Coboundaries in $L_{0}^{\infty}$

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Abstract

Let $T$ be an ergodic automorphism of a probability space, $f$ a bounded measurable function, $S_n(f) = \sum_{k=0}^{n-1} f \circ T^k$. It is shown that the property that the probabilities $\mu(|S_n(f)| > n)$ are of order $n^{-p}$ roughly corresponds to the existence of an approximation in $L_{0}^{\infty}$ of $f$ by functions (coboundaries) $g - g \circ T$, $g \in L^p$. Similarly, the probabilities $\mu(|S_n(f)| > n)$ are exponentially small iff $f$ can be approximated by coboundaries $g - g \circ T$ where $g$ have finite exponential moments.

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1. Introduction and results

Let \((\Omega, \mathcal{A}, \mu)\) be a probability space and \(T : \Omega \to \Omega\) a bijective, bimeasurable and measure preserving mapping. Throughout the paper we shall suppose that \(T\) is ergodic and aperiodic. For a measurable function \(f\) on \(\Omega\) we denote
\[
S_n(f) = \sum_{i=0}^{n-1} f \circ T^i.
\]

\(L^p_0\) denotes the space of all \(f \in L^p\) with zero mean, \(1 \leq p \leq \infty\). If \(f = g - g \circ T\) with \(g\) a measurable function, then we say that \(f\) is a coboundary. The function \(g\) is then called the cobounding function. We shall study the approximation of functions from \(L^\infty_0\) by coboundaries. The main results are presented in Theorems 1, 2, and 3; they show a relationship between the moments of the cobounding function \(g\) of the approximating coboundary and probabilities of large deviations of the stochastic process \((f \circ T^i)\).

It is well known that for \(1 \leq p < \infty\), the coboundaries with a cobounding function in \(L^p\) are a dense subset of \(L^p_0\) (because \(L^\infty\) is a dense subset of \(L^p\), the coboundaries \(g - g \circ T\) with \(g\) bounded thus form a dense subset of all \(L^p_0\), \(1 \leq p < \infty\)). As an immediate consequence of the density of the sets of coboundaries in \(L^p\) spaces we get the von Neumann’s ergodic theorem in these spaces:
\[
\frac{1}{n} S_n(f) - Ef \to 0 \quad \text{for any } f \in L^p, 1 \leq p < \infty \text{ (cf. e.g. [10, p. 21]).}
\]

For \(p = \infty\) the things are more complicated:

**Theorem A.** Let \(f \in L^1_0\) and \(\varepsilon > 0\). Then there exists a measurable function \(g\) such that
\[
\|f - (g - g \circ T)\|_\infty < \varepsilon.
\]

The theorem follows from [7, Corollary 3] ([8] in English). For completeness, we shall show a proof the idea of which is due to Michael Keane.

There exist, however, bounded functions which cannot be (in \(L^\infty\)) approximated by any coboundary with an integrable cobounding function \(g\):

**Theorem B.** Let \(\varphi\) be a positive real function, \(\lim_{t \to -\infty} \varphi(t) = \lim_{t \to \infty} \varphi(t) = \infty\). Then there exists a function \(f \in L^\infty_0\) with \(\|f\|_\infty = 1\) such that for each measurable function \(g\) with \(\int \varphi \circ g \, d\mu < \infty\),
\[
\|f - (g - g \circ T)\|_\infty \geq 1/2.
\]

In particular, for any \(p > 0\) there exists \(f\) with \(\|f\|_\infty = 1\) s.t. \(\|f - (g - g \circ T)\|_\infty \geq 1/2\) for each \(g \in L^p\).

This result is not completely new either; a version of it can be found in the work of A. Katok [6].

The main aim of this paper is to show that the set of \(f \in L^\infty_0\) which can be approximated by coboundaries whose cobounding functions have finite moments can be characterized by the probabilities of large deviations. The first two theorems show that the integrability of \(|f|^p\) is “almost equivalent” to the property that the probabilities \(\mu(|S_k(f)| > xk)\) are of order \(1/k^p\). The third proposition extends the result to functions with exponential moments.

