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ONSAGER–MACHLUP FUNCTIONAL FOR STOCHASTIC EVOLUTION EQUATIONS

FONCTIONNELLE D'ONSAGER–MACHLUP POUR LES ÉQUATIONS D'ÉVOLUTION STOCHASTIQUES

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ABSTRACT. – We compute the Onsager–Machlup functional for a stochastic evolution equation with additive noise, using L^2-type techniques. The regularity that has to be imposed to the drift coefficient is a trace type condition on its derivative. The expression of the divergence part of the functional does not depend on the stochastic convolution related to our evolution system.

Keywords: Onsager–Machlup functionals; Stochastic evolution equations

RÉSUMÉ. – Dans cet article, nous calculons la fonctionnelle d'Onsager–Machlup associée à une équation d'évolution stochastique avec bruit additif, en utilisant des techniques de type L^2. Les hypothèses de régularité que l'on impose au coefficient de dérive sont des conditions de trace sur sa dérivée. Notons que l'expression de la partie divergence de la fonctionnelle ne dépend pas de la convolution stochastique associée au système d'évolution.

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1. Introduction

Let H be a real separable Hilbert space, and consider the following stochastic differential equation on H:

$$\begin{cases} dX(t) = [AX(t) + F(t, X(t))]\,dt + B\,dW(t), & t \in [0,1], \\ X(0) = x, \end{cases} \tag{1}$$

where $x \in H$, $A : D(A) \subset H \to H$ generates a C_0-semigroup $\{\exp(tA); t \geqslant 0\}$, F is a Lipschitz function defined on $[0,1] \times H$, B is a non-negative bounded linear operator from H to H, and W is a cylindrical Wiener process in H. It is known that if $\int_0^1 \|\exp(tA)B\|_{HS}^2\,dt$ is finite, then (1) admits a unique solution $X \in L^2([0,1]; H)$, in a sense that will be specified later (see [4] for further details). In the remainder of the paper, we will also denote by W^A the stochastic convolution of A by W, that is the solution to (1) for $F \equiv 0$ and $x = 0$.

This article proposes to study the limiting behaviour of ratios of the form

$$\gamma_\varepsilon(\phi) \equiv \frac{P(\|X - \phi\| \leqslant \varepsilon)}{P(\|W^A\| \leqslant \varepsilon)}$$

when ε tends to 0, where ϕ is a deterministic function satisfying some regularity conditions and $\|\cdot\|$ is a suitable norm defined on the functions from $[0,1]$ to H. When $\lim_{\varepsilon \to 0} \gamma_\varepsilon(\phi) = \exp(J_0(\phi))$ for all ϕ in a reasonable class of functions, then the functional J_0 is called the Onsager–Machlup functional associated to (1) and $\|\cdot\|$. It is worth noting that J_0 can easily be interpreted as a generalized likelihood functional in infinite dimension, which makes its calculation an interesting problem.

For usual stochastic differential equations, namely when $H = \mathbb{R}^d$, $A = 0$, $B = \mathrm{Id}$, the problem of computing the Onsager–Machlup functional for (1) has been widely investigated. Ikeda and Watanabe [9] gave a rigorous proof for the case of any $\phi \in C^2([0,1]; \mathbb{R}^d)$ for the norm $\|\cdot\|_\infty$ defined on $C([0,1]; \mathbb{R}^d)$. This result has been enhanced then in two directions: Shepp and Zeitouni proved first in [13] that the function ϕ could be taken in the Cameron–Martin space $W_0^{1,2}([0,1]; \mathbb{R}^d)$, and Capitaine proved in [2] and [3] (basing this last result on some techniques inspired by the computation of the Onsager–Machlup functional for diffusions on manifolds, see e.g. [8]) that the norm $\|\cdot\|$ could be taken as any euclidian norm on the functions from $[0,1]$ to $\mathbb{R}^d$ making sense for the solution to (1) and dominating the norm on $L^2([0,1]; \mathbb{R}^d)$. It is important to note that in the case of finite dimensional diffusions, the functional J_0 does not depend on the norm considered.

Our current work fits in a more global project of studying the Onsager–Machlup functionals for infinite stochastic systems. The first results in that direction have been obtained by Dembo and Zeitouni [5] for trace class elliptic SPDEs on a bounded domain of $\mathbb{R}^d$, and then by Mayer-Wolf and Zeitouni [11] in the non trace case. We shall use some of their techniques in order to get our main result: if $\|\cdot\|$ is chosen as the Hilbert norm on $L^2([0,1];H)$, then, for ϕ satisfying some suitable hypothesis,

$$J_0(\phi) = -\int_0^1 \frac{1}{2}\left|B^{-1}\left[A\phi(t) + F\bigl(t,\phi(t)\bigr) - \dot{\phi}(t)\right]\right|_H^2 dt - \mathrm{Tr}(\mathcal{S}_{PR^*}),$$

where $\mathcal{S}_{PR^*}$ is a certain bounded linear operator based on $\nabla_x F$ and ϕ. However, in case $\nabla_x F(s,x)$ is a trace class linear operator for all $x \in H$ and $s \in [0,1]$, then $\mathrm{Tr}(\mathcal{S}_{PR^*}) = \frac{1}{2}\int_0^1 [\mathrm{Tr}(\nabla_x F)(s,\phi(s))]\,ds$.

With respect to the finite dimensional case, some differences can already be stressed in this introduction: first, the generality on the kind of norm considered by Capitaine in [3] seems beyond our hopes at this moment. On the other hand, since an independence of the functional J_0 with respect to the norm $\|\cdot\|$ can also be expected in our SPDE case, we chose to work with the Hilbert norm on $L^2([0,1];H)$ for two main reasons:

1. We are working here with the minimal assumptions on A and B under which Eq. (1) has a unique solution in $L^2([0,1];H)$, though we will also make the additional assumption that A and B can be diagonalized in the same orthonormal basis of H.
2. We will be able to use the conditional exponential moments results stated in [11], where the conditioning is over an infinite dimensional Gaussian random variable (these results will be recalled at Section 2). However, the fact that we are dealing with an evolution type equation will force us to delve deeper into the different Karhunen decompositions involved in the application of the results of [11]. We shall give some details about these decompositions at Section 2.

Another relevant difference between the finite and infinite dimensional case is that the normalizing factor in $\gamma_\varepsilon(\phi)$ cannot be a function of the norm of the cylindrical Brownian motion. This leads us to the natural choice of a normalization by $P(\|W^A\| \leqslant \varepsilon)$, which prives us of the rotational invariance type properties of the Brownian motion used by Ikeda and Watanabe ([9, Lemma 8.2]) and by Capitaine ([2, Lemma 4], [3, Lemma 3]) as a fundamental step towards the computation of the Onsager–Machlup functional.

Our paper is organized as follows: in Section 2, we recall some basic results and fix our notations for the stochastic evolution equation considered. We recall a basic lemma of Mayer-Wolf and Zeitouni [11, Lemma 2.5] that we shall use later on, and give the Karhunen expansion of an Ornstein–Uhlenbeck process in dimension one. In Section 3 we obtain the Onsager–Machlup functional: at Section 3.1 we reduce our problem by Girsanov's transform and at Section 3.2 we obtain useful decompositions of W^A and Wiener integrals. Section 3.3 is then devoted to some details about the linear case, that is when F is a linear bounded operator, which will lead us to the general case after Taylor's expansion, and is of independent interest, since the conditions given on F in this case will be more explicit, especially when F can be diagonalized in the same complete orthonormal system than A and B. At Section 3.4, we will deal with the general nonlinear case.

2. Notations and preliminary results

2.1. Stochastic evolution systems

2.1.1. The operators A and B

Let H be a real separable Hilbert space and $A : D(A) \subset H \to H$ an unbounded operator on H. The norm on H will be denoted by $|\cdot|_H$, and the scalar product by $\langle\cdot,\cdot\rangle$. The L^2 norm in $L^2([0,1]; H)$ will be denoted by $\|\cdot\|_2$. Let $\mathcal{L}(H)$ the set of bounded linear operators on H. The norm $\|\cdot\|$ will be the usual operator norm defined on $\mathcal{L}(H)$, that is, $\|T\| = \sup_{x\in H} \frac{|T(x)|_H}{|x|_H}$. For an operator $T \in \mathcal{L}(H)$, $\mathcal{S}_T \equiv \frac{1}{2}(T + T^*)$ will denote the symmetrized operator based on T. We shall suppose

(H1) The operator A generates a self adjoint C_0-semigroup

$$\{\exp(tA); t \geqslant 0\},$$

of negative type. Moreover, there exists a complete orthonormal system $\{e_j; j \geqslant 1\}$ which diagonalizes A. We shall denote by $\{-\alpha_j; j \geqslant 1\}$ the corresponding set of eigenvalues and we assume that $\{\alpha_j; j \geqslant 1\}$ is an increasing sequence of real numbers such that $\alpha_j \geqslant 0$ and $\lim_{j\to\infty} \alpha_j = \infty$. Set $j_0 = \inf\{j, \alpha_j > 0\}$.

We shall also consider an operator $B \in \mathcal{L}(H)$ satisfying

(H2) The operator B is of non-negative type and is diagonal when expressed in the orthonormal basis $\{e_j; j \geqslant 1\}$. We shall denote by $\{\beta_j; j \geqslant 1\}$ the corresponding set of eigenvalues. Furthermore, we shall suppose that $\beta_j > 0$ for all $j \geqslant 1$ and that

$$\sum_{j=1}^{\infty} \frac{\beta_j^2}{1+\alpha_j} < \infty.$$

In our case where A and B can be diagonalized in the same complete orthonormal system, note that the last hypothesis corresponds to the more general one $\int_0^1 \|\exp(tA)B\|_{HS}^2\, dt < \infty$, that can be found in [4] in order to ensure the existence and uniqueness of a solution to (1).

