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On the covering by small random intervals

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Abstract

Consider the random intervals $I_n = \omega_n + (0, \ell_n)$ (modulo 1) with their left points ω_n independently and uniformly distributed over the interval $[0,1) = \mathbb{R}/\mathbb{Z}$ and with their lengths decreasing to zero. We prove that the Hausdorff dimension of the set $\overline{\lim}_n I_n$ of points covered infinitely often is almost surely equal to $1/\alpha$ when $\ell_n = a/n^\alpha$ for some $a > 0$ and $\alpha > 1$.

Résumé

Considérons des intervalles aléatoires $I_n = \omega_n + (0, \ell_n)$ (modulo 1) dont les extrémités gauches ω_n sont indépendantes et uniformément réparties sur l'intervalle $[0,1) = \mathbb{R}/\mathbb{Z}$ et dont les longueurs décroissent vers zéro. Nous montrons que la dimension de Hausdorff de l'ensemble $\overline{\lim}_n I_n$ des points infiniment recouverts est presque sûrement égale à $1/\alpha$ quand $\ell_n = a/n^\alpha$ avec $a > 0$ et $\alpha > 1$.

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1. Introduction

Let $\{\ell_n\}_{n\geqslant 1}$ be a sequence of positive real numbers which is decreasing to zero and let $I_n(\omega) = (\omega_n, \omega_n + \ell_n)$ (modulo 1) be a random interval where $\{\omega_n\}_{n\geqslant 1}$ is a sequence of independent random variables uniformly distributed over the unit interval $I = [0,1)$ which is identified with the circle $\mathbb{R}/\mathbb{Z}$. We consider the set $E_\infty(\omega) = \overline{\lim}_n I_n$ of those points which are covered infinitely often.

It is easy to see that $E_\infty(\omega)$ is almost surely (a.s. for short) a set of Lebesgue measure 0 or 1 according to $\sum_{n=1}^\infty \ell_n < \infty$ or $\sum_{n=1}^\infty \ell_n = \infty$. A. Dvoretzky [2] asked the question when $E_\infty(\omega) = [0,1)$ a.s. or not. There was a series of contributions (for references and related topics see [5] for information before 1985 and [6]

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for recent information). A complete answer was obtained by L. Shepp [7]: $E_\infty(\omega) = [0, 1)$ a.s. if and only if $\sum_{n=1}^{\infty}(1/n^2)\exp(\ell_1 + \ell_2 + \cdots + \ell_n) = \infty$. It is the case for $\ell_n = a/n$ if and only if $a \geqslant 1$.

We address to the study of $E_\infty(\omega)$ when the covering intervals I_n are small in the sense $\sum_{n=1}^{\infty} \ell_n < \infty$. As we mentioned above, in this case the set $E_\infty(\omega)$ is of Lebesgue measure zero. However, it is a.s. of second category in Baire sense (see [5, p. 55]). We will determine the Hausdorff dimension of $E_\infty(\omega)$ in the case $\ell_n = a/n^\alpha$ with $a > 0$, $\alpha > 1$.

Theorem. *Suppose $\ell_n = a/n^\alpha$ for some $a > 0$ and $\alpha > 1$. Then*

$$\dim E_\infty(\omega) = \frac{1}{\alpha} \quad a.s.$$

As we shall see from the proof of the theorem, $\ell_n = O(n^{-\alpha})$ implies $\dim E_\infty(\omega) \leqslant 1/\alpha$ and $n^{-\alpha} = O(\ell_n)$ implies $\dim E_\infty(\omega) \geqslant 1/\alpha$. It follows that $\dim E_\infty(\omega) = 1/\alpha$ a.s. when the following limit exists:

$$1 < \alpha = \lim_{n\to\infty} \frac{-\log \ell_n}{\log n}.$$

We point out that a similar result holds for random coverings on trees [3].

