstable horocycle flows on  $T^1 L$ , respectively. For the definition and basic properties of the geodesic, stable horocycle and unstable horocycle flows on hyperbolic surfaces, see [12]. All these flows are continuous on  $T^1 \mathcal{L}$ , as a consequence of Candel’s theorem (see [3]).

In the lamination  $T^1 \mathcal{L}$  there is a right  $PSL(2, \mathbb{R})$ -action whose orbits are the leaves. As in the case of surfaces, the geodesic and the horocycle flows correspond to the action on  $T^1 \mathcal{L}$  of the one-parameter subgroups  $D_t$ ,  $H^+$  and  $H^-$ , respectively, where

$$D = \left\{ \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \right\}, \quad H^+ = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad H^- = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right\}.$$

Consequently, the joint action of  $D$  and  $H^+$  corresponds to the action of the affine group

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}; a, b \in \mathbb{R}, a > 0 \right\}.$$

### 1.4. A measure invariant under the action of the affine group

Let  $\pi : T^1 \mathcal{L} \rightarrow \mathcal{L}$  be the canonical projection, and consider a harmonic probability measure  $\mu$  on  $\mathcal{L}$ . In [14], there is a construction that produces a measure  $\nu$  on  $T^1 \mathcal{L}$  which is invariant under the stable (or the unstable) horocycle flow and such that  $\pi_* \nu = \mu$ . It will be more convenient for us to consider the measure  $\nu$  which is invariant under the *unstable* horocycle flow and which projects onto  $\mu$ .

**Theorem 1.** *The measure  $\nu$  is invariant under the geodesic flow  $g$ .*

This theorem says that  $\nu$  is, in fact, invariant under both the geodesic and horocycle flows. Therefore, any harmonic measure is the projection of a measure invariant under both flows. This statement and its converse which is quoted in the Introduction constitute our Main Theorem.

**Proof of Theorem 1.** As in [14], two simplifying assumptions shall be made, that imply no loss of generality:

1. That  $\mu$  is ergodic; that is, there is an  $x \in \mathcal{L}$  such that  $\mu = \tilde{\delta}_x$ ; and
2. that the point  $x$  belongs to a leaf which is simply connected. (If it does not, we consider the universal cover of the leaf through  $x$ , as explained in [14], p. 857.)

The construction of  $\nu$  goes as follows:

For any natural number  $n \geq 1$ , let  $\delta_x^{(n)}$  be the Krylov–Bogolyubov sum  $\frac{1}{n} \sum_{t=0}^{n-1} D_t \delta_x$ . Namely,  $\delta_x^{(n)}$  is the probability measure such that, for every continuous function  $f$  in  $\mathcal{L}$ ,

$$\int f d\delta_x^{(n)} = \frac{1}{n} \sum_{t=0}^{n-1} D_t f(x) = \frac{1}{n} \sum_{t=0}^{n-1} \int_{\mathcal{L}} p(x, y, t) f(y) dy.$$

With this notation,  $\mu = \lim_n \delta_x^{(n)}$ .

Let  $L_x$  be the leaf of the lamination  $\mathcal{L}$  passing through  $x$ . It is a hyperbolic plane. Let  $R : L_x \setminus \{x\} \rightarrow T^1 \mathcal{L}$  be the unit radial vector field pointing outwards; i.e.  $R(y) = (\gamma(s), \dot{\gamma}(s))$  if  $\gamma$  is the geodesic of unit speed such that  $\gamma(0) = x$  and  $\gamma(s) = y$ . (Here  $s$  is, of course, the distance from  $x$  to  $y$  measured on  $L_x$ .)

Define  $\mu_n = R_* \delta_x^{(n)}$ . This gives a sequence of probability measures on the compact space  $T^1 \mathcal{L}$ . Let  $\nu$  be any limit point of the sequence  $(\mu_n)$  in the sense of the weak-\* topology. (It is not difficult to see that in fact  $\nu$  is the limit of the  $\mu_n$ , but this is unessential for our argument.) Then, as was proved in [14],  $\nu$  is invariant under the unstable horocycle flow. And  $\pi_* \nu = \mu$  since  $\pi_*$  is a continuous map from the space of finite measures on  $T^1 \mathcal{L}$  to that of finite measures on  $\mathcal{L}$ .