2.1.2. Stochastic evolution equations

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a stochastic basis and $\{W^j(t); t \in [0,1], j \geqslant 1\}$ a sequence of mutually independent Brownian motions adapted to $\mathcal{F}_t$. The cylindrical Brownian motion on H is defined by the formal series

$$W(t) = \sum_{j=1}^{\infty} W^j(t) e_j, \tag{2}$$

where $\{e_j; j \geqslant 1\}$ is the complete orthonormal system of H introduced at Section 2.1.1. Note that the series (2) does not converge in H, but for any $h \in H$, $\{\langle W(t), h\rangle; t \in [0,1]\}$ is a linear Brownian motion with covariance $|h|_H^2$ (see [4]). We shall consider the stochastic evolution equation (1), where A and B have been defined at Section 2.1.1 and $F : [0,1] \times H \to H$ is a Lipschitz function with a linear growth condition in x uniformly in t (some further hypothesis on F will be made at Section 3). Eq. (1) is only formal, and has to be interpreted in the usual *mild* sense: we will say that $X = \{X(t); t \in [0,1]\}$ is a solution to (1) if it is an H-valued $\mathcal{F}_t$-adapted square integrable process such that

$$X(t) = \exp(tA)x + \int_0^t \exp\big((t-s)A\big) F\big(s, X(s)\big)\, ds + \int_0^t \exp((t-s)A) B\, dW(s), \tag{3}$$

for all $t \in [0,1]$, where the last integral is of Itô's type. The following proposition is then a particular case of [4, Theorem 7.4].

PROPOSITION 2.1. – *Suppose that* (H1) *and* (H2) *are satisfied. Then there exists a unique solution X to* (3), *such that* $X \in L^2(\Omega \times [0,1]; H)$.

2.2. Some conditioned exponential inequalities

We will recall here some basic lemmae that we shall use for the computation of our Onsager–Machlup functional. The first one is a crucial, though elementary, inequality that can be found in [9, p. 537].

LEMMA 2.2. – *For a fixed $n \geqslant 1$, let $z_1, \dots, z_n$ be n random variables defined on $(\Omega, \mathcal{F}, P)$ and $\{A_\varepsilon; \varepsilon > 0\}$ a family of sets in $\mathcal{F}$. Suppose that for any $c \in \mathbb{R}$ and any $i = 1, \dots, n$, we have*

$$\limsup_{\varepsilon \to \infty} E\big[\exp(cz_i) \mid A_\varepsilon\big] \leqslant 1.$$

Then

$$\lim_{\varepsilon \to \infty} E\left[\exp\left(\sum_{i=1}^n z_i\right) \middle| A_\varepsilon\right] = 1.$$

In the sequel of the paper, we shall use some inequalities involving trace class operators. Let us state now the notion of trace that we shall consider.

DEFINITION 2.3. – *Let K be a separable Hilbert space. Let $T : K \to K$ be a compact symmetric operator. Let $\{\tau_i; i \geqslant 1\}$ be the eigenvalues of the operator T. We will say that T is a trace class operator if $\sum_{i=1}^\infty |\tau_i|$ is finite.*

If T is trace class, we define the trace of T, $\mathrm{Tr}(T)$, *as $\sum_{i=1}^\infty \langle Te_i, e_i\rangle$ for any basis $\{e_i; i \geqslant 1\}$. In particular* $\mathrm{Tr}(T) = \sum_{i=1}^\infty \tau_i$.

We finish this subsection recalling two technical lemmas from [11]: the first one is a particular case of [11, Lemma 2.4], and the second one is a slight variation of [11, Lemma 2.5]. We denote here by ℓ^2 the set of sequences of real numbers $\{\eta_i; i \geqslant 1\}$ such that $\sum_{i \geqslant 1} \eta_i^2 < \infty$.

LEMMA 2.4. – *Let $\{z_i; i \geqslant 1\}$ be a sequence of independent $\mathcal{N}(0,1)$ random variables defined on $(\Omega, \mathcal{F}, P)$, and $\{\eta_i; i \geqslant 1\}$ and $\{\nu_i; i \geqslant 1\}$ two ℓ^2 sequences of real numbers such that $\eta_i \neq 0$ for any $i \geqslant 1$. Then*

$$\lim_{\varepsilon \to 0} E\left[\exp\left(\sum_{i=1}^\infty z_i \nu_i\right) \middle| \sum_{i=1}^\infty \eta_i^2 z_i^2 \leqslant \varepsilon\right] = 1.$$

LEMMA 2.5. – *Let* $\{z_i; i \geqslant 1\}$ *be a sequence of independent* $\mathcal{N}(0,1)$ *random variables defined on* $(\Omega, \mathcal{F}, P)$, *and* $\{\eta_i; i \geqslant 1\}$ *a* ℓ^2 *sequence of numbers such that* $\eta_i \neq 0$ *for any* $i \geqslant 1$. *Let* $T : \ell^2 \to \ell^2$ *be a Hilbert–Schmidt operator,* $\{m_i; i \geqslant 1\}$ *a complete orthonormal system of* ℓ^2, *and denote* $\langle Tm_i, m_j\rangle$ *by* $T_{i,j}$.

1. *If* $\mathcal{S}_T \equiv \frac{1}{2}(T + T^*)$ *is trace class, then*

$$\lim_{\varepsilon \to 0} E\left[\exp\left(\sum_{i,j=1}^{\infty} z_i z_j T_{i,j} \right) \,\middle|\, \sum_{i=1}^{\infty} \eta_i^2 z_i^2 \leqslant \varepsilon \right] = 1.$$

2. *If* $\sum_{i=1}^{\infty} T_{i,i} = +\infty$ (*respectively* $-\infty$), *then*

$$\lim_{\varepsilon \to 0} E\left[\exp\left(\sum_{i \neq j} z_i z_j T_{i,j} + \sum_{i=1}^{\infty} (z_i^2 - 1) T_{i,i} \right) \,\middle|\, \sum_{i=1}^{\infty} \eta_i^2 z_i^2 \leqslant \varepsilon \right] = 0$$

(*respectively* $+\infty$).

Proof. – We refer the reader to the proof of [11, Lemma 2.5]. Note only that for any $j, i \geqslant 1$,

$$z_i z_j (T_{i,j} + T_{j,i}) = z_i z_j \big((\mathcal{S}_T)_{i,j} + (\mathcal{S}_T)_{j,i} \big),$$

where $(\mathcal{S}_T)_{i,j} = \langle \mathcal{S}_T m_i, m_j \rangle$. □

2.3. The Karhunen–Loève expansion

We compute here the Karhunen–Loève expansion for a class of one-dimensional Ornstein–Uhlenbeck processes that will appear in the decomposition of W^A. The following lemma is presumably fairly standard, but we include it for the sake of completeness.

LEMMA 2.6. – *Let* β *a standard Brownian motion and* $\lambda \geqslant 0$. *Then, the process* $X = \{X(t) = \int_0^t \exp(-\lambda(t-s))\, d\beta(s),\ 0 \leqslant t \leqslant 1\}$ *has the following Karhunen–Loève expansion*:

$$X(t) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda^2 + x_k^2}} Y_k\, g_k(t)$$

where, for each $k \geqslant 1$, x_k *is the unique positive solution of the equation* $\tan(x) = -\frac{1}{\lambda} x$ *in the interval* $[(2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2})$, $\{g_k(t) = A_k \sin(x_k t),\ k \geqslant 1\}$ *is an orthonormal basis of* $L^2([0,1])$ *with the normalizing constants* A_k *satisfying* $\sup_k |A_k| \leqslant 2$, *and* $\{Y_k,\ k \geqslant 1\}$ *is a family of orthogonal Gaussian random variables with mean* 0 *and variance* 1, *defined by* $Y_k = \sqrt{\lambda^2 + x_k^2} \int_0^1 X(t)\, g_k(t)\, dt$, *for all* $k \geqslant 1$.

Proof. – Since the case $\lambda = 0$ is well known, we will assume that $\lambda > 0$. Note that X is a Gaussian process with covariance function

$$K(t,s) = \int_0^{t \wedge s} e^{-\lambda(t-u)} e^{-\lambda(s-u)}\, du = \frac{1}{2\lambda} \big(e^{-\lambda(t \vee s - t \wedge s)} - e^{-\lambda(t+s)} \big), \tag{4}$$

$s, t \in [0,1]$. To find the eigenvalues and the eigenvectors of the symmetric operator in $\mathcal{L}(L^2([0,1]))$ associated with the kernel K, we have to solve the equation

$$\int_0^1 K(t,s)g(t)\,dt = \mu g(s), \quad 0 \leqslant s \leqslant 1, \tag{5}$$

that is, for $0 \leqslant s \leqslant 1$,

$$\frac{1}{2\lambda}\left(\mathrm{e}^{-\lambda s}\int_0^s \mathrm{e}^{\lambda t} g(t)\,dt + \mathrm{e}^{\lambda s}\int_s^1 \mathrm{e}^{-\lambda t} g(t)\,dt - \mathrm{e}^{-\lambda s}\int_0^1 \mathrm{e}^{-\lambda t} g(t)\,dt\right) = \mu g(s).$$

Differentiating twice, it is easy to check that g satisfies

$$\left(\lambda^2\mu - 1\right)g(s) = \mu g''(s), \quad 0 \leqslant s \leqslant 1, \tag{6}$$

with initial conditions $g(0) = 0$ and $\lambda g(1) = -g'(1)$. Observe that Eq. (6) clearly implies that $\mu \neq 0$.

Set $\alpha_{\lambda,\mu} = \frac{\mu}{\lambda^2\mu - 1}$. Then $\alpha_{\lambda,\mu}$ is well-defined and strictly negative. Indeed, suppose that $\lambda^2\mu - 1 = 0$. Then $g''(s) = 0$, $0 \leqslant s \leqslant 1$, and the initial conditions imply $g \equiv 0$. Finally, suppose that $\alpha_{\lambda,\mu} > 0$. In this case, the solution of the differential equation (6) is of the form

$$g(s) = c_1 \exp\left(\frac{s}{\sqrt{\alpha_{\lambda,\mu}}}\right) + c_2 \exp\left(-\frac{s}{\sqrt{\alpha_{\lambda,\mu}}}\right),$$

where c_1, c_2 are real constants. Then, the initial conditions $g(0) = 0$ and $\lambda g(1) = -g'(1)$ yield

$$\tanh\left(\frac{1}{\sqrt{\alpha_{\lambda,\mu}}}\right) = -\frac{1}{\lambda}\frac{1}{\sqrt{\alpha_{\lambda,\mu}}}$$

and this equation has no solution.

