

LIMIT THEOREMS FOR SUBCRITICAL BRANCHING PROCESSES IN RANDOM ENVIRONMENT

THÉORÈMES LIMITES POUR DES PROCESSUS DE BRANCHEMENT SOUS-CRITIQUES EN ENVIRONNEMENT ALÉATOIRE

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ABSTRACT. – Let $(Z_n)_{n \geq 0}$ be a branching process in random environment represented by a sequence of i.i.d. generating functions $(f_n)_{n \geq 0}$. In the subcritical case, $E \log f'(1) < 0$, the non-extinction probability at generation n decays exponentially fast, the rate depending on whether $E[f'(1) \log f'(1)]$ is less, equal or greater than 0. We determine the exact asymptotic of the non-extinction probability $P(Z_n > 0)$ in all three cases under suitable integrability assumptions. Moreover, we show that Z_n conditioned on $Z_n > 0$ has a non-degenerate limit law.

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RÉSUMÉ. – Soit $(Z_n)_{n \geq 0}$ un processus de branchement dans un environnement aléatoire représenté par une suite $(f_n)_{n \geq 0}$ de fonctions génératrices i.i.d.. Dans le cas sous-critique, $E \log f'(1) < 0$, la probabilité de survie à la génération n décroît exponentiellement, à un taux différent selon que $E[f'(1) \log f'(1)]$ est négative, nulle ou positive. Nous déterminons le comportement asymptotique exact de la probabilité de survie $P(Z_n > 0)$ dans les trois cas sous certaines conditions d'intégrabilité. En plus, nous montrons que Z_n conditionnellement à $Z_n > 0$ a une loi limite non-dégénérée.

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1. Introduction and main results

In this paper we obtain asymptotics for a branching process $(Z_n)_{n \geq 0}$ in random environment specified by a sequence of generating functions $(f_n)_{n \geq 0}$. In such a process it is assumed that, conditioned on the environment, particles reproduce as in a Galton–Watson branching process in varying environment, i.e., particles reproduce independently of each other and the offspring of a particle at generation n has generating function f_n . (For details and background on branching processes in random environment we refer the reader to [4,5,19].) If Z_n denotes the number of particles at generation n , then Z_{n+1} is the sum of Z_n independent random variables, each of which has generating function f_n , i.e.,

$$E(s^{Z_{n+1}} | Z_0, \dots, Z_n; f_0, f_1, \dots) = f_n(s)^{Z_n}, \quad 0 \leq s \leq 1. \tag{1.1}$$

In the following we assume that the process starts with a single founding ancestor, $Z_0 = 1$. Then the conditional probability generating function of Z_n given the environment sequence $(f_n)_{n \geq 0}$ is

$$E(s^{Z_n} | f_0, f_1, \dots) = f_0(f_1(\dots f_{n-1}(s) \dots)), \quad 0 \leq s \leq 1. \tag{1.2}$$

In particular, the conditional mean generation size and the conditional non-extinction probability at n given the environment are

$$E(Z_n | f_0, f_1, \dots) = f'_0(1)f'_1(1) \dots f'_{n-1}(1) \tag{1.3}$$

and

$$P(Z_n > 0 | f_0, f_1, \dots) = 1 - f_0(f_1(\dots f_{n-1}(0) \dots)). \tag{1.4}$$

If the random generating functions f_n are i.i.d. and if $E \log f'_0(1)$ exists, then, by (1.3) and the law of large numbers,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log E(Z_n | f_0, f_1, \dots) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \log f'_{i-1}(1) \stackrel{\text{a.s.}}{=} E \log f'(1),$$

where f denotes a random generating function with the common distribution of the f_n , and $f'(1)$ is the conditional mean number of children per particle.

Here we study the *subcritical case* $E \log f'(1) < 0$, where the conditional mean generation size at n decays exponentially for almost every environment. We determine the exact asymptotic of the non-extinction probability at n and show that Z_n has a non-degenerate conditional limit law. For convenience we assume throughout that $P(f'(1) = 0) = 0$. [Note that if $P(f'(1) \neq 0) = q < 1$, then $\mathcal{L}(Z_n) = (1 - q^n)\delta_0 + q^n \mathcal{L}(Z_n^\circ)$, where $(Z_n^\circ)_{n \geq 0}$ is a branching process in i.i.d. random environment $(f_n^\circ)_{n \geq 0}$ with $\mathcal{L}(f^\circ) = \mathcal{L}(f | f'(1) \neq 0)$. In particular, $P(Z_n > 0) = q^n P(Z_n^\circ > 0)$ and $\mathcal{L}(Z_n | Z_n > 0) = \mathcal{L}(Z_n^\circ | Z_n^\circ > 0)$.]

As was first observed by Afanasyev [1] and later independently by Dekking [7], the decay rate of the non-extinction probability of a subcritical branching process in

i.i.d. random environment depends on the sign of $E[f'(1) \log f'(1)]$. In the *strongly subcritical case*, where $E[f'(1) \log f'(1)] < 0$, the first moment estimate $P(Z_n > 0) \leq EZ_n = (Ef'(1))^n$ gives the right decay of the non-extinction probability up to a constant and the conditional limit law of Z_n has finite mean.

THEOREM 1.1 (Strongly subcritical case). – *Let*

$$E[f'(1) \log f'(1)] < 0 \tag{1.5}$$

and assume that

$$E[Z_1 \log^+ Z_1] < \infty. \tag{1.6}$$

Then, as $n \rightarrow \infty$,

$$P(Z_n > 0) \sim c_1 (Ef'(1))^n \tag{1.7}$$

for some $0 < c_1 \leq 1$. Moreover,

$$\lim_{n \rightarrow \infty} P(Z_n = k \mid Z_n > 0) = q_1(k), \quad k \geq 1, \tag{1.8}$$

where

$$\sum_{k=1}^{\infty} q_1(k) = 1 \quad \text{and} \quad \sum_{k=1}^{\infty} kq_1(k) < \infty.$$

The asymptotic (1.7) is due to Guivarc’h and Liu (Theorem 1.2(a) in [14]). It was originally proved by D’Souza and Hambly under an extra moment assumption. Observe that, by means of Jensen’s inequality, condition (1.5) implies $Ef'(1) < 1$ and subcriticality.

In the *intermediate subcritical case*, where $E[f'(1) \log f'(1)] = 0$, the first moment estimate still gives the right exponential rate of decay of the non-extinction probability at n , but differs from the exact asymptotic by a factor of order $n^{-1/2}$.

THEOREM 1.2 (Intermediate subcritical case). – *Let*

$$E \log f'(1) < 0, \quad E[f'(1) \log f'(1)] = 0 \tag{1.9}$$

and assume that the following integrability conditions are satisfied,

$$E[f'(1) \log^2 f'(1)] < \infty, \quad E[(1 + \log^- f'(1)) f''(1)] < \infty. \tag{1.10}$$

Then, as $n \rightarrow \infty$,

$$P(Z_n > 0) \sim c_2 n^{-1/2} (Ef'(1))^n \tag{1.11}$$

for some $0 < c_2 < \infty$. Moreover,

$$\lim_{n \rightarrow \infty} P(Z_n = k \mid Z_n > 0) = q_2(k), \quad k \geq 1, \tag{1.12}$$

where $\sum_{k=1}^{\infty} q_2(k) = 1$.

In the *weakly subcritical case*, where $E[f'(1) \log f'(1)] > 0$, the situation is different. The exponential rate of decay of the non-extinction probability at n is strictly less than $E f'(1)$ (which might possibly be > 1). Let

$$\gamma := \inf_{0 \leq \theta \leq 1} E f'(1)^\theta$$

and let $\alpha \in [0, 1]$ be defined through

$$\gamma = E f'(1)^\alpha.$$

For simplicity we assume the following aperiodicity condition: The distribution of $\log f'(1)$ is not supported by any non-centered lattice, i.e.,

$$P(\log f'(1) \in x + \lambda\mathbb{Z}) < 1, \quad \forall 0 < x < \lambda. \tag{1.13}$$

THEOREM 1.3 (Weakly subcritical case). – *Let*

$$E \log f'(1) < 0 \quad \text{and} \quad 0 < E[f'(1) \log f'(1)] < \infty. \tag{1.14}$$

Assume that (1.13) holds and that the following integrability conditions are satisfied,

$$E \frac{f''(1)}{f'(1)^{1-\alpha}} < \infty, \quad E \frac{f''(1)}{f'(1)^{2-\alpha}} < \infty. \tag{1.15}$$

Then, as $n \rightarrow \infty$,

$$P(Z_n > 0) \sim c_3 n^{-3/2} \gamma^n \tag{1.16}$$

for some $0 < c_3 < \infty$. Moreover,

$$\lim_{n \rightarrow \infty} P(Z_n = k \mid Z_n > 0) = q_3(k), \quad k \geq 1, \tag{1.17}$$

where $\sum_{k=1}^\infty q_3(k) = 1$.

Note that condition (1.14) implies $\gamma < (1 \wedge E f'(1))$ and $0 < \alpha < 1$.

In the special case, where the f_n are linear fractional with probability one, the asymptotic behavior of $P(Z_n > 0)$ has been determined by Afanasyev [1] under similar integrability assumptions (see Lemma 11 in [11]). By a comparison argument due to Agresti [3], asymptotics for the linear fractional case imply upper and lower bounds for general i.i.d. environment. Assuming $E f'(1) < \infty$ Liu [18] showed that $\lim_{n \rightarrow \infty} P(Z_n > 0)^{1/n} = \gamma$ for a branching process in i.i.d. environment with countable state space, extending a result of Dekking [7]. D’Souza and Hambly [9] obtained this conclusion for branching processes in certain stationary and ergodic environments. In the linear fractional case also some functional limit theorems and results for the reduced process have been derived; see [2,11].

The starting point of our analysis is a formula for the conditional non-extinction probability at n in terms of a random walk which has been obtained in [13]. This

result and some frequently used estimates are collected in Section 2. Sections 3 and 4 demonstrate Theorems 1.1 to 1.3. The result in the strongly subcritical case follows rather easily from a change-of-measure argument. In the intermediate and the weakly subcritical case we need to study the behavior of the associated random walk when conditioned on rare behavior.

2. Preliminaries

In this section we state some preliminary results. We introduce notations

$$\begin{aligned}
 f_{k,\ell} &:= \begin{cases} f_k \circ f_{k+1} \circ \dots \circ f_{\ell-1}, & k < \ell; \\ f_{k-1} \circ f_{k-2} \circ \dots \circ f_{\ell}, & k > \ell; \\ \text{id}, & k = \ell. \end{cases} \\
 g_k(s) &:= \frac{1}{1 - f_k(s)} - \frac{1}{f'_k(1)(1 - s)}, \quad 0 \leq s < 1. \\
 \eta_{k,n}(s) &:= g_k(f_{k+1,n}(s)), \quad 0 \leq s < 1, \quad 0 \leq k \leq n - 1. \\
 S_n &:= \log f'_{0,n}(1), \quad n \geq 0.
 \end{aligned} \tag{2.1}$$

Note that if the f_n are assumed i.i.d., then $(S_n)_{n \geq 0}$ is a random walk started at 0 with increments $X_n := \log f'_{n-1}(1), n \geq 1$.

The following formula for the conditional generating function of Z_n has been obtained in [13].

LEMMA 2.1. – *Let $f_k \neq 1, 0 \leq k \leq n - 1$. Then, for every $0 \leq s < 1$,*

$$1 - f_{0,n}(s) = \left(\frac{\exp(-S_n)}{1 - s} + \sum_{k=0}^{n-1} \eta_{k,n}(s) \exp(-S_k) \right)^{-1}. \tag{2.2}$$

In particular,

$$\mathbb{P}(Z_n > 0) = \mathbb{E} \left(\sum_{k=0}^n \eta_{k,n} \exp(-S_k) \right)^{-1}, \tag{2.3}$$

where $\eta_{k,n} := \eta_{k,n}(0), 0 \leq k \leq n - 1$, and $\eta_{n,n} := 1$.

