Heat flow, Brownian motion and Newtonian capacity

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Abstract

Let $K$ be a compact, non-polar set in $\mathbb{R}^m (m \geq 3)$ and let $u$ be the unique weak solution of $\Delta u = \frac{\partial u}{\partial t}$ on $\mathbb{R}^m \setminus K \times (0, \infty)$, $u(x; 0) = 0$ on $\mathbb{R}^m \setminus K$ and $u(x; t) = 1$ for all $x$ on the boundary of $K$ and for all $t > 0$. The asymptotic behaviour of $u(x; t)$ as $t$ tends to infinity is obtained up to order $O(t^{-m/2})$.

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1. Introduction

Let $K$ be a compact, non-polar set in Euclidean space $\mathbb{R}^m (m \geq 3)$ with boundary $\partial K$ and let $u : \mathbb{R}^m \setminus K \times [0, \infty) \to \mathbb{R}$ be the unique weak solution of

$$
\Delta u = \frac{\partial u}{\partial t}, \quad x \in \mathbb{R}^m \setminus K, \quad t > 0,
$$

with boundary condition

$$
u(x; t) = 1, \quad x \in \partial K, \quad t > 0,
$$

and initial condition

$$u(x; 0) = 0, \quad x \in \mathbb{R}^m \setminus K.
$$

It is well known that

$$
\lim_{t \to \infty} u(x; t) = h_K(x), \quad x \in \mathbb{R}^m \setminus K,
$$

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where \( h_K \) is the unique function which is harmonic on \( \mathbb{R}^m \setminus K \), which equals 1 on the regular points of \( K \), and which vanishes at infinity.

S.C. Port [8], [10, pp. 64, 65] proved that if \( K \) is a compact and non-polar set in \( \mathbb{R}^m (m \geq 3) \) then for \( t \to \infty \)

\[
  u(x; t) = h_K(x) - \left( \frac{m}{2} - 1 \right)^{-1}(4\pi)^{-m/2}C(K)(1 - h_K(x))t^{(2-m)/2} + o(t^{(2-m)/2}),
\]

(5)

where \( C(K) \) is the Newtonian capacity of \( K \).

Formula (5) was first proved by A. Joffe [7] in the special case where \( m = 3 \) and where \( K \) has positive Lebesgue measure \( |K| \). Subsequently F. Spitzer [12, p. 114] proved formula (5) for arbitrary compact, non-polar sets in \( \mathbb{R}^3 \) and obtained the asymptotic behaviour of the total amount of heat \( E_K(t) \) in \( \mathbb{R}^m \setminus K \) at time \( t \) defined by

\[
  E_K(t) = \int_{\mathbb{R}^m \setminus K} u(x; t) \, dx.
\]

He showed that for \( m = 3 \) and \( t \to \infty \)

\[
  E_K(t) = C(K)t + \frac{1}{2\pi^{3/2}}C(K)^2t^{1/2} + o(t^{1/2}).
\]

(7)

J.-F. Le Gall [4–6] and Port [11] obtained refinements of (7) and extensions to \( m \geq 4 \) and \( m = 2 \) without the use of (5). Port also obtained the large \( t \) behaviour of \( u \) in the case where \( K \) is a non-polar compact set in \( \mathbb{R}^2 \) [9].

The main result of this paper concerns the analysis of the remainder estimate \( o(t^{(2-m)/2}) \) in (5). For \( m \geq 5 \) we show that this remainder can be improved to \( O(t^{-m/2}) \). A new term of order \( (\log t)/t^2 \) shows up for \( m = 4 \) before we recover the remainder \( O(t^{-2}) \). A remarkable cancellation of two terms of order \( t^{-1} \) and four terms of order \( (\log t)/t^{5/2} \) takes place for \( m = 3 \), resulting in the sharp remainder \( O(t^{-3/2}) \).

**Theorem 1.** Let \( K \) be a compact and non-polar set in \( \mathbb{R}^m \).

(i) If \( m = 3, 5, 6, \ldots \) then for \( x \in \mathbb{R}^m \setminus K \) and \( t \to \infty \)

\[
  u(x; t) = h_K(x) - \left( \frac{m}{2} - 1 \right)^{-1}(4\pi)^{-m/2}C(K)(1 - h_K(x))t^{(2-m)/2} + O(t^{-m/2}).
\]

(8)

(ii) If \( m = 4 \) then for \( x \in \mathbb{R}^4 \setminus K \) and \( t \to \infty \)

\[
  u(x; t) = h_K(x) - (4\pi)^{-2}C(K)(1 - h_K(x))t^{-1} + 2(4\pi)^{-4}C(K)^2(1 - h_K(x)) \frac{\log t}{t^2} + O(t^{-2}).
\]

(9)

(iii) The remainder in (8) is sharp for a ball in \( \mathbb{R}^3 \).

(iv) The remainder \( O(t^{-m/2}) \) in (8) and (9) is uniform in \( x \) on compact subsets of \( \mathbb{R}^m \setminus K \).

The results described in Theorem 1 have an equivalent probabilistic formulation. Let \( (B(s), s \geq 0; \mathbb{P}_x, x \in \mathbb{R}^m) \) be a Brownian motion with generator \( \Delta \). For \( x \in \mathbb{R}^m \) we define the first hitting time of \( K \) by

\[
  T_K = \inf\{s \geq 0: B(s) \in K\},
\]

(10)

and \( T_K = +\infty \) if the infimum is taken over the empty set. It is a classical result that \( u(x; t) = \mathbb{P}_x[T_K < t], \quad x \in \mathbb{R}^m, \ t > 0, \)

(11)

where we have extended both \( u \) and \( h_K \) to all of \( \mathbb{R}^m \) by putting \( u \equiv h_K \equiv 1 \) on \( K \). For \( x \in \mathbb{R}^m (m \geq 3) \) we define the last exit time of \( K \) by

\[
  L_K = \sup\{s \geq 0: B(s) \in K\},
\]

(12)

and \( L_K = +\infty \) if the supremum is taken over the empty set. The law of \( L_K \) is given by [10, p. 61]

\[
  \mathbb{P}_x[L_K < t] = \int_0^t ds \int \mu_K(dy) p(x, y; s),
\]

(13)
where
\[ p(x, y; s) = (4\pi s)^{-m/2} e^{-|x-y|^2/(4s)}, \] (14)
and where \( \mu_K \) is the equilibrium measure supported on \( K \) with
\[ \int \mu_K(\text{dy}) = C(K). \] (15)
It follows that
\[ h_K(x) = \mathbb{P}_x[T_K < \infty] = \mathbb{P}_x[L_K < \infty] = c_m \int \mu_K(\text{dy}) |x-y|^2 \] (16)
where
\[ c_m = 4^{-1} \pi^{-m/2} \Gamma((m-2)/2). \] (17)
Since
\[ \mathbb{P}_x[T < L_K < \infty] = \int_t^\infty ds \int \mu_K(\text{dy}) p(x, y; s), \] (18)
and
\[ (4\pi s)^{-m/2}(1 - |x-y|^2/(4s)) \leq p(x, y; s) \leq (4\pi s)^{-m/2}, \] (19)
we have that
\[ \mathbb{P}_x[T < L_K < \infty] = \left( \frac{m}{2} - 1 \right)^{-1} (4\pi)^{-m/2} C(K)t^{(2-m)/2} + O(t^{-m/2}). \] (20)
Using (11), (16) and (20) we can rewrite (8), (9) as follows.

**Proposition 2.** Let \( K \) be a compact and non-polar set in \( \mathbb{R}^m \).

(i) If \( m = 3, 5, 6, \ldots \) then for \( x \in \mathbb{R}^m \setminus K \) and \( t \to \infty \)
\[ \mathbb{P}_x[T_K < \infty] = \mathbb{P}_x[L_K < \infty] = c_m \int \mu_K(\text{dy}) |x-y|^2 \] (21)

(ii) If \( m = 4 \) then for \( x \in \mathbb{R}^4 \setminus K \) and \( t \to \infty \)
\[ \mathbb{P}_x[T_K < \infty] = \mathbb{P}_x'[T_K = \infty] \mathbb{P}_x[t < L_K < \infty] - 2(4\pi)^{-4} C(K)^2 \mathbb{P}_x'[T_K = \infty] \left( \log t \right) t^{-2} + O(t^{-2}). \] (22)

It is well known [4, p. 392] that if \( m = 3 \) and \( K = B(0; R) \) (the closed ball with center 0 and radius \( R \)) then for \( |x| \geq R \)
\[ \mathbb{P}_x[T < T_B(0; R) < \infty] = \int_{t}^\infty ds \left( 4\pi s^3 \right)^{-1/2} \frac{R(|x| - R)}{|x|} e^{-((|x|-R)^2)/(4s)}. \] (23)

Moreover for a ball \( B(0; R) \) in \( \mathbb{R}^3 \) the corresponding equilibrium measure is concentrated on \( \partial B(0; R) \) and proportional to the surface measure, with constant of proportionality equal to \( R^{-1} \). This gives by (18)
\[ \mathbb{P}_x[T_B(0; R) = \infty] = \left( \frac{|x| - R}{|x|} \right), \] (24)
and
\[ \mathbb{P}_x[T < L_B(0; R) < \infty] = \int_{t}^\infty ds \left( 4\pi s^3 \right)^{-1/2} |x|^{-1} \left( 1 - e^{-|x| R/s} \right) e^{-((|x|-R)^2)/(4s)}. \] (25)
It is a straightforward computation to show that, by (23)–(25), for $m = 3$
\[
\mathbb{P}_x [t < T_{B(0;R)} < \infty] = \mathbb{P}_x [T_{B(0;R)} = \infty]\mathbb{P}_x [t < L_{B(0;R)} < \infty] + \frac{1}{6\pi^{1/2}} \mathbb{P}_x [T_{B(0;R)} = \infty]|x|R^2 t^{-3/2} + O(t^{-5/2}).
\] (26)
This proves the assertion in Theorem 1(iii).

