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Limit shapes of Gibbs distributions on the set of integer partitions: The expansive case

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Abstract. We find limit shapes for a family of multiplicative measures on the set of partitions, induced by exponential generating functions with expansive parameters, $a_k \sim Ck^{p-1}$, $k \to \infty$, $p > 0$, where C is a positive constant. The measures considered are associated with the generalized Maxwell–Boltzmann models in statistical mechanics, reversible coagulation–fragmentation processes and combinatorial structures, known as assemblies. We prove a central limit theorem for fluctuations of a properly scaled partition chosen randomly according to the above measure, from its limit shape. We demonstrate that when the component size passes beyond the threshold value, the independence of numbers of components transforms into their conditional independence (given their masses). Among other things, the paper also discusses, in a general setting, the interplay between limit shape, threshold and gelation.

Résumé. Nous trouvons des formes limites pour une famille de mesures multiplicatives sur l'ensemble des partitions, induites par des fonctions génératrices exponentielles avec des paramètres d'expansion $a_k \sim Ck^{p-1}$, $k \to \infty$, $p > 0$, où C est une constante positive. Les mesures considérées sont associées aux modèles Maxwell–Boltzmann généralisés de la mécanique statistique, des processus de coagulation–fragmentation réversibles et des structures combinatoires connues sous le nom d'assemblées. Nous prouvons un théorème de limite centrale pour les fluctuations d'une partition qui est mise à l'echelle convenablement et choisie aléatoirement selon la mesure ci-dessus. Nous démontrons que, quand la taille des composantes dépasse la valeur seuil, l'indépendance des nombres de composants se transforme en leur indépendance conditionnelle. Entre autres, cet article traite, dans un cadre général, des relations entre la forme limite, le seuil et la congélation.

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1. Introduction and summary

Given a sequence of probability measures $\{\mu_N, N \geq 1\}$ on the sets of unordered partitions of integers $N \geq 1$, the limit shape, provided it exists, defines the limiting structure, as $N \to \infty$ of properly scaled partitions chosen randomly according to the above sequence of measures. The study of the asymptotic structure of random partitions is stimulated by applications to combinatorics, statistical mechanics, stochastic processes, etc. Our paper focuses on limit shapes for a class of measures associated with the generalized Maxwell–Boltzmann models in statistical mechanics, reversible coagulation–fragmentation processes and combinatorial structures called assemblies. In the course of the asymptotic analysis of the above class of measures μ_N, we reveal some interesting phenomenon that, as we expect, will be also seen in models related to other multiplicative measures on the set of partitions.

We describe now the context of the present paper. In Section 2, we deal with an arbitrary probability measure on the set of partitions. We study here the linkage between the following three important concepts in statistical mechanics and combinatorics: threshold, gelation and limit shape. As a by-product of this study, we establish the nonexistence of limit shapes for some known models. Section 3 gives a definition of a multiplicative measure and outlines the four main fields of applications of these measures. Section 4 contains the statements of our main results that are related to multiplicative measures induced by exponential generating functions with expansive parameters, $a_k \sim Ck^{p-1}$, $k \to \infty$, $p > 0$, where C is a positive constant. Namely, Theorems 4.1 and 4.3 determine the asymptotics of component counts of sizes that are of order $N^{1/(p+1)}$ (=the threshold value), while Theorem 4.6 accomplishes the same for components of small sizes (=$o(N^{1/(p+1)})$). In particular, in Theorem 4.3 we obtain limit shapes of Young diagrams under the measures considered. As a corollary of the above three theorems, we reveal that when the component size passes beyond the above threshold value, the asymptotic independence of the component counts transforms into their conditional independence, specified in Theorem 4.1. Section 5 provides proofs that are based on a far reaching generalization of Khitchine's probabilistic method. In the last Section 6, we discuss the limit shapes for our models versus the ones obtained by Vershik for the generalized models of Bose–Einstein and Fermi–Dirac. The two latter models correspond to a class of multiplicative measures induced by Euler type generating functions.

2. The interplay between limit shape, threshold and gelation

We shall work with the set $\Omega_N = \{\eta\}$ of all unordered partitions $\eta = (n_1, \ldots, n_N)$: $\sum_{k=1}^N k n_k = N$, of an integer N. Here $n_k = n_k(\eta)$ is the number of summands (=components) equal to k in a partition $\eta \in \Omega_N$. Each $\eta \in \Omega_N$ can be depicted by its Young diagram (see, e.g. [1]). The boundary of a Young diagram (shortly, Young diagram) of $\eta \in \Omega_N$ is a nonincreasing step function $\nu = \nu(\bullet; \eta)$ which is given by

$$\nu(u) = \nu(u; \eta) = \sum_{k=u}^{N} n_k(\eta), \quad u \geq 0, \eta \in \Omega_N, \tag{2.1}$$

where we set $n_0(\eta) \equiv 0$, $\eta \in \Omega_N$. By the above definition (2.1), for any integer u, the decrement $\nu(u-0) - \nu(u+0)$ equals the number (possibly 0) of components of size u, $\nu(0)$ gives the total number of components, whereas the largest u with $\nu(u) > 0$ equals the size of the largest component. Obviously, $\int_0^\infty \nu(u)\,\mathrm{d}u = N$, $\eta \in \Omega_N$. Let now $r_N > 0$ and for a given partition $\eta \in \Omega_N$ define the scaled Young diagram $\widetilde{\nu} = \widetilde{\nu}(\bullet; \eta)$ with the scaling (scaling factor) r_N:

$$\widetilde{\nu}(u) = \frac{r_N}{N} \nu(r_N u), \quad u \geq 0, \eta \in \Omega_N. \tag{2.2}$$

When a partition $\eta \in \Omega_N$ is chosen randomly according to a given probability measure on Ω_N, it is natural to ask if there exists a scaling $r_N = o(N)$, such that the random curve $\widetilde{\nu}(u)$ converges, as $N \to \infty$, to some nonrandom curve. Such a curve, if it exists, is called the limit shape (of a random Young diagram). To give a formal definition, we will need some more notations. Let $\mathcal{L} = \{l(\cdot)\}$ be the space of nonnegative nonincreasing functions l on $[0, \infty)$ with $\int_0^\infty l(u)\,\mathrm{d}u = 1$. Clearly, $\widetilde{\nu} \in \mathcal{L}$, for all $\eta \in \Omega_N$. We supply $\mathcal{L}$ with the topology of uniform convergence on compact sets in $[0, \infty)$. For a given r_N denote by ρ_{r_N} the mapping $\eta \to \tilde{\nu}$ of Ω_N onto $\mathcal{L}$. Given a probability measure μ_N on Ω_N, the mapping ρ_{r_N} induces the measure $\rho_{r_N}\mu_N$ on $\mathcal{L}$, $(\rho_{r_N}\mu_N)(l) := \mu_N(\rho_{r_N}^{-1} l)$, $l \in \mathcal{L}$.

In this section, we refer to μ_N as an arbitrary probability measure on Ω_N. The definition of a limit shape given below follows the one by Vershik in [44].

Definition 2.1. *A continuous curve $l \in \mathcal{L}$ is called the* limit shape *of a random Young diagram w.r.t. a sequence of measures $\{\mu_N, N \geq 1\}$ on $\{\Omega_N, N \geq 1\}$ (the limit shape of μ_N) under the scaling r_N: $r_N = o(N)$, $N \to \infty$ if the sequence of measures $\{\rho_{r_N}\mu_N, N \geq 1\}$ on $\mathcal{L}$ weakly converges, as $N \to \infty$, to the delta measure which is concentrated on the curve l.*

Clearly, the weak convergence of the sequence of measures $\{\rho_{r_N}\mu_N, N \geq 1\}$ in Definition 2.1 is equivalent to convergence in probability of $\widetilde{\nu}(u)$, $u \geq 0$ w.r.t. the sequence $\{\mu_N, N \geq 1\}$. Namely, the continuous function $l(\cdot) \in \mathcal{L}$ is the limit shape of the measure μ_N under a scaling r_N if for any $0 < a < b < \infty$ and $\varepsilon > 0$,

$$\lim_{N\to\infty} \mu_N \Big\{ \eta \in \Omega_N \colon \sup_{u\in[a,b]} \big|\widetilde{\nu}(u) - l(u)\big| < \varepsilon \Big\} = 1. \tag{2.3}$$

Equation (2.3) expresses the law of large numbers for the random process $\widetilde{\nu}(u)$, $u \geq 0$.

Note, that if a measure μ_N has a limit shape $l \in \mathcal{L}$ under a scaling r_N, then for any $c > 0$ the function $cl(cu)$, $u \geq 0$, is a limit shape of the same measure μ_N under the scaling cr_N.

We will say that a measure has no limit shape if there is no scaling $r_N = \mathrm{o}(N)$, that provides (2.3) for some $l \in \mathcal{L}$.

A sketch of the history of limit shapes

The evolution of shapes of random ensembles of particles, as the number of particles goes to infinity, was studied for a long time in a variety of applied fields: statistical mechanics (the Wulf construction for the formation of crystals, see [38,39]), stochastic processes on lattices (the Richardson model, see [10]), biology (growth of colonies), etc. A special study was concentrated on limit shapes for random structures on the set of partitions, in view of applications to statistical mechanics, combinatorics, representation theory, and additive number systems. In 1977, two independent teams of researchers, Vershik and Kerov [42] and Shepp and Logan [29], derived the limit shape of a Young diagram w.r.t. the Plancherel measure. Following this seminal result, Pittel [35] found the limit shape of Young tableaux w.r.t. a uniform measure. Since the number of Young tableaux corresponding to a given partition (Young diagram) is known to be equal to the degree of the irreducible representation associated with the partition, the above uniform measure, as well as the Plancherel measure, is related to the hook formula. It should be mentioned that the research in this direction revealed also a deep linkage to the random matrix theory, which is now a rapidly growing subject (see [32]). Parallel to this line of research, Vershik [44] developed a general theory of limit shapes for a class of measures he called multiplicative and which are discussed in Section 3. These measures encompass a wide scope of models from statistical mechanics and combinatorics, but do not include the measures associated with the hook formula. The results on limit shapes of multiplicative measures obtained by Vershik and Yakubovich [43–45,48,50] during the last decade concern measures induced by Euler type generating functions. (We refer to some details of this research in the course of the present paper.) Extending these results, Romik [36] derived limit shapes for multiplicative measures corresponding to some restricted integer partitions. Note that the limit shape of the uniform measure on the set of partitions (which is a multiplicative measure) was firstly obtained via a heuristic argument, by Temperley [41]. A comprehensive study of this case was done by Pittel in [34].

In contrast, the multiplicative measures μ_N considered in our paper are associated with exponential generating function.

This being said, it should be stressed that the results on limit shapes of multiplicative measures were stimulated by the remarkable papers of Erdös, Turan and Szalay as well as by other researchers, on statistics related to integer partitions (for more details see [16,34,35,44]). In the course of research on limit shapes of multiplicative measures, links of the subject to various fields of mathematics were revealed. In particular, recently the application of probabilistic methods to the study of logical limit laws was implemented in [19] and [40]. This link is based on fundamental theorems of Compton that were extended and deepen by Burris and Bell [6,7]. The place of limit shapes in this latter field is not yet understood.

It turns out that the existence of a limit shape is closely related to two other phenomena known in statistical mechanics, which are gelation and threshold. We recall

Definition 2.2. *Let $q_N(\eta)$ be the size of the largest component in a partition $\eta = (n_1, \ldots, n_N) \in \Omega_N$, i.e.* $q_N(\eta) = \max\{i \colon n_i > 0\}$.

(i) *We say that a measure μ_N exhibits gelation if for some* $0 < \alpha < 1$,

$$\limsup_{N\to\infty} \mu_N \big\{ \eta \in \Omega_N \colon q_N(\eta) > \alpha N \big\} > 0. \tag{2.4}$$

(ii) *A sequence* $\bar{q}_N = \mathrm{o}(N)$, $N \to \infty$, *is called a threshold for the size of the largest component*, *under a measure* μ_N (*shortly*, *threshold of a measure* μ_N), *if*

$$\lim_{N\to\infty} \mu_N\{\eta \in \Omega_N\colon q_N(\eta) \le \bar{q}_N\} = 1, \tag{2.5}$$

while for any sequence v_N, *s.t.* $v_N = \mathrm{o}(\bar{q}_N)$, $N \to \infty$,

$$\limsup_{N\to\infty} \mu_N\{\eta \in \Omega_N\colon q_N(\eta) \le v_N\} < 1. \tag{2.6}$$

Remark 2.3. (i) *In physics and chemistry*, *gelation is viewed as a formation of a gel which is a two-phase system consisting of a solid and a liquid in more solid form than a solution. In combinatorics*, *equivalent names for gelation are connectedness of components and appearance of a giant component. We notice that in* [24], *in contrast to our definition* (2.4), *the definition of gelation* (=*formation a giant component*) *requires that*

$$\lim_{N\to\infty} \mu_N\{\eta \in \Omega_N\colon q_N(\eta) > \alpha N\} = 1, \tag{2.7}$$

for some $\alpha > 0$. *In* [24], *some sufficient conditions for the absence of gelation* (*in the above sense*) *are given for combinatorial models encompassed into Kolchin's generalized allocation scheme* (*see* [28]).

(ii) *It is common* (*see e.g.*, [14]) *to seek a threshold* $\bar{q}_N$, *if it exists*, *in the form* $\bar{q}_N = N^{\bar{\beta}}$, *where the exponent* $\bar{\beta} := \inf\{\beta\colon \mu_N(\eta \in \Omega_N\colon q_N(\eta) \le N^{\beta}) = 1\}$.

We write (L), (G), (T) to abbreviate the statements: "There exists a limit shape/there exists gelation/there exists threshold," respectively, and write $(\bar{\bullet})$ to denote the negations of the above statements.

Proposition 2.4. *Let* μ_N *be a measure on* Ω_N. *Then*

$$(\bar{G}) \quad \Longleftrightarrow \quad (T) \quad \text{and} \quad (L) \quad \Longrightarrow \quad (\bar{G}). \tag{2.8}$$

Moreover, *if* μ_N *has a limit shape under a scaling* r_N, *then* μ_N *has a threshold* $\bar{q}_N \ge \mathrm{O}(r_N)$, $N \to \infty$.

