Large deviation principle for enhanced Gaussian processes

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Abstract

We study large deviation principles for Gaussian processes lifted to the free nilpotent group of step $N$. We apply this to a large class of Gaussian processes lifted to geometric rough paths. A large deviation principle for enhanced (fractional) Brownian motion, in Hölder- or modulus topology, appears as special case.

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1. Introduction

We prove large deviation principles for $d$-dimensional Gaussian processes lifted to the free nilpotent group of step $N$. The example we have in mind is a class of zero mean Gaussian processes subject to certain conditions on the covariance, [5,17].

After recalls on nilpotent groups and Wiener Itô chaos, we give simple conditions under which a large deviation principle holds. To illustrate the method, we quickly check these conditions for enhanced Brownian motion (see [15,10] for earlier approaches). We then apply our methodology to the class of enhanced Gaussian processes considered in [5,17]. Enhanced fractional Brownian motion appears as a special case and we strengthen the results in [18,4].

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1.1. The free nilpotent group and rough paths

Let $N \geq 0$. The truncated tensor algebra of degree $N$ is given by the direct sum

$$T^N(\mathbb{R}^d) = \mathbb{R} \oplus \mathbb{R}^d \oplus \cdots \oplus (\mathbb{R}^d)^{\otimes N}.$$  

With the usual scalar product, vector addition, and tensor product $\otimes$ the space $T^N(\mathbb{R}^d)$ is an algebra. Let $\pi_i$ denote the canonical projection from $T^N(\mathbb{R}^d)$ onto $(\mathbb{R}^d)^{\otimes i}$.

Let $q \in [1,2)$ and $x \in C^q_0([0,1], \mathbb{R}^d)$, the space of continuous $\mathbb{R}^d$-valued paths of bounded $q$-variation started at the origin. The lift $S_N(x): [0,1] \to T^N(\mathbb{R}^d)$ is defined as

$$S_N(x)_t = 1 + \sum_{i=1}^{N} \int_{0<s_1<\cdots<s_i<t} dx_{s_i} \otimes \cdots \otimes dx_{s_i},$$

the iterated integrals are Young integrals. Then

$$G^N(\mathbb{R}^d) = \{ g \in T^N(\mathbb{R}^d) : \exists x \in C^1_{0\mathrm{-var}}([0,1], \mathbb{R}^d) : g = S_N(x)_1 \}$$

is a submanifold of $T^N(\mathbb{R}^d)$ and, in fact, a Lie group with product $\otimes$, called the free nilpotent group of step $N$. The dilation operator $\delta: \mathbb{R} \times G^N(\mathbb{R}^d) \to G^N(\mathbb{R}^d)$ is defined by

$$\pi_i(\delta_{\lambda}(g)) = \lambda^i \pi_i(g), \quad i = 0, \ldots, N,$$

and a continuous norm on $G^N(\mathbb{R}^d)$, homogeneous with respect to $\delta$, is given

$$\|g\| = \inf\{ \text{length}(x) : x \in C^1_{0\mathrm{-var}}([0,1], \mathbb{R}^d), S_N(x)_1 = g \}.$$  

By equivalence of such norms there exists a constant $K_N$ such that

$$\frac{1}{K_N} \max_{i=1,\ldots,N} |\pi_i(g)|^{1/i} \leq \|g\| \leq K_N \max_{i=1,\ldots,N} |\pi_i(g)|^{1/i}.$$  

(1.1)

The norm $\| \cdot \|$ induces a metric on $G^N(\mathbb{R}^d)$ known as Carnot–Caratheodory metric,

$$d(g, h) = \|g^{-1} \otimes h\|.$$  

Let $x, y \in C_0([0,1], G^N(\mathbb{R}^d))$, the space of continuous $G^N(\mathbb{R}^d)$-valued paths started at the neutral element of $(G^N(\mathbb{R}^d), \otimes)$. We define a supremum distance,

$$d_{\infty}(x, y) = \sup_{t \in [0,1]} d(x_t, y_t),$$

a Hölder distance, $\alpha \in [0,1]$,

$$d_{\alpha\text{-Hölder}}(x, y) = \sup_{0 \leq s, t \leq 1} \frac{d(x_s^{-1} \otimes x_t, y_s^{-1} \otimes y_t)}{|t - s|^\alpha},$$

write

$$\|x\|_{\alpha\text{-Hölder}} = \sup_{0 \leq s, t \leq 1} \frac{\|x_s^{-1} \otimes x_t\|}{|t - s|^\alpha} = \sup_{0 \leq s, t \leq 1} \frac{d(x_s, x_t)}{|t - s|^\alpha},$$

and $\text{Lip}$ instead of $1$-Hölder. Similarly, one defines $p$-variation regularity and distance for $G^N(\mathbb{R}^d)$-valued paths. We refer to [11] for a more detailed discussion of these topics.