**Theorem 1.** Let \(f \in L^\infty_0\), \(p \geq 1\). If for every \(\delta > 0\) there exists a \(g \in L^p\) with \(\|f - (g - g \circ T)\|_\infty < \delta\) then

\[
\mu(|S_k(f)| > xk) \leq Cx^{-p} \quad \text{for some } C > 0 \text{ and } x > 0.
\]
(i) For each \( \epsilon > 0 \)
\[
\sum_{k=1}^{\infty} k^{p-1} \mu(|S_k(f)| > \epsilon k) < \infty,
\]

(ii) For each \( \epsilon > 0 \) there exists a \( c_\epsilon > 0 \) such that
\[
\mu(|S_k(f)| > \epsilon k) < c_\epsilon \cdot \frac{1}{k^p} \quad \text{for all } k = 1, 2, \ldots
\]

**Theorem 2.** Let \( f \in L_0^\infty \), \( p \geq 1 \). If for every \( x > 0 \) there exists a \( 0 < c_x < \infty \) such that for all \( k \),
\[
\mu(|S_k(f)| > xk) < c_x k^{-p}
\]
then for all \( \epsilon > 0 \) and every measurable function \( v: \mathbb{R}^+ \to \mathbb{R}^+ \) such that
\[
\sum_{j=1}^{\infty} \frac{1}{v(a_j)} < \infty \quad \text{for all } a > 0 \quad \text{and} \quad v(x), x^p v(x) \text{ are increasing}
\]
there exists a measurable function \( g \) such that
\[
E\left( \frac{|g|^p}{v(|g|)} \right) < \infty \quad \text{and} \quad \| f - (g - g \circ T) \|_{\infty} < \epsilon.
\]
In particular, for any \( \delta > 0 \) we can find \( g \in L^{p-\delta} \).

**Theorem 3.** Let \( f \in L_0^\infty \). It is equivalent

(i) For every \( \epsilon > 0 \) there exists a \( c_\epsilon > 0 \) and \( n_\epsilon \in \mathbb{N} \) such that \( \mu(|S_n(f)| > \epsilon n) < e^{-c_\epsilon n} \) for all \( n \geq n_\epsilon \).

(ii) For every \( \epsilon > 0 \) there exists a measurable function \( g \) and \( c > 0 \) such that \( Ee^{c|g|} < \infty \) and
\[
\| f - (g - g \circ T) \|_{\infty} < \epsilon.
\]

For sequences of independent and weakly dependent random variables \( X_i \), the probabilities of \( \{ \sum_{i=1}^{n} X_i > xn \} \) have been analyzed in detail before. For example, by Azuma’s inequality [4] an exponential bound like in (i) of Theorem 3 exists whenever \( (X_i) \) is a uniformly bounded sequence of martingale differences. Therefore, if \( f \in L_0^\infty \) and \( (f \circ T^i) \) is a martingale difference sequence (or even a sequence of mutually independent random variables) then \( f \) can be approximated by coboundaries whose transfer functions have finite exponential moments.

Generic properties of sets of functions \( f \) for which the probabilities \( \mu(S_n(f) > xn) \) have a particular asymptotic behaviour are studied in the paper [9].

2. Proofs

**Proof of Theorem A.** By the Birkhoff (almost sure) Ergodic Theorem for any \( \eta > 0 \) there exists \( A \in \mathcal{A} \) with \( \mu(A) > 0 \) and \( n_0 \in \mathbb{N} \) such that for each \( \omega \in A \) we have
\[
\left| \frac{1}{n} S_n(f)(\omega) \right| < \eta \quad \text{for all } n \geq n_0.
\]

Let \( A^k \) be the set of points whose return time to \( A \) is \( k \): \( A^k = \{ \omega \in A: T^k \omega \in A, T^i \omega \notin A \text{ for } 0 < i < k \} \), \( k = 1, 2, \ldots \). Let \( A = \bigcup_{k=1}^{\infty} A^k \). Because \( T \) is ergodic, the set \( \bigcup_{k=1}^{\infty} T^kA^k \) has measure 1; without loss of generality we can suppose that it equals \( \Omega \).
It is known that we can also suppose that $A^k = \emptyset$ for $1 \leq k \leq n_0 - 1$. This is the case if $A, TA, \ldots, T^{n_0-1}A$ is a Rokhlin tower. If not, we can recursively find an adequate subset of $A$ of strictly positive measure because for any $A$ of positive measure and $k \geq 0$, $B \subset A$, $0 < \mu(B) < \mu(A)$, we by ergodicity have $\mu(B \setminus T^{k+1}B) > 0$, hence $B, \ldots, T^{k+1}B$ are disjoint.