Back to the theorem. The inequality $\dim E_\infty(\omega) \leqslant 1/\alpha$ is easy to see. It even holds for every ω. Because $\{I_n(\omega)\}_{n\geqslant N}$ is a δ-cover of $E_\infty(\omega)$ with $\delta = \ell_N$ and for any $\varepsilon > 0$

$$\sum_{n=N}^{\infty} |I_n|^{1/\alpha+\varepsilon} = a^{1/\alpha+\varepsilon} \sum_{n=N}^{\infty} n^{-1-\varepsilon\alpha} < \infty.$$

In order to prove the inverse inequality, we will construct a random Cantor subset of $E_\infty(\omega)$ by using known results due to D.A. Darling on random spacings of uniform random samples. Before our proof of the theorem, let us give some preliminaries including Darling's results and a construction of Cantor set.

2. Preliminaries

Let $X_1, X_2, \ldots, X_n$ $(n \geqslant 2)$ be a set of independent random variables uniformly distributed over the unit interval $I = [0, 1)$. We call it a random sample of size n. Reordering the n points $X_1, X_2, \ldots, X_n$ in their natural order from left to right, we get n new random variables which will be denoted by $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$. The intervals $[X_{(k)}, X_{(k+1)})$, $0 \leqslant k \leqslant n$, are called the subspacings and their lengths are denoted by L_k, $0 \leqslant k \leqslant n$ (by convention, $X_{(0)} = 0$ and $X_{(n+1)} = 1$). There is a vast literature on the distributions of $(L_0, L_1, \ldots, L_n)$ and related statistics. We will only need the following results among others due to D.A. Darling [1]. Suppose $h : I \to \mathbf{R}$. Let

$$W_n = \sum_{j=0}^{n} h(L_j).$$

The first two moments of W_n are expressed by the following Darling formulas:

$$\mathbb{E}W_n = n(n+1)\int_0^1 (1-t)^{n-1} h(t)\,dt. \tag{1}$$

$$\mathbb{E}W_n^2 = n(n+1)\int_0^1 (1-t)^{n-1} h^2(t)\,dt + n^2\left(n^2-1\right)\iint_D (1-x-y)^{n-2} h(x)h(y)\,dx\,dy, \tag{2}$$

where $D = \{(x, y): x \geqslant 0, y \geqslant 0, x + y \leqslant 1\}$. We need to know how many subspacings with given length fall into a fixed subinterval. Let $J \subset I$ be a subinterval of length ℓ and let $0 < s_1 < s_2 < 1$. We denote by $M = M_n(\ell, s_1, s_2)$ the number of subspacings in J whose lengths are between s_1 and s_2. Using the Darling formulas, J. Hawkes [4] obtained explicit expressions of the first two moments of M:

$$\mathbb{E}M = n\ell\left(\left(1 - \frac{s_1}{\ell}\right)(1 - s_1)^{n-1} - \left(1 - \frac{s_2}{\ell}\right)(1 - s_2)^{n-1}\right) + (1 - s_1)^n - (1 - s_2)^n. \tag{3}$$

$$\mathbb{E}M^2 = \mathbb{E}M + S(n, \ell, s_1) + S(n, \ell, s_2) - 2S\big(n, \ell, (s_1 + s_2)/2\big) \tag{4}$$

where

$$S(n, \ell, s) = n(n-1)\ell^2\left(1 - \frac{2s}{\ell}\right)^2(1 - 2s)^{n-2} + 2n\ell\left(1 - \frac{2s}{\ell}\right)(1 - 2s)^{n-1}.$$

Proposition 1. *Suppose $0 < c_1 \leqslant c_2 < 1/2$, $0 < \ell < 1$ and $n \geqslant 3$. Let $J \subset [0, 1)$ be a subinterval of length ℓ and let n be the sample size. Denote by M the number of subspacings in J having length in $[\frac{c_1 \log n}{n}, \frac{c_2 \log n}{n}]$. Then there exist constants γ and C only depending on c_1 and c_2 (independent of ℓ and n) such that*

$$\mathbf{P}\big(M < \gamma \ell n^{1-c_1}\big) \leqslant \frac{C}{n^{c_2 - c_1}}$$

for all n such that $n^{1-2(c_2-c_1)}\ell \geqslant \log^4 n$.