Proof. By the definition of gelation,

$$(\bar{G}) \quad \Longleftrightarrow \quad \lim_{N\to\infty} \mu_N\{\eta \in \Omega_N\colon q_N(\eta) > \varepsilon N\} = 0, \tag{2.9}$$

for any $\varepsilon > 0$. Therefore,

$$(\bar{G}) \quad \Longleftrightarrow \quad \lim_{N\to\infty} \mu_N\{\eta \in \Omega_N\colon q_N(\eta) \le \varepsilon_N N\} = 1, \tag{2.10}$$

for some sequence $\varepsilon_N \to 0$, $N \to \infty$. This proves the existence of a threshold when there is no gelation. For the proof of the second implication in (2.8), our strategy will be to show that (L) implies that for large N the major part of the total mass N is partitioned among component sizes of order $\mathrm{O}(r_N)$. For this purpose, we employ the following argument. For given $d \ge 1$, $u_1 > 0$, let $u_0 = 0$, $u_i = iu_1$, $i = 1, \ldots, d$, be $d + 1$ equidistant nodes, and assume that $l \in \mathcal{L}$ is a limit shape of μ_N. Then

$$\frac{1}{N}\sum_{k=u_1 r_N}^{u_d r_N - 1} k n_k(\eta) = \frac{1}{N}\sum_{i=1}^{d-1}\sum_{k=u_i r_N}^{u_{i+1} r_N - 1} k n_k(\eta) \ge \sum_{i=1}^{d-1} u_i\left(\frac{r_N}{N}\sum_{k=u_i r_N}^{u_{i+1} r_N - 1} n_k(\eta)\right), \quad \eta \in \Omega_N. \tag{2.11}$$

Next, (2.3) gives for all $1 \le i \le d - 1$

$$\lim_{N\to\infty} \mu_N\left\{\eta \in \Omega_N\colon u_i \frac{r_N}{N}\sum_{k=u_i r_N}^{u_{i+1} r_N - 1} n_k(\eta) > u_i\big(l(u_i) - l(u_{i+1}) - \varepsilon_i\big)\right\} = 1, \tag{2.12}$$

for arbitrary $\varepsilon_i > 0$, $1 \le i \le d-1$. Consequently, substituting $u_i = iu_1$, $1 \le i \le d$, gives

$$\lim_{N\to\infty} \mu_N \left\{ \eta \in \Omega_N \colon \frac{1}{N} \sum_{k=u_1 r_N}^{u_d r_N - 1} k n_k(\eta) > \sum_{i=1}^{d-1} \bigl(l(u_i) - l(u_{i+1})\bigr) u_i - u_1 \sum_{i=1}^{d-1} i\varepsilon_i \right\} = 1. \tag{2.13}$$

Now we write

$$\begin{aligned}
\sum_{i=1}^{d-1} \bigl(l(u_i) - l(u_{i+1})\bigr) u_i &= \sum_{i=1}^{d-1} (u_i - u_{i-1}) l(u_i) - l(u_d) u_{d-1} \\
&= \sum_{i=1}^{d-1} (u_{i+1} - u_i) l(u_i) - l(u_d) u_{d-1} > \int_{u_1}^{u_d} l(t)\,\mathrm{d}t - l(u_d) u_{d-1},
\end{aligned} \tag{2.14}$$

where the last equation is due to the fact that the points u_i, $i = 0, \ldots, d$, are equidistant. Since $\int_0^\infty l(t)\,\mathrm{d}t = 1$ and the function l is nonincreasing and continuous on $[0, \infty)$, we have that for a sufficiently large d and a sufficiently small $u_1 > 0$,

$$\int_{u_1}^{u_d} l(t)\,\mathrm{d}t > 1 - \frac{\varepsilon}{3}, \qquad l(u_d) u_{d-1} < \frac{\varepsilon}{3}, \tag{2.15}$$

for any $\varepsilon > 0$. Now it is left to couple u_1 with the above ε_i, $1 \le i \le d-1$, and ε, by setting, say, $\varepsilon_i = i^{-3}$ and $0 < u_1 < \frac{\varepsilon}{3}(\sum_{i=1}^\infty i^{-2})^{-1}$, to conclude from (2.13) that for any $\varepsilon > 0$

$$\lim_{N\to\infty} \mu_N \left\{ \eta \in \Omega_N \colon \frac{1}{N} \sum_{k=u_1 r_N}^{u_d r_N - 1} k n_k(\eta) > 1 - \varepsilon \right\} = 1. \tag{2.16}$$

Since $r_N = \mathrm{o}(N)$, $N \to \infty$, this immediately implies $\bar{G}$ and, therefore, T. We also derive from (2.16),

$$\lim_{N\to\infty} \mu_N \bigl\{ \eta \in \Omega_N \colon q_N(\eta) < u_1 r_N \bigr\} = 0, \tag{2.17}$$

which completes the proof. □

The expression (2.16) exposes the following meaning of the scaling factor r_N, that is not immediately seen from the law of large numbers (2.3). When N is large, almost all mass N is partitioned into component sizes in the range $[u_1 r_N, u_d r_N]$, for a sufficiently small $u_1 > 0$ and a sufficiently large $u_d < \infty$. Consequently, the boundary of the major part of the random Young diagram scaled by r_N acquires, as $N \to \infty$, the shape of a nonrandom curve l.

Proposition 2.4 is applied in Corollary 3.1, in the next section to prove the nonexistence of limit shapes for certain classes of measures on the set Ω_N.

3. Gibbs distributions and multiplicative measures

3.1. *Mathematical setting*

The subject of the present paper will be the following class of measures μ_N on the set Ω_N of integer partitions of N:

$$\mu_N(\eta) = (c_N)^{-1} \frac{a_1^{n_1} a_2^{n_2} \cdots a_N^{n_N}}{n_1! n_2! \cdots n_N!}, \quad \eta = (n_1, \ldots, n_N) \in \Omega_N. \tag{3.1}$$

Here $a = \{a_k\}$ is a positive function on the set of integers, called a parameter function, and $c_N = c_N(a_1, \ldots, a_N)$ is the partition function of the measure μ_N:

$$c_0 = 1, \qquad c_N = \sum_{\eta \in \Omega_N} \frac{a_1^{n_1} a_2^{n_2} \cdots a_N^{n_N}}{n_1! n_2! \cdots n_N!}, \quad N \ge 1. \tag{3.2}$$

Following Pitman (see [8,33]), we call the measures of the form (3.1), (3.2) Gibbs distributions. Gibbs distributions are incorporated into the following general construction formulated by Vershik [44] who was motivated by applications to statistical mechanics. Let $\Omega = \bigcup_{N\geq 1} \Omega_N$ be the set of all integer partitions $\eta = (n_1, \ldots, n_k, \ldots)$ and let $s = \{s_k\}_1^\infty$ be a sequence of positive functions on the set of integers. We associate with a partition $\eta \in \Omega_N$ the function $F_N(\eta) = \prod_{k=1}^N s_k(n_k)$ and define the measure μ_N on Ω_N by $\mu_N(\eta) = (c_N)^{-1} F_N(\eta)$, where $c_N = \sum_{\eta\in\Omega_N} F_N(\eta)$ is the partition function of μ_N. Next, the family of probability measures $\mu^{(x)}$ on Ω, depending on a parameter $x > 0$ is constructed in such a way that for each $N \geq 1$ the conditional probability of $\mu^{(x)}$ given Ω_N, is the aforementioned measure μ_N, for all $x > 0$ from the domain of definition of the function $\mathcal{F}(x) = \sum_{N\geq 1} c_N x^N$. Explicitly, $\mu^{(x)}$ is given by

$$\mu^{(x)}(\eta) = x^{N(\eta)}\big(\mathcal{F}(x)\big)^{-1} F_{N(\eta)}(\eta), \quad \eta \in \Omega, 0 < x\colon \mathcal{F}(x) < \infty,$$

where $N(\eta) = \sum_{k\geq 1} k n_k$ denotes the number N which is partitioned by an $\eta \in \Omega$ and $\mathcal{F}(x)$ is the partition function of $\mu^{(x)}$, or equivalently $\mathcal{F}$ is the generating function for the sequence $\{c_N\}_1^\infty$. It is not hard to verify that the measure $\mu^{(x)}$ possesses the required conditioning property which formally reads as follows:

$$\mu_N(\eta) := \big(\mu^{(x)}|\Omega_N\big)(\eta) = \frac{\mu^{(x)}(\eta)}{\mu^{(x)}(\Omega_N)}, \quad \eta \in \Omega_N, \tag{3.3}$$

for all x in the domain of definition of the function $\mathcal{F}$. Moreover, it follows from the above definitions that the generating function $\mathcal{F}$ is expressed as the Cauchy product of the generating functions $\mathcal{F}_k(x) = \sum_{r\geq 0} s_k(r)x^r$, $k \geq 1$, for the sequences $\{s_k(r)\}_{r\geq 0}$, $k \geq 1$:

$$\mathcal{F}(x) = \prod_{k\geq 1} \mathcal{F}_k\big(x^k\big). \tag{3.4}$$

Consequently,

$$\mu^{(x)}(\eta) = x^{N(\eta)} \frac{\prod_{k=1}^{N(\eta)} s_k(n_k)}{\prod_{k\geq 1} \mathcal{F}_k(x^k)}, \quad \eta \in \Omega, 0 < x\colon \mathcal{F}(x) < \infty, \tag{3.5}$$

and

$$\mu_N(\eta) = (c_N)^{-1} \prod_{k=1}^N s_k(n_k), \quad \eta \in \Omega_N. \tag{3.6}$$

In view of the representation (3.5), Vershik calls the family (with respect to x) of measures $\mu^{(x)}$ multiplicative. Following [45], we will preserve the same name for measures μ_N induced by multiplicative families of measures $\mu^{(x)}$, via (3.3). The representation (3.5) tells us the important fact that the random counts n_k, $k \geq 1$, are independent with respect to the probability product measures $\mu^{(x)}$.

From the above formulae, the following relation between the two families of measures $\mu^{(x)}$ and μ_N holds:

$$\mu^{(x)} = \sum_{N\geq 1} x^N \big(\mathcal{F}(x)\big)^{-1} c_N \mu_N, \quad 0 < x\colon \mathcal{F}(x) < \infty. \tag{3.7}$$

This says that measures $\mu^{(x)}$ can be viewed as a Poissonization of measures μ_N, which is a standard way to deduce the equivalence, as $N \to \infty$, of canonical and microcanonical ensembles in statistical mechanics (see [9,47] and references therein).

In the above setting, the following three particular forms of the functions s_k, $k \geq 1$, and corresponding to them generating functions $\mathcal{F}$ are of great interest:

Case 1.

$$s_k(r) = \frac{a_k^r}{r!}, \quad r \geq 0, a_k > 0, k \geq 1, \qquad \mathcal{F}(x) = \exp\Bigg(\sum_{k\geq 1} a_k x^k\Bigg), \quad 0 < x\colon \sum_{k\geq 1} a_k x^k < \infty. \tag{3.8}$$

By (3.5),

$$\mu^{(x)}(\eta)=\prod_{k\ge1}\left(\frac{(a_kx^k)^{n_k}}{n_k!}\exp\left(-a_kx^k\right)\right),\quad \eta=(n_1,\dots,n_k,\dots)\in\Omega,\tag{3.9}$$

i.e. $\mu^{(x)}$ is the probability product measure induced by the sequence of Poisson $(a_kx^k,\ k\ge1)$ random variables. It follows from (3.6) that in this case the measures μ_N, $N\ge1$, are the Gibbs distributions defined by (3.1), (3.2). The associated generating function $\mathcal{F}$ is called exponential.

Case 2.

$$\begin{aligned}&s_k(r)=\binom{m_k+r-1}{r},\quad r\ge0, m_k\ge1, k\ge1,\\&\mathcal{F}(x)=\prod_{k\ge1}\frac{1}{(1-x^k)^{m_k}},\quad 0<x\colon \sum_{k\ge1}m_kx^k<\infty.\end{aligned}\tag{3.10}$$

Consequently, (3.5), (3.6) give

$$\begin{aligned}&\mu^{(x)}(\eta)=x^{N(\eta)}\prod_{k\ge1}\left(1-x^k\right)^{m_k},\quad \eta\in\Omega,\\&\mu_N(\eta)=(c_N)^{-1}\prod_{k=1}^{N}\binom{m_k+n_k-1}{n_k},\quad \eta\in\Omega_N,\end{aligned}\tag{3.11}$$

which says that $\mu^{(x)}$ is the probability product measure induced by the negative binomial random variables $\mathrm{NB}(m_k,x^k)$, $k\ge1$, with a free parameter $0<x<1$. The function $\mathcal{F}$ is called the Euler type generating function, because in the case $m_k\equiv1$, $k\ge1$, it conforms to the standard Euler generating function for integer partitions. Note that in the latter case μ_N is the uniform measure on Ω_N.

Case 3.

$$\begin{aligned}&s_k(r)=\binom{m_k}{r},\quad 0\le r\le m_k, m_k\ge1, k\ge1,\\&\mathcal{F}(x)=\prod_{k\ge1}\left(1+x^k\right)^{m_k},\quad 0<x\colon \sum_{k\ge1}m_kx^k<\infty.\end{aligned}\tag{3.12}$$

In view of (3.12), the following notation for the sets of restricted partitions is needed:

$$\begin{aligned}&\Omega(m_1,\dots,m_k,\dots)=\left\{\eta=(n_1,\dots,n_k,\dots)\in\Omega\colon\ 0\le n_k\le m_k, k\ge1\right\},\\&\Omega_N(m_1,\dots,m_N)=\left\{\eta=(n_1,\dots,n_N)\in\Omega_N\colon\ 0\le n_k\le m_k, 1\le k\le N\right\}.\end{aligned}\tag{3.13}$$

We have from (3.5), (3.6)

$$\begin{aligned}&\mu^{(x)}(\eta)=x^{N(\eta)}\frac{\prod_{k=1}^{N(\eta)}\binom{m_k}{n_k}}{\prod_{k\ge1}(1+x^k)^{m_k}},\quad \eta\in\Omega(m_1,\dots,m_k,\dots),\\&\mu_N(\eta)=(c_N)^{-1}\prod_{k=1}^{N}\binom{m_k}{n_k},\quad \eta\in\Omega_N(m_1,\dots,m_N).\end{aligned}\tag{3.14}$$

Thus, $\mu^{(x)}$ is the product measure induced by the binomial random variables $\mathrm{Bin}(m_k,x^k)$, $k\ge1$, with a free parameter $0<x<1$. In particular, if $m_k\equiv1$, $k\ge1$, then $\Omega(1,\dots,1,\dots)$, $\Omega_N(1,\dots,1)$ are the sets of integer partitions with distinct components, i.e. each n_k is either 0 or 1, and μ_N is the uniform measure on $\Omega_N(1,\dots,1)$.

We see that in all the aforementioned three cases the sequence $\{\mu_N, N \geq 1\}$ of multiplicative measures is given by a single parameter function, which is $a = \{a_k\}$ in the first case and $m = \{m_k\}$ in the two other cases. It is known that for a great scope (but not all) of applied models, the asymptotics of a parameter function can be described by a power law with the exponent $(p-1) \in \mathbb{R}$, by which we mean that the parameter function behaves asymptotically as Ck^{p-1}, $C > 0$, $k \to \infty$. From the point of view of the asymptotics of the corresponding models, the following three ranges of p should be distinguished (see [4]): the logarithmic case, $p = 0$, the expansive case, $p > 0$, and the convergent case, $p < 0$. It was found (see [3] and [4], respectively) that in the logarithmic and the convergent cases, the measures μ_N exhibit gelation. In view of this, Proposition 2.4 leads to the following:

Corollary 3.1. *In the logarithmic and the convergent cases, multiplicative measures have not limit shapes.*

Therefore, multiplicative measures may have limit shapes in the expansive case only.

3.2. *Applications*

We outline the four main fields of applications of multiplicative measures.

- *Coagulation–fragmentation processes (CFP's).* Given an integer N, a CFP is a continuous-time Markov chain on the set $\Omega_N = \{\eta\}$ of all partitions of N. Here N codes the total population of indistinguishable particles partitioned into n_j groups of size j, $j = 1, \ldots, N$. The possible infinitesimal (in time) transitions are coagulation of two groups (=clusters) of sizes i and j into one group of size $i + j$ and fragmentation of one group of size $i + j$ into two groups of sizes i and j. If the ratio of these rates is of the form

$$\frac{a_{i+j}}{a_i a_j}, \quad 2 \leq i + j \leq N, \tag{3.15}$$

  where $a = \{a_j\}$ is a positive function on the set of integers, called a parameter function of a CFP, then the process conforms to the classic mean-field reversible model of clustering formulated in the 1970s by Kelly and Whittle. (For more details and history see [11,12,14,18,25,49].) It is known (see [11,25]) that given a parameter function a, the invariant measure μ_N of the corresponding CFP is exactly the Gibbs distribution (3.1). In particular, under the parameter function $a_k = k^{p-1}$, $k \geq 1$, among the possible rates of coagulation and fragmentation are the ones given by the kernels $K(i, j) = (ij)^{p-1}$ and $F(i, j) = (i + j)^{p-1}$, respectively. The cases $p = 1$, constant rates, and $p = 2$ are common in chemical applications (see [11,22]).
- *Decomposable random combinatorial structures.* A decomposable structure of size N is a union of indecomposable components, so that the counts $n_1, \ldots, n_N$ of components of sizes $1, \ldots, N$, respectively, form an integer partition of N. Given a sequence of integers $m = \{m_k\}$, it is assumed that each component of size k belongs to one of the m_k types. Three classes of decomposable structures: assemblies, multisets and selections, encompass the whole universe of classical combinatorial objects. Assemblies are structures composed of labeled elements, so that $a_k = \frac{m_k}{k!}$, $k \geq 1$, multisets are formed from unlabeled elements and selections are multisets with distinct components. Supposing that a structure is chosen randomly from the finite set of a certain class of structures with size N, the random partition $\mathbf{K}^{(N)}$ of an integer N, called component spectrum is induced:

$$\mathbf{K}^{(N)} = \left(K_1^{(N)}, \ldots, K_N^{(N)}\right)\colon \sum_{k=1}^{N} k K_k^{(N)} = N, \quad N \geq 1,$$

  where the random variable $K_k^{(N)}$ represents the number of components of size k in the random structure. The distribution of $\mathbf{K}^{(N)}$ is described by a probability measure on the set Ω_N of integer partitions of N. A remarkable fact is (see [3]) that the distributions of $\mathbf{K}^{(N)}$ corresponding to assemblies, multisets and selections are just the multiplicative measures μ_N induced by Poisson, negative binomial and binomial distributions, respectively. Based on the combinatorial context, it is also common to treat distributions μ_N of $\mathbf{K}^{(N)}$ as the ones generated by the conditioning relation (see [3,18]), a frame which is close (but not identical) to Vershik's formalism.

