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COUPLED MAP LATTICES WITH ASYNCHRONOUS UPDATINGS

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ABSTRACT. – We consider on $M = (S^1)^{\mathbb{Z}^d}$ a family of continuous local updatings, of finite range type or Lipschitz-continuous in all coordinates with summable Lipschitz-constants. We show that the infinite-dimensional dynamical system with independent identically Poisson-distributed times for the individual updatings is well-defined. We then consider the setting of analytically coupled uniformly expanding, analytic circle maps with weak, exponentially decaying interaction. We define transfer operators for the infinite-dimensional system, acting on Banach-spaces that include measures whose finite-dimensional marginals have analytic, exponentially bounded densities. We prove existence and uniqueness (in the considered Banach-space) of a probability measure and its exponential decay of correlations. © 2001 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – On considère sur $M = (S^1)^{\mathbb{Z}^d}$ une famille de mises-à-jour continues et locales, de type distance finie ou Lipschitzienne-continue sur toutes les coordonnées, les constantes Lipschitziennes étant de somme finie. On montre que le système dynamique à dimension infinie avec une distribution de Poisson identique et indépendante des instants de mise-à-jour est bien défini. Ensuite on considère le cas des applications du circle, analytiques, couplées entre elles analytiquement et à expension uniforme, à faible interaction exponentiellement décroissante. On définit des opérateurs de transfert pour le système à dimension infinie, agissant sur des espaces de Banach incluant des mesures dont les projections à dimensions finies ont des densités analytiques bornées exponentiellement. On montre l'existence et l'unicité (dans l'espace de Banach considéré) d'une mesure probabiliste et la décroissance exponentielle de ses corrélations. © 2001 Éditions scientifiques et médicales Elsevier SAS

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0. Introduction

In this paper we study coupled map lattices with indepentent identically (i.i.) Poisson-distributed updatings at the individual sites.

A deterministic coupled map lattice (CML) is given by a $\mathbb{Z}^d$-lattice with a copy of the same Riemannian manifold at each lattice point (i.e. the state space is the product of these manifolds with index set $\mathbb{Z}^d$) and a map on the infinite space that can be decomposed into an uncoupled map that acts individually on each component and an 'interaction step' where the change of each coordinate depends also on the other sites.

L.A. Bunimovich and Y.G. Sinai prove in [8] (cf. also the remarks on this in [5]) the existence and uniqueness of an invariant measure and its exponential decay of correlations for a one-dimensional lattice of interval maps with weak coupling. By constructing a Markov partition they relate the system to a two-dimensional spin system whose Gibbs measure corresponds to the invariant measure of the CML.

G. Keller and M. Künzle prove in [21] the existence and uniqueness of an invariant measure for periodic or infinite one-dimensional lattices of weakly coupled interval maps by studying the transfer operators on the space of measures whose finite-dimensional marginals have densities of bounded variation. For small perturbation of the uncoupled map any invariant measure is 'close' to the one they found.

J. Bricmont and A. Kupiainen extend in [4] and [5,6] the result of Bunimovich and Sinai [8] to $\mathbb{Z}^d$-lattices of weakly coupled circle maps with analytic and Hölder-continuous interaction, respectively.

They represent the iterates of the Perron–Frobenius operator for finite-dimensional subsystems (over $\Lambda \subset \mathbb{Z}^d$) by a 'polymer'- or 'cluster'-expansion that gives rise to a representation of the corresponding invariant measure in terms of a $(d+1)$-dimensional spin system. The weak limit (as $\Lambda \to \mathbb{Z}^d$) of these measures is the unique (in a certain class) invariant probability measure and it is exponentially mixing with respect to (w.r.t.) spatio-temporal shifts.

C. Maes and A. Van Moffaert introduce in [25] for a similar setting as in [4] a simplified 'cluster'-expansion for the truncated Perron–Frobenius operator and show stochastic stability of the CML under stochastic perturbation.

In [2] V. Baladi, M. Degli Esposti, S. Isola, E. Järvenpää and A. Kupiainen define Frechet spaces, and, for $d = 1$, a Banach space and transfer operators for the infinite-dimensional systems, considered by Bricmont and Kupiainen in [4], and study the spectral properties of these operators.

In [13] we consider analytically coupled circle maps (uniformly expanding and analytic) on the $\mathbb{Z}^d$-lattice with exponentially decaying interaction and introduce Banach spaces for the infinite-dimensional system that include measures whose finite-dimensional marginals have analytic, exponentially bounded densities. We define transfer operators on these spaces, get a unique (in the considered Banach spaces) probability measure and prove its exponential decay of correlations.