Proof. We claim that

$$\mathbb{E}M = \ell n^{1-c_1} + \mathrm{O}\big(\ell n^{1-c_2}\big) \tag{5}$$

$$\mathbb{E}M^2 = \mathbf{E}M + \ell^2 n^{2(1-c_1)} + \mathrm{O}\big(\ell^2 n^{2(1-c_2)}\big), \tag{6}$$

where, and in the sequel, the constants involved in $\mathrm{O}(1)$ depend only on c_1 and c_2 and is independent with ℓ and n. First notice that

$$\left(1 - \frac{c \log n}{n}\right)^n = \frac{1}{n^c}\left(1 + \mathrm{O}\left(\frac{\log^2 n}{n}\right)\right), \tag{7}$$

$$\frac{1 - \frac{c \log n}{\ell n}}{1 - \frac{c \log n}{n}} = 1 + \mathrm{O}\left(\frac{\log n}{\ell n}\right), \tag{8}$$

$$\left(\frac{1 - \frac{c \log n}{\ell n}}{1 - \frac{c \log n}{n}}\right)^2 = 1 + \mathrm{O}\left(\frac{\log n}{\ell^2 n}\right). \tag{9}$$

The equalities (8) and (9) hold under the condition $n\ell \geqslant c \log n$ which is ensured by the hypothesis made in the proposition. Let $s = \frac{c \log n}{n}$. Using (7) and (8), we get

$$\left(1 - \frac{s}{\ell}\right)(1 - s)^{n-1} = \frac{1}{n^c}\left(1 + \mathrm{O}\left(\frac{\log^2 n}{\ell n}\right)\right).$$

Then, by the formula (3), we obtain

$$\mathbb{E}M = \ell n^{1-c_1} - \ell n^{1-c_2} + \mathrm{O}\big(n^{-c_1} \log^2 n\big) = \ell n^{1-c_1} + \mathrm{O}\big(\ell n^{1-c_2}\big).$$

Thus we have proved (5). Using (7) and (9), we get

$$\left(1-\frac{2s}{\ell}\right)^2(1-2s)^{n-2}=\frac{1}{n^{2c}}\left(1+\mathrm{O}\left(\frac{\log^4 n}{\ell^2 n}\right)\right).$$

Then

$$S(n,\ell,s)=n^{2(1-c)}\ell^2+\mathrm{O}\big(\ell n^{1-2c}\log^4 n\big).$$

Notice that $\ell n^{1-2c}\log^4 n$ is dominated by $n^{2(1-c)}\ell^2$ if $n\ell\geqslant\log^4 n$. So the main term in $S(n,\ell,s)$ is $n^{2(1-c)}\ell^2$. Also notice that $\ell n^{1-2c_1}\log^4 n$ is dominated by $n^{2(1-c_2)}\ell^2$ if $\ell n^{1-2(c_2-c_1)}\geqslant\log^4 n$ (this is the hypothesis). So we get (6).

As a consequence of (5) and (6), we have the following estimate of the variance of M:

$$\mathbf{Var}(M)=\mathbb{E}M+\mathrm{O}\big(\ell^2 n^{2-(c_1+c_2)}\big)=\mathrm{O}\big(\ell^2 n^{2-(c_1+c_2)}\big).$$

By Chebyshev inequality,

$$\mathbf{P}\left(M\leqslant\frac{\mathbb{E}M}{2}\right)\leqslant\mathbf{P}\left(|M-\mathbb{E}M|>\frac{\mathbb{E}M}{2}\right)\leqslant\frac{4\mathbf{Var}(M)}{(\mathbb{E}M)^2}=\mathrm{O}\left(\frac{1}{n^{c_2-c_1}}\right).\qquad\square$$

Consider now a construction of generalized Cantor sets on $[0,1)$. Let $\{n_k\}_{k\geqslant1}$ be a sequence of integers satisfying $n_k\geqslant2$. Let $\{\rho_k\}_{k\geqslant1}$ and $\{d_k\}_{k\geqslant1}$ be two sequences of positive real numbers. Assume that for any $k\geqslant1$, we have a collection $\mathcal{J}_k$ of closed subintervals of $[0,1)$. Each interval in $\mathcal{J}_k$ is called a k-interval. Suppose

(1) Each k-interval is of length ρ_k and contains n_{k+1} intervals;
(2) Each $(k+1)$-interval is contained in some k-interval;
(3) The gap between any two k-intervals is at least d_k.