- *Statistical mechanics.* In this context, the multiplicative measures $\mu^{(x)}$ and μ_N are referred to as macrocanonical and microcanonical ensembles (of particles), respectively. For more details on this topic, see [18,44]. The multiplicative measures induced by Poisson, negative binomial and binomial distributions, with constant parameter functions, provide a mathematical setting for the three classical models of ideal gas, called Maxwell–Boltzmann (MB), Bose–Einstein (BE) and Fermi–Dirac (FD) statistics, respectively. In [44], the multiplicative measures with parameter functions Ck^{p-1}, $C>0$, $p\ge 1$, $k\ge 1$, are called generalized classical statistics.
- *CFP's on set partitions* (see [8,33]). This comparatively new setting is based on Pitman's study of combinatorial models of random set partitions, which developed from the Ewens sampling formula and Kingman's coalescence processes. We assume here that in the preceding setup for CFPs, particles are labeled by $1,\dots,N$, so that the state space of a CFP is the set $\Omega_{[N]}=\{\pi_{[N]}\}$ of all partitions $\pi_{[N]}$ of the set $[N]=\{1,\dots,N\}$. Denoting $|A_j|$ the size of a cluster (block) $A_j\subseteqq[N]$ of $\pi_{[N]}$, we assign to each A_j, a weight $m_{|A_j|}$ which depicts the number of possible inner states of A_j, the states can be, for example, shapes (in the plane or in space), colors, energy levels, and so forth. This says that to a set partition $\pi_{[n],k}$ with k given clusters $A_1,\dots,A_k$ correspond $\prod_{j=1}^k m_{|A_j|}$ different states of the CFP considered, so that the total number of states formed by all partitions of the set $[N]$ into k clusters is equal to

$$\sum_{\pi_{[N],k}\in\Omega_{[N],k}}\prod_{j=1}^{k} m_{|A_j|}:=B_{N,k}, \tag{3.16}$$

where $\Omega_{[N],k}$ is the set of all partitions of the set $[N]$ into k blocks and the number $B_{N,k}$ is known as a Bell polynomial in weights $m_1,\dots,m_{N-k+1}$. Next, for a given $k\ge 1$ the uniform measure $p_{[N],k}$ on the set $\Omega_{[N],k}=\{\pi_{[N],k}\}$ is defined:

$$p_{[N],k}(\pi_{[N],k})=\frac{\prod_{j=1}^k m_{|A_j|}}{B_{N,k}},\quad \pi_{[N],k}\in\Omega_{[N],k}. \tag{3.17}$$

In a more general setting which encompasses a variety of models (see [8,33]), the weights m_j in (3.17) are allowed to be arbitrary nonnegative numbers. By the known combinatorial relation, the distribution $p_{[N],k}$ on $\Omega_{[N],k}$ induces the distribution $p_{N,k}$ of cluster sizes $|A_j|$, $j=1,\dots,k$, on the set $\Omega_{N,k}$ of integer partitions of N into k positive summands:

$$p_{N,k}(\eta)=(B_{N,k})^{-1}\prod_{j=1}^{N}\left(\frac{m_j}{j!}\right)^{n_j}\frac{1}{(n_j)!},\quad \eta=(n_1,\dots,n_N)\in\Omega_{N,k}, \tag{3.18}$$

where the partition function $B_{N,k}$ defined as in (3.16) can be rewritten in the following form:

$$B_{N,k}=\sum_{\eta\in\Omega_{N,k}}\prod_{j=1}^{k}\left(\frac{m_j}{j!}\right)^{k_j}\frac{1}{(k_j)!}. \tag{3.19}$$

Pitman calls the measures $p_{N,k}$, $p_{[N],k}$ microcanonical Gibbs distributions on the sets $\Omega_{N,k}$ and $\Omega_{[N],k}$ respectively. The linkage of $p_{N,k}$ to the Gibbs distributions μ_N in (3.1) is expressed via the conditioning relation:

$$p_{N,k}=\mu_N|\Omega_{N,k},$$

where μ_N is the Gibbs distribution given by (4.1) with $a_j=\frac{m_j}{j!}$, $j\ge 1$.

However, the generic model associated with set partitions of $[N]$ involves a wealth of specific problems (see [8]) arising from treating $p_{[N],k}$, $k=1,\dots,N$, as marginal distributions at any time t, of irreversible time continuous Markov processes of pure fragmentation (or pure coagulation) on the state space $\Omega_{[N]}$.

Finally note that Gnedin and Pitman [17] and Kerov [26] studied exchangeable Gibbs distributions related to the Ewens sampling formula (the case $p=0$).

4. Statement of main results

Our forthcoming asymptotic analysis is devoted exclusively to Gibbs distributions in the expansive case, that is, when the parameter function a of the measure μ_N in (3.1) is of the form:

$$a_k \sim Ck^{p-1}, \quad C > 0,\, p > 0,\, k \to \infty. \tag{4.1}$$

We already mentioned in the previous section that this case conforms to the generalized MB statistics and to reversible CFPs with certain transition rates. To explain the relevance of the case considered to random combinatorial structures, recall that in the combinatorial context, Gibbs distributions with the parameter function (4.1) describe assemblies with the number of indecomposable components of size k, equal $m_k = a_k k! \sim Ck^{p-1}k!$, $C > 0$, $k \to \infty$. Examples of such structures with $p = 1, 2$ were given in [12]. Generally speaking, we can think that in an initial assembly having, say, $\tilde{m}_k$ indecomposable components on k labeled items, each one of the $\tilde{m}_k$ components is distinguished by one of b_k types. For example, in permutations, each of $(k-1)!$ cycles on k labelled items is colored by one of b_k colors; in forests, each of k^{k-2} trees on k labeled vertices belongs to one of b_k species, etc. In this setting, the resulting assembly is composed of $m_k = b_k \tilde{m}_k$ components of size k. Thus, under $b_k = k^\alpha$, where $\alpha > 0$ in the first example and $\alpha > 3/2$ in the second example, the resulting assembly satisfies (4.1).

Throughout the rest of the paper, we assume that all random variables considered are induced by the Gibbs distributions μ_N with the parameter function (4.1).

We note that Vershik's approach [44–46] to asymptotic analysis of the generalized BE and FD models is based on the expression (3.3), which allows to replace the measure μ_N by the product measure $\mu^{(x)}$, with a properly chosen $x \to 1$. In contrast to it, our strategy is the straightforward study of the asymptotics of μ_N, as $N \to \infty$, using stratifications of the integer N and the total number of components. However, both approaches are based on common ideas rooted in statistical mechanics and the saddle point method.

We first introduce some more notations. Consider two sets of random variables ν_j and K_j, $j = 0, \ldots, q \geq 1$, that determine for a given $\eta \in \Omega_N$ stratifications of the total number of components $\nu = \nu(\eta) = \sum_{k=1}^N n_k(\eta)$ and of the total mass $N = \sum_{k=1}^N k n_k(\eta)$, respectively. Namely, for given integers $q \geq 1$ and $0 = M_0 < M_1 < \cdots < M_q < N < M_{q+1} = N+1$ and a given $\eta \in \Omega_N$, we set

$$\begin{aligned} &\nu_0 = \nu_0(M_1, \eta) := \sum_{k=M_0}^{M_1-1} n_k(\eta), \qquad \nu_j = \nu_j(M_j, \eta) := \sum_{k=M_j}^{N} n_k(\eta), \\ &K_0 = K_0(M_1, \eta) := \sum_{k=M_0}^{M_1-1} k n_k(\eta) = N - K_1, \qquad K_j = K_j(M_j, \eta) := \sum_{k=M_j}^{N} k n_k(\eta), \quad j = 1, \ldots, q, \end{aligned} \tag{4.2}$$

and denote

$$\overrightarrow{\nu} = (\nu_0, \ldots, \nu_q), \qquad \overrightarrow{K} = (K_1, \ldots, K_q). \tag{4.3}$$

We will refer to M_j, $j = 1, \ldots, q$, as stratification points. The random vectors $\overrightarrow{\nu}$, $\overrightarrow{K}$ induce the decompositions $\overrightarrow{\nu}^\star$ and $\overrightarrow{K}^\star$ of the random variables ν and $N - K_0$ respectively, into sums of $q+1$ and q disjoint parts:

$$\begin{aligned} &\overrightarrow{\nu}^\star = \left(\nu_0^\star, \ldots, \nu_q^\star\right) := (\nu_0, \nu_1 - \nu_2, \ldots, \nu_{q-1} - \nu_q, \nu_q), \\ &\overrightarrow{K}^\star = \left(K_1^\star, \ldots, K_q^\star\right) := (K_1 - K_2, \ldots, K_{q-1} - K_q, K_q). \end{aligned} \tag{4.4}$$

We also set $K_0^\star = K_0$. Clearly,

$$\nu_j^\star = \sum_{k=M_j}^{M_{j+1}-1} n_k, \qquad K_j^\star = \sum_{k=M_j}^{M_{j+1}-1} k n_k, \quad j = 0, \ldots, q, \quad \sum_{j=0}^{q} K_j^\star = N. \tag{4.5}$$

Formally, $\overrightarrow{\nu}^{\star}$ and $\overrightarrow{K}^{\star}$ can be viewed as the 1–1 linear transformations $\boldsymbol{\mathcal{B}}:\mathbb{R}^{q+1}\to\mathbb{R}^{q+1}$ and $\boldsymbol{\mathcal{B}}_1:\mathbb{R}^{q}\to\mathbb{R}^{q}$ of the random vectors $\overrightarrow{\nu}$ and $\overrightarrow{K}$, respectively:

$$\overrightarrow{\nu}^{\star}=\boldsymbol{\mathcal{B}}\overrightarrow{\nu}, \qquad \overrightarrow{K}^{\star}=\boldsymbol{\mathcal{B}}_1\overrightarrow{K}. \tag{4.6}$$

It is known from [14] that in the expansive case (4.1), the measure μ_N has a threshold $N^{1/(p+1)}$, $p>0$, and that the major part of the total mass N, when N is large, is concentrated on components of sizes of order $N^{1/(p+1)}$. Therefore, in view of the note after Proposition 2.4, we take throughout the rest of the paper

$$r_N=h^{-1}N^{1/(p+1)}, \quad h=\bigl(C\Gamma(p+1)\bigr)^{1/(p+1)} \tag{4.7}$$

as the scaling factor for the limit shape in question. Here the constant h is chosen in order to simplify the expression for the limit shape.

To establish the desired limit theorems, it is required to define proper scalings of the two pairs of random vectors in (4.3) and (4.4). We denote by $\hat{\bullet}$ the corresponding scaled quantities. The explicit expressions for $\hat{\bullet}$ will be chosen in accordance with the asymptotic problem considered.

First, in the forthcoming Theorems 4.1, 4.3 and Corollary 4.5, we will study the asymptotics of the random quantities in (4.2), (4.5) in the vicinity of the r_N. In this case, we consider the stratifications induced by the points $M_j=[u_jr_N]$, $j=0,\dots,q$, $M_{q+1}=N+1$, where $0=u_0<u_q<u_{q+1}=\infty$ do not depend on N, and take the scalings $\hat{\bullet}$ in the form

$$\begin{aligned}\hat{\nu}^{\star}_j&=\sqrt{f_0(p-1)}\frac{\nu^{\star}_j-S^{\star}_j}{\sqrt{S_0}},\\ \hat{K}^{\star}_j&=\sqrt{f_0(p+1)}\frac{K^{\star}_j-E^{\star}_j}{\sqrt{V_0}}, \quad j=0,\dots,q,\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}\hat{\nu}_j&=\sqrt{f_0(p-1)}\frac{\nu_j-S_j}{\sqrt{S_0}},\\ \hat{K}_j&=\sqrt{f_0(p+1)}\frac{K_j-E_j}{\sqrt{V_0}}, \quad j=0,\dots,q.\end{aligned} \tag{4.9}$$

Here

$$f_j(s)=C\int_{u_j}^{u_{j+1}}x^s\mathrm{e}^{-x}\,\mathrm{d}x, \quad j=0,\dots,q, s\ge 0, \tag{4.10}$$

whereas the quantities S_j, E_j, V_j depend on the parameter $\delta=\delta_N=(r_N)^{-1}$:

$$\begin{aligned}S_0&=\sum_{k=M_0}^{M_1-1}a_k\mathrm{e}^{-\delta k}, & E_0&=\sum_{k=M_0}^{M_1-1}ka_k\mathrm{e}^{-\delta k}, & V_0&=\sum_{k=M_0}^{M_1-1}k^2a_k\mathrm{e}^{-\delta k},\\ S_j&=\sum_{k=M_j}^{N}a_k\mathrm{e}^{-\delta k}, & E_j&=\sum_{k=M_j}^{N}ka_k\mathrm{e}^{-\delta k}, & V_j&=\sum_{k=M_j}^{N}k^2a_k\mathrm{e}^{-\delta k}, \quad j=1,\dots,q.\end{aligned} \tag{4.11}$$

Finally, the starred quantities $S^{\star}_j,\ E^{\star}_j$ in the RHS's of (4.8) are defined respectively as the transformations $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{B}}_1$ of the corresponding nonstarred quantities $S_j,\ E_j$. Also, we will use the following abbreviation for the conditional probabilities

$$\rho\bigl(\hat{\nu}^{\star}_j\big|\hat{K}^{\star}_j\bigr):=\mathbb{P}\bigl(\hat{\nu}^{\star}_j=\bullet_{1,j}\big|\hat{K}^{\star}_j=\bullet_{2,j}\bigr),$$

when the particular values $\bullet_{1,j}, \bullet_{2,j}$ are not important.

The theorem below that deals with the asymptotic behavior of the scaled quantities defined by (4.8), will be a source of our subsequent results on limit shapes.

Theorem 4.1. *Under $N \to \infty$ and a given $q \ge 1$,*

(i) *The random variables $\hat{\nu}_j^\star$, $j = 0, \dots, q$, are conditionally independent given $\hat{K}_j^\star, j = 0, \dots, q$, s.t. $\sum_{j=0}^q K_j^\star = N$. Moreover,*

$$\rho\big(\overrightarrow{\hat{\nu}^\star} \big| \overrightarrow{\hat{K}^\star}\big) = \prod_{j=0}^{q} \rho\big(\hat{\nu}_j^\star \big| \hat{K}_j^\star\big). \tag{4.12}$$

(ii) *The $(2q+1)$-dimensional random vector $(\overrightarrow{\hat{\nu}^\star}, \overrightarrow{\hat{K}^\star}) = (\hat{\nu}_0^\star, \dots, \hat{\nu}_q^\star, \hat{K}_1^\star, \dots, \hat{K}_q^\star)$ converges weakly to the $(2q+1)$-dimensional Gaussian random vector with zero mean and the covariance matrix $\{\vartheta_{mk}^\star(q),\ m, q = 0, \dots, 2q\}$ given by*

$$\vartheta_{mk}^\star(q) = \begin{cases} f_m(p-1)\mathbf{1}_{m=k} - \frac{f_m(p) f_k(p)}{\Gamma(p+2)} & \text{if } 0 \le m, k \le q, \\ f_m(p)\mathbf{1}_{m=k-q} - \frac{f_m(p) f_{k-q}(p+1)}{\Gamma(p+2)} & \text{if } 0 \le m \le q, q+1 \le k \le 2q, \\ f_k(p)\mathbf{1}_{k=m-q} - \frac{f_k(p) f_{m-q}(p+1)}{\Gamma(p+2)} & \text{if } 0 \le k \le q, q+1 \le m \le 2q, \\ f_{m-q}(p+1)\mathbf{1}_{m=k} - \frac{f_{m-q}(p+1) f_{k-q}(p+1)}{\Gamma(p+2)} & \text{if } q+1 \le m, k \le 2q, \end{cases} \tag{4.13}$$

where $\mathbf{1}_{i=j}$ is the Kroneker symbol.