Consequently, we can assume $\frac{\mu}{\lambda^2\mu - 1} < 0$, and the solution of (6) is of the form

$$g(s) = c_1 \sin\left(\frac{s}{\sqrt{|\alpha_{\lambda,\mu}|}}\right) + c_2 \cos\left(\frac{s}{\sqrt{|\alpha_{\lambda,\mu}|}}\right).$$

The condition $g(0) = 0$ implies $c_2 = 0$, and $\lambda g(1) = -g'(1)$ yields

$$\tan\left(\frac{1}{\sqrt{|\alpha_{\lambda,\mu}|}}\right) = -\frac{1}{\lambda}\frac{1}{\sqrt{|\alpha_{\lambda,\mu}|}}.$$

Set $x = |\alpha_{\lambda,\mu}|^{-1/2}$. The relation $\alpha_{\lambda,\mu} = \frac{\mu}{\lambda^2\mu - 1}$ implies that the eigenvalues of the operator K form a family $\{\mu_n; n \geqslant 1\}$, where $\mu_n = \frac{1}{\lambda^2 + x_n^2}$ and x_n is the solution of the equation $\tan(x) = -\frac{1}{\lambda}x$ in the interval $[(2n-1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2})$; and the orthonormalized eigenfunctions are of the form $g_n(s) = A_n \sin(x_n s)$, $n \geqslant 1$. An easy computation shows that $|A_n| \leqslant 2$ for all n.

The classical Karhunen–Loève Theorem (see e.g. [10]) finishes the proof. □

3. Onsager–Machlup functional

In this section we compute the Onsager–Machlup functional for our equation, following the usual scheme used for both finite and infinite cases: we apply first the Girsanov transform (Section 3.1) in order to reduce our problem to the evaluation of a functional of the stochastic convolution W^A, up to some easily controlled correction terms (Section 3.2). We are then left with the evaluation of the conditional exponential moments of a stochastic integral with respect to the cylindrical Brownian motion, that can be performed explicitely when F is a linear operator (Section 3.3). The general case for F can be deduced then by Taylor's expansion (Section 3.4).

3.1. Application of Girsanov's transform

Fix a function $h \in L^2([0,1]; H)$. Let ϕ^h be the solution of the infinite dimensional equation

$$\begin{cases} d\phi^h(t) = A\phi^h(t)\,dt + Bh(t)\,dt, & t \in [0,1], \\ \phi^h(0) = x. \end{cases} \tag{7}$$

We will compute the Onsager–Machlup functional on $L^2([0,1]; H)$ at points of the form ϕ^h. The assumption $h \in L^2([0,1]; H)$ is required in order to apply Girsanov's transform.

Assume also the following conditions:

(h1) For any $t \in [0,1]$, $F(t, X(t)) \in \operatorname{Im}(B)$ a.s. and one of the two following relations holds: for some $\delta > 0$,

$$\sup_{t \in [0,1]} E\left[\exp\left(\delta \left|B^{-1}F\left(t, X(t)\right)\right|_H^2\right)\right] < +\infty,$$

or

$$E\left[\exp\left(\frac{1}{2}\int_0^1 \left|B^{-1}F\left(t, X(t)\right)\right|_H^2 dt\right)\right] < +\infty,$$

(h2) there exist a positive constant K such that

$$\sup_{t \in [0,1]} \left|B^{-1}F(t,x) - B^{-1}F(t,y)\right|_H \leqslant K|x-y|_H$$

and

$$\sup_{t \in [0,1]} \left|B^{-1}F(t,x)\right|_H \leqslant K\left(1 + |x|_H\right),$$

for any $x, y \in H$.

Then, using (h1) we can apply Girsanov's transformation (see [4, Theorem 10.14 and Proposition 10.17]) with $\widehat{W}(t) = W(t) + \int_0^t B^{-1}F(s, X(s))\,ds$ and we obtain for any $\varepsilon > 0$

$$P\left(\left\|X - \phi^h\right\|_2 \leqslant \varepsilon\right) = E\Bigg[\exp\Bigg(\int_0^1 \left\langle B^{-1}F\left(s, W^A(s) + \mathrm{e}^{sA}x\right), dW(s)\right\rangle - \frac{1}{2}\int_0^1 \left|B^{-1}F\left(s, W^A(s) + \mathrm{e}^{sA}x\right)\right|_H^2 ds\Bigg)\mathbf{1}_{\{\|\mathrm{e}^{\cdot A}x + W^A - \phi^h\|_2 \leqslant \varepsilon\}}\Bigg].$$

Note that to simplify the notation we have denoted $\widehat{W}$ by W. Since $h \in L^2([0,1]; H)$ we can apply again Girsanov's transformation, now with $\overline{W}(t) = W(t) - \int_0^t h(s)\,ds$. We get, for any $\varepsilon > 0$,

$$\begin{aligned} P\left(\left\|X - \phi^h\right\|_2 \leqslant \varepsilon\right) = E\Bigg[\exp\Bigg(&\int_0^1 \left\langle B^{-1}F\left(s, W^A(s) + \phi^h(s)\right), dW(s)\right\rangle \\ &+ \int_0^1 \left\langle B^{-1}F\left(s, W^A(s) + \phi^h(s)\right), h(s)\right\rangle ds \\ &- \frac{1}{2}\int_0^1 \left|B^{-1}F\left(s, W^A(s) + \phi^h(s)\right)\right|_H^2 ds \\ &- \int_0^1 \left\langle h(s), dW(s)\right\rangle - \frac{1}{2}\int_0^1 \left|h(s)\right|_H^2 ds\Bigg)\mathbf{1}_{\{\|W^A\|_2 \leqslant \varepsilon\}}\Bigg]. \end{aligned}$$

Then

$$\gamma_\varepsilon\left(\phi^h\right) = \frac{P(\|X - \phi^h\|_2 \leqslant \varepsilon)}{P(\|W^A\|_2 \leqslant \varepsilon)}$$

can be written as

$$\exp(\Lambda)E\left[\exp\left(\sum_{i=1}^3 T_i\right) \,\Big|\, \left\|W^A\right\|_2 \leqslant \varepsilon\right], \tag{8}$$

with

$$\begin{aligned} \Lambda &:= -\int_0^1 \frac{1}{2}\left|B^{-1}\left[A\phi^h(t) + F\left(t, \phi^h(t)\right) - \dot{\phi}^h(t)\right]\right|_H^2 dt, \\ T_1 &:= \int_0^1 \left\langle B^{-1}F\left(s, W^A(s) + \phi^h(s)\right), dW(s)\right\rangle, \\ T_2 &:= \int_0^1 \left\langle B^{-1}F\left(s, W^A(s) + \phi^h(s)\right) - B^{-1}F\left(s, \phi^h(s)\right), h(s)\right\rangle ds \end{aligned}$$

$$-\frac{1}{2}\int_0^1 \left(\left|B^{-1}F\left(s, W^A(s)+\phi^h(s)\right)\right|_H^2 - \left|B^{-1}F\left(s,\phi^h(s)\right)\right|_H^2\right) ds$$

and

$$T_3 := -\int_0^1 \langle h(s), dW(s)\rangle.$$

3.2. Study of W^A and Wiener integrals

In this subsection we obtain expressions of W^A and Wiener integrals that will be useful to study the terms obtained in the Girsanov expansion. We apply these expressions to reduce the study of the Onsager–Machlup functional to the study of T_1.

Assuming hypothesis (H1) and (H2), by Lemma 2.6 and decomposition (2) we have

$$\begin{aligned} W^A(t) &= \sum_{j=1}^{\infty}\left(\int_0^t \beta_j\, \mathrm{e}^{-\alpha_j(t-s)}\, dW^j(s)\right) e_j \\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \mu_{k,j} Y_{k,j}(g_{k,j}\otimes e_j)(t), \end{aligned} \tag{9}$$

where $\mu_{k,j} = \beta_j/\sqrt{\alpha_j^2 + x_{k,j}^2}$, and $x_{k,j}$, $Y_{k,j}$ and $g_{k,j}$ are the x_k, Y_k and g_k defined in Lemma 2.6 when $\lambda = \alpha_j$. Moreover, $\{Y_{k,j},\ k \geqslant 1,\ j \geqslant 1\}$ is a family of independent centered random variables with variance 1.

Given $f \in L^2([0,1])$ and $e \in H$, we denote by $f \otimes e$ the function of $L^2([0,1]; H)$ such that $(f\otimes e)(s) = f(s)e$. Note then that $\{g_{k,j}\otimes e_j,\ k \geqslant 1,\ j \geqslant 1\}$ is an orthonormal basis of $L^2([0,1]; H)$ such that for any $j,k \geqslant 1$: $\mathrm{Cov}(\langle W^A, g_{k,j}\otimes e_j\rangle_{L^2([0,1];H)}) = \mu_{k,j}^2$. Thus,

$$\|W^A\|_2^2 = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \mu_{k,j}^2 Y_{k,j}^2,$$

with $\sum_{j,k=1}^{\infty} \mu_{k,j}^2 < +\infty$. Indeed, if $j < j_0$

$$\sum_{k=1}^{\infty} \mu_{k,j}^2 = \beta_j^2 \sum_{k=1}^{\infty} \frac{1}{x_{k,j}^2} = C\beta_j^2.$$

On the other hand, when $j \geqslant j_0$,

$$\begin{aligned} \sum_{k=1}^{\infty} \mu_{k,j}^2 &= \sum_{k=1}^{\infty} \frac{\beta_j^2}{\alpha_j^2 + x_{k,j}^2} \leqslant \frac{\beta_j^2}{\alpha_j^2}\sum_{k=1}^{\infty}\frac{1}{1+\frac{(2k-1)^2\pi^2}{4\alpha_j^2}} < \frac{\beta_j^2}{\alpha_j^2}\int_0^{\infty}\frac{1}{1+\frac{\pi^2}{4\alpha_j^2}x^2}\,dx \\ &= \frac{\beta_j^2}{\alpha_j}\int_0^{\infty}\frac{1}{1+\frac{\pi^2}{4}x^2}\,dx \leqslant C\frac{\beta_j^2}{\alpha_j}. \end{aligned} \tag{10}$$