The following bound for the random coefficients $\eta_{k,n}$ in (2.3) is from Lemma 2.1 in [13] (recall (2.1)).

LEMMA 2.2. – *Let $f \neq 1$ be a probability generating function with $f''(1) < \infty$. Then, for every $0 \leq s < 1$,*

$$0 \leq g(s) \leq \frac{f''(1)}{f'(1)^2}, \tag{2.4}$$

where

$$g(s) = \frac{1}{1 - f(s)} - \frac{1}{f'(1)(1 - s)}.$$

We finally state the following monotonicity property.

LEMMA 2.3. – *Let $(f_n)_{n \geq 0}$ be a sequence of probability generating functions. Then, for every $k \geq 0$ and $0 \leq s \leq 1$,*

$$\exp(-S_{k+1})(1 - f_{k+1,0}(s)) \leq \exp(-S_k)(1 - f_{k,0}(s)) \leq 1 - s. \tag{2.5}$$

In particular,

$$\lim_{n \rightarrow \infty} \exp(-S_n)(1 - f_{n,0}(s)) \text{ exists} \tag{2.6}$$

for every $0 \leq s \leq 1$.

Proof. – By convexity of f_k we have for every $0 \leq s \leq 1$,

$$\begin{aligned} 1 - f_{k+1,0}(s) &= 1 - f_k(f_{k,0}(s)) \\ &\leq f'_k(1)(1 - f_{k,0}(s)) = \exp(S_{k+1} - S_k)(1 - f_{k,0}(s)). \end{aligned}$$

For the second inequality recall that $f_{0,0} = \text{id}$.

Remark. – Observe that

$$\frac{\exp(S_k)}{1 - f_{0,k}(0)} = \frac{\mathbb{E}(Z_k \mid f_0, f_1, \dots)}{\mathbb{P}(Z_k > 0 \mid f_0, f_1, \dots)} = \mathbb{E}(Z_k \mid Z_k > 0; f_0, f_1, \dots).$$

Hence, taking $s = 0$ in (2.5) we see that, given the environment $(f_{k-j})_{0 \leq j \leq k}$ and non-extinction at $k + 1$, the conditional mean generation size at $k + 1$ is larger than the conditional mean generation size at k given the environment $(f_{k-1-j})_{0 \leq j \leq k-1}$ and non-extinction at k . In fact, a stronger statement holds: The first conditional generation size stochastically dominates the second. This monotonicity is an immediate consequence of the backward construction of the conditional family tree produced by the branching process (see [12] for the special case of classical Galton–Watson processes).

3. Strongly subcritical and intermediate subcritical cases

The main objective of this section is to prove Theorem 1.2. However, we will also demonstrate the second part of Theorem 1.1 and give a representation of the constant c_1 in the asymptotic (1.7). Our first step is to extract the exponential term of the non-extinction probability at generation n .

Suppose that $\mathbb{E}f'(1) < \infty$. Then we can introduce the random probability generating function \bar{f} with distribution given by

$$\mathbb{E}\psi(\bar{f}) = \frac{\mathbb{E}[f'(1)\psi(f)]}{\mathbb{E}f'(1)} \tag{3.1}$$

for every non-negative measurable function ψ on

$$\Delta := \left\{ f(s) = \sum_{k=0}^{\infty} p_k s^k : p_k \geq 0, f(1) = 1 \right\}.$$

We identify the set of functions Δ with the Polish space of probability measures on $\mathbb{N} := \{0, 1, 2, \dots\}$. If we refer to the mean of the corresponding offspring distribution as the “size” of a probability generating function, then the law of \bar{f} is what is called the *size-biased* distribution of f . We remark that the law of \bar{f} can also be viewed as the measure on Δ induced by tilting the law of $\log f'(1)$. Observe also that the conditional distribution of \bar{f} given $\bar{f}'(1)$ is the same as the conditional distribution of f given $f'(1)$ (see the monograph [16] for properties of probability measures on Polish spaces as existence of regular conditional distributions). If $\bar{f}_0, \bar{f}_1, \dots$ are i.i.d. copies of the random generating function \bar{f} , then

$$E\psi(\bar{f}_0, \dots, \bar{f}_{n-1}) = \frac{E[\exp(S_n)\psi(f_0, \dots, f_{n-1})]}{(E f'(1))^n} \tag{3.2}$$

for every non-negative measurable ψ on $\Delta^n, n \geq 1$. (We note that the change of measure in (3.1) and (3.2) is the same as in [14].)

Using the exchangeability of the f_n and taking $\psi(f_0, \dots, f_{n-1}) = \exp(-S_n)(1 - f_{n,0}(s))$ in (3.2) we obtain

$$1 - E f_{0,n}(s) = (E f'(1))^n E[\exp(-\bar{S}_n)(1 - \bar{f}_{n,0}(s))]. \tag{3.3}$$

(We use notation $\bar{S}_n, \bar{f}_{n,0}, \bar{g}_k$, etc., for the analogues of $S_n, f_{0,n}$ and g_k defined in terms of the $\bar{f}_j, j \geq 0$; e.g., $\bar{S}_n := \sum_{j=1}^n \log \bar{f}'_{j-1}(1)$). In the sequel we will also introduce random generating functions \bar{f}_j, \hat{f}_j and $\check{f}_j, j \geq 0$. Notation $\bar{S}_n, \bar{g}_k, \bar{\eta}_{k,n}(s)$, etc., will then be used for the corresponding random quantities without further mentioning.)

By (2.2), replacing f_k by \bar{f}_{n-k-1} ,

$$\begin{aligned} \exp(-\bar{S}_n)(1 - \bar{f}_{n,0}(s)) &= \left(\frac{1}{1-s} + \sum_{k=0}^{n-1} \exp(\bar{S}_{n-k}) \bar{g}_{n-1-k}(\bar{f}_{n-1-k,0}(s)) \right)^{-1} \\ &= \left(\frac{1}{1-s} + \sum_{k=1}^n \bar{\zeta}_{k-1}(s) \exp(\bar{S}_k) \right)^{-1}, \end{aligned} \tag{3.4}$$

where

$$\bar{\zeta}_k(s) := \bar{g}_k(\bar{f}_{k,0}(s)), \quad 0 \leq s < 1, k \geq 0. \tag{3.5}$$

Combining (3.3) and (3.4) gives

$$1 - E f_{0,n}(s) = (E f'(1))^n E \left(\frac{1}{1-s} + \sum_{k=1}^n \bar{\zeta}_{k-1}(s) \exp(\bar{S}_k) \right)^{-1}, \quad 0 \leq s < 1. \tag{3.6}$$

Take $s = 0$ in (3.6) and abbreviate $\bar{\zeta}_{k-1} := \bar{\zeta}_{k-1}(0)$ to obtain

$$P(Z_n > 0) = (E f'(1))^n E \left(1 + \sum_{k=1}^n \bar{\zeta}_{k-1} \exp(\bar{S}_k) \right)^{-1}. \tag{3.7}$$

Proof of Theorem 1.1. – The asymptotic (1.7) was proved in [14]. Note that, by (3.7),

$$c_1 = E \left(1 + \sum_{k=1}^{\infty} \bar{\zeta}_{k-1} \exp(\bar{S}_k) \right)^{-1}.$$

The second part of the theorem is an easy consequence of (1.7). Observe that the conditional generating function of Z_n given non-extinction at n is

$$\begin{aligned} E(s^{Z_n} | Z_n > 0) &= \frac{E s^{Z_n} - P(Z_n = 0)}{P(Z_n > 0)} \\ &= 1 - \frac{1 - E f_{0,n}(s)}{P(Z_n > 0)}, \quad 0 \leq s \leq 1. \end{aligned} \tag{3.8}$$

By (1.7) and (3.6), we have

$$\lim_{n \rightarrow \infty} E(s^{Z_n} | Z_n > 0) = 1 - c_1^{-1} E \left(\frac{1}{1-s} + \sum_{k=1}^{\infty} \bar{\zeta}_{k-1}(s) \exp(\bar{S}_k) \right)^{-1} \tag{3.9}$$

for every $0 \leq s < 1$. Hence,

$$q_1(k) := \lim_{n \rightarrow \infty} P(Z_n = k | Z_n > 0) \tag{3.10}$$

exists for every $k \geq 1$. Use (1.7) again to deduce

$$\limsup_{n \rightarrow \infty} E(Z_n | Z_n > 0) = \limsup_{n \rightarrow \infty} \frac{(E f'(1))^n}{P(Z_n > 0)} = c_1^{-1} < \infty, \tag{3.11}$$

which implies tightness of $\mathcal{L}(Z_n | Z_n > 0)$, $n \geq 0$. Consequently, $(q_1(k))_{k \geq 1}$ is a proper probability measure on the positive integers.

Finally, use (3.10), Fatou’s lemma and (3.11) to conclude

$$\begin{aligned} \sum_{k=1}^{\infty} k q_1(k) &= \sum_{k=1}^{\infty} \liminf_{n \rightarrow \infty} k P(Z_n = k | Z_n > 0) \\ &\leq \liminf_{n \rightarrow \infty} E(Z_n | Z_n > 0) \leq \limsup_{n \rightarrow \infty} E(Z_n | Z_n > 0) < \infty, \end{aligned}$$

which completes the proof of Theorem 1.1.

We now begin preparations of the proof of Theorem 1.2. By (1.4) and (3.3), to prove (1.11) it suffices to show

$$E[\exp(-\bar{S}_n)(1 - \bar{f}_{n,0}(0))] \sim c_2 n^{-1/2} \quad \text{as } n \rightarrow \infty \tag{3.12}$$

for some $0 < c_2 < \infty$. The key ingredient to establish (3.12) will again be formula (3.4). Our arguments will follow closely those in [13] where it is shown that in the critical case the non-extinction probability at n is asymptotically $cn^{-1/2}$ for some $0 < c < \infty$. (Observe that in the intermediate subcritical case the random walk $(\bar{S}_k)_{k \geq 0}$ has mean

zero. In particular, the random quantity in (3.12), resp. (3.4), resembles the non-extinction probability at n given the random environment $\bar{f}_0, \bar{f}_1, \dots$. However, note the opposite sign of the random walk and the difference in the random coefficients.) The basic idea to prove (3.12) is that only random walk paths $(\bar{S}_k)_{0 \leq k \leq n}$ which have a low maximum $\bar{M}_n := \max\{\bar{S}_k: 0 \leq k \leq n\}$ give a substantial contribution to the expectation in (3.12) (compare (3.4)). To make this intuition precise let $0 =: \bar{\sigma}_0 < \bar{\sigma}_1 < \bar{\sigma}_2 < \dots$ denote the strict ascending ladder epochs of the random walk $(\bar{S}_k)_{k \geq 0}$,

$$\bar{\sigma}_{j+1} := \min\{k > \bar{\sigma}_j: \bar{S}_k > \bar{S}_{\bar{\sigma}_j}\}, \quad j \geq 0,$$

and let

$$h(x) := \sum_{j=0}^{\infty} P(\bar{S}_{\bar{\sigma}_j} \leq x), \quad x \in \mathbb{R}.$$

The renewal function h is harmonic for the random walk $(-\bar{S}_k)_{k \geq 0}$ (see [6]; and Chapter 12 in [10] for background material on fluctuation theory of random walks), i.e., if $\bar{X} \stackrel{d}{=} \log \bar{f}'(1)$, then

$$h(x) = E h(x - \bar{X}), \quad x \geq 0. \tag{3.13}$$

By (1.9), the random walk $(\bar{S}_k)_{k \geq 0}$ has mean zero. The first part of assumption (1.10) and relation (3.1) imply that the walk has positive finite variance,

$$0 < E \bar{X}^2 = E \log^2 \bar{f}'(1) = (E f'(1))^{-1} E[f'(1) \log^2 f'(1)] < \infty.$$

Hence, $h(x) \sim cx$ as $x \rightarrow \infty$ for some $c > 0$ by the elementary renewal theorem.