(iii) *Moreover, moments of the random vector $(\overrightarrow{\hat{\nu}^\star}, \overrightarrow{\hat{K}^\star})$ converge to the corresponding moments of the Gaussian random vector in* (ii). *In particular,*

$$\vartheta_{mk}^\star(q) = \lim_{N \to \infty} \begin{cases} \mathrm{Cov}\big(\hat{\nu}_m^\star, \hat{\nu}_k^\star\big) & \text{if } 0 \le m, k \le q, \\ \mathrm{Cov}\big(\hat{\nu}_m^\star, \hat{K}_{k-q}^\star\big) & \text{if } 0 \le m \le q, q+1 \le k \le 2q, \\ \mathrm{Cov}\big(\hat{K}_{m-q}^\star, \hat{K}_{k-q}^\star\big) & \text{if } q+1 \le k, m \le 2q. \end{cases} \tag{4.14}$$

(iv) *Let the stratifications be induced by the equidistant points u_j, $j = 0, \dots, q$, and let $k = m + s$, $s > 0$, $0 \le m$, $k \le 2q$. Then the absolute value of the covariance, $|\vartheta_{mk}^\star(q)|$, monotonically decreases in s. In particular, if $q \to \infty$, $s \to \infty$, while m is fixed, then*

$$\big|\vartheta_{mk}^\star(q)\big| = \begin{cases} \mathrm{O}\big(u_k^p \exp(-u_k)\big) & \text{if } 0 \le m < k \le q, \\ \mathrm{O}\big(u_{k-q}^{p+1} \exp(-u_{k-q})\big) & \text{if } q < k \le 2q, s - q \to \infty. \end{cases} \tag{4.15}$$

Remark 4.2. *The first part of* (4.15) *expresses the exponential decay of correlations between the scaled counts of components of different sizes, as the "distance" s between the sizes goes to infinity. Phenomena of such kind are widely known in equilibrium statistical mechanics* (*see e.g.*, [20,21] *and references therein*).

Let $l \in \mathcal{L}$ be a limit shape of a measure μ_N and let $\tilde{\nu}(u)$ be defined as in (2.2). We call

$$\Delta(u) := \tilde{\nu}(u) - l(u), \quad u \ge 0, \tag{4.16}$$

the random fluctuation of μ_N from its limit shape at a point u. To state the next theorem, denote

$$b_r(u) = C\Gamma(r+1, u), \quad u \ge 0, r > -1, \tag{4.17}$$

where $\Gamma(r+1, u) = \int_u^\infty \mathrm{e}^{-x} x^r \,\mathrm{d}x$, $u \ge 0$, $r > -1$, is the incomplete Gamma function.

Theorem 4.3 (Limit shape of μ_N and the cental limit theorem for scaled fluctuations from the limit shape). *Under $N \to \infty$ and a given $q \ge 1$,*

(i) *The random vector $(\hat{\nu}_1, \dots, \hat{\nu_q})$ defined by* (4.9) *weakly converges to the q-dimensional Gaussian random vector with zero mean and the covariance matrix $\{e_{mk}\}_1^q$ given by*

$$e_{mk} = b_{p-1}(u_s) - \frac{b_p(u_m)b_p(u_k)}{\Gamma(p+2)}, \quad m, k = 1, \dots, q, \tag{4.18}$$

where $s = max(k, m)$.

(ii) *Setting $l(u) = \frac{\Gamma(p,u)}{\Gamma(p+1)}$, $u \ge 0$, in* (4.16), *the relation between the scaled quantities $\hat{\nu}_1, \dots, \hat{\nu_q}$ and the random fluctuations $\Delta(u_j)$ is given by*

$$\hat{\nu}_j \sim \left(C\Gamma(p+1)Nr_N^{-1}\right)^{1/2}\Delta(u_j), \quad j = 1, \dots, q, N \to \infty, \tag{4.19}$$

for all $\eta \in \Omega_N$.

(iii) *The measure μ_N has the limit shape*

$$l_{p-1}(u) = \frac{\Gamma(p,u)}{\Gamma(p+1)}, \quad u \ge 0, p > 0,$$

under the scaling r_N given by (4.7).

Remark 4.4. (a) *Recall that the measure μ_N considered has the threshold $N^{1/(p+1)}$. By* (ii) *of the above theorem and* (4.7), *the threshold turns out to be of the same order as the scaling r_N for the limit shape. In view of this, we believe that the following stronger form of the second part of* (2.8) *in Proposition* 2.4 *is valid: For a wide class of multiplicative measures, the existence of a threshold $\bar{q}_N$ implies the existence of a limit shape under a scaling $r_N = \mathrm{O}(\bar{q}_N)$, $N \to \infty$.*

(b) *In* [45], *an analog of our cental limit theorem for fluctuations was established for the uniform measure on the set of unordered partitions with distinct summands (=classical FD statistics). Recall that this multiplicative measure is associated with the generating function $\prod_{k\ge1}(1 + x^k)^{-1}$. The structure of the covariance matrix in* [45] *is similar to the one in* (4.18). *In particular, the exponential decay of covariances is also seen there.*

Corollary 4.5 (The functional central limit theorem for the scaled number of components). *Let in the above stratification scheme, $q = 1$, $u_1 = u$ and denote $\hat{\nu}(u) = \hat{\nu}_1$. Then, for all $u \ge 0$, the scaled random variable $\hat{\nu}(u)$ weakly converges to $N(0, e_{11})$, where $e_{11} = e_{11}(u)$, $u \ge 0$, is as in* (4.18).

Recalling that $\nu(0)$ is equal to the total number of components in a random partition, the particular case $u = 0$ of the above corollary recovers the central limit theorem for $\nu(0)$ established in [12].

We now turn to the asymptotic behavior of the counts of components of sizes $\mathrm{o}(\bar{q}_N)$ (=small sizes in comparison to the threshold), s.t. $\mathrm{o}(\bar{q}_N) \to \infty$, $N \to \infty$, where $\bar{q}_N = N^{1/(p+1)}$. For this problem, we make use of the stratification points $M_j = M_j(N)$, s.t.

$$\begin{aligned} &M_j = \mathrm{o}\left(N^{1/(p+1)}\right) \to \infty, \quad N \to \infty, \qquad \lim_{N\to\infty}\frac{M_j}{M_{j+1}} := \rho_j < 1, \quad j = 1, \dots, q, \\ &M_0 = 0, \qquad M_{q+1} = N + 1, \end{aligned} \tag{4.20}$$

while the scaled quantities $\hat{\nu}_j^\star$, $\hat{K}_j^\star$ are taken in the form slightly different from the one in (4.8):

$$\begin{aligned} \hat{\nu}_j^\star &= \frac{\nu_j^\star - S_j^\star}{\sqrt{S_j^\star}}, \quad j = 0, \dots, q, \\ \hat{K}_j^\star &= \frac{K_j^\star - E_j^\star}{\sqrt{V_j^\star}}, \quad j = 0, \dots, q-1. \end{aligned} \tag{4.21}$$

Theorem 4.6 (Asymptotic independence and the central limit theorem for the counts of components of small sizes). *Let the stratification points satisfy* (4.20) *and* $\overrightarrow{\hat{\nu}^\star} = (\hat{\nu}_0^\star, \dots, \hat{\nu}_q^\star)$, $\overrightarrow{\hat{K}^\star} = (\hat{K}_1^\star, \dots, \hat{K}_q^\star)$ *be given by* (4.21). *Then, as* $N \to \infty$,

(i) *The coordinates of each one of the two random vectors* $\overrightarrow{\hat{\nu}^\star}$ *and* $\overrightarrow{\hat{K}^\star}$ *are independent random variables.*

(ii) *The random vector* $\overrightarrow{\chi} = (\hat{\nu}_0^\star, \hat{K}_0^\star, \hat{\nu}_1^\star, \hat{K}_1^\star, \dots, \hat{\nu}_{q-1}^\star, \hat{K}_{q-1}^\star, \hat{\nu}_q^\star)$ *weakly converges to the* $(2q+1)$*-dimensional Gaussian vector with zero mean and the covariance matrix having a diagonal block structure*, *with q blocks*

$$B_j^{-1} := \begin{pmatrix} 1 & \alpha_j \\ \alpha_j & 1 \end{pmatrix}, \quad j = 0, \dots, q-1, \qquad B_q^{-1} = 1 - \alpha_q^2, \tag{4.22}$$

where

$$\begin{aligned} \alpha_j &= \frac{\sqrt{p(p+2)}}{p+1} \frac{1-\rho_j^{p+1}}{\sqrt{(1-\rho_j^{p+2})(1-\rho_j^p)}} = \lim_{N\to\infty} \operatorname{Cov}(\hat{\nu}_j^\star, \hat{K}_j^\star), \quad j = 0, \dots, q-1, \\ \alpha_q^2 &= 1 - \lim_{N\to\infty} \operatorname{Var}(\hat{\nu}_q^\star) = \frac{\Gamma^2(p+1)}{\Gamma(p)\Gamma(p+2)} = \frac{p}{p+1}. \end{aligned} \tag{4.23}$$

Remark 4.7. *It was proven in* [14] *that in the model considered*, *the counts of components of fixed sizes* (=*the random variables* $n_{k_1}, \dots, n_{k_l}$, $0 \le k_1 < \dots < k_l < \infty$, $l > 1$) *are asymptotically independent. Combining this result with our Theorems* 4.1 *and* 4.6 *says that when the component size passes beyond the threshold value* $N^{1/(p+1)}$, *the asymptotic independence of component counts transforms into their conditional independence* (*given masses*). *In this respect*, *the threshold value can be also viewed as the critical value for the independence of component counts in the model considered.*

5. Proofs

5.1. *Khintchine-type representation formula*

As in [12–14] and [19], our tool for the asymptotic problems considered will be the probabilistic method by Khintchine, that was introduced in the 1950s in his book [27]. The idea of the method is to construct the representation of the quantity of interest via the probability function of a sum of independent integer valued random variables depending on a free parameter, and then to implement a local limit theorem. In the course of time, this method, sometimes without mentioning Khintchine's name, was applied to investigation of a large scope of asymptotic problems arising in statistical mechanics and in enumeration combinatorics (see e.g., [2,12–15,19,28,30,31] and references therein). The first three subsections of the present section contain preparatory asymptotic analysis toward the proof of our main results.

Clearly, the distributions of the random vectors $\overrightarrow{\nu}$ and $\overrightarrow{K}$ are completely determined by the measure μ_N. Explicitly, for any given vectors $\overrightarrow{L} = (L_0, \dots, L_q)$ and $\overrightarrow{N} = (N_1, \dots, N_q)$,

$$\mathbb{P}(\overrightarrow{\nu} = \overrightarrow{L}) = \sum_{\overrightarrow{N}} \mathbb{P}(\overrightarrow{\nu} = \overrightarrow{L}, \overrightarrow{K} = \overrightarrow{N}) = \sum_{\overrightarrow{N^\star}} \mathbb{P}(\overrightarrow{\nu^\star} = \overrightarrow{L^\star}, \overrightarrow{K^\star} = \overrightarrow{N^\star}) := \sum_{\overrightarrow{N^\star}} R(\overrightarrow{L^\star}, \overrightarrow{N^\star}), \tag{5.1}$$

where, in accordance with (4.4), (4.6), we denoted

$$\begin{aligned} \overrightarrow{L^\star} &= (L_0^\star, \dots, L_q^\star) = \boldsymbol{\mathcal{B}} \overrightarrow{L}, \\ \overrightarrow{N^\star} &= (N_1^\star, \dots, N_q^\star) = \boldsymbol{\mathcal{B}}_1 \overrightarrow{N}. \end{aligned} \tag{5.2}$$

(4.4) implies that for any vectors $\overrightarrow{L^\star}$ and $\overrightarrow{N^\star}$,

$$R(\overrightarrow{L^\star}, \overrightarrow{N^\star}) = \mathbb{P}(\overrightarrow{\nu} = \overrightarrow{L}, \overrightarrow{K} = \overrightarrow{N}). \tag{5.3}$$

Consequently, it follows from (3.1), (3.2) that

$$R\big(\overrightarrow{L^\star}, \overrightarrow{N^\star}\big) = c_N^{-1} \sum_{\eta \in \Omega_N} \left(\prod_{k=1}^{N} \frac{a_k^{n_k}}{n_k!} \mathbf{1}_A \right), \tag{5.4}$$

where

$$A = A\big(\overrightarrow{L^\star}, \overrightarrow{N^\star}\big) = \big\{\eta \in \Omega_N\colon\ \overrightarrow{\nu^\star}(\eta) = \overrightarrow{L^\star}, \overrightarrow{K^\star}(\eta) = \overrightarrow{N^\star}\big\}. \tag{5.5}$$

Our first goal will be to derive the Khintchine type representation for the probability $R(\overrightarrow{L^\star}, \overrightarrow{N^\star})$. Setting $a_0 = 0$, we construct the array of discrete random variables $\beta_l^{(j)}$ defined by

$$\mathbb{P}\big(\beta_l^{(j)} = k\big) = \frac{a_k \mathrm{e}^{-\delta_j k}}{S_j^\star(\delta_j)}, \quad k = M_j, \dots, M_{j+1} - 1, j = 0, \dots, q, l \ge 1, \tag{5.6}$$

where the stratification points $M_0, \dots, M_{q+1}$ are as in (4.2), $\delta_0 > 0, \dots, \delta_q > 0$ are free parameters and

$$S_j^\star = S_j^\star(\delta_j) = \sum_{k=M_j}^{M_{j+1}-1} a_k \mathrm{e}^{-\delta_j k}, \quad j = 0, \dots, q, \tag{5.7}$$

in accordance with (4.11). We assume that for a given $0 \le j \le q$, the random variables $\beta_l^{(j)}$, $l \ge 1$, are i.i.d. and for a given $l \ge 1$, the random variables $\beta_l^{(j)}$, $0 \le j \le q$, are independent.

Distributions of the type (5.6) are widely used in the asymptotic analysis related to combinatorial structures, [3, 28,30]. These distributions firstly appeared in the papers of Goncharov (1944) (for references see [28]) and in the monograph [27] by Khintchine (1950).

Lemma 5.1 (Khintchine's type representation for the probabilities $R(\overrightarrow{L^\star}, \overrightarrow{N^\star})$). *Denote*

$$Y^{(j)} = \sum_{l=1}^{L_j^\star} \beta_l^{(j)}, \quad j = 0, \dots, q, \tag{5.8}$$

where $\beta_l^{(j)}$, $l \ge 1$, are i.i.d. random variables given by (5.6), (5.7). *Then*

$$R\big(\overrightarrow{L^\star}, \overrightarrow{N^\star}\big) = c_N^{-1} \prod_{j=0}^{q} \frac{(S_j^\star)^{L_j^\star}}{(L_j^\star)!} \exp\big(\delta_j N_j^\star\big) \mathbb{P}\big(Y^{(j)} = N_j^\star\big), \tag{5.9}$$

where

$$N_0^\star := N - \sum_{j=1}^{q} N_j^\star. \tag{5.10}$$

Proof. It follows from (5.6)–(5.8) and the multinomial enumeration formula for the number of ways to distribute $L_j^\star$ numbered balls over $L_j^\star$ numbered urns, that

$$\prod_{j=0}^{q} \mathbb{P}\big(Y^{(j)} = N_j^\star\big) = \left(\prod_{j=0}^{q} \frac{(L_j^\star)!}{(S_j^\star)^{L_j^\star}} \exp\big(-\delta_j N_j^\star\big) \right) \sum_{\eta \in \Omega_N} \left(\prod_{k=1}^{N} \frac{a_k^{n_k}}{n_k!} \mathbf{1}_A \right). \tag{5.11}$$

By (5.4), (5.5) this gives (5.9). □

Note that in contrast to the standard one parameter scheme of Khintchine's method, the representation (5.9) is based on $(q+1)$ free parameters $\delta_0, \dots, \delta_q$.