CMLs with multi-dimensional local systems of hyperbolic type have been studied by Ya.B. Pesin and Ya.G. Sinai [26], M. Jiang [17,18], M. Jiang and A. Mazel [19], M. Jiang and Ya.B. Pesin [20] and D.L. Volevich [29,30].

For detailed reviews on mathematical results on CMLs we refer to [2], [5], [7] and [20].

An interacting particle system (IPS) is given by an infinite lattice with a copy of the same state space (that is usually a finite or countable set but can also be a Riemannian manifold) at each site. The updating at an individual site is a deterministic or stochastic map (e.g. in the case of finite local state spaces it is given by a stochastic matrix with transition probabilities as its coefficients) that is applied with ‘exponential waiting times’, i.e. like the waiting times for jumps in a Poisson process. The jump rates for the updating depends also on the other sites. R.J. Glauber introduces in [14] (a special case of) the stochastic Ising model as a model for magnetism. The total state space $\{-1,+1\}^{\mathbb{Z}^d}$ represents the spins of the atoms at all sites. The rate for a flip of an individual spin depends on the states of the neighbour sites. F. Spitzer [27,28] and R.L. Dobrushin [9,10] study more general systems where the individual jump rates do not only depend on the nearest neighbours.

A basic problem is to establish the existence of infinite systems with asynchronous updatings. The infinitely many jumps in a finite time-interval cannot be ‘listed’, i.e. there is no order preserving bijection between the time-ordered set of jumps and $\mathbb{N}$.

R.L. Dobrushin obtains in [9] the infinite system as the limit of subsystems over finitely many sites.

By using a percolation argument T.E. Harris proves in [15] that for systems of finite range interaction and a sufficiently small time interval the history of an individual particle depends on only finitely many sites, and so he provides a natural definition of the infinite system. With probability 1 the set $\mathbb{Z}^d$ splits into finite clusters such that each site is affected at most by sites in the same cluster.

R. Holley shows in [16] for generators, corresponding to one-dimensional models, and T.M. Liggett in [23] for the d-dimensioanal case, that these operators generate, in fact, a semigroup, acting on continuous functions.

Here we have only mentioned different methods to establish the existence of the infinite systems. For detailed reviews on IPSs and results on invariant measures, mixing properties, phase transitions and applications to physics and other sciences we refer to [11] and [24].

In this paper we consider the infinite topological product $M=(S^1)^{\mathbb{Z}^d}$ and continuous updating maps for the individual coordinates that are of finite range or Lipschitz-continuous w.r.t. all coordinates with a summable family of Lipschitz constants (cf. Section 2.2 for the definition). The times for the updatings at the individual sites are independently Poisson-distributed with the same constant rate $\lambda>0$. For the finite range case we show that with probability 1 the updatings at any finite set of sites and any finite time-interval depend on only finitely many sites. Our proof uses time- and space-oriented percolation and is different from the one in [15]. This result provides a natural definition of the infinite dynamical system.

For the systems with infinite range interaction we show that with probability 1 it is well-defined as the net-limit of its finite-dimensional subsystems with arbitrary boundary conditions. We combine standard estimates for error growth with ideas from percolation theory. The limit of the corresponding Markov kernels, acting on continuous functions, exists and provides a definition of the infinite process, different from the widely used generator approach.

Our proofs still work if we replace S^1 by any compact Riemannian manifold or stochastic systems with finite state spaces. The assumption of having the same constant jump rate at all sites is by no means essential and can be weakened to the case of upper bounded individual jump rates that depend on other states as well. However, we do not consider these generalizations in this paper.

In a setting similar to that of [13], i.e. for analytically coupled circle maps (uniformly expanding and analytic) on the $\mathbb{Z}^d$-lattice with weak, exponentially decaying interaction but with asynchronous updatings as described above, we define transfer operators for the Markov kernels of the infinite system. The operators act on the Banach space $\mathcal{H}_\vartheta$ (introduced in [13]) that includes measures whose finite-dimensional marginals have analytic, exponentially bounded densities. Using 'cluster-expansion'-like techniques, we represent these integral operators in terms of configurations and prove the existence and uniqueness (in $\mathcal{H}_\vartheta$) of an invariant probability measure and its exponential decay of correlations.