The probability that the random walk path $(\bar{S}_k)_{0 \leq k \leq n}$ stays below x decays as $n^{-1/2}$. More precisely, if we write $\bar{m}_n(x) := P(\bar{M}_n \leq x)$ then there exist positive finite constants c' and c'' such that (see Theorem A in [17]), as $n \rightarrow \infty$,

$$\bar{m}_n(x) \sim c' h(x) n^{-1/2}, \quad x \geq 0; \tag{3.14}$$

$$\bar{m}_n(x) \leq c'' h(x) n^{-1/2}, \quad x \geq 0, \quad n \geq 1. \tag{3.15}$$

The asymptotic behavior of the initial piece of the sequence $(\bar{f}_k)_{k \geq 0}$, given the associated random walk has a low maximum until n , is described by the following lemma.

LEMMA 3.1. – *Let $x \geq 0$ and suppose that*

$$E \log \bar{f}'(1) = 0, \quad 0 < E \log^2 \bar{f}'(1) < \infty. \tag{3.16}$$

Then,

$$\begin{aligned} & \lim_{n \rightarrow \infty} E(\psi(\bar{f}_0, \dots, \bar{f}_{k-1}) \mid \bar{M}_n \leq x) \\ & = h(x)^{-1} E[\psi(\bar{f}_0, \dots, \bar{f}_{k-1}) h(x - \bar{S}_k); \bar{M}_k \leq x] \end{aligned} \tag{3.17}$$

for every bounded measurable function ψ on Δ^k , $k \geq 1$.

Proof. – Fix $x \geq 0$ and let $1 \leq k \leq n$. With no loss of generality we assume $0 \leq \psi \leq 1$. Conditioning on $\bar{f}_0, \dots, \bar{f}_{k-1}$ shows that

$$\begin{aligned} & \mathbb{E}[\psi(\bar{f}_0, \dots, \bar{f}_{k-1}); \bar{M}_n \leq x] \\ &= \mathbb{E}[\psi(\bar{f}_0, \dots, \bar{f}_{k-1})\bar{m}_{n-k}(x - \bar{S}_k); \bar{M}_k \leq x]. \end{aligned} \tag{3.18}$$

By Fatou’s lemma and (3.14),

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \mathbb{E}(\psi(\bar{f}_0, \dots, \bar{f}_{k-1}) \mid \bar{M}_n \leq x) \\ & \geq \mathbb{E} \left[\psi(\bar{f}_0, \dots, \bar{f}_{k-1}) \liminf_{n \rightarrow \infty} \frac{\bar{m}_{n-k}(x - \bar{S}_k)}{\bar{m}_n(x)}; \bar{M}_k \leq x \right] \\ & = h(x)^{-1} \mathbb{E}[\psi(\bar{f}_0, \dots, \bar{f}_{k-1})h(x - \bar{S}_k); \bar{M}_k \leq x]. \end{aligned} \tag{3.19}$$

Replacing ψ by $1 - \psi$ in (3.19) gives (3.17).

From (3.13) and the fact that $h(x) = 0$ for $x < 0$ we see that

$$h(x)^{-1} \mathbb{E}[h(x - \bar{S}_k); \bar{M}_k \leq x] = 1$$

for every $x \geq 0$ and $k \in \mathbb{N}$. Hence, for each $x \geq 0$, the right-hand side of (3.17) in particular specifies a probability measure \mathcal{L}_x on $\Delta^{\mathbb{N}} := \{(f_0, f_1, \dots) : f_i \in \Delta\}$. We write $(\tilde{f}_k)_{k \geq 0}$ for a sequence of random generating functions with distribution \mathcal{L}_x and denote expectation with respect to \mathcal{L}_x by E_x , i.e.,

$$E_x \psi(\tilde{f}_0, \dots, \tilde{f}_{k-1}) = h(x)^{-1} \mathbb{E}[\psi(\bar{f}_0, \dots, \bar{f}_{k-1})h(x - \bar{S}_k); \bar{M}_k \leq x] \tag{3.20}$$

for every non-negative measurable function ψ on $\Delta^k, k \geq 1$. We remark that, for each $x \geq 0$, the process $(\tilde{Y}_k)_{k \geq 0}$ with increments $\tilde{Y}_{k+1} - \tilde{Y}_k := -\log \tilde{f}'_k(1), k \geq 0$, is a time-homogeneous Markov chain under \mathcal{L}_x . If we choose the initial distribution $\mathcal{L}_x(\tilde{Y}_0)$ to be δ_x , then the chain has state space $[0, \infty)$ and its transition kernel does not depend on x .

Our next lemma describes the limit of the conditional expectation of the quantity of interest in (3.4) given the event that the random walk path $(\bar{S}_k)_{k \geq 0}$ stays below x until some time in the near future. For $x \geq 0$ and $0 \leq s < 1$ let (compare (2.6) and (3.4))

$$\begin{aligned} \beta(x, s) &:= \lim_{n \rightarrow \infty} E_x [\exp(\tilde{Y}_n - x)(1 - \tilde{f}_{n,0}(s))] \\ &= E_x \left(\frac{1}{1-s} + \sum_{k=1}^{\infty} \tilde{\zeta}_{k-1}(s) \exp(x - \tilde{Y}_k) \right)^{-1} \end{aligned} \tag{3.21}$$

and let $[r]$ denote the integer part of $r \in \mathbb{R}_0^+$.

LEMMA 3.2. – Assume (3.16) and suppose

$$\mathbb{E} \left[(1 + \log^- \bar{f}'(1)) \frac{\bar{f}''(1)}{\bar{f}'(1)} \right] < \infty. \tag{3.22}$$

Then, for every $x \geq 0$ and $0 \leq s < 1$, and any $\rho > 1$,

$$\lim_{n \rightarrow \infty} \mathbb{E}(\exp(-\bar{S}_n)(1 - \bar{f}_{n,0}(s)) \mid \bar{M}_{[\rho n]} \leq x) = \beta(x, s). \tag{3.23}$$

Proof. – Fix $x \geq 0$, $0 \leq s < 1$ and $\rho > 1$. By (3.4),

$$\exp(-\bar{S}_n)(1 - \bar{f}_{n,0}(s)) = \left(\frac{1}{1-s} + \sum_{k=1}^n \bar{\zeta}_{k-1}(s) \exp(\bar{S}_k) \right)^{-1} \leq 1 - s.$$

Hence, by Lemma 3.1, (3.20) and (3.21),

$$\begin{aligned} & \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{E}(\exp(-\bar{S}_m)(1 - \bar{f}_{m,0}(s)) \mid \bar{M}_{\lfloor \rho n \rfloor} \leq x) \\ &= \mathbb{E}_x \left(\frac{1}{1-s} + \sum_{k=1}^{\infty} \tilde{\zeta}_{k-1}(s) \exp(x - \tilde{Y}_k) \right)^{-1} = \beta(x, s). \end{aligned} \tag{3.24}$$

Now let $1 \leq m \leq n$. Use first $a^{-1} - b^{-1} \leq b - a$ for $1 \leq a \leq b$ and relation (3.18), and then (3.14), (3.15) and (3.20) to estimate

$$\begin{aligned} & \mathbb{E}(\exp(-\bar{S}_m)(1 - \bar{f}_{m,0}(s)) - \exp(-\bar{S}_n)(1 - \bar{f}_{n,0}(s)) \mid \bar{M}_{\lfloor \rho n \rfloor} \leq x) \\ & \leq \mathbb{E} \left(\sum_{k=m+1}^n \bar{\zeta}_{k-1}(s) \exp(\bar{S}_k) \mid \bar{M}_{\lfloor \rho n \rfloor} \leq x \right) \\ & = \frac{1}{\bar{m}_{\lfloor \rho n \rfloor}(x)} \mathbb{E} \left[\sum_{k=m+1}^n \bar{\zeta}_{k-1}(s) \exp(\bar{S}_k) \bar{m}_{\lfloor \rho n \rfloor - n}(x - \bar{S}_n); \bar{M}_n \leq x \right] \\ & \leq \bar{c} \left(\frac{\rho}{\rho - 1} \right)^{1/2} h(x)^{-1} \mathbb{E} \left[\sum_{k=m+1}^n \bar{\zeta}_{k-1}(s) \exp(\bar{S}_k) h(x - \bar{S}_n); \bar{M}_n \leq x \right] \\ & = \bar{c} \left(\frac{\rho}{\rho - 1} \right)^{1/2} \mathbb{E}_x \left(\sum_{k=m+1}^n \tilde{\zeta}_{k-1}(s) \exp(x - \tilde{Y}_k) \right) \end{aligned} \tag{3.25}$$

for some finite constant \bar{c} . In view of (3.24) assertion (3.23) will follow if we show that the right-hand side of (3.25) tends to 0 as first n and then m goes to ∞ . Now recall from Lemma 2.2 that

$$\sup_{0 \leq s < 1} \tilde{\zeta}_k(s) \leq \sup_{0 \leq s < 1} \tilde{g}_k(s) \leq \frac{\tilde{f}_k''(1)}{\tilde{f}_k'(1)^2} =: \tilde{\eta}_k, \quad k \geq 0. \tag{3.26}$$

Hence, to complete the proof of (3.23) it suffices to verify the claim of the following lemma. \square

LEMMA 3.3. – Assume (3.16) and (3.22). Then, for every $x \geq 0$,

$$\mathbb{E}_x \left(\sum_{k=1}^{\infty} \tilde{\eta}_{k-1} \exp(-\tilde{Y}_k) \right) < \infty. \tag{3.27}$$

Proof. – Fix $x \geq 0$ and let $k \geq 1$. By (3.20),

$$\exp(x) \mathbb{E}_x [\tilde{\eta}_{k-1} \exp(-\tilde{Y}_k)] = h(x)^{-1} \mathbb{E}[\tilde{\eta}_{k-1} \exp(\bar{S}_k) h(x - \bar{S}_k); \bar{M}_k \leq x]. \tag{3.28}$$

The fact that the renewal process $(\bar{S}_{\bar{\sigma}_j})_{j \geq 0}$ is zero delayed implies $h(x) - h(y) \leq h(x - y)$ for every $x, y \geq 0$. Hence, by the elementary renewal theorem,

$$h(x) - h(y) \leq c(1 + (x - y)^+)$$

for some $1 \leq c < \infty$. Plugging this estimate into (3.28) gives

$$\begin{aligned} & \exp(x)E_x[\tilde{\eta}_{k-1} \exp(-\tilde{Y}_k)] \\ & \leq h(x)^{-1}E[\tilde{\eta}_{k-1} \exp(\bar{S}_k)(h(x - \bar{S}_{k-1}) + c(1 + \bar{X}_k^-)); \bar{M}_k \leq x] \\ & \leq 2ch(x)^{-1}E[\tilde{\eta}_{k-1} \exp(\bar{S}_k)h(x - \bar{S}_{k-1})(1 + \bar{X}_k^-); \bar{M}_{k-1} \leq x], \end{aligned} \tag{3.29}$$

where for the second inequality recall that $c \geq 1$ and $h(y) \geq 1$ for $y \geq 0$. Using first the independence of the \tilde{f}_k and then (3.20) we obtain

$$\begin{aligned} & \exp(x)E_x[\tilde{\eta}_{k-1} \exp(-\tilde{Y}_k)] \\ & \leq 2cE[\tilde{\eta}_{k-1} \exp(\bar{X}_k)(1 + \bar{X}_k^-)]h(x)^{-1}E[\exp(\bar{S}_{k-1})h(x - \bar{S}_{k-1}); \bar{M}_{k-1} \leq x] \\ & = 2c \exp(x)E[\tilde{\eta}_0 \tilde{f}'_0(1)(1 + \log^- \tilde{f}'_0(1))]E_x[\exp(-\tilde{Y}_{k-1})]. \end{aligned}$$

The first expectation being finite by assumption (3.22), assertion (3.27) follows from Lemma 3.1 in [13] (they use notation $\hat{E}_x \exp(-S_k)$ for $E_x[\exp(x - \tilde{Y}_k)]$). Hence, we have established Lemmas 3.3 and 3.2.