To finish the proof of Theorem 1 we shall show that  $(g_s)_* \nu = \nu$  for all  $s \in [0, 1]$ . It is clearly enough to prove the following:

(\*) For every continuous real-valued function  $f$  on  $T^1 \mathcal{L}$  and every  $s \in [0, 1]$ ,

$$\lim_{t \rightarrow +\infty} \left| \int_{T^1 \mathcal{L}} f d(R_* D_t \delta_x) - \int_{T^1 \mathcal{L}} f \circ g_s d(R_* D_t \delta_x) \right| = 0.$$

Let  $S_r = S_r(x) = \{y \in L_x : d(x, y) = r\}$ , where  $d$  is the hyperbolic distance measured on the leaf  $L_x$ . Its normalized Lebesgue measure can be pushed forward by  $R$  to get a measure, that we call  $\lambda_r$ , supported on the curve  $R(S_r) \subset T^1 \mathcal{L}$ . We can write the integral of  $f$  with respect to  $R_* D_t \delta_x$  as

$$\int_{\mathbb{R}^+} p_t(r) \left( \int f d\lambda_r \right) dr,$$

where  $p_t(r) = p(x, y, t) \times (\text{length}(S_r))$  for any point  $y$  in  $L_x$  such that  $d(x, y) = r$ . If  $s > 0$ , the geodesic flow at time  $s$  takes  $R(S_r)$  to  $R(S_{r+s})$ , and  $(g_s)_* \lambda_r = \lambda_{r+s}$ . Writing  $u(r) = \int f d\lambda_r$ , the statement labeled (\*) follows from:

(\*\*) For every continuous bounded function  $u : \mathbb{R}^+ \rightarrow \mathbb{R}$  and every  $s \in [0, 1]$ ,

$$\lim_{t \rightarrow \infty} \left| \int_{\mathbb{R}^+} p_t(r) u(r) dr - \int_{\mathbb{R}^+} p_t(r+s) u(r) dr \right| = 0.$$

Claim (\*\*) is implied by

**Theorem 2.** For any  $s \in [0, 1]$ ,

$$\lim_{t \rightarrow \infty} \int_{\mathbb{R}^+} |p_t(r) - p_t(r+s)| dr = 0. \tag{1}$$

Section 2 is devoted to the proof of this theorem. □

**Remark 1.** Theorem 1 says that any harmonic measure on  $\mathcal{L}$  is the projection of a measure invariant under the action of the lower triangular group in  $T^1 \mathcal{L}$ . Of course we could do the same thing for the upper triangular group, considering the unit inward radial vector field on  $L_x \setminus \{x\}$ , and then taking  $s$  in  $[-1, 0]$ .

## 2. The regularity of the heat kernel

Though an explicit expression for  $p_t(r)$  is well-known, see e.g. [5], p. 246, it is not easy to use that formula directly. We use a stochastic calculus approach instead. This method can be generalized to higher dimensions.

We start with the following rotationally invariant Riemannian metric on the plane  $\mathbb{R}^2$  in polar coordinates:

$$ds^2 = dr^2 + \sinh^2(r) d\theta^2$$

which turns it into a hyperbolic plane (see [5], Chapter X).

In our study of the Brownian motion on the hyperbolic plane we shall refer to [11] and [15] for basic facts on Brownian motion and stochastic calculus.

Let  $B_t$  be the standard Brownian motion on this hyperbolic plane (a stochastic process with generator given by  $\Delta/2$  where  $\Delta$  is the Laplace–Beltrami operator associated with  $ds^2$ ). Its radial component  $X_t = d(0, B_t)$  satisfies the following stochastic differential equation:

$$dX_t = dW_t + \frac{1}{2} \coth(X_t) dt, \quad (2)$$

where  $W_t$  is a standard Wiener process. This can be easily derived using the explicit expression for the Laplace–Beltrami operator in polar coordinates  $(r, \theta)$ , see e.g. [13]. Equation (2) should be understood in the integral sense:

$$X_t - X_0 = W_t + \frac{1}{2} \int_0^t \coth(X_s) ds. \quad (3)$$

This equation defines a family of transition probability kernels:

$$P_t(x, dy) = \mathbb{P}\{X_t \in dy | X_0 = x\}, \quad t > 0, x \in \mathbb{R}_+. \quad (4)$$

If  $x \in \mathbb{R}_+$ , then  $P_t(x, \cdot) = 1$  is concentrated on  $\mathbb{R}_+$  due to a singularity of the drift term at 0. (This singularity point is of “entrance and non-exit” type according to the classification in [9].)