So

$$\sum_{j,k=1}^{\infty} \mu_{k,j}^2 \leqslant C\left(\sum_{j=1}^{j_0-1} \frac{\beta_j^2}{\alpha_j+1} + \sum_{j=j_0}^{\infty} \frac{\beta_j^2}{\alpha_j+1}\frac{\alpha_j+1}{\alpha_j} \right)$$
$$\leqslant C\left(1+\frac{1}{\alpha_{j_0}}\right)\sum_{j=1}^{\infty}\frac{\beta_j^2}{\alpha_j+1} < \infty.$$

Consider also

$$h_{k,j}(s) := \frac{1}{\mu_{k,j}} \int_s^1 \beta_j \mathrm{e}^{-\alpha_j(t-s)} g_{k,j}(t)\,dt,$$

$j \geqslant 1, k \geqslant 1$. Then, for any $j \geqslant 1$, $\{h_{k,j},\ k \geqslant 1\}$ is an orthonormal basis of $L^2([0,1])$. Indeed, the $h_{k,j}$ form an orthogonal family since, by (4) and (5),

$$\begin{aligned}
&\langle h_{m,j}, h_{n,j}\rangle_{L^2[0,1]} \\
&= \frac{\beta_j^2}{\mu_{m,j}\mu_{n,j}} \int_0^1 \left(\int_s^1 \mathrm{e}^{-\alpha_j(t-s)} g_{m,j}(t)\,dt \right)\left(\int_s^1 \mathrm{e}^{-\alpha_j(u-s)} g_{n,j}(u)\,du \right) ds \\
&= \frac{\beta_j^2}{\mu_{m,j}\mu_{n,j}} \int_0^1\int_0^1 K_j(t,u) g_{m,j}(t) g_{n,j}(u)\,dt\,du \\
&= \langle g_{m,j}, g_{n,j}\rangle_{L^2[0,1]},
\end{aligned} \tag{11}$$

for any $m,n \geqslant 1$, where K_j denotes the covariance function defined at (4) with $\lambda = \alpha_j$. Thus, in order to prove that $\{h_{k,j},\ k \geqslant 1\}$ is a basis, it is sufficient to show that, if $h \in L^2([0,1])$ satisfies $\langle h_{k,j}, h\rangle_{L^2[0,1]} = 0$ for all $k \geqslant 1$, then $h \equiv 0$. But this follows easily from the fact that if for all $k \geqslant 1$,

$$0 = \langle h_{k,j}, h\rangle_{L^2[0,1]} = \frac{1}{\mu_{k,j}} \int_0^1 \left(\int_s^1 \beta_j\, \mathrm{e}^{-\alpha_j(t-s)} g_{k,j}(t)\,dt \right) h(s)\,ds$$
$$= \frac{\beta_j}{\mu_{k,j}} \langle g_{k,j}, \varphi^h \rangle_{L^2[0,1]},$$

then $\varphi^h \equiv 0$ with $\varphi^h(t) = \int_0^t \mathrm{e}^{-\alpha_j(t-s)} h(s)\,ds$, and of course $h \equiv 0$.

Furthermore,

$$\begin{aligned}
Y_{k,j} &= \sqrt{\alpha_j^2 + x_{k,j}^2} \int_0^1 \left(\int_0^t \mathrm{e}^{-\alpha_j(t-s)}\,dW^j(s) \right) g_{k,j}(t)\,dt \\
&= \int_0^1 \left(\frac{\beta_j}{\mu_{k,j}} \int_s^1 \mathrm{e}^{-\alpha_j(t-s)} g_{k,j}(t)\,dt \right) dW^j(s) \\
&= I_j(h_{k,j}),
\end{aligned}$$

where $I_j(l)$ denotes the Wiener integral of l with respect to W^j, that is $I_j(l) = \int_0^1 l(s)\,dW^j(s)$.

From these considerations, we get the following lemma:

LEMMA 3.1. – *Let $l \in L^2([0,1]; H)$, and set $U = \int_0^1 \langle l(s), dW(s)\rangle$. Then*

$$\lim_{\varepsilon\to 0} E\left[\exp(U) \mid \left\|W^A\right\|_2 \leqslant \varepsilon\right] = 1.$$

Proof. – Since $l \in L^2([0,1]; H)$, we have

$$l = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \pi_{k,j}(h_{k,j} \otimes e_j),$$

with $\pi_{k,j} = \langle l, h_{k,j} \otimes e_j\rangle_{L^2([0,1];H)}$ and $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \pi_{k,j}^2 < \infty$. Furthermore, we have

$$U = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \pi_{k,j} I_j(h_{k,j}) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \pi_{k,j} Y_{k,j}.$$

Hence, the result follows easily by Lemma 2.4. □

Applying Lemma 3.1, since $h \in L^2([0,1])$ we directly get that

$$\lim_{\varepsilon\to 0} E\left[\exp(cT_3) \mid \left\|W^A\right\|_2 \leqslant \varepsilon\right] = 1,$$

for any $c \in \mathbb{R}$. On the other hand, on the set $\{\|W^A\|_2 \leqslant \varepsilon\}$ using (h2) it is easy to check that $|T_2| \leqslant C\varepsilon$, and hence

$$\limsup_{\varepsilon\to 0} E\left[\exp(cT_2) \mid \left\|W^A\right\|_2 \leqslant \varepsilon\right] \leqslant 1,$$

for any $c \in \mathbb{R}$. Thus, using Lemma 2.2, the only point remaining in order to determine $\lim_{\varepsilon\downarrow 0} \gamma_\varepsilon(\phi^h)$ is the study of the term $E[\exp(T_1) \mid \|W^A\|_2 \leqslant \varepsilon]$.

3.3. The linear case

In this subsection we discuss the case of a linear function F that not depend on t. We obtain the Onsager–Machlup functional and we study carefully the particular case where F can be diagonalized in the same basis as the operators A and B.

The main theorem of this part states as follows:

THEOREM 3.2. – *Assume that* (H1) *and* (H2) *are satisfied, $h : [0,1] \to H$ is a function such that $h \in L^2([0,1]; H)$, ϕ^h is defined by* (7) *and $F \in \mathcal{L}(H)$ is such that $\widehat{P} = B^{-1}F$ is a bounded operator. Denote by P and R the linear operators defined on $L^2([0,1]; H)$ such that for any $f \in L^2([0,1])$, $P(f \otimes e_j) = f \otimes \widehat{P}(e_j)$ and $R(f \otimes e_j) = R_j(f) \otimes e_j$ with*

$$(R_j f)(s) := \int_s^1 \beta_j \,\mathrm{e}^{-\alpha_j (t-s)} f(t)\,dt.$$

Then, for any $j,k\geqslant 1$,

$$\big\langle (R^*R)(g_{k,j}\otimes e_j), g_{k,j}\otimes e_j\big\rangle_{L^2([0,1];H)}=\mathrm{Cov}\big(\big\langle W^A, g_{k,j}\otimes e_j\big\rangle_{L^2([0,1];H)}\big)$$

and

(i) *if* $\mathcal{S}_{PR^*}\equiv\frac{1}{2}(PR^*+(PR^*)^*)$ *is trace class, then*

$$\lim_{\varepsilon\downarrow 0}\gamma_\varepsilon(\phi^h)=\exp\left(-\int_0^1\frac{1}{2}\big|B^{-1}\big[(A+F)\phi^h(t)-\dot\phi^h(t)\big]\big|_H^2\,dt-\mathrm{Tr}(\mathcal{S}_{PR^*})\right),$$

(ii) *if* $\sum_{j,k}\langle PR^*(g_{k,j}\otimes e_j), g_{k,j}\otimes e_j\rangle=+\infty$ (*respectively* $-\infty$), *then*

$$\lim_{\varepsilon\downarrow 0}\gamma_\varepsilon(\phi^h)=0$$

(*respectively* $+\infty$).

Proof. –

Step 1. Reduction to a stochastic integral involving W^A and W.

Since $\widehat{P}$ is a bounded operator, condition (h2) is clearly satisfied. On the other hand, since $F\in\mathcal{L}(H)$ we have that $(A+F)$ generates a C_0-semigroup such that $\int_0^1\|\mathrm{e}^{t(A+F)}B\|_{HS}^2\,ds<\infty$ (see e.g. Goldberg [7, Chapter 5.1]). Thus, following the proof of [4, Theorem 10.20] we get

$$\begin{aligned}
&E\left[\exp\left(\frac{1}{2}\int_0^1\big|B^{-1}F\big(X(t)\big)\big|_H^2\,dt\right)\right]\\
&\quad\leqslant\exp\left(\frac{C}{2}\int_0^1\big(1+\big\|\mathrm{e}^{t(A+F)}\big\|^2|x|_H^2\big)\,dt\right)E\left[\exp\left(\frac{C}{2}\int_0^1\big|W^{A+F}(t)\big|_H^2\,dt\right)\right].\\
&\quad<+\infty
\end{aligned}$$

and (h1) is clearly satisfied.

Hence, it is sufficient to study $\lim_{\varepsilon\downarrow 0}E[\exp(T_1)\big|\|W^A\|_2\leqslant\varepsilon]$.

Since we are in the linear case, we can write $T_1:=T_{1(a)}+T_{1(b)}$ with

$$T_{1(a)}=\int_0^1\big\langle B^{-1}F\big(W^A(s)\big), dW(s)\big\rangle,$$

$$T_{1(b)}=\int_0^1\big\langle B^{-1}F\big(\phi^h(s)\big), dW(s)\big\rangle.$$

Since $B^{-1}F$ is a bounded operator, $B^{-1}F(\phi^h)\in L^2([0,1];H)$ and $T_{1(b)}$ can be handled using Lemma 3.1.

The study of the term $T_{1(a)}$ will follow the ideas presented by Mayer-Wolf and Zeitouni [11].

Step 2. Expression of $T_{1(a)}$ in terms of Skorohod integrals.