The main stratagem which permeates the proof of Proposition 2 is to replace $T_K$ by $L_K$ at “every possible opportunity” and to use the strong Markov property to control terms like $\mathbb{P}_x [T_K < t < L_K]$. For a different application of these techniques we refer to the study of the expected volume of a Wiener sausage in $\mathbb{R}^3$ associated to the compact set $K$ [4]. There Spitzer’s formula (7) was improved up to order $O(t^{-1/2})$ proving a conjecture by M. Kac. See [1–3,13] for more recent applications.

It turns out that a single application of the strong Markov property (Proposition 4) supplemented by additional estimates (Lemma 3) is sufficient to prove Proposition 2 for $m \geq 5$. However, for $m = 4$ or $m = 3$ the strong Markov property has to be applied twice respectively six times (Propositions 5 and 8). The reason is that for $m = 3$ two non-trivial terms of order $t^{-1}$ and four non-trivial terms of order $(\log t)/t^{3/2}$ contribute to $\mathbb{P}_x [t < T_K < \infty]$. Lengthy calculations using the above techniques finally result in the cancellation of these non-trivial terms. Such a cancellation does not take place for $m = 4$, and this results in the $(\log t)/t^2$ contribution in (9).

The analysis of the $O(t^{-m/2})$ remainder in Proposition 2 is complicated since the distribution of the random variable $B(T_K)$ on the regular part of $\partial K$ enters at each application of the strong Markov property. Unlike the special case of a ball in $\mathbb{R}^3$ we do not expect a simple improvement of the remainder.

This paper is organized as follows. In Section 2 we prove some basic estimates (Lemma 3) which will be used throughout the paper. Proposition 4 is the key estimate from which Proposition 2 follows for $m \geq 5$. In Section 3 we use Proposition 4 to obtain a further refinement (Proposition 5) from which Proposition 2 follows for $m = 4$. Finally in Section 4 we complete the proof of Proposition 2 for $m = 3$ by refining Proposition 5 (Proposition 8). The proof of Proposition 8 follows the same strategy as the proof of Proposition 5, and has been omitted.

2. Proof of Proposition 2 for $m \geq 5$

It is convenient to introduce some further notation. For $c \in \mathbb{R}^m$ and $K$ compact in $\mathbb{R}^m$ we define
\[
R(c) = \inf \{ \rho > 0 : K \subset B(c; \rho) \},
\] (27)
where $B(c; \rho)$ is the closed ball with center $c$ and radius $\rho$. Let
\[
R = \inf \{ R(c) : c \in \mathbb{R}^m \}.
\] (28)
The infima in (27) and (28) are attained and we may assume without loss of generality that the latter is attained at $c = 0$.

**Lemma 3.** Let $K$ be a compact and non-polar set in $\mathbb{R}^m (m \geq 3)$. Then for $0 < s < t < \infty$
\[
\mathbb{P}_x [t < T_K < \infty] \leq \mathbb{P}_x [t < L_K < \infty] \leq 1 \wedge \left( \frac{m}{2} - 1 \right)^{-1} (4\pi)^{-m/2} C(K) t^{(2-m)/2},
\] (29)
\[
\mathbb{P}_x [s < L_K < t] \leq 1 \wedge \left( \frac{m}{2} - 1 \right)^{-1} (4\pi)^{-m/2} C(K) (s^{(2-m)/2} - t^{(2-m)/2}),
\] (30)
and for $z \in K$
\[
|\mathbb{P}_z [t < L_K < \infty] - \mathbb{P}_z [t < L_K < \infty]| \leq 1 \wedge C_{x,K} t^{-m/2},
\] (31)
where
\[
C_{x,K} = (|x| + R)(|x| + 3R) C(K).
\] (32)
For any Borel set $E$ of $[0, t]$
\[
\int_E ds \int \mu_K(dy) p(x, y; t-s) \leq 1.
\] (33)

Let $T > 0$ be arbitrary. There exists a constant $C$ depending on $T$ and on $K$ such that for all $t > T$, $0 < s < t$ and $x \in \mathbb{R}^m$
\[
\mathbb{P}_x[s < T_K < t] \leq C(T(t-T)^{-m/2} \vee (t-s)s^{-m/2}).
\] (34)

**Proof.** Estimate (29) follows immediately from the fact that $L_K \geq T_K$ and (18), (19).

Estimate (30) follows from
\[
\mathbb{P}_x[s < L_K < t] = \int_s^t d\tau \int \mu_K(dy)p(x, y; \tau),
\] (35)
and the bound in the right-hand side of (19).

To prove (31) we note that by (18)
\[
\big| \mathbb{P}_x[t < L_K < \infty] - \mathbb{P}_z[t < L_K < \infty] \big| \leq \int_{s=0}^{\infty} ds \int (4\pi s)^{-m/2} \mu_K(dy) \left( e^{-|x-y|^2/(4s)} - e^{-|z-y|^2/(4s)} \right)
\]
\[
\leq \int_{s=0}^{\infty} ds \int (4\pi s)^{-m/2}(4s)^{-1} \mu_K(dy) \left( |x-y|^2 - |z-y|^2 \right)
\]
\[
\leq t^{-m/2} \int \mu_K(dy) \left( |x| + |z| \right) \left( |x| + |z| + 2|y| \right)
\]
\[
\leq C_{x,K} t^{-m/2}
\] (36)
since both $y$ and $z \in K \subset B(0; R)$.

Since $p$ is non-negative
\[
\int_E ds \int \mu_K(dy) p(x, y; t-s) \leq \int_{[0,t]} ds \int \mu_K(dy) p(x, y; t-s)
\]
\[
= \mathbb{P}_x[L_K < t] \leq 1.
\] (37)

This proves (33).

The proof of (34) relies on the following [4,11,12]. For $m \geq 3$
\[
\int_{\mathbb{R}^m} dy \mathbb{P}_y[T_K < t] = C(K)t + o(t), \quad t \to \infty.
\] (38)

Hence there exists $T_1$ such that for all $t \geq T_1$
\[
\int_{\mathbb{R}^m} dy \mathbb{P}_y[T_K < t] \leq 2C(K)t.
\] (39)

By the Markov property at time $s$ we have that
\[
\mathbb{P}_x[s < T_K < t] = \int_{\mathbb{R}^m} dy \, p_{\mathbb{R}^m \setminus K}(x, y; s) \mathbb{P}_y[T_K < t-s],
\] (40)

where $p_{\mathbb{R}^m \setminus K}(\cdot, \cdot; \cdot)$ is the Dirichlet heat kernel for the open set $\mathbb{R}^m \setminus K$ (i.e. the transition density of Brownian motion with killing on $K$). By domain monotonicity of the Dirichlet heat kernel
\[
p_{\mathbb{R}^m \setminus K}(x, y; s) \leq p(x, y; s) \leq (4\pi s)^{-m/2}.
\] (41)
We first consider the case $t - s > T_1$. Then by (39)–(41)

$$
P_x[s < T < t] \leq (4\pi s)^{-m/2} \int_{\mathbb{R}^m} dy \mathbb{P}_y[T < t - s]$$

$$\leq 2(4\pi s)^{-m/2} C(K)(t - s). \tag{42}$$

Next suppose that $T < T_1$ and $t - s \in [T, T_1]$. Then by monotonicity

$$\int_{\mathbb{R}^m} dy \mathbb{P}_y[T < t - s] \leq \int_{\mathbb{R}^m} dy \mathbb{P}_y[T < T_1]$$

$$\leq 2C(K)T_1 \leq 2C(K) \frac{T_1}{T}(t - s), \tag{43}$$

and

$$P_x[s < T < t] \leq 2(4\pi s)^{-m/2} C(K) \frac{T_1}{T}(t - s). \tag{44}$$

Combining (42) and (44) we obtain that

$$P_x[s < T < t] \leq Cs^{-m/2} \left(1 \vee \frac{T_1}{T}\right), \tag{45}$$

with $C$ given by

$$C = 2(4\pi)^{-m/2} C(K) \left(1 \vee \frac{T_1}{T}\right). \tag{46}$$

By (45)

$$P_x[s < T < t] \leq \mathbb{P}_x[t - T < T < t] \leq CT(t - T)^{-m/2}, \quad t - s \leq T, \tag{47}$$

and (34) follows from (45)–(47). \quad \square

**Proposition 4.** Let $K$ be a compact and non-polar set in $\mathbb{R}^m (m \geq 3)$. Then for $t \to \infty$

$$P_x[t < T < \infty] = P_x[T = \infty] P_x[t < L_K < \infty] + P_x[t < T < \infty] P_x[t < L_K < \infty]$$

$$- \int_0^t ds \mathbb{P}_x[s < T < t] \int_{\mu_K(dy)p(x, y; t - s)} + O(t^{-m/2}). \tag{48}$$