These kernels are, in fact, absolutely continuous with respect to the Lebesgue measure  $dy$  on  $\mathbb{R}_+$ , and the associated density  $p_t(x, y)$ ,  $x, y \in \mathbb{R}_+$  satisfies the following forward Kolmogorov (or Fokker–Planck) equation (see [11], Eqs (1.6) and (1.8) on p. 282):

$$\frac{\partial p_t(x, y)}{\partial t} = \frac{1}{2} \frac{\partial^2 p_t(x, y)}{\partial y^2} - \frac{\partial}{\partial y} \left( \frac{1}{2} \coth(y) p_t(x, y) \right), \quad x, y \in \mathbb{R}_+. \quad (5)$$

For  $y \in \mathbb{R}_+$  we shall denote  $p_t(y) = p_t(0, y)$  which is consistent with the definition of  $p_t(\cdot)$  given in Section 1.4, since the heat kernel  $p(0, \cdot, t)$  is rotationally invariant and coincides with the transition density of the hyperbolic Brownian motion.

**Remark 2.** In probabilistic terms, Eq. (1) from the statement of Theorem 2 is equivalent to

$$\lim_{t \rightarrow \infty} \|P_t(\cdot) - P_t(\cdot + s)\|_{\text{TV}} = 0,$$

where  $P_t(\cdot) = P_t(0, \cdot)$ ,  $P_t(\cdot + s)$  means  $P_t(\cdot)$  translated by  $s$ , and  $\|\mu_1 - \mu_2\|_{\text{TV}}$  denotes the total variation distance between measures  $\mu_1$  and  $\mu_2$ , see [16], Section 5.3.

**Proof of Theorem 2.** Our strategy will be to compare  $p_t(y, z)$  to the transition density of the Brownian motion with constant drift  $1/2$ . This will be possible due to Girsanov’s theorem and closeness of the drift term  $\coth(x)/2$  in Eq. (2) to  $1/2$  for large values of  $x$ .

We shall need the following Kolmogorov–Chapman equation (see [15], Chapter III):

$$P_t(x, A) = \int_{\mathbb{R}_+} P_{t'}(x, dy) P_{t-t'}(y, A), \quad t' \in (0, t), \quad (6)$$

or, equivalently,

$$p_t(x, z) = \int_{\mathbb{R}_+} P_{t'}(x, dy) p_{t-t'}(y, z), \quad t' \in (0, t). \tag{7}$$

Let us fix  $s \in [0, 1]$  and denote the integral under limit on the l.h.s. of (1) by  $I_t$ . Equation (7) implies

$$I_t \leq \int_{\mathbb{R}_+} P_{t'}(dy) \int_{\mathbb{R}_+} |p_{t-t'}(y, z) - p_{t-t'}(y, z + s)| dz, \quad t' \in (0, t). \tag{8}$$

Next,

$$\begin{aligned} \int_{\mathbb{R}_+} |p_{t-t'}(y, z) - p_{t-t'}(y, z + s)| dz &\leq \int_{\mathbb{R}_+} |p_{t-t'}(y, z) - p_{t-t'}^*(y, z)| dz \\ &\quad + \int_{\mathbb{R}_+} |p_{t-t'}(y, z + s) - p_{t-t'}^*(y, z + s)| dz \\ &\quad + \int_{\mathbb{R}_+} |p_{t-t'}^*(y, z) - p_{t-t'}^*(y, z + s)| dz \\ &= I_1(y, t - t') + I_2(y, t - t') + I_3(y, t - t'), \end{aligned} \tag{9}$$

where

$$p_t^*(y, z) = \frac{1}{\sqrt{2\pi t}} e^{-(z-y-t/2)^2/(2t)} \tag{10}$$

is the density of  $Y_t = y + W_t + t/2$  which is a Wiener process with constant drift  $1/2$  started at  $y$ . In fact,  $Y_t$  is a solution of the following SDE:

$$dY_t = dW_t + \frac{1}{2} dt. \tag{11}$$

Since  $s \in [0, 1]$ , a straightforward estimate based on (10) implies that for some constant  $K$  and all  $y$ ,

$$I_3(y, t - t') \leq \frac{K}{\sqrt{t - t'}}. \tag{12}$$

Let us estimate  $I_1(y, t)$  for large values of  $y$ . Namely, let us fix a number  $R > 0$  to be specified later and assume that  $y > R$ .