The paper is organized as follows. Section 1 provides definitions, notation and some propositions about stochastic processes and metric spaces. In Section 2 we define the infinite-dimensional systems for finite range (Section 2.1) and infinite range interaction (Section 2.2) and the corresponding Markov kernels (Section 2.3). In Section 3 we study the transfer operators for a specific class of interactions. In Section 4 we prove the mixing properties of the invariant measure (found in Section 3) w.r.t. spatio-temporal shifts.

1. Basic definitions and examples

In this section we present definitions from probability theory and topology and also introduce most of the notation used in this paper. We have taken most definitions and statements on probability theory from [3].

DEFINITION 1.1. – $\mathbb{N}$ *denotes the set of natural numbers including zero. Let* $(E, \mathcal{A}_2)$ *be a measurable space,* $(\Omega, \mathcal{A}_1, P)$ *a probability space and* $(X_t)_{t\in I}$ *a family* (*with index set* $I \neq \emptyset$) *of random variables on* $(\Omega, \mathcal{A}_1, P)$ *with values in* E.

- *Then* $(\Omega, \mathcal{A}_1, P, (X_t)_{t\in I})$ *is called a* stochastic process with values in $(E, \mathcal{A}_2)$.
- *If* $I = \mathbb{R}^{\geqslant 0}$, $[0, T]$ *or* $[0, T)$ *for some* $T > 0$ *the process is called a* continuous time stochastic process.
- *For fixed* $\omega \in \Omega$ *the map* $t \mapsto X_t(\omega)$ *is called the* trajectory *of* ω. *It is also denoted by* $X_\bullet(\omega)$.
- *We consider the set* $\mathbb{N}$ *as measurable space with the discrete* σ*-algebra. For any set* Λ *we denote by* $\mathbb{N}^\Lambda$ *the product space, equipped with the product* σ*-algebra.*
  A continuous time stochastic process with values in $\mathbb{N}^\Lambda$ *and with index set* I *and* P*-a.a. of whose trajectories are non-decreasing* (*i.e. the functions* $t \mapsto \pi_q \circ X_t(\omega)$ *are non-decreasing for all* $q \in \Lambda$ *and* P*-a.a.* $\omega \in \Omega$. 'π_q' *denotes the projection on the* q*th coordinate.*), *is called a* counting process *with values in* $\mathbb{N}^\Lambda$ *We say that such a process is* of finite expectation *if for all* $t \in I$ *the random variable* $\omega \mapsto \sum_{q\in\Lambda} \pi_q \circ X_t(\omega)$ *has finite expectation.*

Remark 1.2. –

1. We will also use the short-hand-notation $X_\bullet$ for a stochastic process if Ω, $\mathcal{A}_1$ and P are obvious from the context.
2. The term *path* seems to be more common than *trajectory* but we will denote something else later on by *path*.
3. Finite expectation means that with probability 1 there are only finitely many jumps (cf. Definition 1.3 below) in every finite time-interval.

DEFINITION 1.3 (cf. [3]). – *Let $(\Omega, \mathcal{A}_1, P, (X_t)_{t\in I})$ be a continuous time counting process with values in $\mathbb{N}$ as in Definition 1.1 and $\omega \in \Omega$. We define*

$$X_t^+(\omega) \stackrel{\text{def}}{=} \begin{cases} X_t(\omega) & \text{if } I = [0,t], \\ \lim_{s\searrow t} X_s(\omega) & \text{otherwise,} \end{cases} \tag{1}$$

$$X_t^-(\omega) \stackrel{\text{def}}{=} \begin{cases} \lim_{s\nearrow t} X_s(\omega) & t > 0, \\ X_0(\omega) & t = 0. \end{cases} \tag{2}$$

We say that $X_\bullet(\omega)$ jumps at time $t \geqslant 0$ if $X_t^-(\omega) < X_t^+(\omega)$. Then $X_t^+(\omega) - X_t^-(\omega)$ is called the size of the jump.

Let $X_\bullet(\omega)$ be a continuous time counting process with values in $\mathbb{N}^\Lambda$ and $\omega \in \Omega$. We say that $X_\bullet(\omega)$ jumps at time t and site $q \in \Lambda$ if $\pi_q \circ X_\bullet(\omega)$ jumps at t. Then we also say that ω jumps at (q,t).