Notice first that for any $j \geqslant 1, k \geqslant 1$, $h_{k,j} = \frac{1}{\mu_{k,j}} R_j g_{k,j}$.

Then, using the decompositions given in Section 3.2 (see (9)) we have

$$\begin{aligned} T_{1(a)} &= \int_0^1 \langle \widehat{P}(W^A(s)), dW(s) \rangle \\ &= \sum_{i,j,k=1}^{\infty} \mu_{k,j} \int_0^1 Y_{k,j} \langle g_{k,j}(s) \otimes \widehat{P}(e_j), e_i \rangle \, dW^i(s) \\ &= \sum_{i,j,k=1}^{\infty} \mu_{k,j} \langle \widehat{P} e_j, e_i \rangle \int_0^1 Y_{k,j}\, g_{k,j}(s) \, dW^i(s). \end{aligned} \tag{12}$$

It is worth noting that the random variables $Y_{k,j}$ are $\mathcal{F}_1$-measurable. Hence, some of the stochastic integrals appearing in (12) are anticipating. When they are of this kind, we have taken them in the Skorohod sense, and we switch from Itô's integrals to Skorohod's ones using the fact that they coincide on the set L_a^2 of square integrable adapted processes (see for instance [12] for an account on Skorohod's integrals). Moreover, using [12, Eq. (1.45)], observe that when $j = i$

$$\int_0^1 Y_{k,j} g_{k,j}(s) \, dW^j(s) = Y_{k,j} \sum_{m=1}^{\infty} \langle g_{k,j}, h_{m,j} \rangle_{L^2[0,1]} I_j(h_{m,j}) - \langle h_{k,j}, g_{k,j} \rangle_{L^2[0,1]},$$

and when $j \neq i$

$$\int_0^1 Y_{k,j} g_{k,j}(s) \, dW^i(s) = Y_{k,j} \sum_{m=1}^{\infty} \langle g_{k,j}, h_{m,i} \rangle_{L^2[0,1]} I_i(h_{m,i}).$$

Using the fact that $h_{k,j} = \mu_{k,j}^{-1} R_j g_{k,j}$ and $Y_{k,j} = I_j(h_{k,j})$, we can write (12) in the following way:

$$\begin{aligned} T_{1(a)} = &\sum_{(k,j) \neq (m,i)} \frac{\mu_{k,j}}{\mu_{m,i}} Y_{k,j} Y_{m,i} \langle \widehat{P} e_j, e_i \rangle \langle g_{k,j}, R_i g_{m,i} \rangle_{L^2[0,1]} \\ &+ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (Y_{k,j}^2 - 1) \langle \widehat{P} e_j, e_j \rangle \langle g_{k,j}, R_j g_{k,j} \rangle_{L^2[0,1]}. \end{aligned} \tag{13}$$

Step 3. Expression of $T_{1(a)}$ in terms of P and R^*.

Define now the operator $T : \ell^2_{\mathbb{N}^2} \to \ell^2_{\mathbb{N}^2}$ by

$$T_{(k,j),(m,i)} = \frac{\mu_{k,j}}{\mu_{m,i}} \langle \widehat{P} e_j, e_i \rangle \langle g_{k,j}, R_i g_{m,i} \rangle_{L^2[0,1]}, \quad (k,j),(m,i) \in \mathbb{N}^2. \tag{14}$$

Then

$$T_{1(a)} = \sum_{(k,j)\neq(m,i)} T_{(k,j),(m,i)} Y_{k,j} Y_{m,i} + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} T_{(k,j),(k,j)}\big(Y_{k,j}^2 - 1\big). \tag{15}$$

Let Λ be the linear operator such that $\Lambda(g_{k,j} \otimes e_j) = \mu_{k,j}(g_{k,j} \otimes e_j)$. Observe that, for any $j, k \geqslant 1$,

$$\begin{aligned}(R^*R)(g_{k,j} \otimes e_j) &= \sum_{n=1}^{\infty} \big\langle (R_j^* R_j) g_{k,j}, g_{n,j} \big\rangle_{L^2[0,1]} (g_{n,j} \otimes e_j)\\ &= \sum_{n=1}^{\infty} \langle R_j g_{k,j}, R_j g_{n,j} \rangle_{L^2[0,1]} (g_{n,j} \otimes e_j),\end{aligned}$$

and by (11) this last quantity equals $\mu_{k,j}^2(g_{k,j} \otimes e_j)$.

Hence $\Lambda^2 = R^*R$. Define $U = R\Lambda^{-1} = \frac{R}{|R|}$, which is clearly a bounded operator defined in $L^2([0,1]; H)$. We get

$$\begin{aligned}T_{(k,j),(m,i)} &= \big\langle P\Lambda(g_{k,j} \otimes e_j), R\Lambda^{-1}(g_{m,i} \otimes e_i)\big\rangle_{L^2([0,1];H)}\\ &= \big\langle U^* P R^* U(g_{k,j} \otimes e_j), (g_{m,i} \otimes e_i)\big\rangle_{L^2([0,1];H)}.\end{aligned} \tag{16}$$

Then, when $\mathcal{S}_{PR^*}$ is a trace class operator, $\sum_{k,j} T_{(k,j),(k,j)} = \mathrm{Tr}(\mathcal{S}_{PR^*})$. Indeed, $U(g_{k,j} \otimes e_j) = h_{k,j} \otimes e_j$ so that by (16)

$$T_{(k,j),(k,j)} = \big\langle PR^*(h_{k,j} \otimes e_j), h_{k,j} \otimes e_j\big\rangle_{L^2([0,1];H)}.$$

The conclusion follows as $\{h_{k,j} \otimes e_j, k \geqslant 1, j \geqslant 1\}$ is an orthonormal basis of $L^2([0,1], H)$.

Step 4. Application of Lemma 2.5.

Using Lemma 2.2, (8) and Eq. (15), in order to see part (i) it is sufficient to show that

$$E\left[\exp\left(c \sum_{i,k,j,m} Y_{k,j} Y_{m,i} T_{(k,j),(m,i)}\right) \,\middle|\, \big\|W^A\big\|_2 \leqslant \varepsilon\right] \to 1$$

when ε goes to 0, for any $c \in \mathbb{R}$.

Since by assumption $\mathcal{S}_{PR^*}$ is a trace class operator, by (16), the decomposition of W^A given in (9), and applying part 1) in Lemma 2.5 we finish easily the proof of part (i).

To prove (ii), we should proceed with the same computations. From (14) we get

$$T_{(k,j),(k,j)} = \big\langle PR^*(g_{k,j} \otimes e_j), g_{k,j} \otimes e_j\big\rangle_{L^2([0,1];H)}.$$

Applying part 2) of Lemma 2.5 we obtain easily the desired result. □

We finish this subsection with an important corollary where we study the case where F is a trace class operator. We also examine the diagonal case.

COROLLARY 3.3. – *Assume that* (H1) *and* (H2) *are satisfied, $h : [0,1] \to H$ is a function such that $h \in L^2([0,1]; H)$, ϕ^h is defined by* (7) *and F is a trace class operator* *such that $\widehat{P} = B^{-1}F$ is a bounded operator. Then, if $\mathcal{S}_{PR^*}$ is trace class*

$$\lim_{\varepsilon\downarrow 0}\gamma_\varepsilon(\phi^h) = \exp\left(-\int_0^1 \frac{1}{2}\left|B^{-1}\left[(A+F)\phi^h(t) - \dot{\phi}^h(t)\right]\right|_H^2 dt - \frac{1}{2}\operatorname{Tr}(F)\right).$$

Proof. – Using Theorem 3.2 it is enough to prove that if F is a trace class operator then $\operatorname{Tr}(\mathcal{S}_{PR^*}) = \frac{1}{2}\operatorname{Tr}(F)$.

We have to study the eigenvalues of the operator $\mathcal{S}_{PR^*}$. Denote by $\widehat{R}_j$ the operator $\beta_j^{-1}R_j$. Set $V_j \equiv \frac{1}{2}(\widehat{R}_j^* + \widehat{R}_j)$. First, we can check trivially that for all $j \geqslant 1$, V_j is a linear operator on $L^2([0,1])$ given by a kernel denoted by $\widehat{K}_j$: indeed, for any $h \in L^2([0,1])$

$$\left[V_j(h)\right](s) = \int_0^1 \widehat{K}_j(s,t)h(t)\,dt, \quad s \in [0,1],$$

with $\widehat{K}_j : [0,1]^2 \to [0,1]$ defined by $\widehat{K}_j(s,t) = \frac{1}{2}\mathrm{e}^{-\alpha_j|s-t|}$.

Consider now an operator V on $L^2([0,1])$ given by a kernel $\hat{K}(s,t) = \frac{1}{2}\mathrm{e}^{-\lambda|s-t|}$, $\lambda > 0$, an let us show that V is a non-negative: indeed, if $\widehat{R}$ is the Volterra operator on $L^2([0,1])$ defined by $\widehat{R}h = g$ and

$$g(t) = \int_0^t \mathrm{e}^{-\lambda(t-s)}h(s)\,ds, \quad t \in [0,1],$$

then it is readily seen that

$$Vh = \frac{1}{2}\langle h, f\rangle_{L^2[0,1]} f + \lambda \widehat{R}^*\widehat{R}h,$$

where $f(t) = \mathrm{e}^{-\lambda(1-t)}$ for all $t \in [0,1]$. Therefore,

$$\langle Vh, h\rangle_{L^2[0,1]} = \frac{1}{2}\langle h, f\rangle_{L^2[0,1]}^2 + \lambda\langle \widehat{R}h, \widehat{R}h\rangle_{L^2[0,1]},$$

which shows the positivity.

Thus, V is a positive Hilbert–Schmidt operator given by a continuous kernel on $[0,1]^2$. If we denote by $\{\nu_k, k \geqslant 1\}$ the eigenvalues of V, it is then well-known [6, Proposition 10.1] that

$$\sum_{k=1}^\infty \nu_k = \int_0^1 \widehat{K}(t,t)\,dt = \frac{1}{2}.$$

Note that this value does not depend on λ.