Girsanov's theorem (see [15], Chapter VIII) implies that the measures on paths generated by solutions of (2) and (11) emitted from the same initial point  $y$  are absolutely continuous with respect to the Wiener measure on paths  $W$  with

$$\int_0^t \coth^2(y + W_r) dr < \infty$$

(i.e. on paths not crossing 0), and the densities are given by

$$Z_t(y + W) = \exp \left\{ \int_0^t \frac{\coth(y + W_r)}{2} dW_r - \frac{1}{2} \int_0^t \left( \frac{\coth(y + W_r)}{2} \right)^2 dr \right\}$$

and

$$Z_t^*(y + W) = \exp \left\{ \int_0^t \frac{1}{2} dW_r - \frac{1}{2} \int_0^t \left( \frac{1}{2} \right)^2 dr \right\} = \exp \left\{ \frac{1}{2} W_t - \frac{t}{8} \right\},$$

respectively.

Itô's formula (see [11], p. 149) gives

$$\ln \sinh(y + W_t) - \ln \sinh(y) = \int_0^t \coth(y + W_r) dW_r + \frac{1}{2} \int_0^t (1 - \coth^2(y + W_r)) dr$$

which implies

$$Z_t(y + W) = \left( \frac{\sinh(y + W_t)}{\sinh(y)} \right)^{1/2} \exp \left\{ -\frac{t}{4} + \frac{1}{8} \int_0^t \coth^2(y + W_r) dr \right\}.$$

Therefore

$$\begin{aligned} \frac{Z_t(y + W)}{Z_t^*(y + W)} &= \left( \frac{\sinh(y + W_t)}{\sinh(y)e^{W_t}} \right)^{1/2} \exp \left\{ \frac{1}{8} \int_0^t \coth^2(y + W_r) dr - \frac{t}{8} \right\} \\ &= \left( \frac{1 - e^{-2(y+W_t)}}{1 - e^{-2y}} \right)^{1/2} \exp \left\{ \frac{1}{8} \int_0^t (\coth^2(y + W_r) - 1) dr \right\} \end{aligned}$$

for all paths  $y + W$  not crossing 0.

So, if  $y \in [R, \infty)$ , and the entire path  $y + W$  lies in  $[R/2, \infty)$ , then

$$(1 - e^{-R})^{1/2} \leq \frac{Z_t(y + W)}{Z_t^*(y + W)} \leq \frac{\exp\{t/(8 \sinh^2(R/2))\}}{(1 - e^{-2R})^{1/2}}. \tag{13}$$

Let us now split the densities  $p_t^*(y, z)$  and  $p_t(y, z)$  as follows:

$$\begin{aligned} p_t^*(y, z) &= q_t^*(y, z) + \psi_t^*(y, z), \\ p_t(y, z) &= q_t(y, z) + \psi_t(y, z). \end{aligned}$$

Here  $q_t^*(y, z)$  (respectively,  $q_t(y, z)$ ) denotes the contribution to  $p_t^*(y, z)$  (respectively,  $p_t(y, z)$ ) from paths  $y + W$  connecting  $y$  and  $z$  and staying within  $[R/2, \infty)$ .

From (13) we obtain

$$\frac{1}{C(R, t)} \leq \frac{q_t(y, z)}{q_t^*(y, z)} \leq C(R, t),$$

where

$$C = C(R, t) = \frac{\exp\{t/(8 \sinh^2(R/2))\}}{(1 - e^{-R})^{1/2}}.$$

Then,

$$I_1(y, t) \leq \int_{\mathbb{R}_+} |q_t(y, z) - q_t^*(y, z)| dz + \int_{\mathbb{R}_+} \psi_t(y, z) dz + \int_{\mathbb{R}_+} \psi_t^*(y, z) dz. \tag{14}$$

The first term is bounded by

$$\begin{aligned} &\int_{z \geq R, q_t(y, z) > q_t^*(y, z)} (Cq_t^*(y, z) - q_t^*(y, z)) dz + \int_{z \geq R, q_t(y, z) \leq q_t^*(y, z)} (Cq_t(y, z) - q_t(y, z)) dz \\ &\leq (C - 1) \int_{z \geq R} q_t(y, z) dz \leq C - 1. \end{aligned}$$