**Proof.** Note that

$$P_x[t < T < \infty] = P_x[t < L_K < \infty] - P_x[T_K < t < L_K]. \tag{49}$$

By the strong Markov property

$$P_x[T_K < t < L_K] = E_x \left\{ \int_0^t 1_{T_K < s} \mathbb{P}_{B(T_K)}[t - s < L_K < \infty] \right\}. \tag{50}$$

Using Lemma 3, (31) with $z = B(T_K)$

$$|\mathbb{P}_{B(T_K)}[t - s < L_K < \infty] - P_x[t - s < L_K < \infty]| \leq 1 \vee C_{x,K}(t - s)^{-m/2}. \tag{51}$$

If we can show that

$$E_x \left\{ \int_0^t 1_{T_K < s} \left(1 \vee C_{x,K}(t - s)^{-m/2}\right) \right\} = O(t^{-m/2}), \tag{52}$$

then, by (50)–(52),
\[ \mathbb{P}_x[T_K < t < L_K] = E_x \left\{ \int_0^t 1_{T_K \in ds} \mathbb{P}_x[t - s < L_K < \infty] \right\} + O(t^{-m/2}) \]
\[ = \int_0^t ds \frac{d}{ds} \left( \mathbb{P}_x[T_K < s] - \mathbb{P}_x[T_K < t] \right) \mathbb{P}_x[t - s < L_K < \infty] + O(t^{-m/2}) \]
\[ = \mathbb{P}_x[T_K < t] \mathbb{P}_x[t < L_K < \infty] \]
\[ + \int_0^t ds \mathbb{P}_x[s < T_K < t] \frac{d}{ds} \mathbb{P}_x[t - s < L_K < \infty] + O(t^{-m/2}). \]

This implies Proposition 4 since, by (18),
\[ \frac{d}{ds} \mathbb{P}_x[t - s < L_K < \infty] = \int \mu_K(dy) p(x, y; t - s). \]

To prove (52) we note that
\[ E_x \left\{ \int_0^t T_K \in ds \right\} \]
\[ = \int_0^t ds \frac{d}{ds} \left( \mathbb{P}_x[T_K < s] - \mathbb{P}_x[T_K < t] \right) \mathbb{P}_x[t - s < L_K < \infty] \]
\[ = \mathbb{P}_x[T_K < t] \mathbb{P}_x[t < L_K < \infty] \]
\[ + \int_0^t ds \mathbb{P}_x[s < T_K < t] \frac{d}{ds} \mathbb{P}_x[t - s < L_K < \infty] + O(t^{-m/2}). \]

where
\[ t^* = (t - T) \lor 0, \]
and
\[ T = C_{x,K}^{2/m}. \]

The first term in the right-hand side of (55) is \( O(t^{-m/2}) \). To estimate the second term in the right-hand side of (55) we suppose that \( t > T \) and use Lemma 3 with \( T = C_{x,K}^{2/m} \) to obtain that
\[ \int_0^{t^*} ds \mathbb{P}_x[s < T_K < t](t - s)^{-m/2} \]
\[ \leq \int_0^{(t - T)/2} ds (t - s)^{-m/2} + \int_{(t - T)/2}^{t - T} ds C s^{-m/2} (t - s)^{-m/2} \]
\[ \leq \left( (t + T)/2 \right)^{-m/2} + C \left( (t - T)/2 \right)^{-m/2} \int_0^{t - T} ds (t - s)^{-m/2} = O(t^{-m/2}). \]

We conclude this section with the proof of Proposition 2 for \( m \geq 5 \). By Lemma 3, (29), the second term in the right-hand side of (48) is \( O(t^{2-m}) \) and hence is \( O(t^{-m/2}) \) for \( m \geq 4 \). By (19) and (15) we have for any \( z \in \mathbb{R}^m \)
\[ \int \mu_K(dy) p(z, y; t - s) \leq C(K) t^{-m/2}, \quad s \in [0, t/2]. \]
Hence, by (29) and (59), we have for \( m \geq 5 \)
\[
\int_0^{t/2} \text{d}s \; \mathbb{P}_x[ s < T_K < t ] \int \mu_K(\text{d}y) p(x, y; t - s) \leq C(K) t^{-m/2} \int_0^{t/2} \text{d}s \left( 1 \wedge C(K) s^{1-m/2} \right) = O(t^{-m/2}). \tag{60}
\]
By (34) we have for \( m \geq 5 \)
\[
\int_{t/2}^{t-T} \text{d}s \; \mathbb{P}_x[ s < T_K < t ] \int \mu_K(\text{d}y) p(x, y; t - s) \leq C \int_{t/2}^{t-T} \text{d}s s^{-m/2} C(K) (t - s)^{1-m/2} = O(t^{-m/2}). \tag{61}
\]
By (34) and (33) for \( E = [t - T, t] \) we have
\[
\int_{t-T}^{t} \text{d}s \; \mathbb{P}_x[ s < T_K < t ] \int \mu_K(\text{d}y) p(x, y; t - s) \leq C T(t - T)^{-m/2} = O(t^{-m/2}). \tag{62}
\]
By (60)–(62) and Proposition 4 we conclude that (21) holds for \( m \geq 5 \). \( \square \)

3. Proof of Proposition 2 for \( m = 4 \)

The proof of Proposition 2 for \( m = 4 \) and \( m = 3 \) relies on the asymptotic analysis of the third term in the right-hand side of (48).

**Proposition 5.** Let \( K \) be a compact and non-polar set in \( \mathbb{R}^m (m \geq 3) \). Then for \( t \rightarrow \infty \)
\[
\int_0^{t} \text{d}s \; \mathbb{P}_x[ s < T_K < t ] \int \mu_K(\text{d}y) p(x, y; t - s)
= \mathbb{P}_x[ T_K = \infty ] \int_0^{t} \text{d}s \; \mathbb{P}_x[ s < L_K < t ] \int \mu_K(\text{d}y) p(x, y; t - s) + \sum_{i=1}^{4} A_i + O(t^{-m/2}), \tag{63}
\]
where
\[
A_1 = \int_0^{t} \text{d}s \; \mathbb{P}_x[ s < T_K < t ] \mathbb{P}_x[ t - s < L_K < \infty ] \int \mu_K(\text{d}y) p(x, y; t - s), \tag{64}
\]
\[
A_2 = \int_0^{t} \text{d}s \; \mathbb{P}_x[ s < T_K < \infty ] \mathbb{P}_x[ s < L_K < t ] \int \mu_K(\text{d}y) p(x, y; t - s), \tag{65}
\]
\[
A_3 = \int_0^{t} \text{d}s \int_s^t \text{d}\tau \; \mathbb{P}_x[ \tau < T_K < t ] \int \mu_K(\text{d}z) p(x, z; t - \tau) \int \mu_K(\text{d}y) p(x, y; t - s), \tag{66}
\]
\[
A_4 = \int_0^{t} \text{d}s \int_s^t \text{d}\tau \; \mathbb{P}_x[ \tau < T_K < s ] \int \mu_K(\text{d}z) \left( p(x, z; t - \tau) - p(x, z; s - \tau) \right) \int \mu_K(\text{d}y) p(x, y; t - s). \tag{67}
\]

**Proof.** Since
\[
\mathbb{P}_x[ s < T_K < t ] = \mathbb{P}_x[ s < L_K < t ] + \mathbb{P}_x[ T_K < t < L_K ] - \mathbb{P}_x[ T_K < s < L_K ], \tag{68}
\]
we have that the left-hand side of (63) equals
\[ \int_0^t ds \mathbb{P}_x [s < L_K < t] \int \mu_K(dy) p(x, y; t - s) + \int_0^t ds \left( \mathbb{P}_x [T_K < t < L_K] - \mathbb{P}_x [T_K < s < L_K] \right) \int \mu_K(dy) p(x, y; t - s). \] (69)

By the strong Markov property we can write the second term in (69) as

\[ \int_0^t ds E_x \left\{ \int_0^t \mathbb{1}_{T_K \in d\tau} \mathbb{P}_B(T_K) [t - \tau < L_K < \infty] - \int_0^s \mathbb{1}_{T_K \in d\tau} \mathbb{P}_B(T_K) [s - \tau < L_K < \infty] \right\} \int \mu_K(dy) p(x, y; t - s). \] (70)

First we show that we can replace \( B(T_K) \) in (70) by \( x \) at a cost \( O(t^{-m/2}) \). By (52)

\[ \int_0^t ds E_x \left\{ \int_0^t \mathbb{1}_{T_K \in d\tau} (1 \wedge C_{x,K}(t - \tau)^{-m/2}) \right\} \int \mu_K(dy) p(x, y; t - s) \leq E_x \left\{ \int_0^t \mathbb{1}_{T_K \in d\tau} (1 \wedge C_{x,K}(t - \tau)^{-m/2}) \right\} = O(t^{-m/2}). \] (71)

Moreover by (55)

\[ \int_0^t ds E_x \left\{ \int_0^s \mathbb{1}_{T_K \in d\tau} (1 \wedge C_{x,K}(s - \tau)^{-m/2}) \right\} \int \mu_K(dy) p(x, y; t - s) = \int_0^t ds \mathbb{P}_x [T_K < s] (1 \wedge C_{x,K}s^{-m/2}) \int \mu_K(dy) p(x, y; t - s) \]

\[ + \int_0^t ds \int_0^{s^*} d\tau \mathbb{P}_x [\tau < T_K < s] C_{x,K} \frac{m}{2} (s - \tau)^{-(m+2)/2} \int \mu_K(dy) p(x, y; t - s), \] (72)

where

\[ s^* = (s - T) \vee 0. \] (73)