The interest for all this comes from T. Lyons’ rough path theory, [16,17], a deterministic theory of control differential equations (“Rough Differential Equation”), driven by a continuous $G^{1[p]}(\mathbb{R}^d)$-valued path of finite $p$-variation. Brownian motion and Lévy’s area can be viewed as a $G^{1[p]}(\mathbb{R}^d)$-valued path of finite $p$-variation with $p > 2$ and the corresponding RDE solution turns out to be a classical Stratonovich SDE solution. This gives a general recipe how to make sense of differential equations driven by an arbitrary stochastic process $X$ with continuous sample path of finite $p$-variation: find an appropriate lift of $X$ to a $G^{1[p]}(\mathbb{R}^d)$-valued path $X$ with finite $p$-variation use Lyons’ machinery.
1.2. Gaussian process

The Wiener space $C_0([0, 1], \mathbb{R}^d)$ is the space of continuous function started at 0; the sup norm induces a topology and hence a Borel $\sigma$-algebra. We assume there is a probability measure $\mathbb{P}$ such that the coordinate process $X$, defined by $X(\omega)_t = \omega(t)$, is a centered Gaussian process with covariance $c(s, t)$. We also define $\mathcal{H}$ to be the reproducing kernel Hilbert space associated to $\mathbb{P}$, i.e. the closure of the set of the linear span of the functions $s \rightarrow c(t_i, s)$, under the Hilbert product

$$\langle c(t_i, \cdot), c(t_j, \cdot) \rangle = c(t_i, t_j).$$

Let $i$ denote the canonical injection from the reproducing kernel Hilbert space to the Wiener space. We define $C_n(B)$ the $B$-valued homogeneous Wiener chaos of degree $n$, where $(B, |\cdot|_B)$ is a given Banach space. We also define $C_n(B)$ the sum of the first $n$ homogeneous Wiener chaos. We refer to [14,1–3] for more details on Gaussian spaces and Wiener chaos.

1.3. Enhanced Gaussian process

We make the assumption that there exists a lift of the Gaussian process $X$ to a process $\tilde{X}$ with values in $G^N(\mathbb{R}^d)$ for some fixed $N \geq 1$. Such lifts have been constructed first by Coutin and Qian [5] as a.s. limits in $p$-variation of (canonically lifted) dyadic piecewise linear approximations of $X$. We shall be less specific here and extract those properties of the approximations that we need in the sequel. We make the following

Assumption. There exists a sequence of continuous linear maps $\{\Phi_m\}$ from $C([0, 1], \mathbb{R}^d)$ (with uniform topology) onto the space of bounded variation path from $[0, 1]$ into $\mathbb{R}^d$ (with 1-variation topology) such that

1. $\lim_{m \to \infty} \Phi_m(x) = x$ for all continuous paths $x$ with respect to uniform topology;
2. the uniform distance between $S_N \circ \Phi_m(X)$ and $X$ converges to 0 in probability.

In fact, Condition (2) is equivalent to the seemingly stronger condition of $L^p$-convergence.

Lemma 1. Convergence of $d_\infty(S_N \circ \Phi_m(X), X)$ to zero in probability is equivalent to convergence in $L^p$ for all $p \in [1, \infty)$.

Proof. Theorem III.2 in [20] and Eq. (1.1).

Remark 1. Assumption 2 is always satisfied when $N = 1$. In the applications discussed in later sections, $N$ will be related to the regularity of $X$.

Example 1. Let $\{\Phi_m\}$ be the piecewise linear approximations based on a sequence of dissections $\{D_m\}$ with mesh $|D_m| \to 0$. If $X$ denotes fractional Brownian motion of parameter $H > \frac{1}{4}$, then the lift $\tilde{X}$ for $N = [1/H]$ and was first constructed in [5], see also [4].

Remark 2. As is well known (e.g. Theorem 1 in [13]) any continuous Gaussian process $X$ on $[0, 1]$ is the uniform limit of

$$X^{(m)}(t) \equiv \sum_{j=1}^{m} \xi_j(\omega) \psi_j(t)$$

where $\{\psi_j\}$ is an orthonormal basis for the reproducing Kernel Hilbert space $\mathcal{H}$ and the $\{\xi_j\}$ are i.i.d. standard Gaussian given by the image of $\psi_j$ under the isometric isomorphism $\theta : \mathcal{H} \to L^2(\Omega; t \in [0, 1])$, the closure of $\{X_t : [0, 1]\}$ in $L^2(\mathbb{P})$. The Hilbert structure of $\mathcal{H}$ implies that for all $h \in \mathcal{H}$,

$$h^{(m)} \equiv \sum_{j=1}^{m} \langle \psi_j, h \rangle \psi_j \to h \quad \text{in } \mathcal{H}.$$
and hence uniformly. The construction of $X$ can be based on such (or similar) approximations, see [6, 11, 19, 8], but they do not satisfy our assumption. It may be possible to adapt the subsequent proofs, aimed to establish a large deviation principle for $X$, to such approximations. As this requires further work without improving the results we shall not pursue this further here.