The last term in (14) is bounded by

$$\mathbb{P}\left\{\min_r\left(y + W_r + \frac{r}{2}\right) < \frac{R}{2}\right\} \leq \mathbb{P}\left\{\min_r\left(W_r + \frac{r}{2}\right) < -\frac{R}{2}\right\},$$

and decays to 0 as  $R \rightarrow \infty$ . Since  $\coth(x) > 1$  for all  $x > 0$ , the second term in (14) is less than the third one and decays to 0 as well. Therefore, inequality (14) implies

$$I_1(y, t) \leq C(R, t) - 1 + Q(R), \tag{15}$$

where  $Q(R)$  is a function satisfying  $\lim_{R \rightarrow \infty} Q(R) = 0$ .

Notice that  $I_2(y, t) < I_1(y, t)$ . Therefore, Eqs (8), (12), (15) imply that

$$I_t \leq 2(C(R, t - t') - 1 + Q(R)) + \frac{K}{\sqrt{t - t'}} + 2P_{t'}((0, R)). \tag{16}$$

Notice that for any  $R > 0$  and any  $\varepsilon > 0$  one can choose  $t_0$  such that  $P_{t'}((-\infty, R)) < \varepsilon$  for all  $t' > t_0$ . In fact, this is obviously true with  $P_{t'}$  replaced by  $P_{t'}^*$ . Our claim follows since

$$X_{t'} = W_{t'} + \frac{1}{2} \int_0^{t'} \coth(X_r) dr > W_{t'} + \frac{t'}{2},$$

and  $P_{t'}^*$  is dominated by  $P_{t'}$ .

Let us now take an arbitrary  $\varepsilon > 0$ . First, we choose  $\tau > 0$  so that  $K/\sqrt{\tau} < \varepsilon/3$ . After that, we choose  $R$  to be large enough to ensure that  $2(C(R, \tau) - 1 + Q(R)) < \varepsilon/3$ . Finally, we choose  $t_0$  such that  $P_{t'}((-\infty, R)) < \varepsilon/3$  for all  $t' > t_0$ . This choice of parameters and (16) allow us to conclude that  $I_t < \varepsilon$  for any  $t > t_0 + \tau$ , and the proof of the theorem is complete.  $\square$

### 3. Uniqueness of the measure invariant under the affine group that projects onto a given harmonic measure

In this section  $\mathbb{D}$  will denote the unit disk in  $\mathbb{C}$ , with its Poincaré metric.

**Proposition 1.** *Let  $U$  be an open disk contained in  $\mathbb{D}$ , and  $T^1U \simeq U \times S^1$  its unit tangent bundle. Call  $p: T^1U \rightarrow U$  the canonical projection. Let  $\mu$  be a probability measure in  $U$ , and assume that there is a measure  $\nu$  in  $T^1U$  which is invariant under the action of (small elements of) the affine group and such that  $p_*\nu = \mu$ . Then there is only one such measure; i.e.  $\nu$  is uniquely determined by  $\mu$ .*

By invariance of  $\nu$  under small elements of the affine group we mean the following: Consider a test function  $f$  that is continuous and has compact support in  $T^1U$ . If  $A$  is a sufficiently small element of the affine group, the composition  $f \circ A$  will still have its support in  $T^1U$ . Under these conditions, the measure  $\nu$  verifies that

$$\int f d\nu = \int (f \circ A) d\nu.$$

**Remark 3.** *Clearly not any measure  $\mu$  is the projection of a measure invariant under the affine group. In particular, it has to be of the form “harmonic function  $\times$  area” (see [14], Section 2.3).*

We will start the proof of this proposition by giving two simple lemmas. One of them relies on the following theorem, which can be found in [8], Ch. 4, p. 67, Theorem 4.1:

Let  $k: \mathbb{D} \times S^1 \rightarrow \mathbb{R}$  be the Poisson kernel of  $\mathbb{D}$ .

**Theorem (Herglotz).** *Let  $\Lambda$  be the set of all positive measures on  $S^1$ . The correspondence*

$$\eta \mapsto h(z) = \int_{S^1} k(z, \theta) d\eta(\theta)$$

is a one-to-one map from  $\Lambda$  to the set of nonnegative harmonic functions on the unit disk  $\mathbb{D}$ .