LEMMA 3.4. – Assume (3.16) and (3.22). Then, for every $x \geq 0$ and $0 \leq s < 1$,

$$\lim_{n \rightarrow \infty} n^{1/2}E[\exp(-\bar{S}_n)(1 - \tilde{f}_{n,0}(s)); \bar{M}_n \leq x] = c'h(x)\beta(x, s). \tag{3.30}$$

Proof. – Fix $x \geq 0$ and $0 \leq s < 1$. By Lemma 3.2 and the asymptotic (3.14) we have

$$\lim_{n \rightarrow \infty} n^{1/2}E[\exp(-\bar{S}_n)(1 - \tilde{f}_{n,0}(s)); \bar{M}_{\lfloor \rho n \rfloor} \leq x] = c'h(x)\beta(x, s)\rho^{-1/2} \tag{3.31}$$

for any $\rho > 1$. To get rid of ρ recall from (2.5) that the integrand is bounded by 1. Hence,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} n^{1/2}E[\exp(-\bar{S}_n)(1 - \tilde{f}_{n,0}(s)); \bar{M}_n \leq x < \bar{M}_{\lfloor \rho n \rfloor}] \\ & \leq \limsup_{n \rightarrow \infty} n^{1/2}(\bar{m}_n(x) - \bar{m}_{\lfloor \rho n \rfloor}(x)) = c'h(x)(1 - \rho^{-1/2}), \end{aligned} \tag{3.32}$$

where the last equality again follows from (3.14). Letting ρ tend to 1 in (3.31) and (3.32), respectively, establishes the claim of Lemma 3.4.

Note that identity (3.30) shows that

$$h(x)\beta(x, s) \text{ increases with } x \text{ for every } 0 \leq s < 1. \tag{3.33}$$

Our final lemma in this section makes precise the statement that only random walks with a very low maximum give a substantial contribution to the expectation in (3.12).

LEMMA 3.5. – Assume (3.16) and (3.22). Then, for every $0 \leq s \leq 1$,

$$\lim_{x \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{1/2}E[\exp(-\bar{S}_n)(1 - \tilde{f}_{n,0}(s)); \bar{M}_n > x] = 0. \tag{3.34}$$

Proof. – Fix $0 \leq s \leq 1$ and let $x \geq 0$. By Lemma 2.3,

$$\begin{aligned} \exp(-\bar{S}_n)(1 - \bar{f}_{n,0}(s)) &= \min_{0 \leq k \leq n} \exp(-\bar{S}_k)(1 - \bar{f}_{k,0}(s)) \\ &\leq \min_{0 \leq k \leq n} \exp(-\bar{S}_k) = \exp(-\bar{M}_n). \end{aligned}$$

Consequently,

$$\begin{aligned} \mathbb{E}[\exp(-\bar{S}_n)(1 - \bar{f}_{n,0}(s)); \bar{M}_n > x] &\leq \mathbb{E}[\exp(-\bar{M}_n); \bar{M}_n > x] \\ &\leq \sum_{k \geq \lfloor x \rfloor} \exp(-k) \mathbb{P}(k < \bar{M}_n \leq k + 1) \\ &\leq \sum_{k \geq \lfloor x \rfloor} \exp(-k) \bar{m}_n(k + 1). \end{aligned}$$

Now use (3.15) to deduce

$$\limsup_{n \rightarrow \infty} n^{1/2} \mathbb{E}[\exp(-\bar{S}_n)(1 - \bar{f}_{n,0}(s)); \bar{M}_n > x] \leq c'' \sum_{k \geq \lfloor x \rfloor} \exp(-k) h(k + 1).$$

Since the renewal function h grows only linearly, the claim of the lemma follows as $x \rightarrow \infty$.

Proof of Theorem 1.2. – Recall from (3.12) that to prove (1.11) it suffices to show that

$$c_2 := \lim_{n \rightarrow \infty} n^{1/2} \mathbb{E}[\exp(-\bar{S}_n)(1 - \bar{f}_{n,0}(0))]$$

is positive and finite. Relation (3.1) shows that under the assumptions of Theorem 1.2 conditions (3.16) and (3.22) are satisfied. Hence, we may apply Lemmas 3.4 and 3.5 to conclude

$$\begin{aligned} &\lim_{n \rightarrow \infty} n^{1/2} \mathbb{E}[\exp(-\bar{S}_n)(1 - \bar{f}_{n,0}(s))] \\ &= \lim_{x \rightarrow \infty} \lim_{n \rightarrow \infty} n^{1/2} \mathbb{E}[\exp(-\bar{S}_n)(1 - \bar{f}_{n,0}(s)); \bar{M}_n \leq x] \\ &= c' \lim_{x \rightarrow \infty} h(x) \beta(x, s) \end{aligned} \tag{3.35}$$

for every $0 \leq s < 1$. Recalling the representation of $\beta(x, s)$ in (3.21) we see from (3.26) and Lemma 3.3 that $\beta(x, s) > 0$ for every $x \geq 0$ and $0 \leq s < 1$. Using the monotonicity (3.33) we see that

$$\lim_{x \rightarrow \infty} h(x) \beta(x, s) > 0$$

for every $0 \leq s < 1$. Finally, note that Lemma 3.4 implies

$$\begin{aligned} &\lim_{n \rightarrow \infty} n^{1/2} \mathbb{E}[\exp(-\bar{S}_n)(1 - \bar{f}_{n,0}(s))] \\ &\leq c' h(x) \beta(x, s) + \limsup_{n \rightarrow \infty} n^{1/2} \mathbb{E}[\exp(-\bar{S}_n)(1 - \bar{f}_{n,0}(s)); \bar{M}_n > x] \end{aligned}$$

for every $x \geq 0$ and $0 \leq s < 1$. Application of Lemma 3.5 shows that c_2 is finite which completes our proof of the first part of Theorem 1.2.

For the second part observe that (recall (3.3), (3.8) and (3.35), and use exchangeability of the f_j)

$$\begin{aligned} \lim_{n \rightarrow \infty} E(s^{Z_n} \mid Z_n > 0) &= 1 - \lim_{n \rightarrow \infty} \frac{E[\exp(-\bar{S}_n)(1 - \bar{f}_{n,0}(s))]}{E[\exp(-\bar{S}_n)(1 - \bar{f}_{n,0}(0))]} \\ &= 1 - \lim_{x \rightarrow \infty} \frac{\beta(x, s)}{\beta(x, 0)} \end{aligned}$$

for every $0 \leq s < 1$. Hence,

$$q_2(k) := \lim_{n \rightarrow \infty} P(Z_n = k \mid Z_n > 0) \text{ exists}$$

for each $k \geq 1$.

We finally have to verify that the $q_2(k)$ sum to one. By (1.11), proving tightness of $\mathcal{L}(Z_n \mid Z_n > 0)$ amounts to show that

$$\lim_{x \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{1/2} (E f'(1))^{-n} P(Z_n \geq x) = 0. \tag{3.36}$$

For $x \geq 0$ and $n \geq 1$ let

$$J_{n,x}(f_0, \dots, f_{n-1}) := P(Z_n \geq x \mid f_0, f_1, \dots).$$

By exchangeability of the f_j and identity (1.4) we have

$$\begin{aligned} P(Z_n \geq x) &= E[J_{n,x}(f_{n-1}, \dots, f_0)] \\ &\leq E[J_{n,x}(f_{n-1}, \dots, f_0); M_n \leq x^{1/2}] + E[1 - f_{n,0}(0); M_n > x^{1/2}]. \end{aligned} \tag{3.37}$$

For the first term on the right-hand side of (3.37) first use Chebyshev’s inequality and then (3.2) and (3.15) to deduce

$$\begin{aligned} E[J_{n,x}(f_{n-1}, \dots, f_0); M_n \leq x^{1/2}] &\leq x^{-1} E[\exp(S_n); M_n \leq x^{1/2}] \\ &= x^{-1} (E f'(1))^n P(\bar{M}_n \leq x^{1/2}) \\ &\leq c'' x^{-1} h(x^{1/2}) n^{-1/2} (E f'(1))^n. \end{aligned} \tag{3.38}$$

Since h grows linearly, we have

$$\lim_{x \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{1/2} (E f'(1))^{-n} E[J_{n,x}(f_{n-1}, \dots, f_0); M_n \leq x^{1/2}] = 0. \tag{3.39}$$

For the second term on the right-hand side of (3.37) use relation (3.2) to obtain

$$E[1 - f_{n,0}(0); M_n > x^{1/2}] = (E f'(1))^n E[\exp(-\bar{S}_n)(1 - \bar{f}_{n,0}(0)); \bar{M}_n > x^{1/2}].$$

Application of Lemma 3.5 gives

$$\lim_{x \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{1/2} (E f'(1))^{-n} E[1 - f_{n,0}(0); M_n > x^{1/2}] = 0. \tag{3.40}$$

Combining (3.39) and (3.40) with (3.37) yields (3.36) completing our proof of Theorem 1.2.

4. The weakly subcritical case

In the weakly subcritical case the transformation which led to formula (3.7) is no longer helpful. Instead our analysis starts from the representation of the conditional generating function of Z_n in (2.2). Our proof of Theorem 1.3 will be similar to the proof of Theorem 1.2 in that the basic idea is that only random walk paths $(-S_k)_{0 \leq k \leq n}$ with a low maximum contribute substantially to the expectation. However, there is one essential difference. In Section 3 we dealt with the mean zero random walk $(\bar{S}_k)_{k \geq 0}$ conditioned to stay below some given level x until time n . In this case the state at n of the conditioned random walk is of order $n^{1/2}$. In terms of the branching process this behavior of the conditioned random walk path has the following interpretation: In a favorable environment the population will not get extinct once it has survived a dangerous initial period. In the weakly subcritical case we study the negative drift random walk $(S_k)_{k \geq 0}$. Here, at time n , the random walk path conditioned to stay above $-x$ returns to the vicinity of $-x$ (see Lemma 4.2 below for the precise statement). This means that in a favorable environment a population having survived the starting period might get extinct in the final generations prior to n , even though it has been very large in the meantime. This makes the analysis more involved.

We begin with recalling some facts about random walks with negative drift conditioned to stay positive (see [6,15,20]). We remark that in what follows next the fact that the random walk $(S_k)_{k \geq 0}$ has an interpretation in terms of the branching process is of no importance.

Let φ be the moment generating function of $X = \log f'(1)$,

$$\varphi(\theta) := E \exp(\theta X) = E f'(1)^\theta, \quad \theta \in \mathbb{R}.$$

Note that assumptions (1.13) and (1.14) from Theorem 1.3 imply

$$\varphi'(0) = EX < 0. \tag{4.1}$$

$$\varphi(\theta) < \infty \quad \text{for } 0 \leq \theta \leq 1. \tag{4.2}$$

$$\varphi'(\alpha) = 0 \quad \text{for some } 0 < \alpha < 1. \tag{4.3}$$

$$\text{The distribution of } X \text{ is not supported by any non-centered lattice.} \tag{4.4}$$

We introduce the random walk $(S_k^*)_{k \geq 0}$ started at 0 which plays the role of $(\bar{S}_k)_{k \geq 0}$ from the previous section. The increments $X_{k+1}^* = S_{k+1}^* - S_k^*, k \geq 0$, of this random walk have distribution

$$P(X^* \in dx) = \gamma^{-1} \exp(\alpha x) P(X \in dx), \tag{4.5}$$

where $\gamma := \varphi(\alpha)$. We note that X^* has finite absolute moments of all orders and that

$$EX^* = \gamma^{-1} \varphi'(\alpha) = 0, \quad 0 < EX^{*2} = \gamma^{-1} \varphi''(\alpha). \tag{4.6}$$

We introduce the following two renewal functions associated with the random walk $(S_k^*)_{k \geq 0}$. For $x \in \mathbb{R}$ let

$$u(x) := \sum_{i=0}^{\infty} P(S_{\tau_i^*}^* \geq -x) \quad \text{and} \quad v(x) := \sum_{j=0}^{\infty} P(S_{\sigma_j^*}^* \leq x), \tag{4.7}$$

where $0 =: \tau_0^* < \tau_1^* < \tau_2^* < \dots$ are the strict descending ladder epochs of the random walk $(S_k^*)_{k \geq 0}$,

$$\tau_{i+1}^* := \min\{n > \tau_i^* : S_n^* < S_{\tau_i^*}^*\}, \quad i \geq 0,$$

and $0 =: \sigma_0^* < \sigma_1^* < \sigma_2^* < \dots$ are the strict ascending ladder epochs of $(S_k^*)_{k \geq 0}$

$$\sigma_{j+1}^* := \min\{n > \sigma_j^* : S_n^* > S_{\sigma_j^*}^*\}, \quad j \geq 0.$$

Note that $u(x) = v(x) = 0$, if $x < 0$, and $u(x), v(x) \geq 1$, else. Under our assumptions both functions grow linearly as $x \rightarrow \infty$.