For each $\omega \in \Omega$ there thus exist unique $k \geq n_0$ and $0 \leq i \leq k - 1$ such that $\omega \in T^iA^k$. Define

$$\xi(\omega) = T^{-i}\omega,$$

$$g(\omega) = f(\omega) - \frac{1}{k} \sum_{j=0}^{k-1} f(T^j \xi(\omega)),$$

$$h(\omega) = \sum_{j=0}^{i} g(T^j \xi(\omega)) = \sum_{j=0}^{i} g(T^j \xi(\omega)).$$

For $\omega \in T^{k-1}A^k$, $h(\omega) = 0$. We thus have

$$g = h - h \circ T^{-1}.$$  

From the definition of the set $A$ it follows that

$$\|f - g\|_{\infty} < \eta. \quad \Box$$

In the proof of Theorem B we shall use the result by A. Alpern (cf. [1–3,5]).

**Theorem C** (A. Alpern). Let $1 \leq n_k$, $k = 1, 2, \ldots$, be positive integers whose least common divisor is 1, $p_k$ positive reals, $\sum_{k=1}^{\infty} p_k n_k = 1$. Then there exist measurable sets $F_k$, such that $\mu(F_k) = p_k$, $T^i F_k$, $0 \leq i \leq n_k - 1$, $k = 1, 2, \ldots$, are pairwise disjoint and $\mu(\bigcup_{k=1}^{\infty} \bigcup_{i=0}^{n_k-1} T^i F_k) = 1$.

**Proof of Theorem B.** Without loss of generality we can suppose that $\varphi$ is an even function. For $k = 1, 2, \ldots$ let $r_k$ be a positive integer such that

$$\sum_{j=1}^{[(k-1)/2]} \varphi(j/2) \geq kr_k;$$

by the assumptions, $r_k \to \infty$. There thus exist (strictly) positive numbers $p_k$ such that

$$\sum_{k=1}^{\infty} kp_k = 1,$$

$$\sum_{k=1}^{\infty} kr_k p_k = \infty.$$

By Theorem C there exist measurable sets $A_k$, $k = 1, 2, \ldots$, such that $\mu(A_k) = p_k$ and $\{A_k, T^{-1}A_k, \ldots, T^{-k+1}A_k\}$, $k = 1, 2, \ldots$, are mutually disjoint Rokhlin towers. Let $A_k = B_k \cup C_k$ where $B_k \cap C_k = \emptyset$ and $\mu(B_k) = \mu(A_k)/2 = \mu(C_k)$, $k = 1, 2, \ldots$. We define

$$f(\omega) = \begin{cases} 1 & \text{for } \omega \in \bigcup_{k=1}^{\infty} \bigcup_{i=0}^{k-1} T^{-i}B_k, \\ -1 & \text{for } \omega \in \bigcup_{k=1}^{\infty} \bigcup_{i=0}^{k-1} T^{-i}C_k, \\ 0 & \text{for } \omega \notin \bigcup_{k=1}^{\infty} \bigcup_{i=0}^{k-1} T^{-i}A_k. \end{cases}$$
Suppose that \( g, h \) are measurable functions, \( \|h\|_\infty < 1/2 \) and
\[
f = g \circ T - g - h.
\]
Then
\[
g \circ T = f + g + h,
\]
\[
g \circ T^2 = f \circ T + g \circ T + h \circ T = f + f \circ T + h + h \circ T + g,
\]
\[
\ldots
\]
\[
g \circ T^n = \sum_{i=0}^{n-1} f \circ T^i + \sum_{i=0}^{n-1} h \circ T^i + g.
\]
We have
\[
E \psi \circ g \geq \sum_{k=1}^\infty \int \sum_{i=0}^{k-1} \psi(g \circ T^i) \, d\mu.
\]

Let us denote \( \psi_j = \sum_{j=0}^{k-1} (f \circ T^i + h \circ T^i), \) \( j = 1, 2, \ldots. \)
For \( \omega \in A_k \) we get \( \sum_{j=0}^{k-1} \psi(g(T^j \omega)) = \sum_{j=0}^{k-1} \psi_j(g + g). \)
We distinguish two possibilities:

1. The numbers \( \psi_j(\omega) + g(\omega), j = 0, \ldots, k - 1, \) are all of the same sign. Then \( \sum_{j=1}^{k-1} \psi_j(g + g) \geq \sum_{j=1}^{k-1} \psi_j(j/2). \)