**Lemma 1.**  $\{\theta \mapsto k(z, \theta)\}_{z \in U}$  generates a dense linear subspace of  $C(S^1)$ .

**Proof.** If this were not the case, there would be a non-zero measure  $\xi$  in  $S^1$  in the annihilator of all the functions  $k(z, \cdot)$ ; namely, such that

$$\int k(z, \theta) \, d\xi(\theta) \equiv 0 \tag{17}$$

in  $U$ , and therefore in  $\mathbb{D}$ . Let  $\xi_+$  and  $\xi_-$  be the positive and negative parts of  $\xi$ , respectively. Define

$$h(z) = \int k(z, \theta) \, d\xi_+(\theta) = \int k(z, \theta) \, d\xi_-(\theta).$$

The fact that the positive harmonic function  $h$  has these two different expressions contradicts Herglotz’s theorem.  $\square$

**Lemma 2.** Assume that  $\varphi$  is an unknown measure on  $S^1$  such that, for every continuous real-valued function  $f$  with compact support in  $U$ ,

$$\int_{S^1} \int_U f(z) k(z, \theta) \, dz \, d\varphi(\theta) \tag{18}$$

is known. Then, for all  $z_0 \in U$ , the integral

$$\int_{S^1} k(z_0, \theta) \, d\varphi(\theta) \tag{19}$$

is also known.

The proof is obtained by considering a sequence of continuous functions  $(f_n)_n$  such that  $f_n(z) \, dz \rightarrow \delta_{z_0} \, dz$  weakly.

**Proof of Proposition 1.** The disk  $\mathbb{D}$  is a model for the hyperbolic plane, and therefore its unit tangent bundle  $T^1\mathbb{D}$  can be identified with  $PSL(2, \mathbb{R})$ . Consider the action of the affine group  $B$  by right translation on  $PSL(2, \mathbb{R})$ . Its orbits, under the identification  $PSL(2, \mathbb{R}) \simeq T^1\mathbb{D}$ , are hyperbolic surfaces that foliate  $T^1\mathbb{D}$ . This foliation is the weakly stable foliation for the geodesic flow on  $T^1\mathbb{D}$ , and its space of leaves can be identified with the set of points at infinity in  $\mathbb{D}$ , which can in turn be identified with the unit tangent space to any point in  $\mathbb{D}$ . Namely, when considering the (trivial) bundle  $T^1\mathbb{D}$ , we can identify  $T^1\mathbb{D} \simeq \mathbb{D} \times S^1$  in such a way that the orbits of the affine group are simply the sets of the form  $\mathbb{D} \times \{\theta\}$ .

Let us consider an open disk  $U \subset \mathbb{D}$ , and the restriction of the aforementioned foliation to  $T^1U \subset T^1\mathbb{D}$ . Under the identification mentioned above,  $T^1U \simeq U \times S^1$ , and the restriction to  $U$  of the foliation induced by the affine group action is  $\{U \times \{\theta\}, \theta \in S^1\}$ . We will use Rokhlin’s decomposition theorem to disintegrate the measure  $\nu$  on the partition given by this foliation, obtaining the following:

$$\int f \, d\nu = \int_{S^1} \int_{U \times \{\theta\}} f(z, \theta) \, d\nu_\theta(z) \, d\varphi(\theta).$$

Here each probability measure  $\nu_\theta$  is the conditional measure on the set  $U \times \{\theta\}$ , and  $\varphi$  is the projection of  $\nu$  onto  $S^1$ .

Each  $\nu_\theta$  is invariant under the action of small elements of the affine group. This determines it completely:  $U \times \{\theta\}$ , being an open subset of an orbit and the action being free, can be thought of an open subset of the affine group itself, and  $\nu_\theta$  must be the restriction to it of the right Haar measure, suitably normalized so that it is a probability measure. This amounts to saying that the  $\nu_\theta$  are independent of  $\nu$ , so  $\nu$  is completely determined by  $\varphi$ . Proving the uniqueness of  $\nu$  is proving the uniqueness of  $\varphi$ , given the fact that  $\nu$  projects onto the measure  $\mu$ , which is fixed.

The measures  $\nu_\theta$  are computed in [14], in Lemma 2.7 and the remark preceding it, using the upper-half plane model of the hyperbolic plane. In the Poincaré disk model, which is more convenient for our purposes, and taking  $U$  to be a small disk around 0, the measure  $\nu_\theta$  is

$$\nu_\theta = k(z, \theta) d\theta,$$

where  $k$  is, up to a multiplicative constant and a rotation in  $\theta$ , the Poisson kernel of the unit disk.