We define the jump set $\Lambda(\omega, \mathbf{t})$ *of ω* at time $\mathbf{t}$ *as the set of all $q \in \Lambda$ such that ω jumps at (q,t).*

DEFINITION 1.4 (cf. [3]). – *Let $I = \mathbb{R}^{\geqslant 0}$ or $I = [0,T)$ for some $T > 0$. A stochastic process $(\Omega, \mathcal{A}, P, (X_t)_{t\in I})$ with values in $\mathbb{N}$ is* called (normalized) Poisson process with parameter $\lambda > \mathbf{0}$ *if the following holds*:

1. *The process has stationary and independent increments which for all $s < t \in I$ satisfy*
$$P\big(\{\omega\colon\ X_t(\omega) - X_s(\omega) = n\}\big) = p_\lambda(t-s,n) \tag{3}$$
with
$$p_\lambda(t,n). \stackrel{\text{def}}{=} \mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!}. \tag{4}$$
2. *P-almost every trajectory $X_\bullet(\omega)$ is a right-continuous, increasing function having at most jumps of size* 1.
3. *At time 0 P-a.a. trajectories have value* 0:
$$P\big(\omega\colon\ X_0(\omega) = 0\big) = 1. \tag{5}$$

THEOREM 1.5 (cf. [3], Satz 41.2). – *For any $\lambda > 0$ and I as in Definition* 1.4 *there exists a* (*normalized*) *Poisson process with parameter λ. Any two such processes are equivalent* (*i.e. if $X_\bullet^1$ and $X_\bullet^2$ are two such processes then for any finite sequence $t_1 < \cdots < t_n$ in I the projections $(X_{t_1}^1, \ldots, X_{t_n}^1)$ and $(X_{t_1}^2, \ldots, X_{t_n}^2)$ have the same distribution*).

DEFINITION 1.6. – *Let Λ be a nonempty set and $(\Omega_q, \mathcal{A}_q, P_q, (X_t^q)_{t\in I})_{q\in\Lambda}$ be a family of stochastic processes with values in $(E_q, \mathcal{A}^q)$, respectively. We set*

$$\Omega \stackrel{\text{def}}{=} \prod_{q\in\Lambda} \Omega_q, \tag{6}$$

$$\tilde{\mathcal{A}} \stackrel{\text{def}}{=} \bigotimes_{q\in\Lambda} \mathcal{A}_q, \tag{7}$$

$$\tilde{P} \stackrel{\text{def}}{=} \bigotimes_{q\in\Lambda} P_q, \tag{8}$$

$$\mathcal{A} \stackrel{\text{def}}{=} \textit{completion of } \tilde{\mathcal{A}} \textit{ w.r.t. } \tilde{P}, \tag{9}$$

$$P \stackrel{\text{def}}{=} \textit{ extension of } \tilde{P} \textit{ to } \mathcal{A} \tag{10}$$

$$\textit{and} \quad X_t \stackrel{\text{def}}{=} \left(X_t^q\right)_{q\in\Lambda}. \tag{11}$$

In (6) *we mean the cartesian product of spaces, in* (7) *the product sigma-algebra and in* (8) *the product measure.*

Then the process $(\Omega, \mathcal{A}, P, (X_t)_{t\in I})$ *with values in* $(\prod_{q\in\Lambda} E_q, \bigotimes_{q\in\Lambda} \mathcal{A}^q)$ *is called* the product of the family of processes.

Remark 1.7. –

1. Products of stochastic processes as in Definition 1.6 exist. For example the existence of the non-completed product measure follows from standard measure theory (cf. [3].)
2. For non-empty, at most countable Λ and a family (indexed by Λ) of Poisson processes two such products $X^1_\bullet$ and $X^2_\bullet$ are equivalent because for all $q \in \Lambda$ the Poisson processes $\pi_q \circ X^1_\bullet$ and $\pi_q \circ X^2_\bullet$ are equivalent (cf. Theorem 1.5). It follows from the definition of the product σ-algebra $\bigotimes_{q\in\Lambda} \mathcal{A}^q$ that $X^1_\bullet$ and $X^2_\bullet$ are equivalent.