2. Wiener chaos and $G^N(\mathbb{R}^d)$-valued paths

The hypercontractivity of the Ornstein–Uhlenbeck semigroup leads to a useful

**Lemma 2.** Let $Z_n$ be a random variable in $C_n(B)$. Assume $1 < p < q < \infty$. Then

$$\|Z_n\|_{L^q(P; B)} \leq \left( \frac{q - 1}{p - 1} \right)^{n/2} \|Z_n\|_{L^p(P; B)}.$$  

**Proof.** Let $P_t$ denote the Ornstein–Uhlenbeck semigroup. Whenever $1 < p < q < \infty$ and $t > 0$ satisfies

$$e^t \geq \left( \frac{q - 1}{p - 1} \right)^{1/2},$$  

then, for all $f \in L^p(P; B)$,

$$\|P_t f\|_{L^q} \leq \|f\|_{L^p}.$$  

We also recall that $P_t Z_n = e^{-nt} Z_n$.

See [14] for this two results. We now choose $t$ such that equality holds in (2.1) and find

$$\left( \frac{q - 1}{p - 1} \right)^{-n/2} \|Z_n\|_{L^q} = e^{-nt} \|Z_n\|_{L^q} = \|P_t Z_n\|_{L^q} \leq \|Z_n\|_{L^p}.$$  

It is well known that $L^p(P; B)$- and $L^q(P; B)$-norms are equivalent on $C_n(B) = C_1(B) + \cdots + C_n(B)$.

The next lemma quantifies this equivalence.

**Lemma 3.** Let $n \in \mathbb{N}$ and $Z$ be a random variable in $C_n(B)$. Assume $2 \leq p \leq q < \infty$. Then there exists a constant $M_n$ such that

$$\|Z\|_{L^p(P; B)} \leq \|Z\|_{L^q(P; B)} \leq M_n(q - 1)^{n/2} \|Z\|_{L^p(P; B)}.$$  

**Proof.** Only the second inequality requires a proof. Write $Z = \sum_{i=0}^n Z_i$ with $Z_i \in C_i$. From Lemma 2, for all $i \leq n$,

$$\|Z_i\|_{L^q(P; B)} \leq \left( \frac{q - 1}{p - 1} \right)^{i/2} \|Z_i\|_{L^p(P; B)}.$$  

The sum $C_n(B) = C_1(B) + \cdots + C_n(B)$ is topological direct in $L^q(P; B)$, see [3, p. 6] for instance, which implies that the projection $Z \mapsto Z_i$ is continuous in $L^2(P; B)$. It follows that $\|Z_i\|_{L^2(P; B)} \leq c\|Z\|_{L^2(P; B)}$ for some constant $c = c(n)$. Then, for $p \geq 2$,

$$\|Z_i\|_{L^p(P; B)} \leq (p - 1)^{i/2} \|Z_i\|_{L^2(P; B)} \leq c(p - 1)^{i/2} \|Z\|_{L^2(P; B)} \leq c(p - 1)^{i/2} \|Z\|_{L^p(P; B)}.$$  

By Hölder’s inequality for finite sums,

$$|Z|^q \leq \sum_{i=0}^n |Z_i|^q.$$  

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1 At least for, $N \leq 2$ it is shown in [6] that reproducing kernel – and piecewise linear approximations yield the same lifted process.
and after taking expectations,
\[
\|Z\|_{L^q(P;B)} \leq n^{1-1/q} \left( \sum_{i=0}^{n} \|Z_i\|_{L^q(P;B)}^q \right)^{1/q}.
\]

Hence, from (2.2) and (2.3),
\[
\|Z\|_{L^q(P;B)} \leq cn^{1-1/q} \left( \sum_{i=0}^{n} (q-1)^{q/2} \|Z_i\|_{L^p(P;B)}^q \right)^{1/q}
\]
\[
\leq cn(q-1)^{q/2} \|Z\|_{L^p(P;B)}. \quad \square
\]

**Corollary 1.** Let \(X, Y\) be two continuous \(G^N(\mathbb{R}^d)\)-valued processes such that, for each \(i = 1, \ldots, N\), the projection \(t \mapsto \pi_i(X_t), \pi_i(Y_t) \in (\mathbb{R}^d) \otimes i\) belongs to \(C_i(B_i) = C_0(B_i) \oplus C_1(B_i) \oplus \cdots \oplus C_n(B_i),\) where
\[B_i = C_0([0, 1], (\mathbb{R}^d) \otimes i)\]
and \(C_j(B_i)\) is the \(B_i\)-valued homogeneous Wiener chaos of degree \(j\). Assume \(2 < q < \infty\). Then there exists a constant \(M_N\) such that
\[
\|d_\infty(X,Y)\|_{L^qN(P)} \leq \|d_\infty(X,Y)\|_{L^qN(P)} \leq \sqrt{q} M_N \|d_\infty(X,Y)\|_{L^2N(P)}.
\]