Remark 5.2. *One can see that* (5.9) *is a generalization of the representation formula for the total number of components which was obtained in* [12].

5.2. *The choice of free parameters*

Firstly, we set in (5.9) $\delta_0 = \delta_1 = \cdots = \delta_q := \delta$ and apply the stratifications

$$\overrightarrow{S} = (S_0, \dots, S_q), \qquad \overrightarrow{E} = (E_0, \dots, E_q), \qquad \overrightarrow{V} = (V_0, \dots, V_q),$$

as defined in (4.11), of the quantities $S := \sum_{k=1}^N a_k e^{-\delta k}$, $E := \sum_{k=1}^N k a_k e^{-\delta k}$ and $V := \sum_{k=1}^N k^2 a_k e^{-\delta k}$ respectively. Next, we take in (5.1) the vectors $\overrightarrow{L} = (L_0, \dots, L_q)$ and $\overrightarrow{N} = (N_1, \dots, N_q)$ in the form

$$\begin{aligned} L_j &= S_j + x_j \sqrt{S_0}, \qquad j = 0, \dots, q, \\ N_j &= E_j + x_{j+q} \sqrt{V_0}, \quad j = 1, \dots, q, \end{aligned} \tag{5.12}$$

where x_j, $j = 0, \dots, 2q$, are arbitrary reals. We also set $N_0 = N - N_1$ and adopt the following notation:

$$x_j^\circ = \begin{cases} x_j & \text{if } j \in \{0, q\}, \\ x_j - x_{j+1} & \text{if } 1 \le j \le q-1, \\ -x_{q+1} & \text{if } j = q+1, \\ x_{j-1} - x_j & \text{if } q+2 \le j \le 2q, \\ x_{2q} & \text{if } j = 2q+1. \end{cases} \tag{5.13}$$

It follows from (5.13) that $\sum_{j=q+1}^{2q+1} x_j^\circ = 0$. By (5.2) and (5.12) we then have

$$L_j^\star = S_j^\star + x_j^\circ \sqrt{S_0}, \quad j = 0, \dots, q, \tag{5.14}$$

and

$$N_j^\star = E_j^\star + x_{j+q+1}^\circ \sqrt{V_0}, \quad j = 1, \dots, q. \tag{5.15}$$

Note that (5.10), (5.15) together with (4.11) imply that

$$N_0^\star = N - \sum_{j=1}^{q} \bigl(E_j^\star + x_{q+j+1}^\circ \sqrt{V_0}\bigr) = N - E_1 + x_{q+1}^\circ \sqrt{V_0}. \tag{5.16}$$

We start by investigating the asymptotic behavior of the probability $R(\overrightarrow{L^\star}, \overrightarrow{N^\star})$ given by (5.9), when $N \to \infty$ and δ is fixed. We adopt from [13] the following representation (5.17) of the partition function c_N defined in (3.2). Consider independent random variables ξ_k, s.t. $k^{-1}\xi_k \sim Po(a_k e^{-\sigma k})$, $k \ge 1$, where $\sigma > 0$ is a free parameter, and let $Z_N = \sum_{k=1}^N \xi_k$. Then

$$c_N = \mathbb{P}(Z_N = N) \exp\bigl(N\sigma + S(\sigma)\bigr), \quad N \ge 1, \tag{5.17}$$

where $S = S(\sigma) = \sum_{k=1}^N a_k e^{-\sigma k} = \sum_{j=0}^q S_j^\star(\sigma)$. We set $\sigma = \delta$. The forthcoming asymptotic analysis relies on the following fact that features the expansive case (4.1) considered. Let $\delta = \delta_N \to 0$, as $N \to \infty$ and let the stratification points be as in Theorem 4.1, that is, $M_j = u_j r_N$, $j = 0, \dots, q$, $0 = u_0 < u_1 < \cdots < u_q$. Then

$$\lim_{N \to \infty} S_j^\star = \infty, \qquad S_j^\star = O(S_0), \quad N \to \infty, j = 0, \dots, q. \tag{5.18}$$

This is proved with the help of the integral test, as it is detailed in (5.33) below. We will see from the proof of Theorem 4.6 that (5.18) fails for the asymptotics of component counts of small sizes.

Lemma 5.3. *Suppose* (5.18) *holds. Then for* $L_0^\star, \dots, L_q^\star$ *given by* (5.14) *and for any* $N_0^\star, \dots, N_q^\star \in \mathbb{N}$: $\sum_{j=0}^q N_j^\star = N$,

$$R\big(\overrightarrow{L^\star}, \overrightarrow{N^\star}\big) \sim \frac{W}{(2\pi)^{(q+1)/2}} \left(\prod_{j=0}^q \frac{1}{\sqrt{S_j^\star}}\right) \exp\left(-\frac{1}{2}\sum_{j=0}^q \big(x_j^\circ\big)^2 \frac{S_0}{S_j^\star}\right), \quad N \to \infty, \tag{5.19}$$

where

$$W = \frac{\prod_{j=0}^q \mathbb{P}(Y^{(j)} = N_j^\star)}{\mathbb{P}(Z_N = N)}. \tag{5.20}$$

Proof. We have from (5.9) and (5.17)

$$R\big(\overrightarrow{L^\star}, \overrightarrow{N^\star}\big) = W \cdot \left(\prod_{j=0}^q \frac{(S_j^\star)^{L_j^\star}}{(L_j^\star)!}\right) \exp(\delta N) \exp\big(-(\delta N + S)\big) = W \exp(-S) \prod_{j=0}^q \frac{(S_j^\star)^{L_j^\star}}{(L_j^\star)!}. \tag{5.21}$$

Next, we apply Stirling's asymptotic formula to estimate $(L_j^\star)!$ under (5.14) and the condition (5.18):

$$\begin{aligned}
\big(L_j^\star\big)! &\sim \sqrt{2\pi L_j^\star}\big(L_j^\star\big)^{L_j^\star} \exp\big(-L_j^\star\big) \sim \sqrt{2\pi S_j^\star}\big(S_j^\star\big)^{L_j^\star} \left(1 + \frac{x_j^\circ \sqrt{S_0}}{S_j^\star}\right)^{L_j^\star} \exp\big(-L_j^\star\big) \\
&= \sqrt{2\pi S_j^\star}\big(S_j^\star\big)^{L_j^\star} \exp\left(L_j^\star \log\left(1 + \frac{x_j^\circ \sqrt{S_0}}{S_j^\star}\right)\right) \exp\big(-L_j^\star\big) \\
&\sim \sqrt{2\pi S_j^\star}\big(S_j^\star\big)^{L_j^\star} \mathrm{e}^{-S_j^\star} \exp\left(\frac{1}{2}\big(x_j^\circ\big)^2 \frac{S_0}{S_j^\star}\right), \quad N \to \infty,\ j = 0, \dots, q.
\end{aligned} \tag{5.22}$$

Substituting in (5.21), the preceding asymptotic expansions yields (5.19). □

We will follow the principle of Khintchine's method that the free parameter δ should be coupled with N so that to make the probabilities in the RHS of (5.20) large. Namely (see also [13,14]) we choose $\delta = \delta_N$ as the solution of the equation:

$$\mathrm{E} Z_N = \sum_{k=1}^N k a_k \mathrm{e}^{-\delta k} = N. \tag{5.23}$$

We see from (4.11) that (5.23) implies $E_0^\star = E_0 = N - E_1$. So, we obtain from (5.16) and (5.13) that

$$N_0^\star = N - N_1 = E_0^\star + x_{q+1}^\circ \sqrt{V_0}, \tag{5.24}$$

which enables us to rewrite (5.15) in a unified way:

$$N_j^\star = E_j^\star + x_{j+q+1}^\circ \sqrt{V_0}, \quad j = 0, \dots, q. \tag{5.25}$$

It is easy to see that if a_k, $k \ge 1$, are positive, Eq. (5.23) has a unique solution for any $N \ge 1$.

Our next goal will be the establishment of local limit theorems needed for the proof of Theorem 4.1.

5.3. *The local limit theorems for $Y^{(j)}$, $j=0,\dots,q$*

Lemma 5.4 (The local limit theorems for $Y^{(j)}$, $j=0,\dots,q$). *Let* (4.1) *hold and let $M_j=[u_j r_N]$, $j=0,\dots,q$, where $0=u_0<u_1<\dots<u_q$, $M_{q+1}=N+1$ and $[\bullet]$ is the integer part of a number. Then for $\overrightarrow{L^\star}$, $\overrightarrow{N^\star}$ as in* (5.14), (5.15) *and for δ given by* (5.23),

$$\mathbb{P}\big(Y^{(j)}=N_j^\star\big)\sim\frac{1}{\sqrt{2\pi\operatorname{Var}Y^{(j)}}}\exp\biggl(-\frac{\kappa_j^2}{2}\biggr),\quad N\to\infty, j=0,\dots,q, \tag{5.26}$$

where

$$\kappa_j=\lim_{N\to\infty}\frac{\mathrm{E}Y^{(j)}-N_j^\star}{\sqrt{\operatorname{Var}Y^{(j)}}},\quad j=0,\dots,q,$$

are positive constants calculated in (5.48) *below.*

Proof. We denote by $\varphi^{(j)}$ the characteristic functions of the random variable $Y^{(j)}$, to obtain

$$\mathbb{P}\big(Y^{(j)}=N_j^\star\big)=(2\pi)^{-1}\int_{-\pi}^{\pi}\varphi^{(j)}(t)\mathrm{e}^{-\mathrm{i}N_j^\star t}\,\mathrm{d}t:=(2\pi)^{-1}I^{(j)},\quad j=0,\dots,q. \tag{5.27}$$

We will focus now on the asymptotics, as $N\to\infty$, of the integrals $I^{(j)}$, $j=0,\dots,q$. For any $0<|t_{0,j}|<\pi$ the integral $I^{(j)}$ can be written in the form:

$$I^{(j)}=I_1^{(j)}+I_2^{(j)},\quad j=0,\dots,q, \tag{5.28}$$

where $I_1^{(j)}=I_1^{(j)}(t_{0,j})$ and $I_2^{(j)}=I_2^{(j)}(t_{0,j})$ are integrals of the integrand of $I^{(j)}$, taken over the sets $[-t_{0,j},t_{0,j}]$ and $[-\pi,-t_{0,j}]\cup[t_{0,j},\pi]$, respectively. Using the technique of [13,14], we will show that for an appropriate choice of $0<t_{0,j}=t_0(N,j)\to 0$, $N\to\infty$, $j=0,\dots,q$, the main contribution to $I^{(j)}$, as $N\to\infty$, comes from $I_1^{(j)}$, i.e. from a specially constructed neighborhood of $t=0$. First, observe that

$$\varphi^{(j)}(t)=\big(\varphi_1^{(j)}(t)\big)^{L_j^\star},\quad t\in\mathbb{R}, j=0,\dots,q, \tag{5.29}$$

where $\varphi_1^{(j)}$ is the characteristic function of the random variable $\beta_j:=\beta_1^{(j)}$ defined by (5.6), (5.7). To derive the asymptotics of the integral $I_1^{(j)}$, under the above choice of $t_{0,j}$, we will look for the approximation of $\varphi^{(j)}(t)$, as $t\to 0$ and $N\to\infty$. For this purpose, we need the asymptotic expressions, as $N\to\infty$, for $\mathrm{E}\beta_j$, $\mathrm{E}\beta_j^2$ and $\mathrm{E}\beta_j^3$. We have

$$\mathrm{E}\beta_j=\frac{E_j^\star}{S_j^\star},\qquad \mathrm{E}\beta_j^2=\frac{V_j^\star}{S_j^\star},\qquad \mathrm{E}\beta_j^3=\frac{H_j^\star}{S_j^\star},\quad j=0,\dots,q, \tag{5.30}$$

where $H_j^\star=H_j^\star(\delta,M_j,M_{j+1})=\sum_{k=M_j}^{M_{j+1}-1}k^3a_k\mathrm{e}^{-\delta k}$, $j=0,\dots,q$. It follows from the definitions of $S_j^\star$, $E_j^\star$, $V_j^\star$ and $H_j^\star$ that in the case considered the problem reduces to estimation of sums of the form

$$\sum_{k=M_j}^{M_{j+1}-1}a_kk^r\mathrm{e}^{-\delta k},\quad a_k\sim Ck^{p-1}, k\to\infty, p>0, C>0, r\ge 0, \tag{5.31}$$

when $N\to\infty$, δ is given as the solution of (5.23) and M_j, $j=0,\dots,q+1$, are as in the statement of the lemma. The asymptotic solution of (5.23) was obtained in [14]:

$$\delta=\delta_N\sim hN^{-1/(p+1)}=(r_N)^{-1},\quad N\to\infty, \tag{5.32}$$

where h, r_N are as in (4.7). Thus, $M_j\delta \to u_j$, $N \to \infty$, $j = 0, \dots, q$. In view of (4.1), it is convenient to write $a_k = Ck^{p-1}G(k)$, where the function G is s.t. $\lim_{k\to\infty} G(k) = 1$. Then, applying the integral test (=Euler summation formula), we get from (5.32)

$$\begin{aligned}\sum_{k=M_j}^{M_{j+1}-1} a_k k^r \mathrm{e}^{-\delta k} &\sim C \int_{M_j}^{M_{j+1}-1} x^{p+r-1} G(x) \mathrm{e}^{-\delta x}\,\mathrm{d}x \\ &= C\delta^{-p-r} \int_{M_j\delta}^{(M_{j+1}-1)\delta} x^{p+r-1} G\left(\frac{x}{\delta}\right) \mathrm{e}^{-x}\,\mathrm{d}x \\ &\sim C\delta^{-p-r} \int_{u_j}^{u_{j+1}} x^{p+r-1} G\left(\frac{x}{\delta}\right) \mathrm{e}^{-x}\,\mathrm{d}x \sim \delta^{-p-r} C \int_{u_j}^{u_{j+1}} x^{p+r-1} \mathrm{e}^{-x}\,\mathrm{d}x \\ &:= \delta^{-p-r} f_j(p+r-1), \quad N \to \infty, j = 0, \dots, q, r \ge 0,\ p > 0, \end{aligned} \tag{5.33}$$

where we set $u_{q+1} = \infty$. Note that by our notation (4.17),

$$f_j(s) = b_s(u_j) - b_s(u_{j+1}), \quad s > -1, j = 0, \dots, q. \tag{5.34}$$

We also obtain from (5.33) and (5.30),

$$\mathrm{E}\beta_j^r = \mathrm{O}\big(\delta^{-r}\big), \quad N \to \infty, r > 0, j = 0, \dots, q, \tag{5.35}$$

and observe that

$$\begin{aligned} &f_j(p+1) f_j(p-1) - \big(f_j(p)\big)^2 > 0, \\ &f_j(p+2) f_j(p-1) - f_j(p+1) f_j(p) > 0, \quad j = 0, \dots, q. \end{aligned} \tag{5.36}$$

The first of these inequalities follows immediately from the Cauchy–Schwarz inequality, while the second can be derived by substituting $f_j(p-1)$ from the first one and then applying again the Cauchy–Schwarz inequality.