By (59)

\[ \int_0^{t/2} ds \mathbb{P}_x [T_K < s] (1 \wedge C_{x,K}s^{-m/2}) \int \mu_K(dy) p(x, y; t - s) \leq C(K)t^{-m/2} \left( \int_0^\infty ds \left( 1 \wedge C_{x,K}s^{-m/2} \right) \right) \]

\[ = O(t^{-m/2}). \] (74)

By (33) with \( E = [t/2, t] \)
The interval in expression in (70) equals

\[
\int_{s-T}^{t} ds \int_{s}^{t} \frac{d}{dt} \left( \mathbb{P}_x[T_K < s] - \mathbb{P}_x[T_K < t] \right) \mathbb{P}_x[t - \tau < L_K < \infty]
\]

To estimate the second term in the right-hand side of (72) we have that the contribution from \( s \in [T, 2T] \) is bounded by

\[
\int_{T}^{2T} ds \int_{0}^{s-T} d\tau \frac{m}{2} (s - \tau)^{-m/2} / \int_{s-T}^{t} \frac{d}{dt} \mu_K(d\tau)p(x, y; t - s)
\]

\[
= \frac{C(K)}{T(t - 2T)^{m/2}}.
\]

The interval \([2T, t/2]\) contributes at most, by (34) and (59),

\[
\int_{2T}^{t/2} ds \int_{0}^{(s-T)/2} d\tau \frac{m}{2} (s - \tau)^{-m/2} / \int_{(s-T)/2}^{t/2} \frac{d}{dt} \mu_K(d\tau)p(x, y; t - s)
\]

\[
= \frac{C(K)C_{x,K} t^{-m/2}}{T(t - 2T)^{m/2}}.
\]

The interval \([t/2, t]\) contributes at most, by (33) and (34),

\[
\sup_{t/2 < s < t} \left\{ \int_{0}^{(s-T)/2} d\tau \frac{m}{2} (s - \tau)^{-m/2} + \int_{(s-T)/2}^{s-T} d\tau \frac{m}{2} (s - \tau)^{-m/2} \right\}
\]

\[
\leq \sup_{t/2 < s < t} \left\{ C_{x,K} \frac{(s + T)/2}{m/2} + 3C_{x,K} \frac{(s - T)/2}{m/2} (2 - m/2) \right\} = O(t^{-m/2}).
\]

By (74)–(78) we conclude that the right-hand side of (72) is \( O(t^{-m/2}) \). Then, by Lemma 3, (31), (71) we have that the expression in (70) equals

\[
\int_{s}^{t} ds \int_{s}^{t} d\tau \left( \mathbb{P}_x[T_K < \tau] - \mathbb{P}_x[T_K < t] \right) \mathbb{P}_x[t - \tau < L_K < \infty]
\]

\[
- \int_{s}^{t} d\tau \left( \mathbb{P}_x[T_K < \tau] - \mathbb{P}_x[T_K < s] \right) \mathbb{P}_x[s - \tau < L_K < \infty] \int_{s-T}^{t} \frac{d}{dt} \mu_K(d\tau)p(x, y; t - s) + O(t^{-m/2})
\]

\[
\mathbb{E} \int_{s}^{t} d\tau \left( \mathbb{P}_x[T_K < \tau] - \mathbb{P}_x[T_K < t] \right) \mathbb{P}_x[t - \tau < L_K < \infty]
\]
\[- \int_0^s \frac{d}{d\tau} \left( \mathbb{P}_x[T_K < \tau] - \mathbb{P}_x[T_K < s] \right) \mathbb{P}_x[s - \tau < L_K < t - \tau] \int \mu_K(dy) p(x, y; t - s) + O(t^{-m/2}) \]

\[= -\mathbb{P}_x[T_K < \infty] \int_0^t ds \mathbb{P}_x[s < L_K < t] \int \mu_K(dy) p(x, y; t - s) + \sum_{i=1}^4 A_i + O(t^{-m/2}), \] (79)

after two integrations by parts. Proposition 5 follows by (69) and (79). \( \square \)

Below we obtain the asymptotic behaviour of the first term in the right-hand side of (63).

**Lemma 6.** Let \( K \) be a compact and non-polar set in \( \mathbb{R}^4 \). Then for \( t \to \infty \)

\[ \int_0^t ds \mathbb{P}_x[s < L_K < t] \int \mu_K(dy) p(x, y; t - s) = 2(4\pi)^{-4} C(K)^2 \frac{\log t}{t^2} + O(t^{-2}). \] (80)

**Proof.** By (35)

\[ \int_0^T ds \mathbb{P}_x[s < L_K < t] \int \mu_K(dy) p(x, y; t - s) \leq \int_0^T ds \int \mu_K(dy) p(x, y; t - s) \leq C(K)T(t - T)^{-2}. \] (81)

By (33)

\[ \int_{t - T}^t ds \mathbb{P}_x[s < L_K < t] \int \mu_K(dy) p(x, y; t - s) \leq \mathbb{P}_x[t - T < L_K < t] \int_{t - T}^t ds \int \mu_K(dy) p(x, y; t - s) \leq C(K)T/(t(t - T)). \] (82)

Furthermore by (35) and (19)

\[ \int_{t - T}^t ds \mathbb{P}_x[s < L_K < t] \int \mu_K(dy) p(x, y; t - s) \leq (4\pi)^{-4} C(K)^2 \int_T^t ds (s^{-1} - t^{-1})(t - s)^{-2} \]

\[= 2(4\pi)^{-4} C(K)^2 \frac{\log t}{t^2} + O(t^{-2}), \] (83)

which proves the upper bound in (80). To prove the lower bound in (80) we have by (35) and (19)

\[ \int_0^t ds \mathbb{P}_x[s < L_K < t] \int \mu_K(dy) p(x, y; t - s) \]

\[\geq \int_T^{t - T} ds \int_s^t d\tau (4\pi \tau)^{-2} \int \mu_K(dz) \left( 1 - \frac{|z - x|^2}{4\tau} \right) \int \mu_K(dy) p(x, y; t - s). \] (84)

Since

\[ \int_T^{t - T} ds \int_s^t d\tau (4\pi \tau)^{-2} \int \mu_K(dz) \frac{|z - x|^2}{4\tau} \int \mu_K(dy) p(x, y; t - s) \]

\[\leq C(K)^2 (|x| + R)^2 \int_T^t ds \int_s^t d\tau \tau^{-3} (t - s)^{-2} = O(t^{-2}), \] (85)
we have that the left-hand side of (84) is bounded below by
\[
\int_{t-T}^{t} ds \int_{s}^{t} d\tau (4\pi \tau)^{-2} C(K) \int \mu_{K}(dy) p(x, y; t-s) + O(t^{-2})
\]
\[
\geq (4\pi)^{-1} C(K)^{2} t^{-2} \int_{s}^{t} ds \int_{s}^{t} d\tau (s^{-1} - t^{-1}) - C(K)^{2} (|x| + R)^{2} \int_{s}^{t} ds \int_{s}^{t} d\tau (s^{-1} - t^{-1}) + O(t^{-2})
\]
\[
= 2(4\pi)^{-1} C(K)^{2} \log \frac{t}{t^2} + O(t^{-2}).
\]
(86)
The lower bound in (80) follows from the estimates in (84)–(86). □

We conclude this section with the proof of Proposition 2 for \( m = 4 \). By (29) we have that the second term in the right-hand side of (48) is \( O(t^{-2}) \). Below we will show that \( A_1 = O(t^{-2}) \) for \( i = 1, \ldots, 4 \) and \( t \to \infty \). This implies Theorem 1 for \( m = 4 \) by Propositions 4, 5 and Lemma 6.

The contribution from \( s \in [0, T] \) to \( A_1 \) in (64) is bounded by \( C(K)T(t-T)^{-2} \). Similarly by (33) with \( E = [t-T, t] \) and (34) the contribution from \( s \in [t-T, t] \) is bounded by
\[
P[s \leq \min\{T, t\}] \int_{s}^{t} ds \int \mu_{K}(dy) p(x, y; t-s) \leq CC(K)T/(t(t-T)).
\]
The contribution from \( s \in [t, t/2] \) is bounded, using (29), by
\[
\int_{t/2}^{t} ds C(K)^{3}s^{-1}(t-s)^{-3} = O\left(\frac{\log t}{t^3}\right),
\]
(88)
and the contribution from \( s \in [t/2, t-T] \) is bounded, using (34) and (29), by
\[
\int_{t/2}^{t-T} ds C(K)^{2}s^{-2}(t-s)^{-2} = O(t^{-2}).
\]
(89)
This proves that \( A_1 = O(t^{-2}) \).

The contribution from \( s \in [0, T] \) to \( A_2 \) is bounded by \( C(K)T(t-T)^{-2} \) and the contribution from \( s \in [t-T, t] \) to \( A_2 \) is bounded, using (29), by \( C(K)^{2}(t-T)^{-2} \).

Finally, the contribution from \( s \in [T, t-T] \) is bounded, using (29), (30), by
\[
\int_{t-T}^{t} C(K)^{3}s^{-1}(s^{-1} - t^{-1})(t-s)^{-2} = O(t^{-2}).
\]
(90)
This proves that \( A_2 = O(t^{-2}) \).