So we have the following expression for the measure  $\nu$ :

$$\int f d\nu = \int_{S^1} \int_U f(z, \theta) k(z, \theta) dz d\varphi(\theta).$$

The fact that it projects onto a measure that we know means that we can compute  $\int f d\nu$  when  $f = f(z)$ , which gives (18). Lemma 2 says that this determines (19), and Lemma 1 says that this uniquely determines  $\varphi$ , and therefore  $\nu$ . □

**Theorem 3.** *As before, let  $\mathcal{L}$  be a compact lamination by hyperbolic Riemann surfaces. Consider a harmonic measure  $\mu$  on  $\mathcal{L}$ . There is a unique way to lift  $\mu$  to a measure in  $T^1\mathcal{L}$  which is invariant under the action of the affine group.*

**Proof.** Consider a flow box  $E \simeq U \times T$  of the lamination  $\mathcal{L}$ , where  $U$  is an open disk in the hyperbolic disk (or upper-half plane) and  $T$  is some topological space. The measure  $\mu$  can be disintegrated in  $E$  using Rokhlin’s theorem; namely, for every continuous function  $f$  with compact support in  $E$ ,

$$\int f d\mu = \int_T \int_{U \times \{t\}} f(z, t) d\mu_t(z) d\hat{\mu}(t), \tag{20}$$

where  $\mu_t$  is a probability measure on the plaque  $U \times \{t\}$  and  $\hat{\mu} = (p_2)_*\mu$ . Here  $p_2 : U \times T \rightarrow T$  is the projection onto the second coordinate. Furthermore, this decomposition is unique, in the sense that the *conditional measures*  $\mu_t$  are  $\hat{\mu}$ -almost everywhere determined. The measure  $\mu$  being harmonic is equivalent to the statement that the conditional measures  $\mu_t$  are absolutely continuous with respect to the hyperbolic area in  $U$ , and that their densities are harmonic measures. (The same is true for the Euclidean area.)

As before, let  $\pi : T^1\mathcal{L} \rightarrow \mathcal{L}$  be the canonical projection. Suppose that  $\nu$  is a measure on  $T^1\mathcal{L}$  which is invariant under the action of the affine group and that projects onto  $\mu$ . Disintegration of  $\nu$  in  $\pi^{-1}(E) \simeq T^1U \times T$  yields, for every continuous  $f$  with compact support in  $\pi^{-1}E$ ,

$$\int f d\nu = \int_T \int_{T^1U \times \{t\}} f(\zeta, t) d\nu_t(\zeta) d\hat{\mu}(t), \tag{21}$$

since  $\hat{\mu} = \hat{\nu}$ . The uniqueness of the disintegration implies that each  $\nu_t$  projects onto each  $\mu_t$ . Furthermore, each  $\nu_t$  is invariant under the action of the affine group on  $T^1U$ . Proposition 1 implies that each  $\nu_t$  is unique, and therefore  $\nu$  is uniquely determined by  $\mu$ . □

#### 4. Some final remarks

A simple application of our Main Theorem consists in proving unique ergodicity for the action of the affine group on the unit tangent bundle of some foliations by hyperbolic Riemann surfaces, simply by quoting some recent results on unique ergodicity for harmonic measures. We obtain the corollaries stated below.

##### 4.1. Transversely conformal foliations by hyperbolic Riemann surfaces

In [6], Deroin and Kleptsyn prove the following theorem.

**Theorem.** *Let  $(M, \mathcal{F}, g)$  be a compact manifold together with a transversally conformal foliation and a Riemannian metric on the leaves that varies continuously in  $M$ . Let  $\mathcal{M}$  be a minimal set for  $\mathcal{F}$  (that is, a closed saturated subset of  $M$  where all leaves are dense.) If  $\mathcal{M}$  has no transverse holonomy-invariant measure, then it has a unique harmonic measure.*

Combining this with our result immediately yields:

**Corollary 1.** *Let  $(M, \mathcal{F})$  be a compact manifold together with a transversally conformal foliation by hyperbolic Riemann surfaces. Let  $\mathcal{M}$  be a minimal set for  $\mathcal{F}$ . If  $\mathcal{M}$  has no holonomy-invariant measures, then the action of the affine group on the unit tangent bundle  $T^1\mathcal{M}$  of the lamination  $\mathcal{M}$  is uniquely ergodic.*