On the other hand, for any $h \in L^2([0,1])$, $j \geqslant 1$ we have

$$\frac{1}{2}\left(PR^* + (PR^*)^*\right)(h \otimes e_j)(s)$$

$$= \frac{1}{2}\left\{ P\left(\left(\int_0^s \mathrm{e}^{-\alpha_j(s-u)}h(u)\,du\right)\beta_j \otimes e_j\right) + R\big[h \otimes F^*\big(\beta_j^{-1}e_j\big)\big](s)\right\}$$

$$= \frac{1}{2}\left\{\left(\int_0^s \mathrm{e}^{-\alpha_j(s-u)}h(u)\,du\right) \otimes \beta_j\big(B^{-1}F\big)e_j \right.$$

$$\left. + \sum_{i=1}^{\infty}\beta_i\beta_j^{-1}\langle F^*e_j, e_i\rangle\left(\int_s^1 \mathrm{e}^{-\alpha_i(u-s)}h(u)\,du\right) \otimes e_i\right\}.$$

Furthermore, observe that if we denote by $\{\hat{f}_{k,j}, k \geqslant 1\}$ the $L^2([0,1])$ orthonormal basis that diagonalizes V_j, then $\{\hat{f}_{k,j} \otimes e_j;\ j \geqslant 1, k \geqslant 1\}$ is an orthonormal basis of $L^2([0,1]; H)$. So, since

$$\big\langle \beta_j\big(B^{-1}F\big)e_j, e_j\big\rangle = \big\langle \big(B^{-1}F\big)e_j, Be_j\big\rangle = \langle Fe_j, e_j\rangle,$$

we get

$$\mathrm{Tr}\left(\frac{1}{2}\big(PR^* + (PR^*)^*\big)\right)$$

$$= \sum_{j,k=1}^{\infty}\left\langle \frac{1}{2}\big(PR^* + (PR^*)^*\big)(\hat{f}_{k,j} \otimes e_j),\, \hat{f}_{k,j} \otimes e_j\right\rangle_{L^2([0,1];H)}$$

$$= \frac{1}{2}\sum_{j,k=1}^{\infty}\langle \widehat{R}_j^*\hat{f}_{k,j}, \hat{f}_{k,j}\rangle_{L^2[0,1]}\langle Fe_j, e_j\rangle + \langle \widehat{R}_j\hat{f}_{k,j}, \hat{f}_{k,j}\rangle_{L^2[0,1]}\langle F^*e_j, e_j\rangle$$

$$= \sum_{j,k=1}^{\infty}\langle Fe_j, e_j\rangle\langle V_j\hat{f}_{k,j}, \hat{f}_{k,j}\rangle_{L^2[0,1]} = \sum_{j=1}^{\infty}\langle Fe_j, e_j\rangle\,\mathrm{Tr}(V_j)$$

$$= \frac{1}{2}\sum_{j=1}^{\infty}\langle Fe_j, e_j\rangle = \frac{1}{2}\,\mathrm{Tr}(F).$$

The proof is now completed. □

Example 3.4. – Consider the case of an operator F which is diagonal when expressed in the orthonormal basis $\{e_j;\ j \geqslant 1\}$. We denote by $\{p_j;\ j \geqslant 1\}$ the corresponding set of eigenvalues. Assume also the other hypothesis of Theorem 3.2. Then:

(i′) the operator $\mathcal{S}_{PR^*}$ is trace class if and only if F is, and in this case, $\mathrm{Tr}(\mathcal{S}_{PR^*}) = \frac{1}{2}\,\mathrm{Tr}(F)$;

(ii′) if $\sum_{j\geqslant 1} p_j = +\infty$ and $\sum_{j,p_j<0}|p_j| < +\infty$, then

$$\lim_{\varepsilon\downarrow 0}\gamma_\varepsilon\big(\phi^h\big) = 0;$$

(iii′) if $\sum_{j\geqslant 1} p_j = -\infty$ and $\sum_{j,p_j>0} p_j < +\infty$, then

$$\lim_{\varepsilon\downarrow 0}\gamma_\varepsilon\big(\phi^h\big) = +\infty.$$

Remark 3.5. –

1. Notice that when $h=0$, $X-\phi^h=W^{A+F}$. So, in the situation (ii′) we have that

$$\lim_{\varepsilon\to 0}\frac{P(\|W^{A+F}\|_2\leqslant\varepsilon)}{P(\|W^{A}\|_2\leqslant\varepsilon)}=0,$$

   and in the situation (iii′)

$$\lim_{\varepsilon\to 0}\frac{P(\|W^{A+F}\|_2\leqslant\varepsilon)}{P(\|W^{A}\|_2\leqslant\varepsilon)}=+\infty.$$

2. Since we need $B^{-1}F$ to be a bounded operator, we assume $\sup_j\frac{|p_j|}{\beta_j}<\infty$.
3. The conditions (i′), (ii′) and (iii′) only involve the operator F, and none of the operators A and B.

Proof. – Since $PR^*(h\otimes e_j)=p_j(\widehat{R}_j^*(h)\otimes e_j)$ and $(PR^*)^*(h\otimes e_j)=p_j(\widehat{R}_j(h)\otimes e_j)$ in our diagonal case, it is easily seen that $\mathcal{S}_{PR^*}$ is of trace class iff $\zeta\equiv\sum_{i,j\geqslant 1}|p_jv_{i,j}|<\infty$, where the family $\{v_{i,j};i\geqslant 1\}$ denotes the set of eigenvalues of the operator V_j. As we have seen in Corollary 3.3, for any $j\geqslant 1$, $\sum_{i\geqslant 1}|v_{i,j}|=\sum_{i\geqslant 1}v_{i,j}=\frac{1}{2}$. It is now easily deduced that $\mathcal{S}_{PR^*}$ is trace class if and only if $\sum_{j\geqslant 1}|p_j|<\infty$.

Assume now that the operator $\mathcal{S}_{PR^*}$ is not trace class. Observe first that

$$\langle PR^*(g_{k,j}\otimes e_j),g_{k,j}\otimes e_j\rangle_{L^2([0,1];H)}=p_j\langle\widehat{R}_jg_{k,j},g_{k,j}\rangle_{L^2[0,1]}.$$

We can compute easily

$$\begin{aligned}\widehat{R}_jg_{k,j}(s)&=\int_s^1\mathrm{e}^{-\alpha_j(t-s)}g_{k,j}(t)\,dt\\&=\frac{A_{k,j}\mathrm{e}^{\alpha_js}}{\alpha_j^2+x_{k,j}^2}\big(\mathrm{e}^{-\alpha_js}\big(x_{k,j}\cos(x_{k,j}s)+\alpha_j\sin(x_{k,j}s)\big)\\&\quad-\mathrm{e}^{-\alpha_j}\big(x_{k,j}\cos(x_{k,j})+\alpha_j\sin(x_{k,j})\big)\big)\\&=\frac{A_{k,j}}{\alpha_j^2+x_{k,j}^2}\big(x_{k,j}\cos(x_{k,j}s)+\alpha_j\sin(x_{k,j}s)\big),\end{aligned}$$

using $x_{k,j}\cos(x_{k,j})=-\alpha_j\sin(x_{k,j})$. Since $\int_0^1A_{k,j}^2\sin^2(x_{k,j}s)\,ds=1$, we then obtain

$$\begin{aligned}\langle\widehat{R}_jg_{k,j},g_{k,j}\rangle_{L^2[0,1]}&=\int_0^1\frac{A_{k,j}^2}{\alpha_j^2+x_{k,j}^2}\big(x_{k,j}\cos(x_{k,j}s)\sin(x_{k,j}s)+\alpha_j\sin^2(x_{k,j}s)\big)\,ds\\&=\frac{1}{\alpha_j^2+x_{k,j}^2}\bigg(\frac{A_{k,j}^2}{2}\sin^2(x_{k,j})+\alpha_j\bigg).\end{aligned}$$

Furthermore, we have that for $j\geqslant j_0$

$$\frac{C_1}{\alpha_j}\leqslant\sum_{k=1}^{\infty}\frac{1}{\alpha_j^2+x_{k,j}^2}\leqslant\frac{C_2}{\alpha_j},$$

for some positive constants C_1 and C_2 not depending on j. Indeed, the second inequality is obtained using similar arguments that in (10). On the other hand,

$$\sum_{k=1}^{\infty}\frac{1}{\alpha_j^2+x_{k,j}^2}\geqslant\frac{1}{\alpha_j^2}\sum_{k=1}^{\infty}\frac{1}{1+(2k+1)^2\frac{\pi^2}{4\alpha_j^2}}\geqslant\frac{1}{\alpha_j^2}\sum_{k=1}^{\infty}\frac{1}{1+k^2\frac{9\pi^2}{4\alpha_j^2}}$$
$$\geqslant\frac{1}{\alpha_j^2}\int_1^{\infty}\frac{1}{1+x^2\frac{9\pi^2}{4\alpha_j^2}}\,dx\geqslant\frac{1}{\alpha_j}\int_{\frac{1}{\alpha_{j_0}}}^{\infty}\frac{1}{1+x^2\frac{9\pi^2}{4}}\,dx=\frac{C_1}{\alpha_j}.$$

Using that $\sup_{k,j}|A_{k,j}|\leqslant 2$, and

$$\sum_{j,k=1}^{\infty}\langle PR^*(g_{k,j}\otimes e_j),g_{k,j}\otimes e_j\rangle_{L^2([0,1];H)}$$
$$=\sum_{j,k=1}^{\infty}\frac{p_j}{\alpha_j^2+x_{k,j}^2}\left(\frac{A_{k,j}^2}{2}\sin^2(x_{k,j})+\alpha_j\right),\tag{17}$$

we can prove that in the situation (ii′) last expression is equal to $+\infty$, since

$$\sum_{j=j_0,k=1}^{\infty}\frac{p_j}{\alpha_j^2+x_{k,j}^2}\alpha_j\geqslant\sum_{j=j_0}^{\infty}C_1p_j+\sum_{j\geqslant j_0,\,p_j<0}(C_2-C_1)p_j=+\infty,$$
$$\sum_{j=j_0,k=1}^{\infty}\frac{p_j}{\alpha_j^2+x_{k,j}^2}\left(\frac{A_{k,j}^2}{2}\sin^2(x_{k,j})\right)\geqslant\frac{2C_2}{\alpha_1}\sum_{j\geqslant j_0,\,p_j<0}p_j>-\infty,$$

and

$$\left|\sum_{j=1}^{j_0-1}\sum_{k=1}^{\infty}\frac{p_j}{\alpha_j^2+x_{k,j}^2}\left(\frac{A_{k,j}^2}{2}\sin^2(x_{k,j})+\alpha_j\right)\right|\leqslant 2\sum_{j=1}^{j_0-1}|p_j|\sum_{k=1}^{\infty}\frac{1}{x_{k,j}^2}<\infty.$$

In the situation (iii′), by the same arguments, we can prove that expression (17) is equal to $-\infty$.