The contribution from \( s \in [0, t/2] \) to \( A_3 \) is bounded, using Lemma 3 and (59), by
\[
t^{-2} \int_{0}^{t/2} ds \left\{ t^{-2} \int_{s}^{t/2} d\tau \frac{C(K)^3}{\tau(t-\tau)^2} + \int_{t/2}^{t-T} d\tau \frac{CC(K)^2}{\tau^2(t-\tau)} + \int_{t-T}^{t} d\tau \frac{CC(K)T}{(t-T)^2} \int \mu_{K}(dz) p(x, z; t-\tau) \right\} = O\left(\frac{\log t}{t^3}\right).
\]
(91)
The contribution from \( s \in [t/2, t] \) to \( A_3 \) is bounded, using (34), by
\[
\int_{t/2}^{t} ds \int_{s}^{t} d\tau \frac{C(t-\tau)}{\tau^2} \int \mu_{K}(dz) p(x, z; t-\tau) \int \mu_{K}(dy) p(x, y; t-s)
\]
\[ + \frac{CT}{(t-T)^2} \int_{t-T}^{t} d\tau \int_{t-T}^{t} d\tau \int \mu_K(d\tau) p(x, z; t-\tau) \int \mu_K(dy) p(x, y; t-s) \]

\[ \leq 4C \left( \int_{-\infty}^{t} d\tau (t-\tau)^{1/2} \int \mu_K(d\tau) p(x, z; t-\tau) \right)^2 + \frac{CT}{(t-T)^2} \]

\[ = O(t^{-2}), \]  

where we have used that for \( m = 4 \)

\[ \int_{0}^{\infty} d\tau \tau^{1/2} \int \mu_K(dy) p(x, y; \tau) = \frac{1}{8\pi^{3/2}} \int \mu_K(dy) |x-y|^{-1} \]

\[ \leq \frac{1}{8\pi^{3/2}} \left( \int \mu_K(dy) |x-y|^{-2} \right)^{1/2} \left( \int \mu_K(dy) \right)^{1/2} \leq C(K)^{1/2}. \]

This proves that \( A_3 = O(t^{-2}) \).

The contribution from \( s \in [0, 2T] \) to \( A_4 \) is bounded by

\[ \frac{2C(K)T}{(t-2T)^2} \left( \int_{-\infty}^{s} d\tau \int \mu_K(dz)p(x, z; s-\tau) + \int_{-\infty}^{t} d\tau \int \mu_K(dy)p(x, y; t-\tau) \right) = O(t^{-2}). \]

The contribution from \( s \in [2T, t/2] \) to \( A_4 \) is bounded by

\[ C(K)t^{-2} \int_{2T}^{t/2} ds \left\{ \int_{0}^{T} d\tau \int \frac{2C(K)}{(s-\tau)^2} + \int_{T}^{s/2} d\tau \int \frac{2C(K)^2}{s} + \int_{s/2}^{s-T} d\tau \int \frac{2CC(K)}{s^2} + \int_{s-T}^{s} d\tau \int \frac{CT}{(s-T)^2} \right\} = O(t^{-2}), \]

where we have used that \( p_x[\tau < T < s] \) is bounded on the intervals \([0, T], [T, s/2], [s/2, s-T], [s-T, s]\) by 1, \( C(K)/\tau, C(s-\tau)/\tau^2 \) and \( CT/(s-T)^2 \) respectively.

To bound the contribution from \( s \in [t/2, t] \) to \( A_4 \) we use that uniformly in \( x, z, s, \tau \) and \( t \)

\[ |p(x, z; s-\tau) - p(x, z; t-\tau)| \leq (s-\tau)^{-2} \land (t-s)(s-\tau)^{-3} \land (t-s)^{1/2}(s-\tau)^{-5/2}. \]

First of all the contribution from the rectangle \( \{ (s, \tau): t/2 < s < t, \ 0 < \tau < T \} \) to \( A_4 \) is bounded by

\[ \int_{t/2}^{t} ds \int_{t/2}^{T} d\tau \int \frac{2C(K)}{(s-\tau)^2} \int \mu_K(dy) p(x, y; t-s) \leq \frac{2C(K)T}{(t/2-2T)^2} \int_{t/2}^{T} ds \int \mu_K(dy) p(x, y; t-s) = O(t^{-2}). \]

Secondly, by Lemma 3 and (96), (93)

\[ \int_{t/2}^{t} ds \int_{t/2}^{s/2} d\tau \mu_K(d\tau)p(x, z; t-\tau) - p(x, z; s-\tau) \int \mu_K(dy) p(x, y; t-s) \]

\[ \leq \int_{t/2}^{s/2} ds \int_{t/2}^{s/2} d\tau \frac{C(K)^2(t-s)^{1/2}}{\tau(s-\tau)^{5/2}} \int \mu_K(dy) p(x, y; t-s) \]

\[ \leq C(K)^2 \left( \frac{1}{4} \right)^{-5/2} \int_{t/2}^{s} d\tau (t-s)^{1/2} \log \left( \frac{s}{2T} \right) \int \mu_K(dy) p(x, y; t-s) \]
\( \leq C(K)^{5/2} \left( \frac{t}{4} \right)^{-5/2} \log \left( \frac{t}{2T} \right) \). (98)

Thirdly, by Lemma 3 and (96), (93)
\[
\int_0^t ds \int_{s/2}^{s-T} d\tau \mathbb{P}_x [\tau < T_K < s] \int \mu_K (dz) |p(x, z; t - \tau) - p(x, z; s - \tau)| \int \mu_K (dy) p(x, y; t - s)
\leq \int_0^t ds \int_{s/2}^{s-T} d\tau \frac{C(K)(t - s)^{1/2}}{\tau^2 (s - \tau)^{3/2}} \int \mu_K (dy) p(x, y; t - s)
\leq 16CC(K)t^{-2} \int_0^t ds (t - s)^{1/2} \int \mu_K (dy) p(x, y; t - s)
\times \int_{-\infty}^{s-T} d\tau (s - \tau)^{-3/2} \leq 32C(K)^{3/2}T^{-1/2}t^{-2}. \] (99)

Finally, by Lemma 3,
\[
\int_0^t ds \int_{s/2}^{s-T} d\tau \mathbb{P}_x [\tau < T_K < s] \int \mu_K (dz) |p(x, z; t - \tau) - p(x, z; s - \tau)| \int \mu_K (dy) p(x, y; t - s)
\leq \int_0^t ds \int_{s/2}^{s-T} d\tau \mu_K (dz) (p(x, z; t - \tau) + p(x, z; s - \tau)) \int \mu_K (dy) p(x, y; t - s)
\leq \frac{8CT}{(t - 2T)^2}. \] (100)

This completes the proof of \( A_4 = O(t^{-2}) \) and hence of Proposition 2 for \( m = 4 \). \( \square \)

4. Proof of Proposition 2 for \( m = 3 \)

Throughout this section we assume that \( m = 3 \). By Propositions 4 and 5
\[
\mathbb{P}_x [t < T_K < \infty] = (1 - \mathbb{P}_x [t < L_K < \infty])^{-1}\mathbb{P}_x [T_K = \infty]\mathbb{P}_x [t < L_K < \infty]
- (1 - \mathbb{P}_x [t < L_K < \infty])^{-1}\mathbb{P}_x [T_K = \infty] \int_0^t ds \mathbb{P}_x [s < L_K < t]
\times \int \mu_K (dy) p(x, y; t - s) - (1 - \mathbb{P}_x [t < L_K < \infty])^{-1} \sum_{i=1}^4 A_i + O(t^{-3/2}). \] (101)

By (20)
\[
(1 - \mathbb{P}_x [t < L_K < \infty])^{-1}\mathbb{P}_x [T_K = \infty]\mathbb{P}_x [t < L_K < \infty]
= \mathbb{P}_x [T_K = \infty]\mathbb{P}_x [t < L_K < \infty] + (16\pi^3)^{-1}C(K)^2 \mathbb{P}_x [T_K = \infty]t^{-1} + O(t^{-3/2}). \] (102)