Let $C_n=\bigcup_{J\in\mathcal{J}_k}J$ and $C_\infty=\bigcap_{k=1}^{\infty}C_n$. We call C_∞ a generalized Cantor set.

Proposition 2. *Consider the generalized Cantor set C_∞ constructed above. Suppose that there is a number $a\geqslant1$ such that $n_{k+1}d_{k+1}\geqslant\rho_k^a$ $(\forall k\geqslant1)$. Then we have*

$$\dim C_\infty\geqslant\liminf_{k\to\infty}\frac{\log(n_1n_2\cdots n_k)}{-a\log\rho_k}.$$

Proof. Define a probability measure μ on $[0,1)$ (concentrated on C_∞) by

$$\mu(J_k)=\frac{1}{n_1n_2\cdots n_k},$$

where J_k represents an arbitrary k-interval contained in C_k. Let s be the lim inf. Since $n_1n_2\cdots n_k\rho_k\leqslant1$, we have $s\leqslant1/a\leqslant1$. Suppose $s>0$. By the Frostman lemma, we have only to prove that for any $0<t<s$ and any open interval U we have $\mu(U)\leqslant2|U|^t$ ($|U|$ denotes the length of U). Without loss of generality, we assume that $n_1n_2\cdots n_k\rho_k^{at}\geqslant1$ for all $k\geqslant1$. Choose k_0 such that $\rho_{k_0+1}\leqslant|U|<\rho_{k_0}$. We distinguish two cases:

(a) The case $|U|<d_{k_0+1}$. Then U intersects with at most one (k_0+1)-interval. So

$$\mu(U)\leqslant\frac{1}{n_1n_2\cdots n_{k_0+1}}\leqslant\rho_{k_0+1}^{at}\leqslant|U|^{at}\leqslant|U|^t.$$

(b) The case $|U|\geqslant d_{k_0+1}$. Then U intersects with at most $\min(n_{k_0+1},\frac{2|U|}{d_{k_0+1}})$ (k_0+1)-intervals. So

$$\begin{aligned}\mu(U) &\leqslant \frac{1}{n_1 n_2 \cdots n_{k_0+1}} \min\left(n_{k_0+1}, \frac{2|U|}{d_{k_0+1}}\right)\\ &\leqslant \frac{1}{n_1 n_2 \cdots n_{k_0+1}} n_{k_0+1}^{1-t}\left(\frac{2|U|}{d_{k_0+1}}\right)^t\\ &\leqslant \frac{2}{n_1 n_2 \cdots n_{k_0}(n_{k_0+1}d_{k_0+1})^t}|U|^t \leqslant \frac{2}{n_1 n_2 \cdots n_{k_0}\rho_{k_0}^{at}}|U|^t \leqslant 2|U|^t. \qquad \square\end{aligned}$$

3. Proof of theorem

We only consider the case $\ell_n = a/n^\alpha$ with $a = 1$. As we shall see, only the order α of n^α plays the role. So, we may also assume that I_n is the closed interval $\omega_n + [0, \ell_n]$.

Fix two constants $0 < c_1 < c_2 < 1/2$ verifying the condition of Proposition 1. Take a large integer Δ. Define $m_k = \triangle^k$ $(k = 1, 2, \ldots)$. For $k \geqslant 1$, let

$$\begin{aligned}&\rho_k = \ell_{2^{m_k+1}}, \qquad d_k = \frac{c_1 m_k \log 2}{2^{m_k}},\\ &n_k = \left[\frac{\gamma \ell_{\rho_{k-1}} 2^{m_k(1-c_1)}}{2}\right] - 1 \quad (\rho_0 = 1),\\ &q_k = 1 - C\frac{\prod_{j=1}^{k-1} n_j}{2^{m_k(c_2-c_1)}}\end{aligned}$$

where $[x]$ denotes the integral part of a real number x and the constants γ and C are those in Proposition 1.