To arrive at the required asymptotic formula for $\varphi^{(j)}$, we first write for a fixed N,

$$\varphi_1^{(j)}(t) = 1 + \mathrm{i}\mathrm{E}\beta_j t - \frac{1}{2}\mathrm{E}\beta_j^2 t^2 + \mathrm{O}\big(\mathrm{E}\beta_j^3 t^3\big), \quad t \to 0, j = 0, \dots, q, \tag{5.37}$$

and then couple t with N by setting

$$t_{0,j} = \delta\big(L_j^\star\big)^{-1/2} \log^2\delta = \mathrm{O}\big(\delta^{1+p/2}\log^2\delta\big), \quad j = 0, \dots, q, N \to \infty, \tag{5.38}$$

where the last equality follows from (5.33) and (5.14). Consequently,

$$\big(\mathrm{E}\beta_j^r\big) t_{0,j}^r = \mathrm{O}\big(\delta^{rp/2}\log^{2r}\delta\big) \to 0, \quad N \to \infty, j = 0, \dots, q, r > 0. \tag{5.39}$$

Now (5.29) and (5.37) together with (5.14), (5.15) give

$$\begin{aligned} \varphi^{(j)}(t)\mathrm{e}^{-\mathrm{i}N_j^\star t} &= \left(1 + \mathrm{i}\mathrm{E}\beta_j t - \frac{1}{2}\mathrm{E}\beta_j^2 t^2 + \mathrm{O}\big(\mathrm{E}\beta_j^3 t^3\big)\right)^{L_j^\star} \mathrm{e}^{-\mathrm{i}N_j^\star t} \\ &\sim \exp\left(\mathrm{i}t\big(L_j^\star \mathrm{E}\beta_j - N_j^\star\big) - \frac{1}{2} L_j^\star \mathrm{E}\beta_j^2 t^2 + \frac{1}{2} L_j^\star t^2 (\mathrm{E}\beta_j)^2 + L_j^\star \mathrm{O}\big(\mathrm{E}\beta_j^3 t^3\big)\right) \\ &= \exp\left(\mathrm{i}t\big(\mathrm{E}Y^{(j)} - N_j^\star\big) - \frac{1}{2} t^2 \operatorname{Var} Y^{(j)} + L_j^\star \mathrm{O}\big(\mathrm{E}\beta_j^3 t^3\big)\right), \\ &|t| \le t_{0,j}, N \to \infty, j = 0, \dots, q. \end{aligned} \tag{5.40}$$

In view of (5.35) and (5.36), the choice (5.38) of $t_{0,j}$ provides

$$\lim_{N\to\infty} t_{0,j}^2 \operatorname{Var} Y^{(j)} = \infty, \quad j = 0,\dots,q, \tag{5.41}$$

and

$$\lim_{N\to\infty} L_j^\star t_{0,j}^3 \mathrm{E}\beta_j^3 = 0, \quad j = 0,\dots,q. \tag{5.42}$$

Remark 5.5. *By* (5.35) *and* (5.36),

$$\operatorname{Var} Y^{(j)} = L_j^\star\big(\mathrm{E}\beta_j^2 - (\mathrm{E}\beta_j)^2\big) = \mathrm{O}\big(\delta^{-2}L_j^\star\big), \quad N\to\infty,\ j=0,\dots,q \tag{5.43}$$

and

$$\begin{aligned}\mathrm{E}\big(Y^{(j)} - \mathrm{E}Y^{(j)}\big)^3 &= L_j^\star \mathrm{E}(\beta_j - \mathrm{E}\beta_j)^3\\ &= L_j^\star\big(\mathrm{E}\beta_j^3 - 3\mathrm{E}\beta_j^2\mathrm{E}\beta_j + 2(\mathrm{E}\beta_j)^3\big) = \mathrm{O}\big(\delta^{-3}L_j^\star\big), \quad N\to\infty,\ j=0,\dots,q.\end{aligned} \tag{5.44}$$

Hence, the following weaker (=*the third moment is not absolute*) *form of Lyapunov's sufficient condition for the convergence to the normal law holds for the sums* $Y^{(j)}$, $j=0,\dots,q$:

$$\lim_{N\to\infty} \frac{\mathrm{E}(Y^{(j)} - \mathrm{E}Y^{(j)})^3}{(\operatorname{Var} Y^{(j)})^{3/2}} = 0, \quad j=0,\dots,q. \tag{5.45}$$

This explains the existence of $t_{0,j}$ *that provides* (5.41) *and* (5.42). *The phenomenon described above is typical in applications of Khintchine's method* (*see* [14,15,19]).

To continue the asymptotic expansion (5.40), we obtain from (5.30), (5.33) and (5.36)

$$\begin{aligned}&E_j^\star \sim f_j(p)\delta^{-p-1}, \qquad V_j^\star \sim f_j(p+1)\delta^{-p-2}, \qquad \operatorname{Var} Y^{(j)} \sim w_j\delta^{-p-2}, \qquad S_j^\star \sim f_j(p-1)\delta^{-p},\\ &N\to\infty,\ j=0,\dots,q,\end{aligned} \tag{5.46}$$

where

$$w_j = f_j(p+1) - \frac{f_j^2(p)}{f_j(p-1)} > 0, \quad j=0,\dots,q. \tag{5.47}$$

The asymptotic relations below are the consequence of (5.14), (5.15) and (5.46):

$$\begin{aligned}\frac{\mathrm{E}Y^{(j)} - N_j^\star}{\sqrt{\operatorname{Var} Y^{(j)}}} &= \frac{(E_j^\star/S_j^\star)L_j^\star - N_j^\star}{\sqrt{\operatorname{Var} Y^{(j)}}} = x_j^\circ \frac{E_j^\star\sqrt{S_0}}{S_j^\star\sqrt{\operatorname{Var} Y^{(j)}}} - x_{j+q+1}^\circ \frac{\sqrt{V_0}}{\sqrt{\operatorname{Var} Y^{(j)}}}\\ &\sim x_j^\circ \frac{f_j(p)}{f_j(p-1)}\sqrt{\frac{f_0(p-1)}{w_j}} - x_{j+q+1}^\circ\sqrt{\frac{f_0(p+1)}{w_j}}\\ &:= \kappa_j\big(x_j^\circ, x_{j+q+1}^\circ\big) = \kappa_j, \quad N\to\infty,\ j=0,\dots,q.\end{aligned} \tag{5.48}$$

Now we are in a position to evaluate the integrals $I_1^{(j)}$, $j=0,\dots,q$. By virtue of (5.40)–(5.42) and (5.48),

$$I_1^{(j)} = \int_{-t_{0,j}}^{t_{0,j}} \varphi^{(j)}(t)\mathrm{e}^{-\mathrm{i}N_j^\star t}\,\mathrm{d}t \sim \int_{-t_{0,j}}^{t_{0,j}} \exp\bigg(\mathrm{i}t\big(\mathrm{E}Y^{(j)} - N_j^\star\big) - \frac{1}{2}t^2\operatorname{Var} Y^{(j)}\bigg)\,\mathrm{d}t \tag{5.49}$$

$$= \exp\left(-\frac{(\mathrm{E}Y^{(j)} - N_j^\star)^2}{2\operatorname{Var} Y^{(j)}}\right) \int_{-t_{0,j}}^{t_{0,j}} \exp\left[-\frac{1}{2}\left(t\sqrt{\operatorname{Var} Y^{(j)}} - \mathrm{i}\frac{\mathrm{E}Y^{(j)} - N_j^\star}{\sqrt{\operatorname{Var} Y^{(j)}}}\right)^2\right] \mathrm{d}t$$

$$\sim \frac{1}{\sqrt{\operatorname{Var} Y^{(j)}}} \exp\left(-\frac{(\mathrm{E}Y^{(j)} - N_j^\star)^2}{2\operatorname{Var} Y^{(j)}}\right) \int_{-\sqrt{\operatorname{Var} Y^{(j)}}t_{0,j}}^{\sqrt{\operatorname{Var} Y^{(j)}}t_{0,j}} \exp\left[-\frac{1}{2}(t - \mathrm{i}\kappa_j)^2\right] \mathrm{d}t$$

$$\sim \frac{1}{\sqrt{\operatorname{Var} Y^{(j)}}} \exp\left(-\frac{(\mathrm{E}Y^{(j)} - N_j^\star)^2}{2\operatorname{Var} Y^{(j)}}\right) \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(t - \mathrm{i}\kappa_j)^2\right] \mathrm{d}t$$

$$\sim \frac{\sqrt{2\pi}}{\sqrt{\operatorname{Var} Y^{(j)}}} \exp\left(-\frac{\kappa_j^2}{2}\right), \quad N \to \infty, j = 0, \dots, q.$$

Next we turn to the estimation, as $N \to \infty$, of the integrals $I_2^{(j)}$, $j = 0, \dots, q$. We start with

$$\left|I_2^{(j)}\right| := 2\left|\int_{t_{0,j}}^{\pi} \varphi^{(j)}(t)\mathrm{e}^{-\mathrm{i}N_j^\star t}\,\mathrm{d}t\right| \le 2\int_{t_{0,j}}^{\pi} \left|\varphi^{(j)}(t)\right| \mathrm{d}t = 4\pi \int_{t_{1,j}}^{1/2} \left|\varphi^{(j)}(2\pi t)\right| \mathrm{d}t$$

$$= 4\pi \int_{t_{1,j}}^{1/2} \left|\sum_{k=M_j}^{M_{j+1}-1} a_k \mathrm{e}^{(2\pi\mathrm{i}t - \delta)k}\right|^{L_j^\star} \left(S_j{}^\star\right)^{-L_j^\star} \mathrm{d}t, \quad j = 0, \dots, q, \tag{5.50}$$

where we set $t_{1,j} = \frac{t_{0,j}}{2\pi} > 0$, $j = 0, \dots, q$. Denote

$$g^{(j)}(t) = \left|\varphi_1^{(j)}(2\pi t)\right|, \quad t \in \left[t_{1,j}, \frac{1}{2}\right], j = 0, \dots, q. \tag{5.51}$$

It is easy to see from the definition of the random variables β_j that $g^{(j)}(1/2) < 1$, $j = 0, \dots, q$. Since the β_j, $j = 0, \dots, q$, have lattice distributions with span 1, this implies (see [10], p. 131, [37], p. 286) that for a fixed N, we have $g^{(j)}(t) < 1$, $t \in [t_{1,j}, 1/2]$, $j = 0, \dots, q$. Moreover, $g^{(j)}(t_{1,j}) \to 1$, $N \to \infty$, because $t_{1,j} \to 0$, $N \to \infty$. We then conclude that for sufficiently large N and any $0 < t \le 1/2$,

$$g^{(j)}(t_{1,j}) \ge g^{(j)}(t), \quad j = 0, \dots, q. \tag{5.52}$$

By the same argument,

$$\varphi_1^{(j)}(2\pi t_{1,j}) - 1 = \left(S_j^\star\right)^{-1} \sum_{k=M_j}^{M_{j+1}-1} a_k \mathrm{e}^{-\delta k}\left(\mathrm{e}^{2\pi\mathrm{i}kt_{1,j}} - 1\right) \to 0, \quad N \to \infty, j = 0, \dots, q. \tag{5.53}$$

This and (5.12) enable us to write

$$\left(g^{(j)}(t_{1,j})\right)^{L_j^\star} = \left|1 + \left(S_j^\star\right)^{-1} \sum_{k=M_j}^{M_{j+1}-1} a_k \mathrm{e}^{-\delta k}\left(\mathrm{e}^{2\pi\mathrm{i}kt_{1,j}} - 1\right)\right|^{L_j^\star}$$

$$\sim \exp\left(-2\left(1 + \mathrm{O}\left(\delta^{p/2}\right)\right) \sum_{k=M_j}^{M_{j+1}-1} a_k \mathrm{e}^{-\delta k} \sin^2(\pi k t_{1,j})\right), \quad N \to \infty, j = 0, \dots, q. \tag{5.54}$$

Denote

$$D_j = 2\sum_{k=M_j}^{M_{j+1}-1} a_k \mathrm{e}^{-\delta k} \sin^2(\pi k t_{1,j}), \quad j = 0, \dots, q, \tag{5.55}$$

and let $\tilde{u}_j \in (u_j, u_{j+1}),\ j = 0, \dots, q$. Then we have

$$\begin{aligned}
D_j &\geq 2 \sum_{k=u_j N^{1/(p+1)}}^{\tilde{u}_j N^{1/(p+1)}} a_k \mathrm{e}^{-\delta k} \sin^2(\pi k t_{1,j}) \\
&\geq 2\Big(\min_{k \in [u_j N^{1/(p+1)}, \tilde{u}_j N^{1/(p+1)}]} \big(a_k \mathrm{e}^{-\delta k}\big)\Big) \sum_{k=u_j N^{1/(p+1)}}^{\tilde{u}_j N^{1/(p+1)}} \sin^2(\pi k t_{1,j}) \\
&= \mathrm{O}\big(\delta^{-p+1}\big) \sum_{k=u_j N^{1/(p+1)}}^{\tilde{u}_j N^{1/(p+1)}} \sin^2(\pi k t_{1,j}), \quad N \to \infty, j = 0, \dots, q.
\end{aligned} \tag{5.56}$$

Denote $\varepsilon_j = \tilde{u}_j - u_j > 0,\ j = 0, \dots, q$. To estimate the last sum in (5.56), we employ the following inequality from [15]:

$$2 \sum_{k=l}^{l+m-1} \sin^2(\pi t k) \geq \frac{m}{2} \min\big\{1, (tm)^2\big\}, \quad |t| \leq \frac{1}{2}, m \geq 2, l \geq 1. \tag{5.57}$$

We apply (5.57) with $l = u_j N^{1/(p+1)}$ and $m = \varepsilon_j N^{1/(p+1)} + 1$ to get from (5.38) and (5.32),

$$\begin{aligned}
2 \sum_{k=u_j N^{1/(p+1)}}^{\tilde{u}_j N^{1/(p+1)}} \sin^2(\pi k t_{1,j}) &\geq \frac{1}{2} \varepsilon_j N^{1/(p+1)} \min\big\{1, \big(\varepsilon_j N^{1/(p+1)} t_{1,j}\big)^2\big\} \\
&= \mathrm{O}\big(\delta^{p-1}\big) \log^4 \delta, \quad N \to \infty, j = 0, \dots, q.
\end{aligned} \tag{5.58}$$

Now we deduce from (5.50), (5.52), (5.56) and (5.58) that

$$\big|I_2^{(j)}\big| \leq \mathrm{O}\big(\exp\big(-\log^4 \delta\big)\big), \quad N \to \infty. \tag{5.59}$$

Comparing this with (5.49) gives

$$I_2^{(j)} = \mathrm{o}\big(I_1^{(j)}\big), \quad N \to \infty, j = 0, \dots, q, \tag{5.60}$$

which together with (5.48), (5.49) proves the lemma. □

5.4. *Completion of the proofs of Theorems* 4.1, 4.3 *and* 4.6

To complete the asymptotic analysis of the probability $R(\overrightarrow{L^\star}, \overrightarrow{N^\star})$ in the vicinity of r_N, it remains to derive the asymptotics of $\mathbb{P}(Z_N = N)$, as $N \to \infty$. It was found in [13] that

$$\mathbb{P}(Z_N = N) \sim \frac{1}{\sqrt{2\pi \operatorname{Var} Z_N}}, \quad N \to \infty, \tag{5.61}$$

where

$$\operatorname{Var} Z_N = \sum_{k=1}^{N} k^2 a_k \mathrm{e}^{-\delta k} \sim \Gamma(p+2) \delta^{-(p+2)}, \quad N \to \infty. \tag{5.62}$$

Substituting in (5.19) the asymptotic expressions (5.26), (5.46), (5.61) and (5.62) we obtain

$$R\big(\overrightarrow{L^\star},\overrightarrow{N^\star}\big)\sim\frac{\sqrt{\operatorname{Var}Z_N}}{(2\pi)^{q+1/2}}\left(\prod_{j=0}^{q}\frac{1}{\sqrt{S_j^\star\operatorname{Var}Y^{(j)}}}\right)\exp\left[-\frac{1}{2}\left(\sum_{j=0}^{q}\big(x_j^\circ\big)^2\frac{f_0(p-1)}{f_j(p-1)}+\kappa_j^2\right)\right]$$
$$\sim\delta^{(q(p+1)+p/2)}\frac{\sqrt{\Gamma(p+2)}}{(2\pi)^{q+1/2}}\left(\prod_{j=0}^{q}\frac{1}{\sqrt{w_jf_j(p-1)}}\right)\exp\left[-\frac{1}{2}\left(\sum_{j=0}^{q}\big(x_j^\circ\big)^2\frac{f_0(p-1)}{f_j(p-1)}+\kappa_j^2\right)\right],$$
$$N\to\infty. \tag{5.63}$$

Next we employ the scaled quantities $\hat{\nu}_j^\star$, $\hat{K}_j^\star$ constructed in (4.8) and denote

$$\begin{aligned}\overrightarrow{\hat{\nu}^\star}&=\big(\hat{\nu}_0^\star,\dots,\hat{\nu}_q^\star\big), &\overrightarrow{\hat{K}^\star}&=\big(\hat{K}_1^\star,\dots,\hat{K}_q^\star\big),\\ \overrightarrow{x^\circ}&=\big(x_0^\circ,\dots,x_q^\circ\big), &\overrightarrow{y^\circ}&=\big(x_{q+2}^\circ,\dots,x_{2q+1}^\circ\big).\end{aligned}\tag{5.64}$$

Note that $\hat{K}_0^\star=-\sum_{j=1}^{q}\hat{K}_j^\star$, by (5.23). We get from (5.63)

$$\begin{aligned}\mathbb{P}\big(\overrightarrow{\hat{\nu}^\star}=\overrightarrow{x^\circ},\overrightarrow{\hat{K}^\star}=\overrightarrow{y}^\circ\big)&\sim\frac{\sqrt{\Gamma(p+2)}}{(2\pi)^{(2q+1)/2}}\left(\frac{f_0(p-1)}{S_0}\right)^{(1/2)(q+1)}\left(\frac{f_0(p+1)}{V_0}\right)^{(1/2)q}\\ &\quad\times\left(\prod_{j=0}^{q}\frac{1}{\sqrt{w_jf_j(p-1)}}\right)\exp\left[-\frac{1}{2}\sum_{j=0}^{q}\frac{(x_j^\circ)^2}{f_j(p-1)}\right]\\ &\quad\times\exp\left[-\frac{1}{2}\sum_{j=0}^{q}\left(x_j^\circ\frac{f_j(p)}{f_j(p-1)\sqrt{w_j}}-x_{q+j+1}^\circ\frac{1}{\sqrt{w_j}}\right)^2\right],\quad N\to\infty,\end{aligned}\tag{5.65}$$

where $x_{q+1}^\circ=-\sum_{j=1}^{q}x_{q+j+1}^\circ$.