Lemma 7. Let \( K \) be a compact and non-polar set in \( \mathbb{R}^3 \). Then for \( t \to \infty \)
\[
\int_0^t ds \mathbb{P}_x [s < L_K < t] \int \mu_K (dy) p(x, y; t - s) = (16\pi^3)^{-1}C(K)^2t^{-1} + O(t^{-3/2}). \] (103)
Proof. By (19)
\[ \int \mu_K(dy) p(x, y; t - s) \leq (4\pi)^{-3/2} C(K)(t - s)^{-3/2}, \] (104)
so that by (104) and (30)
\[ \int_0^t ds \mathbb{P}_x[s < L_K < t] \int \mu_K(dy) p(x, y; t - s) \leq (32\pi^3)^{-1} C(K)^2 t^{-1} \int_0^1 ds \frac{1}{s^{1/2}} (1 + s^{1/2})^{-1} (1 - s)^{-1/2} \]
\[ = (16\pi^3)^{-1} C(K)^2 t^{-1}, \] (105)
where the integral with respect to \( s \in [0, 1] \) is evaluated by the change of variable \( s = (\sin \theta)^2 \). To prove the lower bound in Lemma 7 we have
\[ \int \mu_K(dy) p(x, y; t - s) \geq (4\pi)^{-3/2} C(K)(t - s)^{-3/2} - (4\pi)^{-3/2} C(K)(t - s)^{-3/2} \left( 1 - e^{-\frac{(|x| + R)^2}{4(t-s)}} \right). \] (106)
Since
\[ \int_0^t ds \mathbb{P}_x[s < L_K < t](t - s)^{-3/2} \left( 1 - e^{-\frac{(|x| + R)^2}{4(t-s)}} \right) \leq C(K) \int_0^{\pi/2} d\theta \left( 1 - e^{-\frac{(|x| + R)^2}{4(t-s)}} \right) \]
\[ = 2C(K)t^{-1} \int_0^{\pi/2} d\theta \left( 1 - e^{-\frac{(|x| + R)^2}{4(t-s)}} \right) \]
\[ \leq 2C(K)t^{-1} \int_0^{\pi/2} d\theta \left( 1 - e^{-\frac{(|x| + R)^2}{2t}} \right) = O(t^{-3/2}), \] (107)
we have that the left-hand side of (103) is bounded from below by
\[ (4\pi)^{-3/2} C(K) \int_0^t ds \mathbb{P}_x[s < L_K < t](t - s)^{-3/2} + O(t^{-3/2}). \] (108)
Since
\[ \mathbb{P}_x[s < L_K < t] \geq \int \mu_K(dy) \int_0^t d\tau \frac{1}{s^{3/2}} e^{-\frac{(|x| + R)^2}{4t}} \]
\[ = (4\pi^3)^{-1} C(K)(s^{-1/2} - t^{-1/2})(1 - (1 - e^{-\frac{(|x| + R)^2}{4t}})), \] (109)
we have by estimates similar to (107) that (108) is bounded from below by
\[ (16\pi^3)^{-1} C(K)^2 t^{-1} - C(K)^2 t^{-1} \int_0^1 ds \frac{1}{s^{1/2}} (1 - s)^{-3/2} \left( 1 - e^{-\frac{(|x| + R)^2}{4t}} \right) + O(t^{-3/2}) \]
\[ = (16\pi^3)^{-1} C(K)^2 t^{-1} + O(t^{-3/2}). \] (110)
This completes, by (106)–(110), the proof of the lower bound in Lemma 7. \( \Box \)

By Lemma 7 we obtain that the term of order \( t^{-1} \) in (102) cancels with the second term in the right-hand side of (101). So the proof of Proposition 2 for \( m = 3 \) is complete if we can show that
\[ \sum_{i=1}^4 A_i = O(t^{-3/2}), \quad t \to \infty. \] (111)
However, it turns out that each of the $A_i$ is (for $m = 3$) of order $(\log t)/t^{3/2}$. So in order to obtain (111) we will show that the sum of the coefficients of $\log t/t^{3/2}$ of the $A_i$'s cancel with remainder $O(t^{-3/2})$. In Proposition 8 we state that $P_x(s < T_K < t)$ in (64) can be replaced by $P_x(s < T_K < t)|P_x[T_K = \infty]$ at a cost of $O(t^{-3/2})$ with similar replacements in (65)–(67) respectively. In Lemma 9 we obtain, using Proposition 8, the desired asymptotic behaviour of each of the $A_i$. This in turn implies (111) and thereby completing the proof of (111) and of Theorem 1.

**Proposition 8.** Let $K$ be a compact, non-polar set in $\mathbb{R}^3$, and let $A_i$ $i = 1, \ldots, 4$ be given by (64)–(67) respectively. Then for $t \to \infty$

$$A_1 = P_x[T_K = \infty] \int_0^t ds \, P_x(s < L_K < t)|P_x(t-s < L_K < \infty) \int \mu_K(\partial_x p(x, y; t-s) + O(t^{-3/2}), \quad (112)$$

$$A_2 = P_x[T_K = \infty] \int_0^t ds \, P_x(s < L_K < \infty)|P_x(s < L_K < t) \int \mu_K(\partial_x p(x, y; t-s) + O(t^{-3/2}), \quad (113)$$

$$A_3 = P_x[T_K = \infty] \int_0^t ds \, \int_0^t \mu_K(\partial_x p(x, z; t-s)) \int \mu_K(\partial_x p(x, z; t-s) + O(t^{-3/2}), \quad (114)$$

$$A_4 = P_x[T_K = \infty] \int_0^t ds \, \int_0^t \mu_K(\partial_x p(x, z; t-s)) \int \mu_K(\partial_x p(x, z; t-s) + O(t^{-3/2}), \quad (115)$$

It is convenient to denote the first term in the right-hand sides of (112)–(115) respectively by $B_1, \ldots, B_4$.

**Lemma 9.** Let $K$ be a compact and non-polar set in $\mathbb{R}^3$. Then for $t \to \infty$

$$B_1 = 2(4\pi)^{-9/2}C(K)^3 P_x[T_K = \infty] \frac{\log t}{t^{3/2}} + O(t^{-3/2}), \quad (116)$$

$$B_2 = 4(4\pi)^{-9/2}C(K)^3 P_x[T_K = \infty] \frac{\log t}{t^{3/2}} + O(t^{-3/2}), \quad (117)$$

$$B_3 = 2(4\pi)^{-9/2}C(K)^3 P_x[T_K = \infty] \frac{\log t}{t^{3/2}} + O(t^{-3/2}), \quad (118)$$

$$B_4 = -8(4\pi)^{-9/2}C(K)^3 P_x[T_K = \infty] \frac{\log t}{t^{3/2}} + O(t^{-3/2}). \quad (119)$$

**Proof.** By (29), (30) and (104)

$$B_1 \leq 4(4\pi)^{-9/2}C(K)^2 P_x[T_K = \infty] \int_0^t ds \left(s^{-1/2} - t^{-1/2}\right)(t-s)^{-2} \int \mu_K(\partial_x p(x, y) e^{-\frac{|x-y|^2}{4t^{3/2}}}. \quad (120)$$

On the other hand, by (35)

$$P_x(t-s < L_K < \infty) \geq 2(4\pi)^{-3/2}C(K)(t-s)^{-1/2}(1 + e^{-\frac{(x+y)^2}{4t^{3/2}}} - 1). \quad (121)$$

Hence by (109) and (121)
\[ B_1 \geq 4(4\pi)^{-9/2}C(K)^2 P_{\alpha} |T_K = \infty | \int_0^t ds \left( s^{-1/2} - t^{-1/2} \right) (t - s)^{-2} \]

\[ \times \int \mu_K(dy) e^{-\frac{|x-y|^2}{4(\alpha t)}} (1 - (1 - e^{-\frac{|x+y|^2}{4(\alpha t)}})) \left( 1 - e^{-\frac{|x+y|^2}{4\alpha}} \right). \] \quad (122)

Below we will compute the leading asymptotic behaviour of the right-hand side of (120). Substitution of \( s = t(\cos \theta)^2 \) in (120) yields that the integral equals

\[ 2t^{-3/2} \int_0^{\pi/2} \mu_K(dy) \int_0^{\pi/2} d\theta (\sin \theta)^{-1} (1 + \cos \theta)^{-1} e^{-\frac{|x-y|^2}{4(\sin \theta)^2}}. \] \quad (123)

Since

\[ (\sin \theta)^{-1}(1 + \cos \theta)^{-1} \leq (2\theta)^{-1} + 4, \quad 0 < \theta < \pi/2, \] \quad (124)

we have that the right-hand side of (123) is bounded from above by

\[ t^{-3/2} \int_0^{\pi/2} \mu_K(dy) \int_0^{\pi/2} d\theta \theta^{-1} e^{-\frac{|x-y|^2}{4\theta^2}} + O(t^{-3/2}) = \frac{1}{2} t^{-3/2} \int_0^{\pi/2} \mu_K(dy) \int_0^{\pi/2} d\theta \theta^{-1} e^{-\theta} + O(t^{-3/2}) \]

\[ = \frac{1}{2} t^{-3/2} \int \mu_K(dy) \log \left( \frac{|x-y|^2}{\pi^2 t} \right) + O(t^{-3/2}). \] \quad (125)

This gives, together with (120), the desired upper bound for the asymptotic behaviour of the right-hand side of (120). The lower bound for the right-hand side of (120) follows similarly, using \((\sin \theta)^{-1}(1 + \cos \theta)^{-1} \geq (2\theta)^{-1}, \quad 0 < \theta < \pi/2.\) Furthermore returning to (122) we have that

\[ \int_0^t ds \left( s^{-1/2} - t^{-1/2} \right) (t - s)^{-2} \int \mu_K(dy) e^{-\frac{|x-y|^2}{4(\alpha t)}} (1 - e^{-\frac{|x+y|^2}{4(\alpha t)}}) \]

\[ \leq 2t^{-3/2} \int_0^{\pi/2} d\theta (\sin \theta)^{-1} \int \mu_K(dy) e^{-\frac{|x-y|^2}{4(\sin \theta)^2}} (1 - e^{-\frac{|x+y|^2}{4(\sin \theta)^2}}) \]

\[ \leq 2t^{-3/2} \int_0^{\pi/2} d\theta (\sin \theta)^{-1} \int \mu_K(dy) \left( \frac{4(t \sin \theta)^2}{|x-y|^2} \right)^{1/2} (1 - e^{-\frac{|x+y|^2}{4(\sin \theta)^2}}) \]

\[ \leq 16\pi t^{-1} \left( \int \mu_K(dy) \frac{1}{4\pi |x-y|} \right) \int_0^{\infty} d\theta (1 - e^{-\frac{\theta^2}{16\theta^2}}} = O(t^{-3/2}), \] \quad (126)

and

\[ \int_0^t ds \left( s^{-1/2} - t^{-1/2} \right) (t - s)^{-2} \int \mu_K(dy) e^{-\frac{|x-y|^2}{4(\alpha t)}} (1 - e^{-\frac{|x+y|^2}{4\alpha}}) \]

\[ \leq \int_0^t ds \left( s^{-1/2} - t^{-1/2} \right) (t - s)^{-2} \int \mu_K(dy) \left( \frac{4(t - s)}{|x-y|^2} \right)^{1/2} (1 - e^{-\frac{|x+y|^2}{4\alpha}}) \]

\[ \leq 8\pi t^{-1} \int_0^t ds s^{-1/2} (t - s)^{-1/2} (1 - e^{-\frac{|x+y|^2}{4\alpha}}) \]
Since we have that (131) is bounded from above by
\[ O(t^{-3/2}). \] (127)

It follows by (126) and (127) that the two remainders in the right-hand side of (122) contribute each at most \( O(t^{-3/2}). \) This completes the proof of (116).