The proofs of the conclusions of (ii′) and (iii′) in our example are now straightforward, invoking the second part of Lemma 2.5. □

3.4. The general case

In this subsection we deal with the case of a general function F, by means of a linearization procedure, which is usual in Onsager–Machlup type results (see e.g. [9]). Let us introduce first some notation: given a differentiable function $S:H\to H$ and $x\in H$ we denote by $D_xS\in\mathcal{L}(H)$ the derivative operator of S at x, and for any differentiable function $T:[0,1]\times H\to H$ and $\xi\in L^2([0,1];H)$ we define by $D_\xi T$ the operator defined on $L^2([0,1];H)$ by $((D_\xi T)(\psi))(s):=(D_{\xi(s)}T(s,\cdot))(\psi(s))$ for any $\psi\in L^2([0,1];H)$.

The main theorem is the following:

THEOREM 3.6. – *Assume that* (H1) *and* (H2) *are satisfied. Suppose h is an element of $L^2([0,1];H)$, and ϕ^h is defined by* (7) *and $F:[0,1]\times H\to H$ is a Lipschitz* *continuous function such that* $\widehat{P} = B^{-1}F$ *is* $\mathcal{C}_b^2$ *in* x *uniformly in* $s \in [0,1]$. *Denote* $B^{-1}F(s,\cdot)$ *by* $\widehat{P}_s$. *Let* R *be the linear operator defined in Theorem* 3.2 *and denote by* $P : L^2([0,1]; H) \to L^2([0,1]; H)$ *the operator given by*

$$\big(P(u \otimes l)\big)(s) = u(s)(D_{\phi^h(s)}\widehat{P}_s)(l), \quad u \in L^2\big([0,1]\big),\ l \in H.$$

Assume finally that $\mathcal{S}_{PR^*}$ *is trace class and there exists* $r > 0$ *and a deterministic trace class operator* $\widehat{T} : L^2([0,1]; H) \to L^2([0,1]; H)$ *such that, for any* ξ *satisfying* $\|\phi^h - \xi\|_2 \leqslant r$,

$$\big|\big\langle (D_\xi \widehat{P} - D_{\phi^h}\widehat{P})R^*\psi, \psi\big\rangle_{L^2([0,1];H)}\big| \leqslant \langle \widehat{T}\psi, \psi\rangle_{L^2([0,1];H)}, \tag{18}$$

for any $\psi \in L^2([0,1]; H)$.

Then

$$\lim_{\varepsilon\downarrow 0}\gamma_\varepsilon(\phi) = \exp\left(-\frac{1}{2}\int_0^1 \big|B^{-1}\big[A\phi^h(t) + F\big(t,\phi^h(t)\big) - \dot{\phi}^h(t)\big]\big|_H^2\,dt - \mathrm{Tr}(\mathcal{S}_{PR^*})\right).$$

Proof. – Since (h1) and (h2) are satisfied, like in the proof of Theorem 3.2 it is enough to study $\lim_{\varepsilon\downarrow 0} E[\exp(T_1) \mid \|W^A\|_2 \leqslant \varepsilon]$.

Taylor's formula for Hilbert space valued functions gives us

$$\begin{aligned}\widehat{P}_s\big(W^A(s) + \phi^h(s)\big) &= \widehat{P}_s\big(\phi^h(s)\big) + (D_{\phi^h(s)}\widehat{P}_s)\big(W^A(s)\big)\\ &\quad + \left(\int_0^1 \big(D_{\phi^h(s)+\lambda W^A(s)}\widehat{P}_s - D_{\phi^h(s)}\widehat{P}_s\big)\,d\lambda\right)\big(W^A(s)\big).\end{aligned}$$

Then $T_1 := T_{1(c)} + T_{1(d)} + T_{1(e)}$ with

$$T_{1(c)} = \int_0^1 \big\langle \widehat{P}_s\big(\phi^h(s)\big), dW(s)\big\rangle,$$

$$T_{1(d)} = \int_0^1 \big\langle (D_{\phi^h(s)}\widehat{P}_s)\big(W^A(s)\big), dW(s)\big\rangle,$$

$$T_{1(e)} = \int_0^1 \left\langle \left(\int_0^1 (D_{\phi^h(s)+\lambda W^A(s)}\widehat{P}_s - D_{\phi^h(s)}\widehat{P}_s)\,d\lambda\right)\big(W^A(s)\big), dW(s)\right\rangle.$$

Since $\widehat{P}$ is $\mathcal{C}_b^2$ uniformly in s, $\widehat{P}(\phi^h) \in L^2([0,1]; H)$ and we can deal with $T_{1(c)}$ applying Lemma 3.1.

The term $T_{1(d)}$ can be handled in much the same way as $T_{1(a)}$ in the proof of Theorem 3.2, the only difference being in the analysis of $T_{1(d)}$ (i.e. the dependence on s of the operator $D_{\phi^h(s)}\widehat{P}_s$), but note that the structure of the proof is still valid. Similarly to (13), $T_{1(d)}$ can be written

$$T_{1(d)} = \sum_{(k,j)\neq(m,i)} \frac{\mu_{k,j}}{\mu_{m,i}} Y_{k,j} Y_{m,i} \langle M^i_{k,j}, R_i g_{m,i} \rangle_{L^2[0,1]} + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (Y^2_{k,j} - 1) \langle M^j_{k,j}, R_j g_{k,j} \rangle_{L^2[0,1]},$$

with $M^i_{k,j}(s) := \langle (P(g_{k,j} \otimes e_j))(s), e_i \rangle$. Since

$$\langle M^i_{k,j}, R_i g_{m,i} \rangle_{L^2[0,1]} = \langle P(g_{k,j} \otimes e_j), R_i g_{m,i} \otimes e_i \rangle_{L^2([0,1];H)},$$

proceeding as in Theorem 3.2 we can then obtain

$$E\left[\exp(T_{1(d)}) \mid \|W^A\|_2 \leqslant \varepsilon\right] = \exp\left(-\operatorname{Tr}(\mathcal{S}_{PR^*})\right) E\left[\exp(T_{1(d,2)}) \mid \|W^A\|_2 \leqslant \varepsilon\right], \quad (19)$$

with

$$T_{1(d,2)} = \sum_{(k,j),(m,i)} Y_{k,j} Y_{m,i} \langle (U^* P R^* U)(g_{k,j} \otimes e_j), g_{m,i} \otimes e_i \rangle_{L^2([0,1];H)}$$

where U is defined in the proof of Theorem 3.2 and by Lemma 2.5

$$\lim_{\varepsilon\to 0} E\left[\exp(T_{1(d,2)}) \mid \|W^A\|_2 \leqslant \varepsilon\right] = 1.$$

Finally we have to study $T_{1(e)}$. We will follow the method used in [11, Theorem 4.1]. Set $\varphi^\lambda = \phi^h + \lambda W^A$. Just like in the case of $T_{1(d)}$, we get

$$\int_0^1 \left\langle \int_0^1 \left((D_{\varphi^\lambda(s)} \widehat{P}_s - D_{\phi^h(s)} \widehat{P}_s)\, d\lambda\right)\left(W^A(s)\right), dW(s) \right\rangle = \int_0^1 \sum_{(k,j),(m,i)} Y_{k,j} Y_{m,i} T^\lambda_{(k,j),(m,i)}\, d\lambda - \int_0^1 \sum_{j,k} T^\lambda_{(k,j),(k,j)}\, d\lambda,$$

where

$$T^\lambda_{(k,j),(m,i)} = \left\langle \left(U^*(D_{\varphi^\lambda} \widehat{P} - D_{\phi^h} \widehat{P}) R^* U\right)(g_{k,j} \otimes e_j), g_{m,i} \otimes e_i \right\rangle_{L^2([0,1];H)}.$$

Since $\widehat{P}_s$ is $\mathcal{C}^2_b$ uniformly in s, we clearly have

$$\left|(D_{\varphi^\lambda(s)} \widehat{P}_s - D_{\phi^h(s)} \widehat{P}_s)(y)\right| \leqslant C \left|\varphi^\lambda(s) - \phi^h(s)\right|_H |y|_H, \quad (20)$$

for any $y \in H$. So, fixed λ, (k,j), (m,i) and $\omega \in \Omega$,

$$\lim_{\varepsilon\to 0} \mathbf{1}_{\{\|W^A\|_2 \leqslant \varepsilon\}} T^\lambda_{(k,j),(m,i)} = 0.$$

Because of (18), by a dominated convergence argument we can prove that

$$\lim_{\varepsilon\to 0} \mathbf{1}_{\{\|W^A\|_2 \leqslant \varepsilon\}} \int_0^1 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} T^\lambda_{(k,j),(k,j)}\, d\lambda = 0.$$

Then, by Lemma 2.2, we shall have established the theorem if we prove that, for all $c \in \mathbb{R}$,

$$\lim_{\varepsilon\to 0} E\left[\exp\left(c\int_0^1 \sum_{(k,j),(m,i)} Y_{k,j}Y_{m,i}T^{\lambda}_{(k,j),(m,i)}\,d\lambda\right)\,\middle|\,\|W^A\|_2 \leqslant \varepsilon\right] \leqslant 1. \tag{21}$$

But, by assumption (18) with $\psi = \sum_{k,j} Y_{k,j}h_{k,j}\otimes e_j$ we get,

$$\begin{aligned} &E\left[\exp\left(c\int_0^1 \sum_{(k,j),(m,i)} Y_{k,j}Y_{m,i}T^{\lambda}_{(k,j),(m,i)}\,d\lambda\right)\,\middle|\,\|W^A\|_2 \leqslant \varepsilon\right] \\ &\quad\leqslant E\left[\exp\left(|c|\sum_{(k,j),(m,i)} Y_{k,j}Y_{m,i}\widehat{T}_{(k,j),(m,i)}\right)\,\middle|\,\|W^A\|_2 \leqslant \varepsilon\right], \end{aligned}$$

where

$$\widehat{T}_{(k,j),(m,i)} = \left\langle \widehat{T}(h_{k,j}\otimes e_j), h_{m,i}\otimes e_i\right\rangle_{L^2([0,1];H)}.$$

Therefore (21) follows from part 1) of Lemma 2.5. □

Likewise in the linear case, in some situations we can express the trace of $\mathcal{S}_{PR^*}$ in terms of $\nabla_x F$ and ϕ. We give this result in the next proposition.