Each term of the sum in the second exponent depends on the two variables x_j° and x_{q+j+1}° only. Hence, under a fixed vector $\overrightarrow{y^\circ}$, the exponents in the RHS of (5.65) factorize into a product of $q+1$ terms, each one depending on only one of the x_j°, $j=0,\dots,q$. This proves the claim (i) of Theorem 4.1.

For the proof the claim (ii) of Theorem 4.1, we first need to deduce from (5.65) the central limit theorem for the $(2q+1)$-dimensional random vector $(\overrightarrow{\hat{\nu}^\star},\overrightarrow{\hat{K}^\star})$. We see that the expression (5.48) of κ_j, $j=0,\dots,q$, that the expression in the product of the two exponents in (5.65) is a quadratic form of the coordinates of the vector $(\overrightarrow{x^\circ},\overrightarrow{y^\circ})$. For a given $q\ge 1$, we denote by $\Theta^\star(q)$ the $(2q+1)\times(2q+1)$ matrix of this quadratic form. Let now $c_j<d_j$, $j=0,\dots,2q$ and

$$(\overrightarrow{c},\overrightarrow{d})=(c_0,d_0)\times(c_1,d_1)\times\cdots\times(c_{2q},d_{2q}).\tag{5.66}$$

We also define the sets of discrete points

$$G_N^{(j)}=\begin{cases}\Big\{z\in(c_j,d_j)\colon\ S_j^\star+z\sqrt{\frac{S_0}{f_0(p-1)}}\in\mathbb{N}\Big\} & \text{if } 0\le j\le q,\\ \Big\{z\in(c_j,d_j)\colon\ E_j^\star+z\sqrt{\frac{V_0}{f_0(p+1)}}\in\mathbb{N}\Big\} & \text{if } q+1\le j\le 2q,\end{cases}\tag{5.67}$$

and $\overrightarrow{G_N}=G_N^{(0)}\times\cdots\times G_N^{(2q)}$.

Lemma 5.6 (The central limit theorem for the vector $(\overrightarrow{\hat{\nu}^\star},\overrightarrow{\hat{K}^\star})$). *As $N\to\infty$, and $q\ge 1$ is fixed,*

$$\mathbb{P}\big(\big(\overrightarrow{\hat{\nu}^\star},\overrightarrow{\hat{K}^\star}\big)\in(\overrightarrow{c},\overrightarrow{d})\big)\sim(2\pi)^{-(2q+1)/2}\sqrt{T_q}\int_{(\overrightarrow{c},\overrightarrow{d})}\exp\left(-\frac{1}{2}\overrightarrow{z}^T\Theta^\star(q)\overrightarrow{z}\right)\mathrm{d}\overrightarrow{z},\tag{5.68}$$

where the matrix $\Theta^{\star}(q)$ *is defined as above, and*

$$T_q = \Gamma(p+2)\left(\prod_{j=0}^{q} \frac{1}{w_j f_j(p-1)}\right). \tag{5.69}$$

Proof. We follow the known technique for passing from a local limit theorem to an integral one, that is exposed in detail in [37], p. 60 (see also [10], p. 80 and [12]). Summing (5.65) over $(\overrightarrow{x^6}, \overrightarrow{y^6}) \in \overrightarrow{G_N}$, we obtain

$$\mathbb{P}\big((\overrightarrow{\hat{\nu}^{\star}}, \overrightarrow{\hat{K}^{\star}}) \in (\overrightarrow{c}, \overrightarrow{d})\big) = (2\pi)^{-(2q+1)/2}\sqrt{T_q}\left(\frac{f_0(p-1)}{S_0}\right)^{(1/2)(q+1)}\left(\frac{f_0(p+1)}{V_0}\right)^{(1/2)q} \times \sum_{\overrightarrow{z} \in \overrightarrow{G_N}}\left[\exp\left(-\frac{1}{2}\overrightarrow{z}^{T}\Theta^{\star}(q)\overrightarrow{z}\right)\big(1+\varepsilon(\overrightarrow{z}, N)\big)\right], \quad N \to \infty. \tag{5.70}$$

From the preceding asymptotic formulae we derive the crucial fact that in (5.70),

$$\sup_{\overrightarrow{z} \in (\overrightarrow{c}, \overrightarrow{d})} \varepsilon(\overrightarrow{z}, N) \to 0, \quad N \to \infty. \tag{5.71}$$

Based on this property of uniform convergence, we treat the RHS of (5.70) as a Riemann sum with the asymptotically equidistant spacings $(|G_N^{(j)}|)^{-1} \sim \sqrt{f_0(p-1)/S_0} \sim \delta^{p/2}$, $j = 0, \dots, q$, and $(|G_N^{(j)}|)^{-1} \sim \sqrt{f_0(p+1)/V_0} \sim \delta^{(p+2)/2}$, $j = q+1, \dots, 2q$, as $N \to \infty$. □

Next, we have to show that the main term in the RHS of (5.70) is indeed the Gaussian distribution. This is equivalent to proving that

$$\det \Theta^{\star}(q) = T_q, \quad q \geq 1, \tag{5.72}$$

where T_q is as in (5.69). By the definition (5.47) of w_j,

$$\frac{1}{f_j(p-1)} + \frac{f_j^2(p)}{f_j^2(p-1)w_j} = \frac{f_j(p+1)}{f_j(p-1)w_j}, \quad j = 0, \dots, q. \tag{5.73}$$

We first prove (5.72) for $q = 1$. Using (5.73) we have from (5.65)

$$\Theta^{\star}(1) = \begin{pmatrix} \frac{f_0(p+1)}{f_0(p-1)w_0} & 0 & \frac{f_0(p)}{f_0(p-1)w_0} \\ 0 & \frac{f_1(p+1)}{f_1(p-1)w_1} & -\frac{f_1(p)}{f_1(p-1)w_1} \\ \frac{f_0(p)}{f_0(p-1)w_0} & -\frac{f_1(p)}{f_1(p-1)w_1} & \frac{1}{w_0} + \frac{1}{w_1} \end{pmatrix}. \tag{5.74}$$

Now the claim for $q = 1$ follows from the identity $f_0(p+1) + f_1(p+1) = C\Gamma(p+2)$, $p > 0$, and some algebra. This shows that the random vector $(\hat{\nu}_0^{\star}, \hat{\nu}_1^{\star}, \hat{K}_1^{\star})$ weakly converges to the Gaussian random vector with zero mean and the covariance matrix $(\Theta^{\star}(1))^{-1}$. This implies (see [10], p. 89, Theorem 2.6) that for $q = 1$ the sequence of the distribution functions of the random vector $(\hat{\nu}_0^{\star}, \hat{\nu}_1^{\star}, \hat{K}_1^{\star})$ is tight under each $u_1 > 0$. Consequently, by the definition of tightness and the form of the stratifications considered, we deduce the tightness of the distribution functions of the random vector $(\overrightarrow{\hat{\nu}^{\star}}, \overrightarrow{\hat{K}^{\star}})$, for all $q > 1$. Finally, by Prohorov's theorem, the limiting distribution of the above vector is the one of a probability measure. This proves (5.72) for $q \geq 1$.

Now we derive the explicit form of the covariance matrix in Theorem 4.1. For a given $q \geq 1$, let $(\Theta^{\star}(q))^{-1} = \{\vartheta_{mk}^{\star}(q)\}_{m,k=0}^{2q}$ be the covariance matrix of the limiting Gaussian distribution in (5.68).

Lemma 5.7.

$$\vartheta^{\star}_{mk}(q) = \begin{cases} f_m(p-1)\mathbf{1}_{m=k} - \frac{f_m(p) f_k(p)}{\Gamma(p+2)} & \text{if } 0 \le m, k \le q, \\ f_m(p)\mathbf{1}_{m=k-q} - \frac{f_m(p) f_{k-q}(p+1)}{\Gamma(p+2)} & \text{if } 0 \le m \le q, q+1 \le k \le 2q, \\ f_k(p)\mathbf{1}_{k=m-q} - \frac{f_k(p) f_{m-q}(p+1)}{\Gamma(p+2)} & \text{if } 0 \le k \le q, q+1 \le m \le 2q, \\ f_{m-q}(p+1)\mathbf{1}_{m=k} - \frac{f_{m-q}(p+1) f_{k-q}(p+1)}{\Gamma(p+2)} & \text{if } q+1 \le m, k \le 2q. \end{cases} \tag{5.75}$$

Proof. To understand the technique behind the inversion of the matrix $\Theta^{\star}(q)$, it is convenient to verify (5.75) first for $q = 2$. This is easy to do with the help of the identity $\sum_{j=0}^{q} f_j(p+1) = C\Gamma(p+2)$, which holds for any $q \ge 1$. Taking into account the structure of the matrix $\Theta^{\star}(q)$, the verification for $q > 2$ can be done in the same way. □

This completes the proof of the statement (ii) of Theorem 4.1. We note that by (5.75), all covariances $\vartheta^{\star}_{mk}(q)$ between the scaled component counts $\hat{\nu}^{\star}_j$, as well as between masses $\hat{K}^{\star}_j$ are negative.

The convergence of moments, as stated in (iii) of Theorem 4.1 results from (5.71). For the proof, it is needed only to replace probabilities with moments in the argument given in [37], p. 61 (see also [23], p. 67). We see from (5.75) that for the proof of the claim (iv) of Theorem 4.1 one has to examine the behavior in $k \to \infty$ of the integrals $f_k(p)$, $f_{k-q}(p+1)$ in the case when u_j, $j = 0, \dots, q$, are equidistant points and $q \to \infty$. Now Theorem 4.1 is completely proved.

We proceed to the proof of Theorem 4.3. First, we recall the scaled quantities $\hat{\nu}_j$, $\hat{K}_j$ in (4.9) and denote

$$\overrightarrow{\hat{\nu}} = (\hat{\nu}_0, \dots, \hat{\nu}_q), \qquad \overrightarrow{\hat{K}} = (\hat{K}_1, \dots, \hat{K}_q). \tag{5.76}$$

It follows from the definition of ν_j and K_j, $j = 0, \dots, q$, that

$$\nu_0 = \nu^{\star}_0, \qquad \nu_j = \sum_{i=j}^{q} \nu^{\star}_i, \quad K_j = \sum_{i=j}^{q} K^{\star}_i, \quad j = 1, \dots, q. \tag{5.77}$$

By Theorem 4.1, (5.77) and (5.75), the random vector $(\overrightarrow{\hat{\nu}}, \overrightarrow{\hat{K}})$ converges weakly to the Gaussian vector with zero mean and the covariance matrix $\Upsilon(q) = \{e_{mk}\}_{m,k=0}^{2q}$ given by:

$$e_{mk} = \begin{cases} f_0(p-1) - \frac{f_0^2(p-1)}{\Gamma(p+2)} & \text{if } m = k = 0, \\ -\frac{f_0(p) b_p(u_k)}{\Gamma(p+2)} & \text{if } m = 0, 0 < k \le q, \\ -\frac{f_0(p) b_{p+1}(u_{k-q})}{\Gamma(p+2)} & \text{if } m = 0, q+1 \le k \le 2q, \\ b_{p-1}(u_{\max(m,k)}) - \frac{b_p(u_m) b_p(u_k)}{\Gamma(p+2)} & \text{if } 0 < m, k \le q, \\ b_p(u_{\max(m,k-q)}) - \frac{b_p(u_m) b_{p+1}(u_{k-q})}{\Gamma(p+2)} & \text{if } 0 < m \le q, q+1 \le k \le 2q, \\ b_{p+1}(u_{\max(m,k)-q}) - \frac{b_{p+1}(u_{m-q}) b_{p+1}(u_{k-q})}{\Gamma(p+2)} & \text{if } q+1 \le m, k \le 2q. \end{cases} \tag{5.78}$$

Applying now (5.78) with $1 < m, k \le q$ gives the claim (i) of Theorem 4.3.

For the proof of the claim (ii) of the theorem, we first recall that in the notation (2.1), $\nu_j = \nu(r_N u_j)$, $j = 1, \dots, q$, with r_N as in (4.7). Thus, letting $q = 1$ and $u_1 = u$, we rewrite the first part of (4.9) as

$$\hat{\nu}_1 = \sqrt{f_0(p-1)} \frac{\nu(r_N u) - S_1}{\sqrt{S_0}} \tag{5.79}$$

to get with the help of (5.33), (5.34), (4.7) and the notation (2.2), the desired relation (4.19) between $\hat{\nu}_j$ and the fluctuations $\Delta(u_j)$ for $j = 1, \dots, q$. By virtue of the convergence stated in (i) of Theorem 4.3, the fact that $N r_N^{-1} \to \infty$, as $N \to \infty$ and the continuity of $\operatorname{Var} \hat{\nu}_1$ in u, we derive that $\hat{\nu}_1$ is stochastically equicontinuous (see e.g., [34], p. 471) in $u \in [c, d]$, where $[c, d]$ is any finite interval. This implies the claim (iii) of the theorem.