2. The numbers \( \psi_j(\omega) + g(\omega), j = 1, \ldots, k - 1, \) are not all of the same sign. Because \( f(T^j \omega) \) are all 1 or all \(-1\) while \( |h| \leq 1/2, \) the sequence \( \psi_1(\omega), \ldots, \psi_{k-1}(\omega) \) is monotone. Hence, there exists \( 1 \leq n \leq k - 1 \) such that \( \sum_{j=1}^{k-1} \psi_j(g + g) = \sum_{j=1}^{n} \psi_j(g + g) + \sum_{j=n+1}^{k-1} \psi_j(g + g) \geq \sum_{j=1}^{(k-1)/2} \psi_j(j/2) \)

where \( \lfloor x \rfloor \) denotes the integer value of \( x. \)

We thus get \( E \psi(g) \geq \sum_{k=1}^\infty p_k \sum_{j=1}^{(k-1)/2} \psi_j(j/2) \geq \sum_{k=1}^\infty kr_k p_k = \infty. \) This finishes the proof. \( \square \)

**Proof of Theorem 1.** Let \( \varepsilon > 0 \) be fixed. We put \( 0 < \delta < \varepsilon/2, \) \( g \) is a function from \( L^p \) with \( \|f - (g \circ T)\|_\infty < \delta. \)

Then
\[
|S_n(f - (g \circ T))| < \delta n < n\varepsilon/2
\]
hence
\[
\mu \left( \left| S_n(f) > \varepsilon k \right| \right) \leq \mu \left( \left| S_n(g - g \circ T) > k\varepsilon/2 \right| \right) \leq 2 \mu \left( |g| > k\varepsilon/4 \right).
\]

Because \( g \in L^p, \)
\[
\sum_{l=1}^\infty \sum_{j=1}^{(l+1)p-1} \mu \left( |g| > l + 1 \right) \leq \sum_{l=1}^\infty \sum_{j=1}^{(l+1)p-1} \mu \left( |g| > j \right) < \infty
\]
hence
\[
\sum_{l=1}^\infty |l|^{p-1} \mu \left( |g| > l \right) < \infty.
\]

The statement (ii) follows from
\[
\mu \left( \left| S_n(g - g \circ T) > \varepsilon n \right| \right) \leq 2 \mu \left( |g| > \frac{\varepsilon n}{2} \right) \leq 2 \int \frac{|g|^p \, d\mu}{(\frac{\varepsilon n}{2})^p} = \frac{c_\varepsilon}{n^p}
\]
where \( c_\varepsilon = \frac{2}{p+1} \int |g/\varepsilon|^p \, d\mu. \) \( \square \)

For the proof of Theorem 2 we shall need the following statement:
Proposition. Let \( f \in L_0^\infty, \varepsilon > 0 \). Then under the assumptions of Theorem 2 there exists an integer \( n_0 \geq 1 \) and a set \( F \) of positive measure with

\[
F_k = \{ \omega \in F | T^k \omega \in F, \forall 1 \leq i \leq k - 1, \ T^i \omega \notin F \}, \quad k = 1, 2, \ldots, \\
F_\infty = \{ \omega \in F | \forall 1 \leq i, \ T^i \omega \notin F \},
\]

such that

(a) for \( \omega \in F_k, 1 \leq k < \infty \),

\[
|S_k(f)(\omega)| \leq k \varepsilon \quad \text{and} \quad |S_j(f)(\omega)| > j \varepsilon \quad \text{for all} \ 1 \leq j \leq k - n_0
\]

and

(b) there exists a \( 0 < c < \infty \) such that for all \( n, \infty \sum_{k=n}^{\infty} \mu(F_k) < cn^{-p-1} \).

Proof. As in the proof of Theorem A we can show that there exists an integer \( n_0 \geq 1 \) and a set \( A \) of positive measure such that for all \( \omega \in A \)

(i) if \( n \geq n_0 \) then \( |\sum_{i=0}^{n-1} f(T^{-i} \omega)| \leq \varepsilon n \),

(ii) if \( 1 \leq k \leq n_0 - 1 \) then \( T^{-k} \omega \notin A \).

For \( \omega \in \Omega \) we define

\[
\psi(\omega) = \begin{cases} \min\{n \geq 2 | |S_n(f)(\omega)| \leq \varepsilon n\} & \text{if } \exists n \geq 2, |S_n(f)(\omega)| \leq \varepsilon n, \\ \infty & \text{otherwise.} \end{cases}
\]