#### 4.2. Riccati foliations

Notice that although our Main Theorem is stated for compact laminations, its proof does not require compactness. Of course, in the non-compact case the measures involved may fail to exist, but in any case the following holds:

**Remark 4.** *Let  $\mathcal{L}$  be a lamination by hyperbolic Riemann surfaces. The projection  $\pi : T^1\mathcal{L} \rightarrow \mathcal{L}$  induces a bijection between harmonic probability measures on  $\mathcal{L}$  and probability measures invariant under the affine group on  $T^1\mathcal{L}$ .*

Let  $S$  be a hyperbolic Riemann surface of finite area. A group homomorphism  $\rho : \pi_1 S \rightarrow PSL(2, \mathbb{C})$  can be suspended to construct a foliation on a bundle by complex projective lines over  $S$ . Such a foliation is called *Riccati foliation*. In [2] and [1] physical measures invariant under the geodesic and horocycle foliated flows on Riccati foliations are described, under assumptions which are generic (see [10]) on the representation  $\rho$ . One of these assumptions is that the image of  $\rho$  leaves no invariant measure on  $\mathbb{C}P^1$ ; that is, that the foliation has no holonomy-invariant measures. Building on [1], Corollary 3.6 in [14] states that there is a unique harmonic measure for the generic Riccati foliation. We therefore have:

**Corollary 2.** *Let  $S$  be a hyperbolic Riemann surface of finite volume, which is not necessarily compact. For any representation  $\rho : \pi_1 S \rightarrow PSL(2, \mathbb{C})$  let  $(M_\rho, \mathcal{F}_\rho)$  be the Riccati foliation associated to  $\rho$ . For a generic  $\rho$ , the action of the affine group on the unit tangent bundle  $T^1\mathcal{F}$  of the foliation is uniquely ergodic.*

Here “generic” means, as in [10], a real Zariski open subset of the representation variety.

#### 4.3. Hilbert modular foliations

Let  $d > 0$  be a square-free integer. Consider the field  $\mathbb{Q}(\sqrt{d})$  and let  $\mathfrak{D}_d$  be its ring of algebraic integers.

Then the image of the homomorphism

$$PSL(2, \mathfrak{D}_d) \rightarrow PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R}),$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \right),$$

where  $\bar{x}$  is the Galois conjugate of  $x \in \mathbb{Q}(\sqrt{d})$ , is an irreducible lattice  $\Gamma$  in  $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ . Notice that, if  $\mathfrak{h}$  denotes the hyperbolic plane,  $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$  acts by orientation-preserving isometries on  $\mathfrak{h} \times \mathfrak{h}$ .

Define

$$M = M_d = \frac{\mathfrak{h} \times \mathfrak{h}}{\Gamma}.$$

This is a complex normal space having a finite number of cusps and singular points. Compactification of the cusps and desingularization give a complex surface known as *Hilbert modular surface*.  $M$  is endowed with a foliation  $\mathcal{F}$  coming

from the “horizontal” foliation in  $\mathfrak{h} \times \mathfrak{h}$  (i.e. the foliation whose leaves are of the form  $\mathfrak{h} \times \{x\}$ , for some  $x \in \mathfrak{h}$ ). This foliation is known as *Hilbert modular foliation*.

In [14], Proposition 3.2, the following is proved:

**Proposition.** *The volume in  $M$  is the unique harmonic measure for the Hilbert Modular foliation.*

The unit tangent bundle of the foliation is  $(T^1\mathfrak{h} \times \mathfrak{h})/\Gamma$ . Its volume is invariant under the  $PSL(2, \mathbb{R})$ -action, and in particular under the action of the affine group. Furthermore, it is ergodic for the geodesic flow and both horocycle flows as a consequence of Moore’s Ergodicity Theorem. (See [14], p. 863. Moore’s Ergodicity Theorem can be found in [17], Ch. II.)

Together with our Main Theorem, the previous proposition implies:

**Corollary 3.** *The volume in  $(T^1\mathfrak{h} \times \mathfrak{h})/\Gamma$  is the unique invariant measure for the action of the affine group on the unit tangent bundle of the Hilbert modular foliation.*

Notice that the horocycle flow alone is not uniquely ergodic; see [14], p. 864.

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