PROPOSITION 3.7. – *Assume the assumptions of Theorem* 3.6. *If for any* $s \in [0,1]$, $\nabla_x F(s,\phi^h(s))$ *is a trace class operator and* $\int_0^1 \mathrm{Tr}[\nabla_x F(s,\phi^h(s))]\,ds < +\infty$, *then*

$$\mathrm{Tr}(\mathcal{S}_{PR^*}) = \frac{1}{2}\int_0^1 \mathrm{Tr}\left[\nabla_x F\left(s,\phi^h(s)\right)\right]ds.$$

Proof. – Notice that $(D_{\phi^h(s)}\widehat{P}_s)e_j = (B^{-1}\nabla_x F(s,\phi^h(s)))e_j$. Following the same computation we did in the proof of Corollary 3.3 we have

$$\begin{aligned} PR^*(f\otimes e_j)(s) &= \left(\int_0^s \mathrm{e}^{-\alpha_j(s-u)}f(u)\,du\right)\beta_j\left(B^{-1}\nabla_x F\left(s,\phi^h(s)\right)\right)e_j \\ &= \sum_{i=1}^{\infty}\left(\int_0^s \mathrm{e}^{-\alpha_j(s-u)}f(u)\,du\right)\beta_j\beta_i^{-1}\left\langle \nabla_x F\left(s,\phi^h(s)\right)e_j, e_i\right\rangle e_i \end{aligned}$$

and

$$\begin{aligned} (PR^*)^*(f\otimes e_j)(s) &= R\left[f(s)\left[\nabla_x F\left(s,\phi^h(s)\right)\right]^*\left(\beta_j^{-1}e_j\right)\right] \\ &= \sum_{i=1}^{\infty}\beta_i\beta_j^{-1}\left(\int_s^1 \mathrm{e}^{-\alpha_i(u-s)}f(u)\left\langle\left[\nabla_x F\left(u,\phi^h(u)\right)\right]^* e_j, e_i\right\rangle du\right)e_i. \end{aligned}$$

Let $\{f_k; k \geqslant 1\}$ be an orthonormal basis of $L^2([0,1])$. Then $\{f_k \otimes e_j;\, j \geqslant 1, k \geqslant 1\}$ is an orthonormal basis of $L^2([0,1]; H)$. Since $\mathcal{S}_{PR^*}$ is a trace class operator

$$\mathrm{Tr}(\mathcal{S}_{PR^*}) = \sum_{j=1}^{\infty} U_j$$

with

$$U_j = \sum_{k=1}^{\infty} \langle M_j f_k, f_k \rangle_{L^2([0,1])}$$

where M_j denotes the operator defined on $L^2([0,1])$ such that

$$\begin{aligned} M_j f(s) &= \frac{1}{2}\Bigg[\int_0^1 \Big(\mathrm{e}^{-\alpha_j(s-u)} f(u)\big\langle \nabla_x F\big(s, \phi^h(s)\big)e_j, e_j\big\rangle \mathbf{1}_{[0,s]}(u) \\ &\qquad + \mathrm{e}^{-\alpha_j(u-s)} f(u)\big\langle \nabla_x F\big(u, \phi^h(u)\big)e_j, e_j\big\rangle \mathbf{1}_{[s,1]}(u)\Big)\,du\Bigg] \\ &= \int_0^1 \widetilde{K}_j(s,u) f(u)\,du \end{aligned}$$

with

$$\widetilde{K}_j(s,u) = \frac{1}{2}\mathrm{e}^{-\alpha_j|s-u|}\big\langle \nabla_x F\big(u \vee s, \phi^h(u \vee s)\big)e_j, e_j\big\rangle.$$

Clearly, since $\mathcal{S}_{PR^*}$ is a trace class operator, M_j is a trace class operator for any $j \geqslant 1$, and as an operator given by a kernel in $[0,1]^2$, its trace is given by the integral of the kernel on the diagonal, that is

$$U_j = \mathrm{Tr}(M_j) = \int_0^1 \widetilde{K}_j(s,s)\,ds = \frac{1}{2}\int_0^1 \big\langle \nabla_x F\big(s, \phi^h(s)\big)e_j, e_j\big\rangle\,ds.$$

So,

$$\mathrm{Tr}(\mathcal{S}_{PR^*}) = \sum_{j=1}^{\infty} \frac{1}{2}\int_0^1 \big\langle \nabla_x F\big(s, \phi^h(s)\big)e_j, e_j\big\rangle\,ds = \frac{1}{2}\int_0^1 \mathrm{Tr}\big[\nabla_x F\big(s, \phi^h(s)\big)\big]\,ds. \qquad \square$$

Remark 3.8. – In order to stick to the finite dimensional case, we proved our result for a function F depending only on $t \in [0,1]$ and $x \in H$. However, the proof would remain the same for a function F satisfying

1. $F : [0,1] \times L^2([0,1]; H) \to L^2([0,1]; H)$.
2. $B^{-1}F(t,.) : L^2([0,1]; H) \to L^2([0,1]; H)$ is $\mathcal{C}_b^2$ uniformly in $t \in [0,1]$.
3. For all $t \in [0,1]$ and $\xi \in L^2([0,1]; H)$, $F(t,\xi) = F(t, \xi\mathbf{1}_{[0,t]})$.

The last condition is imposed to get an adapted solution to the evolution equation, and corresponds to the usual coefficients depending on the whole past of the process.

Example 3.9. – Suppose $H = L^2([0,1])$ with Neumann boundary conditions, that is $u'(0) = u'(1) = 0$, $A = \Delta$, $B = \mathrm{Id}$, and $F : [0,1] \times L^2([0,1]; H) \to L^2([0,1]; H)$ is given by

$$\big[F(t,\xi)\big](t,x) = f(0)\mathbf{1}_{[0,\frac{1}{2}]}(t)\mathbf{1}(x) + f\left(\int_0^{1/2} dv \int_0^1 dy\, \xi(v,y)\right)\mathbf{1}_{[\frac{1}{2},1]}(t)\mathbf{1}(x),$$

where $f : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}_b^2$ function. Then the conditions of Remark 3.8 are satisfied.

Proof. – We shall concentrate on condition (18), the other ones being easy to verify. Under the assumptions of our example, $\widehat{P} = F$, and the operator R^* is given by $[R^*(u \otimes e_j)] = R_j^* u \otimes e_j$ for all $u \in L^2([0,1])$ and $j \geqslant 0$, with

$$[R_j^* u](s) = \int_0^s \mathrm{e}^{-j^2(s-t)} u(t)\, dt, \quad t \in [0,1],$$

and $e_j(y) = \cos(2\pi j y)$ for all $y \in [0,1]$. For $\xi \in L^2([0,1]; H)$, set

$$\ell(\xi) = \int_0^{1/2} dv \int_0^1 dy\, \xi(v,y),$$

which defines a linear functional ℓ on $L^2([0,1]; H)$, and let us denote by $\delta_{\phi^h,\xi}$ the quantity

$$\delta_{\phi^h,\xi} = f'\big(\ell(\xi)\big) - f'\big(\ell\big(\phi^h\big)\big).$$

Then, for any $m \in L^2([0,1]; H)$, we have

$$\begin{aligned}
&\big|\big\langle (D_\xi F - D_{\phi^h} F) R^* m, m\big\rangle_{L^2([0,1];H)}\big| \\
&\quad = \left|\delta_{\phi^h,\xi} \int_0^{1/2}\int_0^1 R^* m(v,y)\, dv\, dy \int_{1/2}^1 \int_0^1 m(v,y)\, dv\, dy\right| \\
&\quad \leqslant 2\|f'\|_\infty \big|\langle \widetilde{T} R^* m, m\rangle_{L^2([0,1];H)}\big|,
\end{aligned}$$

where $\widetilde{T}$ is defined on $L^2([0,1]; H)$ by

$$\widetilde{T}\xi = \ell(\xi)\mathbf{1}_{[0,\frac{1}{2}]} \otimes \mathbf{1}.$$

It is then easily seen that $\widetilde{T}$ is a trace class operator in the sense of [1]. Since R^* is a bounded operator, $\widetilde{T} R^*$ is also trace class, and thus compact. Hence, $\mathcal{S}_{\widetilde{T}R^*}$ can be diagonalized in an orthonormal basis, and setting now $T = |\mathcal{S}_{\widetilde{T}R^*}|$, we get a positive trace class operator such that

$$\big|\big\langle (D_\xi F - D_{\phi^h} F) R^* m, m\big\rangle_{L^2([0,1];H)}\big| \leqslant 2\|f'\|_\infty \langle Tm, m\rangle_{L^2([0,1];H)},$$

which shows that condition (18) is fullfilled. Furthermore, since

$$\big[F(t,\xi)\big](t,x) = f(0)\mathbf{1}_{[0,\frac{1}{2}]}(t)\mathbf{1}(x) + f\big(\ell(\xi)\big)\mathbf{1}_{[\frac{1}{2},1]}(t)\mathbf{1}(x),$$

it is easily seen that $F(t,.)$ is a $C_b^2(L^2([0,1];H))$ function, uniformly in $t \in [0,1]$. □

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