To prove (117) we note that by (29), (30) and (104)
\[
B_2 \leq 2(4\pi)^{-9/2} C(K)^2 \mathbb{P}_x[T_K = +\infty] \int_0^t ds \left( s^{-1/2} - t^{-1/2} \right) (t - s)^{-3/2} \int \mu_K(dy) \int_0^\infty \frac{\mu_K(dy)}{s} d\tau \tau^{-3/2} e^{-\frac{|x-y|^2}{4\tau}}.
\]
\[= 4(4\pi)^{-9/2} C(K)^2 \mathbb{P}_x[T_K = \infty] \int_0^t ds \left( s^{-1/2} - t^{-1/2} \right) (t - s)^{-3/2} \int \mu_K(dy) \int_1^\infty \frac{\mu_K(dy)}{s} d\tau \tau^{-3/2} e^{-\frac{|x-y|^2}{4\tau(\sin \theta)^2}}. \] (128)

On the other hand
\[
B_2 \geq 2(4\pi)^{-9/2} C(K)^2 \mathbb{P}_x[T_K = \infty] \int_0^t ds \left( s^{-1/2} - t^{-1/2} \right) (t - s)^{-3/2} \int \mu_K(dy) \int_1^\infty \frac{\mu_K(dy)}{s} d\tau \tau^{-3/2} e^{-\frac{|x-y|^2}{4\tau(\sin \theta)^2}} \times \left| \int \frac{\mu_K(dy)}{s} d\tau \tau^{-3/2} \right| e^{-\frac{|x-y|^2}{4\tau}} \left( 1 - \left( 1 - e^{-\frac{|x-y|^2}{4\tau}} \right) \right) \right). \] (129)

Below we will compute the leading asymptotic behaviour of the right-hand side of (128). Using the inequality \( (\sin \theta)^{-1} \leq \theta^{-1} + 4, \) \( 0 < \theta < \pi/2, \) we obtain for (128) the upper bound
\[
4(4\pi)^{-9/2} C(K)^2 \mathbb{P}_x[T_K = \infty] \int_0^t ds \left( s^{-1/2} - t^{-1/2} \right) (t - s)^{-3/2} \int \mu_K(dy) \int_1^\infty \frac{\mu_K(dy)}{s} d\tau \tau^{-3/2} e^{-\frac{|x-y|^2}{4\tau(\sin \theta)^2}} + O(t^{-3/2}). \] (130)

and the upper bound follows by a calculation similar to (125). The lower bound for the right-hand side of (128) follows using \( (\sin \theta)^{-1} (1 + \sin \theta)^{-1} \geq \theta^{-1} - 4, \) \( 0 < \theta < \pi/2. \) Furthermore returning to (129) we have a first error term
\[
\int_0^t ds \left( s^{-1/2} - t^{-1/2} \right) (t - s)^{-3/2} \int \mu_K(dy) \int \frac{\mu_K(dy)}{s} d\tau \tau^{-3/2} e^{-\frac{|x-y|^2}{4\tau}}. \] (131)

Since
\[
\left| \int \mu_K(dy) \int \frac{\mu_K(dy)}{s} d\tau \tau^{-3/2} e^{-\frac{|x-y|^2}{4\tau}} \right| \leq \int \mu_K(dy) \int \frac{\mu_K(dy)}{s} d\tau \tau^{-3/2} \left( \frac{4\tau}{|x-y|^2} \right)^{1/4} \leq 4\sqrt{2} s^{-1/4} \int \mu_K(dy) |x-y|^{-1/2} \leq 4\sqrt{2} s^{-1/4} \left( \int \mu_K(dy) |x-y|^{-1} \right)^{1/2} C(K)^{1/2} \leq 8\sqrt{2\pi} s^{-1/4} C(K)^{1/2}, \] (132)

we have that (131) is bounded from above by
\[
8\sqrt{2\pi} C(K)^{1/2} t^{-1} \int_0^t ds s^{-3/4} (t - s)^{-1/2} \left( 1 - e^{-\frac{|x+y|^2}{4s}} \right) \]
Below we will compute the leading asymptotic behaviour of the right-hand side of (135). Firstly, since remainders in the right-hand side of (129) contribute each at most \(O(1)\), we have that the right-hand side of (135) is bounded from below by

\[
\int_0^\pi d\theta (\sin \theta)^{-1/2} \left( 1 - e^{-\frac{|x|+R^2}{4(\sin \theta)^2}} \right) \leq 16\sqrt{2\pi} C(K)^{1/2} t^{-5/4} \int_0^\pi d\theta (\sin \theta)^{-1/2} \left( 1 - e^{-\frac{|x|+R^2}{4(\sin \theta)^2}} \right) = O(t^{-3/2}).
\]

The second error term is bounded by

\[
\int_0^t ds (s^{-1/2} - t^{-1/2}) (t - s)^{-3/2} (1 - e^{-\frac{|x|+R^2}{4(t-s)^2}}) \int \mu_K(dy) \int_0^t d\tau \tau^{-3/2} e^{-\frac{|x-y|^2}{4t}} \leq 8\sqrt{2\pi} C(K)^{1/2} t^{-1} \int_0^t ds s^{-3/4} (t - s)^{-1/2} (1 - e^{-\frac{|x|+R^2}{4(t-s)^2}})
\]

\[
\leq 8\sqrt{2\pi} C(K)^{1/2} (|x| + R)^{1/2} t^{-1} \int_0^t ds s^{-3/4} (t - s)^{-3/4} = O(t^{-3/2}),
\]

where we have used (132) and the inequality \(1 - e^{-\theta} \leq \theta^{1/4}, \theta \geq 0\). It follows by (133) and (134) that the two remainders in the right-hand side of (129) contribute each at most \(O(t^{-3/2})\). This completes the proof of (117).

To prove (118) we note that by (30) and (104)

\[
B_3 \leq 2(4\pi)^{-9/2} C(K)^2 \mathbb{P}_x[T_K = \infty] \int_0^t ds \int \mu_K(dy) e^{-\frac{|x-y|^2}{4(t-s)^2}} (t - s)^{-3/2} \times \int_s^t d\tau (\tau^{-1/2} - t^{-1/2}) (t - \tau)^{-3/2}.
\]

On the other hand

\[
B_3 \geq 2(4\pi)^{-9/2} C(K)^2 \mathbb{P}_x[T_K = \infty] \int_0^t ds \int \mu_K(dy) e^{-\frac{|x-y|^2}{4(t-s)^2}} (t - s)^{-3/2} \int_s^t d\tau (\tau^{-1/2} - t^{-1/2}) (t - \tau)^{-3/2} \times \left( 1 - e^{-\frac{(x+R)^2}{4(t-s)^2}} \right).
\]

Below we will compute the leading asymptotic behaviour of the right-hand side of (135). Firstly, since

\[
\int_s^t d\tau (\tau^{-1/2} - t^{-1/2}) (t - \tau)^{-3/2} = \int_s^t d\tau \tau^{-1/2} \tau^{-1/2} (\tau^{1/2} + t^{1/2})^{-1} (t - \tau)^{-1/2} \geq \frac{(t-s)^{1/2}}{t^{3/2}}
\]

we have that the right-hand side of (135) is bounded from below by

\[
2(4\pi)^{-9/2} C(K)^2 \mathbb{P}_x[T_K = \infty] t^{-3/2} \int_0^t ds \int \mu_K(dy)(t - s)^{-1} e^{-\frac{|x-y|^2}{4(t-s)^2}} \geq 2(4\pi)^{-9/2} C(K)^3 \mathbb{P}_x[T_K = \infty] \int_0^t \log \frac{t}{t^{3/2}} + O(t^{-3/2}).
\]

Secondly, since

\[
\int_s^t d\tau (\tau^{-1/2} - t^{-1/2}) (t - \tau)^{-3/2} \leq \frac{(t-s)^{1/2}}{t^{3/2}} + \frac{2(t-s)^{3/2}}{t^2 s^{1/2}}
\]

(139)
we have that the right-hand side of (135) is bounded from above by
\[
2(4\pi)^{-9/2} C(K)^2 \mathbb{P}_x[T_K = \infty] t^{-3/2} \int_0^t ds \int \mu_K(dy)(t - s)^{-1} e^{-\frac{(y-x)^2}{4(t-s)^2}} + O(t^{-3/2})
\]
\[
\leq 2(4\pi)^{-9/2} C(K)^3 \mathbb{P}_x[T_K = \infty] \log \frac{t}{t^{3/2}} + O(t^{-3/2}).
\] (140)

In order to complete the proof of (118) we have to show that the two error terms in the right-hand side of (136) contribute at most \(O(t^{-3/2})\). Since the right-hand side of (139) is bounded from above by \(3(t-s)^{1/2}t^{-1}s^{-1/2}\) we have that the first of these error terms is bounded by
\[
C(K)^2 t^{-1} \int_0^t ds \int \mu_K(dy)(t - s)^{-1} s^{1/2} e^{-\frac{(y-x)^2}{4(t-s)^2}} (1 - e^{-\frac{(y-x)^2}{4t}})
\]
\[
\leq 8\pi C(K)^2 t^{-1} \int \mu_K(dy)(4\pi |x - y|)^{-1} \int_0^t ds (t - s)^{-1/2} s^{-1/2} (1 - e^{-\frac{(y-x)^2}{4t}})
\]
\[
\leq 16\pi C(K)^2 t^{-1} \int_0^{\pi/2} d\theta (1 - e^{-\frac{(y-x)^2}{4t \sin^2 \theta}}) = O(t^{-3/2}).
\] (141)

The upper bound for the second of these error terms follows by a similar calculation. This completes the proof of (118).