**Proof.** Again, only the second inequality requires a proof. For \(i = 1, \ldots, N\) define
\[Z_i : t \mapsto \pi_i(X_t^{-1} \otimes Y_t) \in (\mathbb{R}^d) \otimes i.
\]
Observe that \(Z_i \in C_i(B_i) \subset C_N(B_i).\) Let \(|\cdot|_\infty\) denote the supremum norm on \((\mathbb{R}^d) \otimes i\)-valued paths. From Eq. (1.1),
\[
\frac{1}{K_N} \max_{i=1,\ldots,N} |Z_i|_\infty^{1/i} \leq d_\infty(X,Y) \leq \frac{1}{K_N} \max_{i=1,\ldots,N} |Z_i|_\infty^{1/i}.
\]
Therefore, for all \(q \geq 2,\)
\[
\frac{1}{K_N} \max_{i=1,\ldots,N} \|Z_i\|_{L^qN(i,P)}^{1/i} \leq \|d_\infty(X,Y)\|_{L^qN(P)} \leq \frac{1}{K_N} \max_{i=1,\ldots,N} \|Z_i\|_{L^qN(i,P)}^{1/i}.
\]
From Lemma 3
\[
\|Z_i\|_{L^qN(i,P)}^{1/i} \leq M_N^{1/i} \sqrt{q} N/i - 1 \|Z_i\|_{L^2N(i,P)}^{1/i}
\]
\[
\leq \sqrt{q} M_N^{1/i} \|d_\infty(X,Y)\|_{L^2N(P)}
\]
where we used (2.5) with \(q = 2\) in the last line. Another look at (2.5) finishes the proof. \(\square\)

3. Large deviation results for \((\delta_\varepsilon X)_\varepsilon>0\)

From general principles, \((\varepsilon X)_\varepsilon>0\) satisfies a large deviation principle with good rate function \(I\) in uniform topology, where \(I\) is given by (see [14] for example):
\[
I(y) = \begin{cases} \frac{1}{2} \|x\|_{\mathcal{H}}^2 & \text{when } y = i(x) \text{ for some } x \in \mathcal{H}, \\ +\infty & \text{otherwise.} \end{cases}
\]
It is clear that
\[ S_N \circ \Phi_m : (C([0, 1], \mathbb{R}^d), |\cdot|_\infty) \to C([0, 1], G^N(\mathbb{R}^d), d_\infty) \]
is continuous. By the contraction principle [7], \( S_N(\varepsilon \Phi_m(X)) \) satisfies a large deviation principle with good rate function
\[ J_m(y) = \inf \{ I(x), x \text{ such that } S_N(\Phi_m(x)) = y \}, \]
the infimum of the empty set being \(+\infty\). Essentially, a large deviation principle for \( \delta_\varepsilon X \) is obtained by sending \( m \) to infinity. To this end we need

**Lemma 4.** Let \( \delta > 0 \) fixed. Then
\[
\lim_{m \to \infty} \lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(d_\infty(S_N(\Phi_m(\varepsilon X)), \delta_\varepsilon X) > \delta) = -\infty.
\]

**Proof.** First observe that
\[ d_\infty(S_N(\Phi_m(\varepsilon X)), \delta_\varepsilon X) = \varepsilon d_\infty(S_N(\Phi_m(X)), X). \]

By standing assumption on \( X \) and Lemma 1 we have
\[ \alpha_m := \| d_\infty(S_N(\Phi_m(X)), X) \|_{L^2} \overset{m \to \infty}{\to} 0. \]

Then, by Corollary 1,
\[ \| d_\infty(S_N(\Phi_m(X)), X) \|_{L^q} \leq M_N^{1/2} \alpha_m \quad \forall q \geq 2N. \quad (3.1) \]

We then estimate
\[
\mathbb{P}(d_\infty(S_N(\Phi_m(\varepsilon X)), \delta_\varepsilon X) > \delta) = \mathbb{P}(d_\infty(S_N(\Phi_m(X)), X) > \frac{\delta}{\varepsilon})
\]
\[
\leq \left( \frac{\delta}{\varepsilon} \right)^{-q} \sqrt{q} \alpha_m^q
\]
\[
\leq \exp \left[ q \log \left( \frac{\varepsilon}{\delta} \alpha_m \sqrt{q} \right) \right],
\]
and after choosing \( q = 1/\varepsilon^2 \) we obtain, for \( \varepsilon \) small enough,
\[
\varepsilon^2 \log \mathbb{P} \left( \sum_{t \in D} d(S_N(\Phi_m(\varepsilon X)_t), \delta_\varepsilon X_t) > \delta \right) \leq \log \left( \frac{\alpha_m}{\delta} \right).
\]