Now we turn to the proof of Theorem 4.6 which will be done in the manner similar to the proof of Theorem 4.1. For a given $q \geq 1$, we consider the stratification points M_j, $j = 0, \dots, q+1$, as in (4.20), and define the quantities $S_j^\star, E_j^\star, V_j^\star$, $j = 0, \dots, q$, as in (5.7) and (4.11). Clearly, Lemma 5.1 remains valid with the above points M_j. In accordance with the problem considered, the vectors $\overrightarrow{L}^\star$ and $\overrightarrow{N}^\star$ are taken now in a slightly different form:

$$L_j^\star = S_j^\star + x_{2j}\sqrt{S_j^\star}, \quad j = 0, \dots, q, \tag{5.80}$$

and

$$N_j^\star = E_j^\star + x_{2j+1}\sqrt{V_j^\star}, \quad j = 0, \dots, q-1, \qquad N_q^\star = N - \sum_{j=0}^{q-1} N_j^\star, \tag{5.81}$$

where $x_0, \dots, x_{2q}$ are arbitrary reals not depending on N. Under $\delta = \delta_N$ chosen as in (5.32), the following analog of Lemma 5.3 is then valid:

$$R\big(\overrightarrow{L^\star}, \overrightarrow{N^\star}\big) \sim \frac{W}{(2\pi)^{(q+1)/2}} \left(\prod_{j=0}^{q} \frac{1}{\sqrt{S_j^\star}}\right) \exp\left(-\frac{1}{2}\sum_{j=0}^{q} x_{2j}^2\right), \quad N \to \infty, \tag{5.82}$$

where W is defined by (5.20). It is important to note that here, in contrast to the previous setting,

$$\lim_{N\to\infty} \delta M_j = 0, \quad j = 0, \dots, q, \tag{5.83}$$

by (5.32) and (4.20). Next we will intend to show that the local limit theorems (5.26) are valid with κ_j replaced with τ_j given by (5.90), (5.91) below. First, by the definition (4.20) of M_j and by (5.83), the integral test gives for $r \geq 0$ and $N \to \infty$,

$$\sum_{k=M_j}^{M_{j+1}-1} a_k k^r \mathrm{e}^{-\delta k} \sim \begin{cases} \frac{M_{j+1}^{p+r} - M_j^{p+r}}{p+r} & \text{if } j = 0, \dots, q-1, \\ \Gamma(p+r)\delta^{-p-r} & \text{if } j = q. \end{cases} \tag{5.84}$$

From this and (5.6), we derive that, under $N \to \infty$ and $r > 0$,

$$\mathrm{E}\beta_j^r = \begin{cases} \mathrm{O}\big(M_{j+1}^r\big) & \text{if } j = 0, \dots, q-1, \\ \mathrm{O}\big(\delta^{-r}\big) & \text{if } j = q. \end{cases} \tag{5.85}$$

In the case considered the analog of (5.38) will be

$$t_{0,j} = \begin{cases} \big(L_j^\star\big)^{-1/2} M_{j+1}^{-1} \log^2 L_j^\star & \text{if } j = 0, \dots, q-1, \\ \delta\big(L_q^\star\big)^{-1/2} \log^2 L_q^\star & \text{if } j = q. \end{cases} \tag{5.86}$$

Thus, we obtain that for any $r \geq 0$ and $N \to \infty$,

$$\big(\mathrm{E}\beta_j^r\big) t_{0,j}^r = \begin{cases} \mathrm{O}\big(M_{j+1}^{-pr/2} \log^{2r} M_{j+1}\big) & \text{if } j = 0, \dots, q-1, \\ \mathrm{O}\big(\delta^{pr/2} \log^{2r} \delta\big) & \text{if } j = q. \end{cases} \tag{5.87}$$

It is easy to see that (5.87) guarantees the conditions (5.41) and (5.42). Next, we have from (5.84)

$$\lim_{N\to\infty} \frac{E_j^\star}{\sqrt{V_j^\star S_j^\star}} = \begin{cases} \lim_{N\to\infty}\big(\frac{\sqrt{p(p+2)}}{p+1}\big) \frac{M_{j+1}^{p+1} - M_j^{p+1}}{\sqrt{(M_{j+1}^{p+2} - M_j^{p+2})(M_{j+1}^p - M_j^p)}} := \alpha_j & \text{if } j = 0, \dots, q-1, \\ \frac{\Gamma(p+1)}{\sqrt{\Gamma(p)\Gamma(p+2)}} := \alpha_q & \text{if } j = q, \end{cases} \tag{5.88}$$

where the limits $0 < \alpha_j < 1,\ j = 0, \ldots, q$, by (4.20) and some algebra. It follows from (5.81) that

$$N_q^\star = E_q^\star - \sum_{j=0}^{q-1} x_{2j+1}\sqrt{V_j^\star}. \tag{5.89}$$

We obtain from (5.84), (4.20), (5.81) and (5.80)

$$\begin{aligned}\frac{N_j^\star - \mathrm{E}Y^{(j)}}{\sqrt{\operatorname{Var} Y^{(j)}}} &= \frac{N_j^\star - (E_j^\star/S_j^\star)L_j^\star}{\sqrt{\operatorname{Var} Y^{(j)}}} = x_{2j+1}\frac{\sqrt{V_j^\star}}{\sqrt{\operatorname{Var} Y^{(j)}}} - x_{2j}\frac{E_j^\star}{\sqrt{S_j^\star \operatorname{Var} Y^{(j)}}} \\ &\sim \frac{x_{2j+1}\sqrt{V_j^\star} - x_{2j}(E_j^\star/\sqrt{S_j^\star})}{\sqrt{V_j^\star - (E_j^\star)^2/S_j^\star}} \to \frac{x_{2j+1} - \alpha_j x_{2j}}{\sqrt{1-\alpha_j^2}} := \tau_j, \quad N \to \infty, j = 0, \ldots, q-1,\end{aligned} \tag{5.90}$$

while for $j = q$ we have

$$\begin{aligned}\frac{N_q^\star - \mathrm{E}Y^{(q)}}{\sqrt{\operatorname{Var} Y^{(q)}}} &= -\frac{\sum_{j=0}^{q-1} x_{2j+1}\sqrt{V_j^\star}}{\sqrt{\operatorname{Var} Y^{(q)}}} - x_{2q}\frac{E_q^\star}{\sqrt{S_q^\star \operatorname{Var} Y^{(q)}}} \\ &\sim -x_{2q}\frac{E_q^\star}{\sqrt{S_q^\star \operatorname{Var} Y^{(q)}}} \to -\frac{\alpha_q x_{2q}}{\sqrt{1-\alpha_q^2}} := \tau_q, \quad N \to \infty.\end{aligned} \tag{5.91}$$

We observe that $|\tau_j| < \infty,\ j = 0, \ldots, q$. Proceeding further along the same lines as in the proof of Lemma 5.4 leads to the desired analog of (5.26). Consequently, (5.63) takes the form

$$R(\overrightarrow{L^\star}, \overrightarrow{N^\star}) \sim \frac{\sqrt{\operatorname{Var} Z_N}}{(2\pi)^{(2q+1)/2}}\left(\prod_{j=0}^{q} \frac{1}{\sqrt{S_j^\star \operatorname{Var} Y^{(j)}}}\right)\exp\left(-\frac{1}{2}\left[\sum_{j=0}^{q} x_{2j}^2 + \tau_j^2\right]\right), \quad N \to \infty. \tag{5.92}$$

Now we define the scaled quantities for the problem considered

$$\begin{aligned}\hat{\nu}_j^\star &= \frac{\nu_j^\star - S_j^\star}{\sqrt{S_j^\star}}, \qquad j = 0, \ldots, q, \\ \hat{K}_j^\star &= \frac{K_j^\star - E_j^\star}{\sqrt{V_j^\star}}, \quad j = 0, \ldots, q-1,\end{aligned}$$

and denote

$$\overrightarrow{\hat{\nu}^\star} = (\hat{\nu}_0^\star, \ldots, \hat{\nu}_q^\star), \quad \overrightarrow{\hat{K}^\star} = (\hat{K}_0^\star, \ldots, \hat{K}_{q-1}^\star). \tag{5.93}$$

(5.88) and (5.62) imply that

$$\operatorname{Var} Z_N \sim V_q^\star$$

and

$$\frac{V_j^\star}{\operatorname{Var} Y^{(j)}} \to \frac{1}{1-\alpha_j^2} < \infty, \quad N \to \infty, j = 0, \ldots, q. \tag{5.94}$$

Substituting (5.94) into (5.92) gives

$$\mathbb{P}\big(\overrightarrow{\hat{\nu}^\star} = \overrightarrow{x}, \overrightarrow{\hat{K}^\star} = \overrightarrow{y}\big) \sim (2\pi)^{-(2q+1)/2}\left(\prod_{j=0}^{q-1}\frac{1}{\sqrt{S_j^\star V_j^\star}}\right)\frac{1}{\sqrt{S_q^\star}}\left(\prod_{j=0}^{q}\frac{1}{\sqrt{1-\alpha_j^2}}\right)$$
$$\times \exp\left(-\frac{1}{2}\left[\sum_{j=0}^{q-1}\left(x_{2j}^2 + \frac{(x_{2j+1}-\alpha_j x_{2j})^2}{1-\alpha_j^2}\right) + \frac{x_{2q}^2}{1-\alpha_q^2}\right]\right), \quad N\to\infty, \tag{5.95}$$

where we set $\overrightarrow{x} = (x_0, x_2, \ldots, x_{2q})$ and $\overrightarrow{y} = (x_1, x_3, \ldots, x_{2q-1})$.

Remark 5.8. *At the first glance, the expression* (5.95) *and its analog* (5.65) *for Theorem* 4.1 *look very much alike. A crucial difference between them is that in* (5.65) *the variables* x_j° *are linked via the relation* $\sum_{j=q+1}^{2q+1} x_j^\circ = 0$, *while in* (5.95) *the variables* x_j *are free. Formally, this is implied by the fact that the value* κ_q *in* (5.48) *depends on* x_q° *and* x_{2q+1}°, *whereas the value* τ_q *in* (5.91) *depends on* x_{2q} *only. As a result of the aforementioned difference, conditional independence in Theorem* 4.1 *transforms to independence in Theorem* 4.3.

From (5.95), the claim (i) of Theorem 4.6 follows immediately. It is left to find explicitly the covariance matrix of the corresponding limiting Gaussian distribution. For the sake of convenience, we write $\vec{z} = (x_0, \ldots, x_{2q})$. Then the matrix, say B, of the quadratic form in the exponent of (5.95) has a diagonal block structure with the blocks B_j of the form

$$B_j = \begin{pmatrix} \frac{1}{1-\alpha_j^2} & -\frac{\alpha_j}{1-\alpha_j^2} \\ -\frac{\alpha_j}{1-\alpha_j^2} & \frac{1}{1-\alpha_j^2} \end{pmatrix}, \quad j = 0, \ldots, q-1, \tag{5.96}$$

and

$$B_q = \frac{1}{1-\alpha_q^2}. \tag{5.97}$$

Thus,

$$\det B = \prod_{j=0}^{q}\det B_j = \prod_{j=0}^{q}\frac{1}{1-\alpha_j^2} \tag{5.98}$$

and

$$B_j^{-1} = \begin{pmatrix} 1 & \alpha_j \\ \alpha_j & 1 \end{pmatrix}, \quad j = 0, \ldots, q-1, \qquad B_q^{-1} = 1-\alpha_q^2. \tag{5.99}$$

In accordance with the above notation for $\vec{z}$, we set $\overrightarrow{\chi} = (\hat{\nu}_0^\star, \hat{K}_0^\star, \hat{\nu}_1^\star, \hat{K}_1^\star, \ldots, \hat{\nu}_{q-1}^\star, K_{q-1}^\star, \hat{\nu}_q^\star)$ which enables to rewrite (5.95) as

$$\mathbb{P}\big(\overrightarrow{\chi} = \overrightarrow{z}\big) \sim \frac{1}{(2\pi)^{(2q+1)/2}\sqrt{\det B}}\left(\prod_{j=0}^{q-1}\frac{1}{\sqrt{S_j^\star V_j^\star}}\right)\frac{1}{\sqrt{S_q^\star}}\exp\left(-\frac{1}{2}\overrightarrow{z} B \overrightarrow{z}^T\right), \quad N\to\infty. \tag{5.100}$$

Hence, the covariance matrix B^{-1} of the Gaussian distribution can be written as the Kronecker product of blocks B_j^{-1} as in (5.99):

$$B^{-1} = B_0^{-1} \otimes B_1^{-1} \otimes \cdots \otimes B_q^{-1}. \tag{5.101}$$

Finally, by the same argument as before, we derive the weak convergence as stated in (ii) of Theorem 4.6.

6. Comparison with limit shapes for the generalized Bose–Einstein and Fermi–Dirac models of ideal gas

Recall (see Section 3) that the generalized BE and the FD models in the title are associated with the generating functions $\mathcal{F}^{(\mathrm{BE})}$, $\mathcal{F}^{(\mathrm{FD})}$ respectively:

$$\mathcal{F}^{(\mathrm{BE})}(x)=\prod_{k\geq 1}\frac{1}{(1-x^k)^{m_k}},\qquad \mathcal{F}^{(\mathrm{FD})}(x)=\prod_{k\geq 1}\left(1+x^k\right)^{m_k},\quad |x|<1, m_k=Ck^{p-1}, p\geq 1, C>0. \tag{6.1}$$

Limit shapes for these models, say $C_{p-1}^{(\mathrm{BE})}$, $C_{p-1}^{(\mathrm{FD})}$ respectively, were obtained by Vershik in [43,44]:

$$C_{p-1}^{(\mathrm{BE})}(u)=\int_u^\infty x^{p-1}\frac{\mathrm{e}^{-c_1x}}{1-\mathrm{e}^{-c_1x}}\,\mathrm{d}x,\qquad C_{p-1}^{(\mathrm{FD})}(u)=\int_u^\infty x^{p-1}\frac{\mathrm{e}^{-c_2x}}{1+\mathrm{e}^{-c_2x}}\,\mathrm{d}x,\quad u\geq 0, p\geq 1, \tag{6.2}$$

where $c_1=c_1(p)>0$, $c_2=c_2(p)>0$ are normalizing constants.

It is interesting to compare these limit shapes with the ones, denoted l_{p-1}, in our Theorem 4.3:

$$l_{p-1}(u)=\frac{\int_u^\infty \mathrm{e}^{-x}x^{p-1}\,\mathrm{d}x}{\Gamma(p+1)},\quad u\geq 0, p>0. \tag{6.3}$$

First, note that the limit shapes for both models are derived under the scalings r_N of the same order $N^{1/(p+1)}$. In view of Proposition 2.4, the immediate explanation of this striking coincidence is that by the result of Vershik and Yakubovich ([47], Section 5), the models (6.1) of ideal gas have the same threshold $N^{1/(p+1)}$ as the models considered in the paper. In a broader context, we point that the generalized BE models and the models (2.9) studied in our paper, are linked with each other via the exponentiation of the generating function $\mathcal{F}^{(\mathrm{BE})}$ (see [5]). Moreover, the Bell–Burris Lemma 5.1 in [7] states that if the parameter function $m=\{m_k\}$ for the generalized BE model (multiset) is such that $\lim_{k\to\infty}\frac{m_{k-1}}{m_k}=y$ with $y<1$, then the corresponding multiplicative measure for the BE model is asymptotically equivalent to the measure μ_N (assembly) given by (2.9) with $a_k\sim m_k$, $k\to\infty$. However, for the generalized BE models (6.1), $\lim_{k\to\infty}\frac{m_{k-1}}{m_k}=1$, which explains the following basic difference in the form of the limit curves $C_{p-1}^{(\mathrm{BE})}$ and l_{p-1}. From (6.3), (6.2) we see that

$$l_{p-1}(0)=p^{-1}<\infty,\quad p>0, \tag{6.4}$$

while

$$C_0^{(\mathrm{BE})}(0)=\lim_{u\to 0^+}-\frac{\sqrt{6}}{\pi}\log\left(1-\exp\left(-\frac{\pi u}{\sqrt{6}}\right)\right)=\infty,\quad C_{p-1}(0)<\infty, p>1. \tag{6.5}$$

By (2.1)–(2.3), the value of a limit shape at $u=0$ "approximates" the random variable $\widetilde{\nu}(0)=$ the total number of components in a random partition, multiplied by the factor $\frac{r_N}{N}$. In [43,44] the phenomenon (6.5) is linked with the Bose–Einstein condensation of energy. According to this interpretation, the finiteness of the limit shape at $u=0$ indicates the appearance of condensation of energy (around the value $\frac{N}{r_N}$), and in view of this the value $p=1$ (=uniform distribution on the set Ω_N) was distinguished as the phase transition point for the generalized BE models (6.1). We now give an analytic explanation of the fact that $C_0^{(\mathrm{BE})}(0)=\infty$ in contrast to $l_0(0)<\infty$. By the seminal result of Erdös and Lehner (1941) (for references see [47]), the number of components in a random partition of N is asymptotically $\frac{2\pi}{\sqrt{6}}\sqrt{N}\log N\gg\sqrt{N}=\frac{N}{r_N}$. On the other hand, it was proven in [12] that the number of components in our model with $p=1$ is asymptotically $\sqrt{N}=h\frac{N}{r_N}$, where the constant h is as in (4.7).

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