Let $L_n := \min\{S_k : 0 \leq k \leq n\}$, $n \geq 0$, denote the successive minima of the original random walk $(S_k)_{k \geq 0}$. The following result provides the analogues of (3.14) and (3.15).

LEMMA 4.1. – *Let conditions (4.1)–(4.4) be satisfied. Then, as $n \rightarrow \infty$,*

$$P(L_n \geq -x) \sim c \exp(\alpha x) u(x) n^{-3/2} \gamma^n, \quad x \geq 0, \tag{4.8}$$

for some $0 < c < \infty$.

Moreover, for each $\theta > \alpha$ there is a finite constant $c_\theta > 0$ such that

$$P(L_n \geq -x) \leq c_\theta \exp(\theta x) n^{-3/2} \gamma^n, \quad x \geq 0, n \geq 1. \tag{4.9}$$

Remark. – If the distribution of $X = \log f'(1)$ is supported by a lattice with maximal span λ , then (4.8) holds for every $x \in \lambda\mathbb{Z}$. For convenience, we will assume in the sequel that the distribution of X is non-lattice. We note, however, that all our arguments extend to the centered lattice case with only minor changes.

Proof. – Assertion (4.8) is the first part of Lemma 7 in [15] which also gives a representation of the constant c in terms of the renewal function v . The third part of the same lemma states that, for each $\theta > \alpha$,

$$E \exp(\theta L_n) \sim c'_\theta n^{-3/2} \gamma^n \quad \text{as } n \rightarrow \infty \tag{4.10}$$

for some $0 < c'_\theta < \infty$. Estimate (4.9) follows from (4.10) by means of Chebyshev’s inequality.

The following result (Lemma 8 in [15]) identifies the limiting distribution of the state of the random walk at n , if conditioned to stay above level $-x$ until time n .

LEMMA 4.2. – *Let $x \geq 0$ and assume that conditions (4.1)–(4.4) are satisfied. Then*

$$\lim_{n \rightarrow \infty} P(S_n + x \leq z \mid L_n \geq -x) = c_0^{-1} \int_0^z v(y) \exp(-\alpha y) dy, \quad z \geq 0, \tag{4.11}$$

where $c_0 := \int_0^\infty v(y) \exp(-\alpha y) dy$.

We will need more detailed information on the asymptotic behavior of the random walk path $(S_k)_{0 \leq k \leq n}$ given the event $\{L_n \geq -x\}$. A refined description of the path is as follows. The initial and the (time-reversed) final piece of the random walk path shifted by x converge in distribution towards independent homogeneous Markov chains on $[0, \infty)$ (the precise statement is in Proposition 4.3 below).

We first construct the limiting objects. The renewal functions u and v are harmonic for the random walks $(S_k^*)_{k \geq 0}$ and $(-S_k^*)_{k \geq 0}$, respectively (see, e.g., Lemma 1 in [15]),

$$u(x) = \mathbb{E}u(x + X^*), \quad x \geq 0; \tag{4.12}$$

$$v(x) = \mathbb{E}v(x - X^*), \quad x \geq 0. \tag{4.13}$$

Hence, we can define transition kernels Q and R on $[0, \infty)$ by

$$Q(y; y + B) := u(y)^{-1} \mathbb{E}[u(y + X^*); X^* \in B],$$

$$R(y; y + B) := v(y)^{-1} \mathbb{E}[v(y - X^*); -X^* \in B]$$

for every $y \geq 0$ and Borel set $B \subset [-y, \infty)$. We introduce two independent Markov chains $(\hat{Y}_k)_{k \geq 0}$ and $(\check{Y}_k)_{k \geq 0}$ with transition kernels Q and R , respectively. We write \mathcal{L}_π for the laws of the chains when started with initial distribution π . If $\pi = \delta_x$, we write \mathcal{L}_x .

The connection of the laws of $(\hat{Y}_k)_{k \geq 0}$ and $(\check{Y}_k)_{k \geq 0}$ with those of the random walks $(S_k^*)_{k \geq 0}$ and $(S_k)_{k \geq 0}$ is easily verified to be as follows (compare (3.20) and (4.5), and recall that $S_0^* = S_0 = 0$): For each $x \geq 0$,

$$\begin{aligned} & \mathbb{E}_x \psi(\hat{Y}_0, \dots, \hat{Y}_k) \\ &= u(x)^{-1} \mathbb{E}[\psi(x, x + S_1^*, \dots, x + S_k^*)u(x + S_k^*); L_k^* \geq -x] \\ &= \gamma^{-k} u(x)^{-1} \mathbb{E}[\psi(x, x + S_1, \dots, x + S_k) \exp(\alpha S_k)u(x + S_k); L_k \geq -x] \end{aligned} \tag{4.14}$$

and, letting ν denote the distribution on the right-hand side of (4.11),

$$\begin{aligned} & \mathbb{E}_\nu \psi(\check{Y}_0, \dots, \check{Y}_k) \\ &= \mathbb{E} \left[\psi(Y, Y - S_1^*, \dots, Y - S_k^*) \frac{v(Y - S_k^*)}{v(Y)}; M_k^* \leq Y \right] \\ &= \gamma^{-k} \mathbb{E} \left[\psi(Y, Y - S_1, \dots, Y - S_k) \exp(\alpha S_k) \frac{v(Y - S_k)}{v(Y)}; M_k \leq Y \right] \end{aligned} \tag{4.15}$$

for every non-negative measurable ψ on \mathbb{R}^{k+1} , $k \geq 0$. Here, the random variable Y in (4.15) is independent of $(S_k^*)_{k \geq 0}$ and $(S_k)_{k \geq 0}$ with distribution ν . The random variables L_n^* and M_n^* , $n \geq 0$, are the successive minima and maxima of the random walk $(S_k^*)_{k \geq 0}$.

The following result is Lemma 10 in [15]. We use \Rightarrow to denote weak convergence.

PROPOSITION 4.3. – *Let $x \geq 0$ and assume that (4.1)–(4.4) hold. Then, as $n \rightarrow \infty$,*

$$\begin{aligned} & \mathcal{L}(S_0 + x, \dots, S_\ell + x; S_n + x, \dots, S_{n-m} + x \mid L_n \geq -x) \\ & \Rightarrow \mathcal{L}_x(\hat{Y}_0, \dots, \hat{Y}_\ell) \otimes \mathcal{L}_\nu(\check{Y}_0, \dots, \check{Y}_m) \end{aligned} \tag{4.16}$$

for every $\ell, m \geq 0$.

Now recall that the random walk $(S_k)_{k \geq 0}$ was defined as a functional of the environment sequence $(f_k)_{k \geq 0}$. Hence, one might expect Proposition 4.3 to have a

correspondence in terms of the f_k . Indeed, the limit law (4.16) induces a conditional limit law for the initial and final pieces of the sequence of generating functions $(f_j)_{0 \leq j \leq n}$. We first define the limiting measure on $\Delta^{\mathbb{N}} \times \Delta^{\mathbb{N}}$ and then state the result. Let $(\hat{f}_k)_{k \geq 0}$ and $(\check{f}_k)_{k \geq 0}$ be two independent sequences of random probability generating functions. For $x \geq 0$ the law \mathcal{L}_x of the first sequence is given by

$$\begin{aligned} & E_x \psi(\hat{f}_0, \dots, \hat{f}_{\ell-1}) \\ &= \gamma^{-\ell} u(x)^{-1} E[\psi(f_0, \dots, f_{\ell-1}) \exp(\alpha S_\ell) u(x + S_\ell); L_\ell \geq -x] \end{aligned} \tag{4.17}$$

for every non-negative measurable function ψ on Δ^ℓ , $\ell \geq 1$. The distribution of the second sequence is characterized through

$$\begin{aligned} & E_v \psi(\check{f}_0, \dots, \check{f}_{m-1}) \\ &= \gamma^{-m} E\left[\psi(f_0, \dots, f_{m-1}) \exp(\alpha S_m) \frac{v(Y - S_m)}{v(Y)}; M_m \leq Y\right] \end{aligned} \tag{4.18}$$

for every non-negative measurable function ψ on Δ^m , $m \geq 1$, where Y is independent of $(f_k)_{k \geq 0}$ with distribution v .

PROPOSITION 4.4. – *Let $x \geq 0$ and assume that (1.13) and (1.14) hold. Then, as $n \rightarrow \infty$,*

$$\begin{aligned} & \mathcal{L}(f_0, \dots, f_{\ell-1}; f_{n-1}, \dots, f_{n-m}; S_n + x \mid L_n \geq -x) \\ & \Rightarrow \mathcal{L}_x(\hat{f}_0, \dots, \hat{f}_{\ell-1}) \otimes \mathcal{L}_v(\check{f}_0, \dots, \check{f}_{m-1}; \check{Y}_0) \end{aligned} \tag{4.19}$$

for every $\ell, m \geq 1$. The joint distribution \mathcal{L}_v of $(\check{f}_j)_{j \geq 0}$ and \check{Y}_0 is characterized through

$$\begin{aligned} & E_v \psi(\check{f}_0, \dots, \check{f}_{m-1}; \check{Y}_0) \\ &= \gamma^{-m} E\left[\psi(f_0, \dots, f_{m-1}; Y) \exp(\alpha S_m) \frac{v(Y - S_m)}{v(Y)}; M_m \leq Y\right] \end{aligned} \tag{4.20}$$

for every non-negative measurable function ψ on $\Delta^m \times \mathbb{R}_0^+$, $m \geq 1$, where Y is independent of $(f_k)_{k \geq 0}$ with distribution v .