Now take the limits \( \lim_{\varepsilon \to 0} \) and \( \lim_{m \to \infty} \) to finish the proof. \( \square \)

The following theorem is a straightforward application of the extended contraction principle [7, Theorem 4.2.23] and Lemma 4. The point is that although \( S_N \) is a only a measurable map, defined as a.s. limit, it has exponentially good approximations given by \( \{ S_N \circ \Phi_m : m \geq 1 \} \).

**Theorem 1.** Assume that \( S_N(h), \) defined as the pointwise limit of \( (S_N \circ \Phi_m)(h) \), when \( m \to \infty \) exists for \( h \) such that \( I(h) < \infty \) (i.e. all \( h \) in the Cameron–Martin space), and that for all \( \Lambda > 0 \),
\[
\lim_{m \to \infty} \sup_{h : I(h) \leq \Lambda} d_\infty([S_N \circ \Phi_m](h), S_N(h)) = 0. \quad (3.2)
\]

Then the family \( (\delta_\varepsilon X)_{\varepsilon > 0} \) satisfies a large deviation principle in uniform topology with good rate function
\[ J(y) = \inf \{ I(x), x \text{ such that } S_N(x) = y \}. \]

The following corollary allows us to strengthen the topology.
Corollary 2. Under the above assumptions and the additional condition
\[ \exists \alpha' > 0, \, c > 0: \mathbb{E}(\exp c \|X\|_{\alpha'-\text{Hölder}}^2) < \infty, \]
the family \((\delta_{\epsilon}X)_{\epsilon > 0}\) satisfies a large deviation principle in \(\alpha\)-Hölder topology for all \(\alpha \in [0, \alpha']\) with good rate function \(J(y)\).

Proof. By the inverse contraction principle [7, Theorem 4.2.4.] it suffices to show that the laws of \((\delta_{\epsilon}X)\) are exponentially tight in \(\alpha\)-Hölder topology. But this follows from the compact embedding of
\[ C^{\alpha'}(0, 1) \subset C^{\alpha}(0, 1), \]
and Gauss tails of \(\|X\|_{\alpha'-\text{Hölder}}\). Indeed, for every \(M > 0\),
\[ K_M = \{ x : \|x\|_{\alpha'-\text{Hölder}} \leq M \} \]
defines a precompact set w.r.t. \(\alpha\)-Hölder topology. Then
\[ e^{-2 \log (\delta_{\epsilon}X)_{\epsilon} \mathbb{P}}(K_M) = e^{-2 \log \mathbb{P}[\|\delta_{\epsilon}X\|_{\alpha'-\text{Hölder}} > M]} \]
\[ = e^{-2 \log \mathbb{P}[\|X\|_{\alpha'-\text{Hölder}} > M/\epsilon]} \]
\[ \leq -M^2. \]

Remark 3. The same proof applies to the (weaker) \(p\)-variation topology and the (stronger) modulus topology considered in [10].

We emphasise the generality of this approach: a large deviation result for an enhanced Gaussian process holds provided (a) the enhanced Gaussian process exists and (b) a condition on the reproducing Kernel of the Gaussian process.

As a warm-up we show how to apply this to enhanced Brownian motion. We will then discuss a wider class of Gaussian process, which contains fractional Brownian motion with Hurst parameter \(H > \frac{1}{4}\).

4. LDP for enhanced Brownian motion

Let \(X = B\) be \(d\)-dimensional Brownian motion. We take \(\Phi_m\) be the piecewise linear approximation over at the dyadic dissection of \([0, 1]\) with mesh \(2^{-m}\). Then \(S_2(\Phi_m(B))\) converges uniformly on \([0, 1]\) and in probability to a process \(B\), see [17,9,10]). Large deviations for \((\delta_{\epsilon}B)_{\epsilon}\) were first obtained in \(p\)-variation topology [15], then in Hölder and modulus topology [10]. The following proof is considerably quicker. There are only two things to check.

1. Gauss tails of \(\|B\|_{\alpha-\text{Hölder}}\) for any \(\alpha \in [0, 1/2]\). There are many different ways to see this. In [10] we use the scaling \(d(B_s, B_t) = |t - s|^{1/2} \|B_{0,1}\|\), noting that \(\|B_{0,1}\|\) has Gauss tails, followed by an application of the Garsia–Rodemich–Rumsey lemma. One could prove the result directly by extending Lemma 1 to \(\alpha\)-Hölder metric.