Consider the random sample of size 2^{m_1} from the uniform distribution over $[0, 1)$: $\omega_{2^{m_1}}, \omega_{2^{m_1}+1}, \ldots, \omega_{2^{m_1+1}-1}$. Applying Proposition 1 with $n = 2^{m_1}$ and $\ell = 1$. Proposition 1 is applicable if Δ is large enough so that $2^{m_1(1-2(c_2-c_1))} \geqslant \log^4 2^{m_1}$. We assume that $\gamma 2^{m_1(1-c_1)} > 7$ so that $n_1 \geqslant 2$. Let $\mathcal{L}_1$ be the set of left points of subspacing intervals contained in $J = [0, 1)$ having length in $[\frac{c_1 m_1 \log 2}{2^{m_1}}, \frac{c_2 m_1 \log 2}{2^{m_1}}]$. By Proposition 1, we have

$$\mathbf{P}\big(\sharp\mathcal{L}_1 \geqslant \gamma 2^{m_1(1-c_1)}\big) > 1 - \frac{C}{2^{m_1(c_2-c_1)}} = q_1.$$

Thus with probability q_1 we can find a set $\mathcal{L}_1^* \subset \mathcal{L}_1$ with n_1 points such that for each point in $\mathcal{L}_1^*$ there is on its right side a point in $\mathcal{L}_1 \setminus \mathcal{L}_1^*$. So, any two points in $\mathcal{L}_1^*$ has a distance at least $2d_1$. Define

$$C_1 = \bigcup_{\omega \in \mathcal{L}_1^*} [\omega, \omega + \rho_1].$$

Notice that there are n_1 ($\geqslant 2$) intervals in C_1 each of which has length ρ_1 and that these intervals are separated by a distance at least d_1.

Suppose that with probability $q_1 q_2 \cdots q_k$ we have successively constructed a nested sequence of sets $C_1 \supset C_2 \supset \cdots \supset C_k$ such that

(i) every C_j $(1 \leqslant j < k)$ is a union of disjoint closed intervals and each such interval in C_j is of length ρ_j and contains n_{j+1} intervals contained in C_{j+1}, and every interval contained in C_{j+1} is a subset of C_j;

(ii) the gap between two intervals contained in C_{j+1} is at least d_{j+1}.

We now construct C_{k+1}. Consider the random sample of size $2^{m_{k+1}}$: $\omega_{2^{m_{k+1}}}, \omega_{2^{m_{k+1}}+1}, \ldots, \omega_{2^{m_{k+1}+1}-1}$. This sample is independent of all preceding random samples in the construction of $C_1, C_2, \ldots, C_k$ since $2^{m_k+1} - 1 < 2^{m_{k+1}}$. Apply Proposition 1 to each interval J contained in C_k with $n = 2^{m_{k+1}}$ and $\ell = \rho_k = \ell_{2^{m_k+1}}$. Notice that

$$2^{m_{k+1}(1-2(c_2-c_1))}\ell_{2^{m_k+1}} = 2^{-\alpha+\Delta^k(\Delta((1-2(c_2-c_1))-\alpha))} \geqslant \log^4 2^{m_{k+1}}$$

if Δ is large enough. So we can really apply Proposition 1. Thus if $\mathcal{L}_{k+1,J}$ denote the set of left points of subspacings contained in J having length in $[\frac{c_1 m_{k+1}\log 2}{2^{m_{k+1}}}, \frac{c_2 m_{k+1}\log 2}{2^{m_{k+1}}}]$, we have

$$\mathbf{P}\big(\sharp\mathcal{L}_{k+1,J} \leqslant \gamma\rho_k 2^{m_{k+1}(1-c_1)} \text{ for some } J \subset C_k\big) \leqslant C \cdot \frac{\prod_{j=1}^{k} n_j}{2^{m_{k+1}(c_2-c_1)}}.$$