For \( \omega \in A \) we recursively define \( \tau_k(\omega), k = 0, 1, \ldots \) by

\[
\tau_0(\omega) = 0, \quad \tau_{k+1}(\omega) = \tau_k(\omega) + \psi(T^{\tau_k(\omega)} \omega)
\]

and put

\[
\varphi(\omega) = \begin{cases} \min\{k \geq 1 | T^k \omega \in A\} & \text{if } \exists n \geq 1, T^n \omega \in A, \\ \infty & \text{otherwise,} \end{cases}
\]

\[
t(\omega) = \begin{cases} \sup\{0 \leq k | \tau_k(\omega) \leq \varphi(\omega)\} & \text{if } \varphi(\omega) < \infty, \\ \infty & \text{otherwise,} \end{cases}
\]

Observation. Let \( \omega \in A \). If \( \varphi(\omega), t(\omega) < \infty \) then

\[
\tau_{t(\omega)}(\omega) \leq \psi(\omega),
\]

\[
\varphi(\omega) - \tau_{t(\omega)}(\omega) \leq n_0 - 1,
\]

\[
t(\omega) \geq 1.
\]

The first inequality follows immediately from the definition of \( t \), the second follows from (i), the third from the preceding ones and (ii).

Let \( \omega \in A \). Define

\[
F = \bigcup_{\omega \in A} \bigcup_{k=0}^{t(\omega)-1} T^{\tau_k(\omega)} \omega.
\]

By the construction, the set \( F \) is measurable and satisfies (a).
Let us prove (b). Let \( \delta > 0 \). If \( 0 \leq i \leq \delta n, k \geq n_0 + n(1 + \delta) \), \( \omega \in F_k \), then by (a)

\[
\left| \sum_{j=0}^{n+i-1} f(T^j \omega) \right| > (n + i)\varepsilon \geq n\varepsilon \quad \text{and} \quad \left| \sum_{j=0}^{i-1} f(T^j \omega) \right| \leq \delta n \| f \|_{\infty}.
\]

Therefore,

\[
\left| \sum_{j=0}^{n-1} f(T^j T^i \omega) \right| \geq \left| \sum_{j=0}^{n+i-1} f(T^j \omega) \right| - \left| \sum_{j=0}^{i-1} f(T^j \omega) \right| \geq n\varepsilon - n\delta \| f \|_{\infty} = n(\varepsilon - \delta \| f \|_{\infty}).
\]

For \( n_0 \) defined at the beginning of the proof, for each \( \delta > 0 \), \( \omega \in \bigcup_{k \geq n_0 + n(1 + \delta)} F_k \) and \( 0 \leq i \leq \delta n \), we have

\[
|S_n(f)(T^i \omega)| \geq n(\varepsilon - \delta \| f \|_{\infty}).
\]

There thus exists a constant \( c \) depending only on \( \varepsilon - \delta \| f \|_{\infty} \),

\[
\left| \sum_{k \geq n_0 + n(1 + \delta)} \mu(F_k) \right| < cn^{-p}
\]

and the inequality (b) follows. \( \square \)

**Proof of Theorem 2.** Let \( 1 \leq k < \infty \), \( \omega \in F_k \), \( 0 \leq i \leq k - 1 \). We define

\[
h(T^i \omega) = f(T^i \omega) - \frac{1}{k} \sum_{j=0}^{k-1} f(T^j \omega),
\]

\[
g(T^i \omega) = \sum_{j=0}^{i-1} h(T^j \omega).
\]

On the rest of \( \Omega \) we define \( g = 0 \). We then have

\[
h = g \circ T - g
\]

and

\[
\| f - (g \circ T - g) \|_{\infty} \leq \varepsilon.
\]

Let us denote \( a = \| h \|_{\infty} \). Using (b) we calculate

\[
E\left( \frac{|g|^p}{v(|g|)} \right) \leq \sum_{k=1}^{\infty} \left( \sum_{j=1}^{k} \frac{(aj)^p}{v(aj)} \right) \mu(F_k) \leq \sum_{k=1}^{\infty} \frac{(aj)^p}{v(aj)} \left( \sum_{k=j}^{\infty} \mu(F_k) \right) \leq c \sum_{j=1}^{\infty} \frac{(aj)^p}{v(aj)} j^{-p-1}
\]

\[
= ca^p \sum_{j=1}^{\infty} \frac{1}{j v(aj)} < \infty. \quad \square
\]

The proof of Theorem 3 is left as an exercise for the reader. For (i) \( \Rightarrow \) (ii) we can use the same construction as in the proof of Theorem 2, (ii) \( \Rightarrow \) (i) follows from \( \mu(|S_n(f)| > \varepsilon) \leq \mu(|S_n(f - (g - g \circ T))| > \varepsilon/2) + \mu(|S_n(g - g \circ T)| > \varepsilon/2) \).

**References**

Further reading