2. Condition (3.2). But this is a simple consequence of the fact that the 1-variation norm of \(h\) is bounded by \(\sqrt{2T(h)}\) and that the 1-variation of \(\Phi_m(h)\) is bounded by the 1-variation of \(h\). From Theorem 1 and Corollary 2 we deduce

Theorem 2. Let \(B = B(\omega) \in C([0, 1], G^2(\mathbb{R}^d))\) denote enhanced Brownian Motion. Let \(\alpha \in [0, 1/2]\). Then \((\delta_{\epsilon}B)_{\epsilon > 0}\) satisfies a large deviation principle in \(\alpha\)-Hölder topology with good rate function
\[ J(y) = \inf \{ I(x), \, x \text{ such that } S_N(x) = y \}. \]

When restricting \(\alpha\) to \(\alpha \in (1/3, 1/2)\) the universal limit theorem gives the Freidlin–Wentzell theorem (in \(\alpha\)-Hölder topology) as corollary. See [15,10,4] for details on this by now classical application of Lyons’ limit theorem.

Remark 4. The case of enhanced fractional Brownian motion, \(H > \frac{1}{4}\), is similar. First, Gauss tails of the \(\alpha\)-Hölder norm, for any \(\alpha \in [0, H]\), follow from scaling just as above. (This has already been used in [11,4].) Secondly, condition (3.2) follows from the fact that the Cameron–Martin space for fBM is embedded in the space of finite \(q\)-variation for
any $q$ less than $1/(1/2 + H)$, see [12]. In any case, enhanced fractional Brownian motion is a special case of the processes considered in the next section.

5. LDP for a class of Gaussian processes

We fix a parameter $H > 0$. We now specialise to zero-mean $\mathbb{R}^d$-valued Gaussian processes $\{X_t: t \in [0, 1]\}$ with independent components $x = X^i, i = 1, \ldots, d$, satisfying

$$E(|x_s|^2) \leq c|t - s|^{2H}, \quad \text{for all } s < t, \quad (5.1)$$

$$|E(x_{s+h}, x_{t+h})| \leq c|t - s|^{2H} \frac{h}{|t - s|^q}, \quad \text{for all } s, t, h \text{ with } h < t - s \quad (5.2)$$

for some constant $c > 0$. Note that such a process has a.s. $1/p$-Hölder continuous sample paths when $p > 1/H$.

A particular example of such a Gaussian process is fractional Brownian motion of Hurst parameter $H$. This condition is the one which allowed Coutin and Qian [5] to construct the lift of $X$ to a rough path. As before $H$ denotes the Cameron–Martin space. We take $\Phi_m$ to be the piecewise linear approximation over at the dyadic dissections of $[0, 1]$ with mesh $2^{-m}$.

**Definition 1.** For $H > 1/4$, we define, according to [5], the enhanced Gaussian process $X$ to be the (a.s. and in $L^q$ for all $q$) limit in $1/p$-Hölder norm of $S[p](\Phi_m(X))$, where $1/p < H$.\footnote{The convergence in was proven in $p$-variation topology, but it is obvious when reading the proof that it also holds in Hölder norm.}

To obtain the large deviation principle for $(\delta_{\varepsilon}X)_{\varepsilon > 0}$ there are, as in the case of EBM, only two things to check.

1. Gauss tails of $\|X\|_{1/p}$-Hölder for $p > 1/H$, that is
   $$E[\exp(\alpha \|X\|_{1/p})] < \infty. \quad (5.3)$$

   First note that for all $q > 1$,
   $$\sup_{0 \leq s < t \leq 1} E\left[ \left( \frac{\|X_{s,t}\|}{|t - s|^H} \right)^q \right] < \infty.$$

   Similar to Corollary 1, for all $s, t$, and $k \geq 1$,
   $$E\left( \frac{\|X_{s,t}\|}{|t - s|^H}^{2k} \right)^{1/k} \leq C \sqrt{k} E\left( \frac{\|X_{s,t}\|}{|t - s|^H} \right)^{1/q},$$

   where $q$ is any fixed real greater than $2[1/H]$. Now expand the exponential in a power series to see that there exists $\alpha > 0$ s.t. $E[\exp(\alpha \|X_{s,t}\|^2/|t - s|^{2H})] < \infty$ for all $s, t$. The proof of 5.3 is then finished using the Garsia–Rodemich–Rumsey lemma, just as in [10].

2. Condition (3.2). We show (a) that for some $q \in [1, 2)$ the $q$-variation norm of a Cameron–Martin path is bounded by a constant times its Cameron–Martin norm and (b) a uniform modulus of continuity for elements in $\{h: I(h) \leq \Lambda\}$. A soft argument, spelled out in Theorem 3 below, will then imply (3.2).

We start by proving that the quadratic variation of Cameron–Martin paths converges exponentially fast to 0.