In other words,

$$\mathbf{P}\big(\sharp\mathcal{L}_{k+1,J} > \gamma\rho_k 2^{m_{k+1}(1-c_1)} \text{ for all } J \subset C_k\big) \geqslant q_{k+1}$$

where J denotes a typical interval in C_k. For each J in C_k, take a set $\mathcal{L}^*_{k+1,J}$ of n_{k+1} points from $\mathcal{L}_{k+1,J}$ such that for each point in $\mathcal{L}^*_{k+1,J}$ there is on its right side a point in $\mathcal{L}_{k+1,J} \setminus \mathcal{L}^*_{k+1,J}$. Then construct

$$C_{k+1} = \bigcup_{J\subset C_k} \bigcup_{\omega\in\mathcal{L}^*_{k+1,J}} [\omega, \omega + \rho_{k+1}]$$

where $J \subset C_k$ means that J is a component of C_k. Thus with probability $q_1 q_2 \cdots q_{k+1}$ we have constructed a nested sequence of sets $C_1 \supset C_2 \supset \cdots \supset C_{k+1}$ which have the properties described by (i) and (ii) (see above, k being replaced by $k+1$). Thus by induction we get an infinite sequence of nested sets C_k and we can construct a Cantor set $C_\infty = \bigcap_{k=1}^\infty C_k$ with probability

$$p = \prod_{k=1}^{\infty} q_k > 0.$$

The positivity of this probability is the consequence of

$$\sum_{k=1}^{\infty}(1 - q_k) \leqslant C \sum_{k=1}^{\infty} \frac{\prod_{j=1}^{k} n_j}{2^{m_{k+1}(c_2-c_1)}} < \infty,$$

because the general term of the series is bounded by

$$\gamma^k \left(\prod_{i=1}^{k-1} \rho_i\right) 2^{(m_1+m_2+\cdots+m_k)(1-c_1)-m_{k+1}(c_2-c_1)} = \mathrm{O}\Big(\gamma^k 2^{\Delta^k(\frac{1-c_1}{\Delta-1}-(c_2-c_1))}\Big).$$

By the construction, with probability $p > 0$ we have $C_\infty \subset E_\infty(\omega)$. Actually C_∞ is infinitely covered by those intervals I_n with $2^{m_k} \leqslant n < 2 \cdot 2^{m_k}$ for some $k \geqslant 1$.

Let us apply Proposition 2 to estimate the Hausdorff dimension of C_∞ from below. Notice that $\rho_k = 2^{-\alpha(\Delta^k+1)}$ and

$$n_{k+1} d_{k+1} \approx m_{k+1} 2^{-\alpha m_k+(1-c_1)m_{k+1}-m_{k+1}} = \Delta^{k+1} 2^{-\Delta^k(\alpha+c_1\Delta)}.$$

For any $a > 1$ and small $c_1 > 0$ so that $c_1\Delta$ is small, the condition $n_{k+1} d^{k+1} \geqslant \rho_k^a$ is satisfied. Also notice that

$$\lim_{k\to\infty} \frac{\log(n_1 n_2 \cdots n_k)}{-\log\rho_k} = \frac{1}{\alpha}(1-c_1)\frac{\Delta}{\Delta-1} - \frac{1}{\Delta-1}.$$

Thus with probability $p > 0$,

$$\dim E_\infty(\omega) \geqslant \dim C_\infty \geqslant \frac{1}{a\alpha}(1-c_1)\frac{\Delta}{\Delta-1} - \frac{1}{\Delta-1}.$$

Since $E_\infty(\omega)$ is a tail event, we have with probability one

$$\dim E_\infty(\omega) \geqslant \frac{1}{a\alpha}(1-c_1)\frac{\Delta}{\Delta-1} - \frac{1}{\Delta-1}.$$

Let $c_1 \to 0$, $\Delta \to \infty$ and then $a \to 1$, we get $\dim E_\infty(\omega) \geqslant 1/\alpha$ a.s.

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