Proof. – Fix $x \geq 0$ and bounded continuous functions ψ_1 and ψ_2 on Δ^ℓ and $\Delta^m \times \mathbb{R}_0^+$, respectively. Write $l_n(x) := P(L_n \geq -x)$ and

$$\psi_1^*(S_0 + x, \dots, S_\ell + x) := E(\psi_1(f_0, \dots, f_{\ell-1}) \mid S_0, \dots, S_\ell), \tag{4.21}$$

$$\psi_2^*(S_n + x, \dots, S_{n-m} + x) := E(\psi_2(f_{n-1}, \dots, f_{n-m}; S_n + x) \mid S_n, \dots, S_{n-m}). \tag{4.22}$$

Let $1 \leq l \leq n - m \leq n - 1$. By independence of the f_j and Proposition 4.3, we have

$$\begin{aligned} & E(\psi_1(f_0, \dots, f_{\ell-1}) \psi_2(f_{n-1}, \dots, f_{n-m}; S_n + x) \mid L_n \geq -x) \\ &= l_n(x)^{-1} E(E(\psi_1(f_0, \dots, f_{\ell-1}) \\ & \quad \times \psi_2(f_{n-1}, \dots, f_{n-m}; S_n + x) \mathbf{1}\{L_n \geq -x\} \mid S_0, \dots, S_n)) \\ &= l_n(x)^{-1} E[\psi_1^*(S_0 + x, \dots, S_\ell + x) \psi_2^*(S_n + x, \dots, S_{n-m} + x); L_n \geq -x] \\ & \rightarrow E_x \psi_1^*(\hat{Y}_0, \dots, \hat{Y}_\ell) E_v \psi_2^*(\check{Y}_0, \dots, \check{Y}_m) \quad \text{as } n \rightarrow \infty. \end{aligned} \tag{4.23}$$

Using (4.14), (4.21) and (4.17) we obtain (recall that $S_0 = 0$)

$$\begin{aligned} & E_x \psi_1^*(\hat{Y}_0, \dots, \hat{Y}_\ell) \\ &= \gamma^{-\ell} u(x)^{-1} E[\psi_1^*(x, x + S_1, \dots, x + S_\ell) \exp(\alpha S_\ell) u(x + S_\ell); L_\ell \geq -x] \\ &= \gamma^{-\ell} u(x)^{-1} E(E(\psi_1(f_0, \dots, f_{\ell-1}) \exp(\alpha S_\ell) u(x + S_\ell) 1\{L_\ell \geq -x\} \mid S_0, \dots, S_\ell)) \\ &= E_x \psi_1(\hat{f}_0, \dots, \hat{f}_{\ell-1}). \end{aligned} \tag{4.24}$$

For the second factor on the right-hand side of (4.23) observe that, by exchangeability of the f_j and independence of the f_j and Y ,

$$\begin{aligned} & \psi_2^*(z_0, \dots, z_m) \\ &= E(\psi_2(f_0, \dots, f_{m-1}; z_0) \mid S_1 = z_0 - z_1, \dots, S_m = z_0 - z_m) \\ &= E(\psi_2(f_0, \dots, f_{m-1}; Y) \mid Y = z_0; Y - S_1 = z_1, \dots, Y - S_m = z_m) \end{aligned} \tag{4.25}$$

for every $z_0 \geq 0$ and $z_i \in \mathbb{R}$, $1 \leq i \leq m$. From (4.25) we see that

$$\psi_2^*(Y, Y - S_1, \dots, Y - S_m) = E(\psi_2(f_0, \dots, f_{m-1}; Y) \mid Y; S_1, \dots, S_m).$$

Hence, using (4.15) and (4.20) we have

$$\begin{aligned} & E_v \psi_2^*(\check{Y}_0, \dots, \check{Y}_m) \\ &= \gamma^{-m} E \left[\psi_2^*(Y, Y - S_1, \dots, Y - S_m) \exp(\alpha S_m) \frac{v(Y - S_m)}{v(Y)}; M_m \leq Y \right] \\ &= \gamma^{-m} E \left(E \left(\psi_2(f_0, \dots, f_{m-1}; Y) \exp(\alpha S_m) \right. \right. \\ &\quad \left. \left. \times \frac{v(Y - S_m)}{v(Y)} 1\{M_m \leq Y\} \mid Y; S_1, \dots, S_m \right) \right) \\ &= E_v \psi_2(\check{f}_0, \dots, \check{f}_{m-1}; \check{Y}_0). \end{aligned} \tag{4.26}$$

Combining (4.24) and (4.26) with (4.23) establishes the weak convergence result (4.19).

We now begin preparations of the proof of Theorem 1.3. For convenience we use the notation

$$\begin{aligned} & \mathcal{L}_{x,v}((\hat{f}_\ell)_{\ell \geq 0}, (\check{f}_m)_{m \geq 0}, (\check{Y}_m)_{m \geq 0}) \\ &:= \mathcal{L}_x((\hat{f}_\ell)_{\ell \geq 0}) \otimes \mathcal{L}_v((\check{f}_m)_{m \geq 0}, (\check{Y}_m)_{m \geq 0}), \end{aligned} \tag{4.27}$$

where

$$\check{Y}_m := \check{Y}_0 - \sum_{i=1}^m \log \check{f}'_{i-1}(1), \quad m \geq 1.$$

Also, we will write

$$\hat{Y}_m := \hat{Y}_0 + \sum_{j=1}^m \log \hat{f}'_{j-1}(1), \quad m \geq 1,$$

with $\mathcal{L}_{x,v}(\hat{Y}_0) = \delta_x$.

The following proposition shows that the conditional probability generating function of the generation size at n has a weak limit when the associated random walk is conditioned to stay above $-x$. The joint convergence stated in (4.28) below will later be used to show that the limit is non-degenerate.

PROPOSITION 4.5. – *Let $x \geq 0$ and suppose that the conditions of Theorem 1.3 are satisfied. Then, as $n \rightarrow \infty$,*

$$\begin{aligned} &\mathcal{L}(f_0, \dots, f_\ell; f_{n-1}, \dots, f_{n-\ell-1}; S_n + x; f_{0,n}(s), \dots, f_{\ell,n}(s) \mid L_n \geq -x) \\ &\Rightarrow \mathcal{L}_{x,v}(\hat{f}_0, \dots, \hat{f}_\ell; \check{f}_0, \dots, \check{f}_\ell; \check{Y}_0; U_0(s), \dots, U_\ell(s)) \end{aligned} \tag{4.28}$$

for every $\ell \geq 0$ and $0 \leq s < 1$. Here, $(U_k(s))_{k \geq 0}$ is a sequence of $[0, 1]$ -valued random variables satisfying

$$U_k(s) = \hat{f}_k(U_{k+1}(s)), \quad k \geq 0. \tag{4.29}$$

Proof. – Fix $x \geq 0$ and $0 \leq s < 1$. Note that we cannot apply Proposition 4.4 directly since the $f_{k,n}(s)$ depend on all $f_j, k \leq j \leq n - 1$. We circumvent this difficulty by justifying a linear approximation to $f_{k,n}(s)$: We will show that for every $k \geq 0$,

$$\lim_{j,m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}(|f_{k,n}(s) - r_{k,n}^{(j,m)}(s)| \mid L_n \geq -x) = 0, \tag{4.30}$$

where

$$r_{k,n}^{(j,m)}(s) := f_{k,j}([1 - f'_{j,n-m}(1)(1 - f_{n-m,n}(s))]^+), \quad k \leq j \leq n - m \leq n.$$

Repeatedly use convexity and monotonicity of the $f_{i,j}$ and the mean-value theorem to estimate

$$\begin{aligned} 0 &\leq f_{k,j}(f_{j,n}(s)) - r_{k,n}^{(j,m)}(s) \\ &\leq f'_{k,j}(1)[f'_{j,n-m}(1)(1 - f_{n-m,n}(s)) - (1 - f_{j,n}(s))] \\ &= \exp(-S_k)[\exp(S_{n-m})(1 - f_{n-m,n}(s)) - \exp(S_j)(1 - f_{j,n}(s))], \end{aligned}$$

where for the last equality recall that $f'_{k,\ell}(1) = \exp(S_\ell - S_k)$ for $\ell \geq k$. By Lemma 2.2, replacing f_i by f_{n-m+i} (respectively, f_{j+i}), the right-hand side above equals

$$\begin{aligned} &\exp(-S_k) \left[\left(\frac{\exp(-S_n)}{1-s} + \sum_{i=n-m}^{n-1} \eta_{i,n}(s) \exp(-S_i) \right)^{-1} \right. \\ &\quad \left. - \left(\frac{\exp(-S_n)}{1-s} + \sum_{i=j}^{n-1} \eta_{i,n}(s) \exp(-S_i) \right)^{-1} \right]. \end{aligned}$$

Use $z^{-1} - (y+z)^{-1} \leq yz^{-2}$ for $y, z > 0$ to bound above this term by

$$\exp(2S_n - S_k) \sum_{i=j}^{n-m-1} \eta_{i,n}(s) \exp(-S_i).$$

Since the difference between two generating functions is bounded by 1 the estimates above show that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \mathbb{E}(|f_{k,n}(s) - r_{k,n}^{(j,m)}(s)| \mid L_n \geq -x) \\ & \leq \limsup_{n \rightarrow \infty} \mathbb{E} \left(\left(\exp(2S_n - S_k) \sum_{i=j}^{n-m-1} \eta_{i,n}(s) \exp(-S_i) \right) \wedge 1 \mid L_n \geq -x \right) \\ & \leq \exp(2y + x) \limsup_{n \rightarrow \infty} \mathbb{E} \left(\sum_{i=j}^{n-m-1} \eta_{i,n}(s) \exp(-S_i) \mid L_n \geq -x \right) \\ & \quad + \limsup_{n \rightarrow \infty} \mathbb{P}(S_n \geq y \mid L_n \geq -x) \end{aligned} \tag{4.31}$$

for each $y \geq -x$. By Lemma 4.6 below, for any y , the first term on the right-hand side of (4.31) tends to 0 as $j, m \rightarrow \infty$. By Lemma 4.2, the second term vanishes as $y \rightarrow \infty$ and (4.30) will follow.

For the asymptotic behavior of the linear approximation $r_{k,n}^{(j,m)}(s)$ we may apply Proposition 4.4: Since $f'_{j,n-m}(1) = \exp(S_{n-m} - S_j)$ depends on $f_0, \dots, f_{j-1}; f_{n-1}, \dots, f_{n-m}$ and S_n only, we have (using continuity of the f_i)

$$\begin{aligned} & \mathcal{L}(f_0, \dots, f_\ell; f_{n-1}, \dots, f_{n-\ell-1}; S_n + x; r_{\ell,n}^{(j,m)}(s) \mid L_n \geq -x) \\ & \Rightarrow \mathcal{L}_{x,v}(\hat{f}_0, \dots, \hat{f}_\ell; \check{f}_0, \dots, \check{f}_\ell; \check{Y}_0; U_\ell^{(j,m)}(s)) \end{aligned} \tag{4.32}$$

as $n \rightarrow \infty$ for every $\ell \geq 0$ and $0 \leq s < 1$, where

$$U_\ell^{(j,m)}(s) = \hat{f}_{\ell,j}([1 - \exp(\check{Y}_m - \hat{Y}_j)(1 - \check{f}_{m,0}(s))]^+). \tag{4.33}$$

We claim that (4.32) implies the weak convergence result (4.28), where

$$\mathcal{L}_{x,v}(\hat{f}_0, \dots, \hat{f}_\ell; \check{f}_0, \dots, \check{f}_\ell; \check{Y}_0; U_\ell(s))$$

is the weak limit of the right-hand side of (4.32) as $j, m \rightarrow \infty$ and the $U_k(s), 0 \leq k \leq \ell - 1$, are recursively defined through (4.29). Indeed, let $\mathcal{L}_{x,v}(\hat{f}_0, \dots, \hat{f}_\ell; \check{f}_0, \dots, \check{f}_\ell; \check{Y}_0; U_\ell(s))$ be the weak limit of the right-hand side of (4.32) along some subsequence $(j_i, m_i)_{i \geq 0}$ with $j_i \wedge m_i \rightarrow \infty$ as $i \rightarrow \infty$ (note that this sequence of distributions is tight since $0 \leq U_\ell^{(j,m)}(s) \leq 1$). Then, by (4.30), it is the weak limit of

$$\mathcal{L}(f_0, \dots, f_\ell; f_{n-1}, \dots, f_{n-\ell-1}; S_n + x; f_{\ell,n}(s) \mid L_n \geq -x)$$

as $n \rightarrow \infty$. Since the latter limit does not depend on the subsequence $(j_i, m_i)_{i \geq 0}$, we have

$$\begin{aligned} & \mathcal{L}(f_0, \dots, f_\ell; f_{n-1}, \dots, f_{n-\ell-1}; S_n + x; f_{\ell,n}(s) \mid L_n \geq -x) \\ & \Rightarrow \mathcal{L}_{x,v}(\hat{f}_0, \dots, \hat{f}_\ell; \check{f}_0, \dots, \check{f}_\ell; \check{Y}_0; U_\ell(s)) \end{aligned} \tag{4.34}$$

as $n \rightarrow \infty$. Assertion (4.28) with the $U_k, 0 \leq k \leq \ell - 1$, satisfying (4.29) now follows from (4.34) and (4.33). Thus, to complete the proof of Proposition 4.5 it remains to establish Lemma 4.6.