**Lemma 5.** Let $H \in (1/4, 1)$. Then there exists a constant $C_{H,c}$ such that for all $n \geq 0$ and all $h \in \mathcal{H}$

$$\sum_{k=0}^{2^n - 1} |h \frac{k}{2^n}, h \frac{k+1}{2^n}|^2 \leq C_{H,c} |h|^2 2^{n(1 - 4H)/2}.$$
Proof. Without loss of generalities, we assume \( d = 1 \). Write \( R(s, t) = \mathbb{E}(X_s X_t) \) for the covariance. The Cameron–Martin space is the closure of functions of form \( h = \sum_i a_i R(t_i, \cdot) \) with respect to the norm

\[
|h|_{\mathcal{H}} = \left| \sum_i a_i R(t_i, \cdot) \right|_{\mathcal{H}}^2 = \sum a_i a_j R(t_i, t_j) = \mathbb{E}(Z^2)
\]

where \( Z \) is the zero-mean Gaussian r.v. given by \( Z = \sum_i a_i X_{t_i} \). Observe that \( h_{s,t} = \mathbb{E}(Z X_{s,t}) \). We keep \( n \) fixed for the rest of the proof.

\[
Q_n := \sum_{k=0}^{2^n - 1} |h_{\frac{k}{2^n}, \frac{k+1}{2^n}}|^2 = \sum_{k=0}^{2^n - 1} \mathbb{E}(X_{\frac{k}{2^n}, \frac{k+1}{2^n}} Z)^2.
\]

Note that if \( U, V \) are two zero-mean Gaussian random variables, then

\[
\mathbb{E}(U^2 V^2) = 2\mathbb{E}(UV)^2 + \mathbb{E}(U^2)\mathbb{E}(V^2) \quad \text{or} \quad \text{Cov}(U^2, V^2) = 2\mathbb{E}(UV)^2.
\]

This allows us to write

\[
Q_n = \frac{1}{2} \sum_{k=0}^{2^n - 1} \mathbb{E}(Z^2 [X_{\frac{k}{2^n}, \frac{k+1}{2^n}}^2 - \mathbb{E}(X_{\frac{k}{2^n}, \frac{k+1}{2^n}}^2)])
\]

\[
= \frac{1}{2} \mathbb{E} \left( Z^2 \sum_{k=0}^{2^n - 1} [X_{\frac{k}{2^n}, \frac{k+1}{2^n}}^2 - \mathbb{E}(X_{\frac{k}{2^n}, \frac{k+1}{2^n}}^2)] \right)
\]

\[
\leq \frac{1}{2} \mathbb{E}(Z^4)^{1/2} \left( \sum_{k,l=0}^{2^n - 1} \text{Cov}(X_{\frac{k}{2^n}, \frac{k+1}{2^n}}, X_{\frac{l}{2^n}, \frac{l+1}{2^n}}) \right)^{1/2}
\]

\[
= \sqrt{\frac{3}{2}} \mathbb{E}(Z^2) \left( \sum_{k,l=0}^{2^n - 1} \mathbb{E}(X_{\frac{k}{2^n}, \frac{k+1}{2^n}} X_{\frac{l}{2^n}, \frac{l+1}{2^n}}) \right)^{1/2}
\]

where we used (5.4) again. The double sum in line just above splits naturally in a diagonal part, where \( k = l \), and twice the sum over \( k < l \). With (5.1) we control the diagonal part,

\[
\sum_{k=0}^{2^n - 1} \mathbb{E}(X_{\frac{k}{2^n}, \frac{k+1}{2^n}}^2) \leq c^2 2^{n(1-4H)}.
\]

The other part is estimated by using (5.2). Indeed,

\[
\sum_{0 \leq k < l \leq 2^n - 1} \mathbb{E}(X_{\frac{k}{2^n}, \frac{k+1}{2^n}} X_{\frac{l}{2^n}, \frac{l+1}{2^n}})^2 = \sum_{k=0}^{2^n - 1} \sum_{m=1}^{2^n - 1} \sum_{m=1}^{2^n - 1} \mathbb{E}(X_{\frac{k}{2^n}, \frac{k+1}{2^n}} X_{\frac{k+m}{2^n}, \frac{k+m+1}{2^n}}) \leq c^2 \sum_{k=0}^{2^n - 1} \sum_{m=1}^{2^n - 1} \left| \frac{m}{2^n} \right|^{4H-4} 2^{-4m}
\]

\[
\leq \left( \sum_{m=1}^{\infty} m^{4H-4} \right) c^2 2^{n(1-4H)}.
\]

Putting everything together yields a constant \( C = C(H, c) \) such that

\[
Q_n \leq C_H \mathbb{E}(Z^2) 2^{n(1-4H)/2} = C_{H,c} |h|_{\mathcal{H}} 2^{n(1-4H)/2}.
\]