To prove (119) we rewrite \(B_4\) as follows.
\[
B_4 = (4\pi)^{-3/2} C(K)^3 \mathbb{P}_x[T_K = \infty] t \int_0^t ds \int \mu_K(dy) p(x, y; t - s)
\]
\[
\times \int_0^s d\tau ((t - \tau)^{-3/2} - (s - \tau)^{-3/2}) \int_\tau^s d\rho \int \mu_K(dw) p(x, w; \rho)
\]
\[
+ (4\pi)^{-3/2} \mathbb{P}_x[T_K = \infty] t \int_0^t ds \int \mu_K(dy) p(x, y; t - s) \int \mu_K(dz)
\]
\[
\times \int_0^s d\tau (t - \tau)^{-3/2} (e^{-\frac{(z-x)^2}{4(t-\tau)^2}} - 1) \int_\tau^s d\rho \int \mu_K(dw) p(x, w; \rho)
\]
\[
+ (4\pi)^{-3/2} \mathbb{P}_x[T_K = \infty] t \int_0^t ds \int \mu_K(dy) p(x, y; t - s) \int \mu_K(dz)
\]
\[
\times \int_0^s d\tau (s - \tau)^{-3/2} (1 - e^{-\frac{(z-x)^2}{4(s-\tau)^2}}) \int_\tau^s d\rho \int \mu_K(dw) p(x, w; \rho).
\] (142)

We first show that the third term in the right-hand side of (142) is bounded in absolute value by \(O(t^{-3/2})\). Note that
\[
\int_\tau^s d\rho \int \mu_K(dw) p(x, w; \rho) \leq 2(4\pi)^{-3/2} (\tau^{-1/2} - s^{-1/2}) \int \mu_K(dw) e^{-\frac{(w-x)^2}{4t}}
\]
\[
\leq (s - \tau)^{-1/2} s^{-1} \int \mu_K(dw) e^{-\frac{(w-x)^2}{4s}} .
\] (143)
Hence the absolute value of this third term is bounded by
\[
C(K) \int_0^t \int_0^\infty \mu_K(dy) p(x, y; t-s) \int_0^{\pi/2} d\theta \left( 1 - e^{-(|x+y|^2/4\tau + R^2)} \right)(s - \tau)^{-1/2} \tau^{-1/2} \int_0^{\pi/2} \mu_K(dw)e^{-|x-w|^2/4r} \]
\[
= 2C(K) \int_0^t \int_0^\infty \mu_K(dy) p(x, y; t-s) s^{-1} \int_0^{\pi/2} d\theta \left( 1 - e^{-(|x+y|^2/4\theta + R^2)} \right) \int_0^{\pi/2} \mu_K(dw)e^{-|x-w|^2/4r} \]
\[
\leq (4\pi)^2 (|x| + R) C(K) \int_0^t \int_0^\infty \mu_K(dy) p(x, y; t-s) \int_0^\infty \mu_K(dw) p(x, w; s) = O(t^{-3/2}). \quad (144)
\]
Since for \(0 < \tau < s < t\)
\[
(t - \tau)^{-3/2} (1 - e^{-(|x|^2/4(t - \tau) + R^2)}) \leq (s - \tau)^{-3/2} (1 - e^{-(|x|^2/4(t - \tau) + R^2)}), \quad (145)
\]
we have that the second term in the right-hand side of (142) is also estimated by (144).

It remains to find the asymptotic behaviour of the first term in the right-hand side of (142). By the first inequality in (143) we have that this term is bounded from below by
\[
2(4\pi)^{-9/2} C(K)^3 \mathbb{P}_{x}[T_K = \infty] \int_0^t ds \int \mu_K(dy)/(t-s)^{-3/2} e^{-|x-y|^2/4(t-s)}
\]
\[
\times \int_0^s d\tau ((t - \tau)^{-3/2} - (s - \tau)^{-3/2})(\tau^{-1/2} - s^{-1/2}) \int_0^{\pi/2} \mu_K(dw)e^{-|x-w|^2/4r}. \quad (146)
\]
A straightforward calculation gives that
\[
\int_0^s d\tau ((t - \tau)^{-3/2} - (s - \tau)^{-3/2})(\tau^{-1/2} - s^{-1/2})
\]
\[
= 2(t - s)^{3/2}(t^{1/2} + (t - s)^{1/2} - s^{-1/2}[(t - s)^{-1} + (t + s + (st)^{1/2})^{-1}(t - s)^{-1/2}(t^{1/2} + s^{1/2})^{-1}]. \quad (147)
\]
Hence (146) equals
\[
-4(4\pi)^{-9/2} C(K)^3 \mathbb{P}_{x}[T_K = \infty] \int_0^t ds \int \mu_K(dy) \int_0^{\pi/2} \mu_K(dw) e^{-|x-y|^2/4(t-s)} - |x-w|^2/4r(t^{1/2} + (t - s)^{1/2})^{-1}s^{-1}
\]
\[
\times \left[ (t - s)^{-1} + (t + s + (st)^{1/2})^{-1}(t - s)^{-1/2}(t^{1/2} + s^{1/2})^{-1} \right]. \quad (148)
\]
The first term in the square brackets of (148) gives the contribution
\[
-6(4\pi)^{-9/2} C(K)^3 \mathbb{P}_{x}[T_K = \infty] \log \frac{t}{t^{3/2}} + O(t^{-3/2}), \quad (149)
\]
and the second term contributes
\[
-2(4\pi)^{-9/2} C(K)^3 \mathbb{P}_{x}[T_K = \infty] \log \frac{t}{t^{3/2}} + O(t^{-3/2}). \quad (150)
\]
By (146)–(150) we conclude that the first term in the right-hand side of (142) is bounded from below by the expression in the right-hand side of (119). Since
\[
\int_\tau^s d\rho \int \mu_K(dw) p(x, w; \rho) \geq 2(4\pi)^{-3/2}(\tau^{-1/2} - s^{-1/2}) \int_0^{\pi/2} \mu_K(dw)e^{-|x-w|^2/4r} \quad (151)
\]
we have, by (143), that the resulting upper bound differs from the lower bound by at most

\[
\int_0^t ds \int \mu_K(dy) p(x, y; t-s) \int_0^s d\tau (s-\tau)^{-3/2} - (t-\tau)^{-3/2} \\
	imes C(K) \left( \frac{|x-w|^2}{4s} - e^{-\frac{|x-w|^2}{4s}} \right) \\
\leq \int_0^t ds \int \mu_K(dy) p(x, y; t-s) \int_0^s d\tau (s-\tau)^{-3/2} e^{-\frac{|x-w|^2}{4s}} \\
	imes C(K) \left( \tau^{-1/2} - s^{-1/2} \right) (1 - e^{-\frac{|x-w|^2}{4s}}(\frac{1}{\tau} - \frac{1}{s})).
\]

(152)

By substituting \( \tau = s(\sin \theta)^2 \) we have that

\[
\int_0^s d\tau \tau^{-1/2} (s-\tau)^{-1/2} (1 - e^{-\frac{|x-w|^2}{4s}}(\frac{1}{\tau} - \frac{1}{s})) \leq 2 \int_0^{\pi/2} d\theta \left( 1 - e^{-\frac{|x-w|^2(\sin \theta)^2}{4s}} \right) \leq 2 \int_0^{\infty} d\theta \left( 1 - e^{-\frac{(|x|+R)^2}{4s}} \right) \\
\leq (4\pi)^{1/2} (|x| + R)s^{-1/2}.
\]

(153)

Then (152) is bounded from above by

\[
(4\pi)^2 C(K) (|x| + R) \int \mu_K(dy) \int \mu_K(dw) \int_0^t d\tau p(x, w; \tau) p(x, y; t-s).
\]

(154)

But (154) has been estimated in (144). This completes the proof of (119), Lemma 9 and Proposition 2 for \( m = 3 \). \( \Box \)

Finally one can show that, by going through the estimates leading to the proof of Proposition 2, the remainder \( O(t^{-m/2}) \) in Theorem 1 is uniform on compact subsets of \( \mathbb{R}^m \setminus K \). This completes the proof of Theorem 1.

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References