LEMMA 4.6. – *Let $x \geq 0$ and suppose that the conditions of Theorem 1.3 are satisfied. Then,*

$$\lim_{\ell, m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left(\sum_{k=\ell+1}^{n-m-1} \eta_{k,n}(s) \exp(-S_k) \mid L_n \geq -x \right) = 0 \tag{4.35}$$

for every $0 \leq s < 1$.

Proof. – Fix $x \geq 0$ and $0 \leq s < 1$, and let $0 \leq k \leq n - 1$. Recall from (2.1) and (2.4) that $\eta_{k,n}(s) = g_k(f_{k+1,n}(s)) \leq f''_k(1)/f'_k(1)^2 =: \eta_k$. Conditioning on f_0, \dots, f_k thus shows that (again we use notation $l_n(x) := \mathbb{P}(L_n \geq -x)$)

$$\mathbb{E}[\eta_{k,n}(s) \exp(-S_k); L_n \geq -x] \leq \mathbb{E}[\eta_k \exp(-S_k) l_{n-k-1}(x + S_{k+1}); L_{k+1} \geq -x].$$

Using $L_{k+1} \leq L_k$ and estimate (4.9) with $\theta = 1$ we bound above the right-hand side by

$$c \exp(x)(n - k)^{-3/2} \gamma^{n-k} \mathbb{E}[\eta_k \exp(X_{k+1}); L_k \geq -x]$$

for some finite constant c . Combining these estimates and using first the independence of the f_j and then (4.9) again, we obtain

$$\begin{aligned} \mathbb{E}[\eta_{k,n}(s) \exp(-S_k); L_n \geq -x] &\leq c \exp(x)(n - k)^{-3/2} \gamma^{n-k} l_k(x) \mathbb{E}[\eta_k f'_k(1)] \\ &\leq c'_x (k(n - k))^{-3/2} \gamma^n \mathbb{E}[\eta_0 f'_0(1)] \end{aligned} \tag{4.36}$$

for some $c'_x < \infty$. Use $z^{-1} \leq z^{-(1-\alpha)} + z^{-(2-\alpha)}$ for $z > 0$ to see that the expectation $\mathbb{E}[\eta_0 f'_0(1)] = \mathbb{E}[f''(1)/f'(1)]$ is finite by assumption (1.15) of Theorem 1.3. Hence, combining (4.36) with (4.8) yields

$$\mathbb{E} \left(\sum_{k=\ell+1}^{n-m-1} \eta_{k,n}(s) \exp(-S_k) \mid L_n \geq -x \right) \leq c''_x \sum_{k=\ell+1}^{n-m-1} \left(\frac{n}{k(n - k)} \right)^{3/2} \tag{4.37}$$

for some $c''_x < \infty$. A simple exercise shows that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} \left(\frac{n}{k(n - k)} \right)^{3/2} = 2 \sum_{k=1}^{\infty} k^{-3/2}.$$

Hence, we deduce from (4.37) that

$$\limsup_{n \rightarrow \infty} \mathbb{E} \left(\sum_{k=\ell+1}^{n-m-1} \eta_{k,n}(s) \exp(-S_k) \mid L_n \geq -x \right) \leq 2c''_x \sum_{k=(\ell \wedge m)+1}^{\infty} k^{-3/2}. \tag{4.38}$$

Letting $\ell, m \rightarrow \infty$ establishes the claim of Lemma 4.6.

Proposition 4.5 demonstrates that the conditional generating function of the generation size at n has a weak limit when conditioned on the event $\{L_n \geq -x\}$. Our next lemma gives a representation of the expectation of the limit (recall (2.1) and (3.5)).

LEMMA 4.7. – *Let $x \geq 0$ and suppose that the conditions of Theorem 1.3 are satisfied. Then, for every $0 \leq s < 1$,*

$$\exp(x) \lim_{n \rightarrow \infty} \mathbb{E}(1 - f_{0,n}(s) \mid L_n \geq -x) = w(x, s), \tag{4.39}$$

where

$$w(x, s) := \mathbb{E}_{x,v} \left(\sum_{i=0}^{\infty} \hat{g}_i(U_{i+1}(s)) \exp(-\hat{Y}_i) + \frac{\exp(-\check{Y}_0)}{1-s} + \sum_{j=1}^{\infty} \check{\zeta}_{j-1}(s) \exp(-\check{Y}_j) \right)^{-1}. \tag{4.40}$$

Proof. – Fix $x \geq 0$ and $0 \leq s < 1$. To prove the lemma we will show that, as $n \rightarrow \infty$,

$$\begin{aligned} \mathcal{L}(\exp(-x)(1 - f_{0,n}(s))^{-1} \mid L_n \geq -x) &\Rightarrow \\ \mathcal{L}_{x,v} \left(\sum_{i=0}^{\infty} \hat{g}_i(U_{i+1}(s)) \exp(-\hat{Y}_i) + \frac{\exp(-\check{Y}_0)}{1-s} + \sum_{j=1}^{\infty} \check{\zeta}_{j-1}(s) \exp(-\check{Y}_j) \right). \end{aligned} \tag{4.41}$$

The assertion of Lemma 4.7 is an immediate consequence of the weak convergence result (4.41) since $1 - f_{0,n}(s)$ is bounded by 1.

For $0 \leq \ell < n - m \leq n$ decompose (recall (2.2))

$$\begin{aligned} (1 - f_{0,n}(s))^{-1} &= \sum_{k=0}^{\ell} \eta_{k,n}(s) \exp(-S_k) + \sum_{k=\ell+1}^{n-m-1} \eta_{k,n}(s) \exp(-S_k) \\ &\quad + \sum_{k=1}^m \eta_{n-k,n}(s) \exp(-S_{n-k}) + \frac{\exp(-S_n)}{1-s}. \end{aligned} \tag{4.42}$$

By (4.42), if $(1 - f_{0,n}(s))^{-1} \leq y$, then

$$\sum_{k=0}^{\ell} \eta_{k,n}(s) \exp(-S_k) + \frac{\exp(-S_n)}{1-s} + \sum_{k=1}^m \eta_{n-k,n}(s) \exp(-S_{n-k}) \leq y$$

for any $y \geq 0$. Hence, by Proposition 4.5 (recall that $\eta_{k,n} = g_k \circ f_{k+1,n}$ and that the g_k are continuous), letting first n and then $\ell, m \rightarrow \infty$, we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{P}((1 - f_{0,n}(s))^{-1} \leq y \mid L_n \geq -x) \\ \leq \mathbb{P}_{x,v} \left(\sum_{i=0}^{\infty} \hat{g}_i(U_{i+1}(s)) \exp(\hat{Y}_0 - \hat{Y}_i) + \frac{\exp(x - \check{Y}_0)}{1-s} \right. \\ \left. + \sum_{j=1}^{\infty} \check{\zeta}_{j-1}(s) \exp(x - \check{Y}_j) \leq y \right). \end{aligned} \tag{4.43}$$

On the other hand, decomposition (4.42) and Proposition 4.5 yield

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \mathbb{P}((1 - f_{0,n}(s))^{-1} < y \mid L_n \geq -x) \\ & \geq \mathbb{P}_{x,v} \left(\sum_{i=0}^{\ell} \hat{g}_i(U_{i+1}(s)) \exp(\hat{Y}_0 - \hat{Y}_i) + \frac{\exp(x - \check{Y}_0)}{1 - s} \right. \\ & \quad \left. + \sum_{j=1}^m \check{\zeta}_{j-1}(s) \exp(x - \check{Y}_j) < y - \varepsilon \right) \\ & \quad - \limsup_{n \rightarrow \infty} \mathbb{P} \left(\sum_{k=\ell+1}^{n-m-1} \eta_{k,n}(s) \exp(-S_k) \geq \varepsilon \mid L_n \geq -x \right) \end{aligned}$$

for any $0 < \varepsilon \leq y$. By Lemma 4.6, letting first $\ell, m \rightarrow \infty$ and then $\varepsilon \rightarrow 0$, we obtain

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \mathbb{P}((1 - f_{0,n}(s))^{-1} < y \mid L_n \geq -x) \\ & \geq \mathbb{P}_{x,v} \left(\sum_{i=0}^{\infty} \hat{g}_i(U_{i+1}(s)) \exp(\hat{Y}_0 - \hat{Y}_i) + \frac{\exp(x - \check{Y}_0)}{1 - s} \right. \\ & \quad \left. + \sum_{j=1}^{\infty} \check{\zeta}_{j-1}(s) \exp(x - \check{Y}_j) < y \right). \end{aligned} \tag{4.44}$$

Since $\mathcal{L}_{x,v}(\hat{Y}_0) = \delta_x$, combining estimates (4.43) and (4.44) gives (4.41).

The following lemma shows that the limiting distribution in (4.41) is proper.

LEMMA 4.8. – *Let $x, y \geq 0$ and suppose that the conditions of Theorem 1.3 are satisfied. Then,*

$$\mathbb{E}_x \left(\sum_{i=0}^{\infty} \hat{\eta}_i \exp(-\hat{Y}_i) \right) + \mathbb{E}_y \left(\sum_{j=1}^{\infty} \check{\eta}_{j-1} \exp(-\check{Y}_j) \right) < \infty. \tag{4.45}$$

In particular, for every $0 \leq s < 1$,

$$\mathbb{P}_{x,v} \left(\sum_{i=0}^{\infty} \hat{g}_i(U_{i+1}(s)) \exp(-\hat{Y}_i) + \frac{\exp(-\check{Y}_0)}{1 - s} + \sum_{j=1}^{\infty} \check{\zeta}_{j-1}(s) \exp(-\check{Y}_j) < \infty \right) = 1. \tag{4.46}$$

Proof. – Fix $x \geq 0$ and let $i \geq 0$. We proceed in very much the same way as in the proof of Lemma 3.3. Using first (4.17) and the estimates that led to (3.29), and then the independence of the f_k and again (4.17), we obtain

$$\begin{aligned} & \exp(x) \mathbb{E}_x [\hat{\eta}_i \exp(-\hat{Y}_i)] \\ & = \gamma^{-(i+1)} u(x)^{-1} \mathbb{E} [\eta_i \exp(-S_i) \exp(\alpha S_{i+1}) u(x + S_{i+1}); L_{i+1} \geq -x] \\ & \leq 2c \gamma^{-(i+1)} u(x)^{-1} \\ & \quad \times \mathbb{E} [\eta_i \exp(\alpha X_{i+1}) \exp((\alpha - 1)S_i) u(x + S_i) (1 + X_{i+1}^+); L_i \geq -x] \\ & = 2c \gamma^{-1} \mathbb{E} [\eta_i \exp(\alpha X_{i+1}) (1 + X_{i+1}^+)] \end{aligned}$$

$$\begin{aligned} & \times \gamma^{-i} u(x)^{-1} \mathbb{E}[\exp((\alpha - 1)S_i)u(x + S_i); L_i \geq -x] \\ & = 2c\gamma^{-1} \exp(x) \mathbb{E} \left[\frac{f''(1)}{f'(1)^{2-\alpha}} (1 + \log^+ f'(1)) \right] \mathbb{E}_x \exp(-\hat{Y}_i). \end{aligned}$$

Use $z^{-(2-\alpha)} \log^+ z \leq z^{-(1-\alpha)}$ for $z > 0$ to see that (1.15) implies finiteness of the first expectation. Hence, application of Lemma 3.1 in [13] gives

$$\mathbb{E}_x \left(\sum_{i=0}^{\infty} \hat{\eta}_i \exp(-\hat{Y}_i) \right) < \infty.$$

The second sum is estimated in a similar fashion. Fix $y \geq 0$ and let $j \geq 1$. Using (4.20) and the arguments from above we deduce

$$\begin{aligned} & \mathbb{E}_y [\check{\eta}_{j-1} \exp(-\check{Y}_j)] \\ & = \gamma^{-j} v(y)^{-1} \exp(-y) \mathbb{E}[\eta_{j-1} \exp((\alpha + 1)S_j)v(y - S_j); M_j \leq y] \\ & \leq 2c\gamma^{-1} \mathbb{E}[\eta_{j-1} \exp((\alpha + 1)X_j)(1 + X_j^-)] \\ & \quad \times \gamma^{-(j-1)} v(y)^{-1} \exp(-y) \mathbb{E}[\exp((\alpha + 1)S_{j-1})v(y - S_{j-1}); M_{j-1} \leq y] \\ & = 2c\gamma^{-1} \mathbb{E} \left[\frac{f''(1)}{f'(1)^{1-\alpha}} (1 + \log^- f'(1)) \right] \mathbb{E}_y \exp(-\check{Y}_{j-1}). \end{aligned}$$

Since $z^{-(1-\alpha)} \log^- z \leq z^{-(2-\alpha)}$ for $z > 0$, the first expectation is finite by (1.15). Again we may apply Lemma 3.1 in [13] to conclude

$$\mathbb{E}_y \left(\sum_{j=1}^{\infty} \check{\eta}_{j-1} \exp(-\check{Y}_j) \right) < \infty,$$

which completes our proof of (4.45). Assertion (4.46) easily follows from (4.45) and the upper bound (3.26).