This finishes the proof. \( \square \)

Corollary 3. Let \( H \in (1/4, 1/2] \) and \( q \in (1 \vee 1/(H + 1/4), 2) \). Then \( \mathcal{H} \) is continuously embedded in \( C_0^{q, \text{var}}([0, 1], \mathbb{R}^d) \). More precisely, there exists a constant \( C_{H,q,c} \) such that for all \( h \in \mathcal{H} \),

\[
|h|_{q, \text{var}} \leq C_{H,q,c} |h|_{\mathcal{H}}.
\]
Proof. We use the inequality
\[
|h|_{q-\text{var}}^q \leq C_{\gamma,q} \sum_{n=1}^{\infty} n^\gamma \sum_{k=0}^{2^n-1} |h_{k,\frac{k+1}{2^n}}|^q,
\]
see [17] for example. From the previous lemma and Hölder’s inequality,
\[
\sum_{k=0}^{2^n-1} |h_{k,\frac{k+1}{2^n}}|^q \leq \left( \sum_{k=0}^{2^n-1} \frac{1}{n} \right)^{q/2} 2^{n(1-q/2)} \leq C |h|_{\frac{q}{2},t}^q 2^{n(1-4H)^q q + 1 - \frac{q}{2}}.
\]
Observe
\[
(1 - 4H)^q q + 1 - \frac{q}{2} < 0 \iff q \left( H + \frac{1}{4} \right) > 1.
\]
Hence, for every \( q \geq 1 \) such that \( q > 1/(H + 1/4) \), we have
\[
|h|_{q-\text{var}} \leq C_{H,q,c} |h|_H. \quad \square
\]

This result is not optimal. Cameron–Martin paths associated to usual Brownian motion are of bounded variation (≡ finite 1-variation) whereas the above corollary applied to \( H = 1/2 \) only gives finite \( q \)-variation for \( q > 4/3 \). In [12], we prove that Cameron–Martin paths associated to fractional Brownian motion are of finite \( q \)-variation for any \( q \) greater than \( 1 \lor 1/(H + 1/2) \).

Lemma 6. Let \( H \in (0, 1) \). Then \( \mathcal{H} \) is continuously embedded in \( C_0^{H-\text{Hölder}}([0, 1]) \). More precisely, there exists a constant \( C_{H,c} \) such that for all \( h \in \mathcal{H} \),
\[
|h|_{H-\text{Hölder}} \leq C_{H,c} |h|_H.
\]

Proof. We use the notation of the above proofs. In particular, \( h_{s,t} = \mathbb{E}(Z \cdot X_{s,t}) \). By Cauchy–Schwarz and (5.1),
\[
|h_{s,t}| \leq \sqrt{\mathbb{E}(Z^2)} \sqrt{\mathbb{E}(X_{s,t}^2)} \leq C |h|_H |t - s|^H. \quad \square
\]

Theorem 3. For any fixed \( N \geq 1 \), for all \( \Lambda > 0 \),
\[
\lim_{m \to \infty} \sup_{|h|: I(h) \leq \Lambda} d_\infty [(S_N \circ \Phi_m)(h), S_N(h)] = 0.
\]

Proof. Observe first that for \( h \) such that \( I(h) \leq \Lambda \), we have
\[
|\Phi_m(h) - h|_\infty \leq \sup_{|t-s| \leq 2^{-m}} |h_{s,t}| \leq \Lambda^{1/2} C_H 2^{-mH}.
\]
Observe also that for \( q \geq 1/(1/4 + H) \),
\[
|\Phi_m(h)|_{q-\text{var}} \leq 3 |h|_{q-\text{var}} \leq 3 C_{q,H} A^{1/2}.
\]
By interpolation, for \( q' > q \), we therefore obtain that
\[
\lim_{m \to \infty} \sup_{|h|: I(h) \leq \Lambda} |\Phi_m(h) - h|_{q'-\text{var}} = 0.
\]
Then using Theorem 2.2.2 in [16] (or just the continuity of the Young integral), we obtain our theorem. \( \square \)

We have therefore the following:
Corollary 4. Assume that for some $H \in (1/4, 1)$, the Gaussian process satisfies condition (5.1) and (5.2) and let $X$ be as in Definition 1. Then, $(\delta_{\varepsilon}X)$ satisfies a large deviation principle in $1/p$-Hölder topology where $p > 1/H$ with good rate function

$$J(y) = \inf \left\{ \frac{1}{2} |x|_{H}, \ x \text{ such that } S[p](x) = y \right\}.$$

Once again, applying the Itô map to $\delta_{\varepsilon}X$, we can obtain a generalisation of the Freidlin–Wentzell theorem, i.e. a large deviation principle for solution of stochastic differential equation driven by the enhanced Gaussian process $X$.

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References


Further reading