Our final lemma shows that the unlikely event of survival can only happen through a fortunate pick of the environment.

LEMMA 4.9. – *Let the conditions of Theorem 1.3 be satisfied. Then,*

$$\lim_{x \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{3/2} \gamma^{-n} \mathbb{P}(Z_n > 0, L_n < -x) = 0. \tag{4.47}$$

Proof. – The first moment estimate

$$\mathbb{P}(Z_k > 0 \mid f_0, f_1, \dots) \leq \mathbb{E}(Z_k \mid f_0, f_1, \dots) = \exp(S_k)$$

implies

$$\mathbb{P}(Z_n > 0 \mid f_0, f_1, \dots) = \min_{0 \leq k \leq n} \mathbb{P}(Z_k > 0 \mid f_0, f_1, \dots) \leq \exp(L_n). \tag{4.48}$$

Use (4.9) to obtain

$$\begin{aligned}
 n^{3/2}\gamma^{-n}\mathbb{P}(Z_n > 0, L_n < -x) &\leq n^{3/2}\gamma^{-n}\mathbb{E}[\exp(L_n); L_n < -x] \\
 &\leq n^{3/2}\gamma^{-n} \sum_{k \geq \lfloor x \rfloor} \exp(-k)\mathbb{P}(-(k+1) \leq L_n < -k) \\
 &\leq c_\theta \exp(\theta) \sum_{k \geq \lfloor x \rfloor} \exp(-k(1-\theta)) \tag{4.49}
 \end{aligned}$$

for any $\theta > \alpha$. Since $\alpha < 1$ we can choose $\theta < 1$. Letting first n and then x go to ∞ in (4.49) establishes the claim of Lemma 4.9.

Proof of Theorem 1.3. – Application of Lemmas 4.1 and 4.7 shows that

$$\lim_{n \rightarrow \infty} n^{3/2}\gamma^{-n}\mathbb{E}[1 - f_{0,n}(s); L_n \geq -x] = \bar{u}(x)w(x, s), \quad x \geq 0, 0 \leq s < 1, \tag{4.50}$$

where $\bar{u}(x) = c \exp((\alpha - 1)x)u(x)$ and $w(x, s)$ as defined in (4.40). Lemma 4.9 and the monotonicity of $f_{0,n}$ imply

$$\begin{aligned}
 \lim_{n \rightarrow \infty} n^{3/2}\gamma^{-n}\mathbb{E}(1 - f_{0,n}(s)) &= \lim_{x \rightarrow \infty} \lim_{n \rightarrow \infty} n^{3/2}\gamma^{-n}\mathbb{E}[1 - f_{0,n}(s); L_n \geq -x] \\
 &= \lim_{x \rightarrow \infty} \bar{u}(x)w(x, s), \quad 0 \leq s < 1, \tag{4.51}
 \end{aligned}$$

where for the existence of the limit on the right-hand side of (4.51) note that (4.50) implies monotonicity of $\bar{u}(x)w(x, s)$ for every s . To prove (1.16) with

$$c_3 := \lim_{x \rightarrow \infty} \bar{u}(x)w(x, 0),$$

it remains to show that c_3 is positive and finite. Use (4.48) and (4.10) with $\theta = 1$ to conclude

$$c_3 = \lim_{n \rightarrow \infty} n^{3/2}\gamma^{-n}\mathbb{P}(Z_n > 0) \leq \limsup_{n \rightarrow \infty} n^{3/2}\gamma^{-n}\mathbb{E}\exp(L_n) < \infty.$$

Since $\bar{u}(x)w(x, 0)$ is increasing, to prove $c_3 > 0$ it suffices to show that $\bar{u}(x)w(x, 0) > 0$ for some $x \geq 0$. However, in view of the representation of $w(x, 0)$ in (4.40) this is an immediate consequence of (4.46).

For the second part of Theorem 1.3 recall (3.8) and use (4.51) to obtain

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \mathbb{E}(s^{Z_n} | Z_n > 0) &= 1 - \lim_{n \rightarrow \infty} \frac{\mathbb{E}(1 - f_{0,n}(s))}{\mathbb{P}(Z_n > 0)} \\
 &= 1 - c_3^{-1} \lim_{x \rightarrow \infty} \bar{u}(x)w(x, s), \quad 0 \leq s < 1. \tag{4.52}
 \end{aligned}$$

Hence,

$$q_3(k) := \lim_{n \rightarrow \infty} \mathbb{P}(Z_n = k | Z_n > 0) \text{ exists}$$

for each $k \geq 1$.

By the first part of the theorem, to prove tightness of $\mathcal{L}(Z_n | Z_n > 0)$ it suffices to show

$$\lim_{x \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{3/2}\gamma^{-n}\mathbb{P}(Z_n \geq x) = 0. \tag{4.53}$$

Clearly,

$$P(Z_n \geq x) \leq P(Z_n > 0, L_n < -x^\delta) + P(Z_n \geq x, L_n \geq -x^\delta) \tag{4.54}$$

for every $\delta \geq 0$. By Lemma 4.9,

$$\lim_{x \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{3/2} \gamma^{-n} P(Z_n > 0, L_n < -x^\delta) = 0 \tag{4.55}$$

for any $\delta > 0$. For the other term on the right-hand side of (4.54) use Chebyshev’s inequality to deduce

$$\begin{aligned} P(Z_n \geq x, L_n \geq -x^\delta) &= E[P(Z_n \geq x \mid f_0, f_1, \dots); L_n \geq -x^\delta] \\ &\leq E[(x^{-1} \exp(S_n)) \wedge 1; L_n \geq -x^\delta]. \end{aligned} \tag{4.56}$$

From (4.54)–(4.56), the asymptotic (4.8) and the fact that u grows linearly, we see that to prove (4.53) it is sufficient to show that

$$\lim_{x \rightarrow \infty} [x^\delta \exp(\alpha x^\delta) \limsup_{n \rightarrow \infty} E((x^{-1} \exp(S_n)) \wedge 1 \mid L_n \geq -x^\delta)] = 0 \tag{4.57}$$

for some $\delta > 0$.

By Lemma 4.2, $\mathcal{L}(S_n \mid L_n \geq -x^\delta) \Rightarrow \mathcal{L}(Y - x^\delta)$, where Y has distribution ν . Hence, by the bounded convergence theorem,

$$\lim_{n \rightarrow \infty} E((x^{-1} \exp(S_n)) \wedge 1 \mid L_n \geq -x^\delta) = E((x^{-1} \exp(Y - x^\delta)) \wedge 1). \tag{4.58}$$

The fact that ν grows linearly implies

$$\begin{aligned} P(x^{-1} \exp(Y - x^\delta) \geq 1) &= c_0^{-1} \int_{x^\delta + \log x}^{\infty} \nu(y) \exp(-\alpha y) dy \\ &\leq c x^{\delta - \alpha} \exp(-\alpha x^\delta) \end{aligned} \tag{4.59}$$

for some $0 < c < \infty$. Finally, note that

$$\begin{aligned} E[x^{-1} \exp(Y - x^\delta); x^{-1} \exp(Y - x^\delta) \leq 1] \\ &= x^{-1} \exp(-x^\delta) c_0^{-1} \int_0^{x^\delta + \log x} \nu(y) \exp((1 - \alpha)y) dy \\ &\leq c' x^{\delta - \alpha} \exp(-\alpha x^\delta) \end{aligned} \tag{4.60}$$

for some $0 < c' < \infty$. Relation (4.58) and estimates (4.59) and (4.60) show that (4.57) holds for all $\delta < \frac{\alpha}{2}$.

REFERENCES

[1] V.I. Afanasyev, Limit theorems for a conditional random walk and some applications. Diss. Cand. Sci., Moscow, MSU, 1980.

- [2] V.I. Afanasyev, Limit theorems for a moderately subcritical branching process in a random environment, *Discrete Math. Appl.* 8 (1998) 35–52.
- [3] A. Agresti, Bounds on the extinction time distribution of a branching process, *Adv. Appl. Probab.* 6 (1974) 322–335.
- [4] K.B. Athreya, S. Karlin, On branching processes with random environments: I, II, *Ann. Math. Stat.* 42 (1971) 1499–1520, 1843–1858.
- [5] K.B. Athreya, P. Ney, *Branching Processes*, Springer, New York, 1972.
- [6] J. Bertoin, R.A. Doney, On conditioning a random walk to stay positive, *Ann. Probab.* 22 (1994) 2152–2167.
- [7] F.M. Dekking, On the survival probability of a branching process in a finite state i.i.d. environment, *Stochastic Processes Appl.* 27 (1988) 151–157.
- [8] R.A. Doney, On the asymptotic behaviour of first passage times for transient random walk, *Probab. Theory Related Fields* 81 (1989) 239–246.
- [9] J.S. D’Souza, B.M. Hambly, On the survival probability of a branching process in a random environment, *Adv. Appl. Probab.* 29 (1997) 38–55.
- [10] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II, Wiley, New York, 1971.
- [11] K. Fleischmann, V.A. Vatutin, Reduced subcritical Galton–Watson processes in a random environment, *Adv. Appl. Probab.* 31 (1999) 88–111.
- [12] J. Geiger, Elementary new proofs of classical limit theorems for Galton–Watson processes, *J. Appl. Probab.* 36 (1999) 301–309.
- [13] J. Geiger, G. Kersting, The survival probability of a critical branching process in random environment, *Teor. Verojatnost. i Primenen.* 45 (2000) 607–615.
- [14] Y. Guivarc’h, Q. Liu, Propriétés asymptotiques des processus de branchement en environnement aléatoire, *C. R. Acad. Sci. Paris Sér. I Math.* 332 (4) (2001) 339–344.
- [15] K. Hirano, Determination of the limiting coefficient for exponential functionals of random walks with positive drift, *J. Math. Sci. Univ. Tokyo* 5 (1998) 299–332.
- [16] O. Kallenberg, *Foundations of Modern Probability*, Springer, New York, 1997.
- [17] M.V. Kozlov, On the asymptotic behavior of the probability of non-extinction for critical branching processes in a random environment, *Theory Probab. Appl.* 21 (1976) 791–804.
- [18] Q. Liu, On the survival probability of a branching process in a random environment, *Ann. Inst. H. Poincaré Probab. Statist.* 32 (1996) 1–10.
- [19] W.L. Smith, W.E. Wilkinson, On branching processes in random environments, *Ann. Math. Stat.* 40 (1969) 814–827.
- [20] N. Veraverbeke, J.L. Teugels, The exponential rate of convergence of the distribution of the maximum of a random walk. Part II, *J. Appl. Probab.* 13 (1976) 733–740.