DEFINITION 1.8. – *Let* $\lambda > 0$ *and* Λ *a nonempty, at most countable set.* A Poisson process on Λ with parameter λ *is the product of a family, indexed by* Λ*, of Poisson processes with parameter* λ*.*

Remark 1.9. –

1. For $\lambda > 0$ the Poisson process on $\mathbb{Z}^d$ with parameter λ is clearly not of finite expectation. In fact, for any $t > 0$ there are P-almost surely infinitely many jumps in $[0, t]$, i.e.
$$P\left(\left\{\omega\colon \sum_{q\in\mathbb{Z}^d} \pi_q \circ X_t(\omega) = \infty\right\}\right) = 1. \tag{12}$$
2. But if $\Lambda_1 \subset \mathbb{Z}^d$ is finite then $\pi_{\Lambda_1} \circ X_\bullet(\omega)$ has finitely many jumps in $[0, t]$ for P-a.a. $\omega \in \Omega$ and any $t > 0$.
3. There are P-almost surely no simultaneous jumps at two different sites:
$$P\left(\{\omega\colon \exists q_1 \neq q_2 \in \mathbb{Z}^d, t \geqslant 0 \text{ such that } \omega \text{ jumps at } (q_1, t) \text{ and } (q_2, t)\}\right) = 0. \tag{13}$$
4. For $0 \leqslant t_0 < t$
$$P\left(\{\omega\colon \omega \text{ jumps at } t_0\}\right) = 0. \tag{14}$$

Proof of Remark 1.9. – We only show (13). The proofs of the other statements are similar. We set

$$A(q_1, q_2, T) \stackrel{\text{def}}{=} \{\omega\colon \exists t \in [0, T) \text{ such that } \omega \text{ jumps at } (q_1, t) \text{ and } (q_2, t)\}. \tag{15}$$

We have to prove that the set

$$\bigcup_{T \in \mathbb{N}} \bigcup_{q_1, q_2 \in \mathbb{Z}^d} A(q_1, q_2, T) \tag{16}$$

has P-measure zero and it is sufficient to show that

$$P\big(A(q_1, q_2, T)\big) = 0 \tag{17}$$

for fixed $q_1 \neq q_2 \in \mathbb{Z}^d$ and $T > 0$. For this we set

$$I_{N,k} \stackrel{\text{def}}{=} \left[(k-1)\frac{T}{N}, k\frac{T}{N}\right) \tag{18}$$

for $N \in \mathbb{N} \setminus \{0\}$ and $1 \leqslant k \leqslant N$. By (4) we have for $i = 1, 2$:

$$P\big(\{\omega\colon \text{ jumps at } (q_i, t) \text{ for some } t \in I_{N,k}\}\big) = 1 - \mathrm{e}^{-\lambda \frac{T}{N}} \tag{19}$$

and so, using the estimate $\mathrm{e}^x \geqslant 1 + x$:

$$\begin{aligned} &P\big(\{\omega\colon \exists k; t_1, t_2 \in I_{N,k} \text{ such that } \omega \text{ jumps at } (q_1, t_1) \text{ and } (q_2, t_2)\}\big) \\ &\quad \leqslant N\left(1 - \mathrm{e}^{-\lambda \frac{T}{N}}\right)^2 \leqslant \lambda^2 T^2 \frac{1}{N} \end{aligned} \tag{20}$$

which converges to 0 as $N \to \infty$. □

The following two definitions prepare Definition 1.13 that we will need in Section 2.

DEFINITION 1.10. – *In view of Definition* 1.4 *and Remark* 1.9 *we define* (*for a given Poisson process like in that remark*) *the set* $\mathcal{N}_1$ *of* P*-measure zero*:

$$\begin{aligned} \mathcal{N}_1 \stackrel{\text{def}}{=} \{\omega\colon\ & X_\bullet(\omega) \text{ is not non-decreasing, has jumps at 0,} \\ & \text{simultaneous jumps or jumps of size greater than } 1\}. \end{aligned} \tag{21}$$

DEFINITION 1.11. – *Let* $\Lambda \subset \mathbb{Z}^d$. *Then we denote its complement by* $\Lambda^C \stackrel{\text{def}}{=} \mathbb{Z}^d \setminus \Lambda$.

DEFINITION 1.12. – *For* $q = (q_1, \ldots, q_n) \in \mathbb{Z}^d$ *we define*

$$\|q\| \stackrel{\text{def}}{=} |q_1| + \cdots + |q_n|. \tag{22}$$

For $R \geqslant 0$

$$B_R(q) \stackrel{\text{def}}{=} \{\tilde{q} \in \mathbb{Z}^d\colon \|q - \tilde{q}\| \leqslant R\} \tag{23}$$

is the set of points that have distance at most R *from* q.

DEFINITION 1.13. – *Let* $a, b \in \mathbb{Z}^d$ *and* $n \geqslant 0$. A path from a to b *is a finite sequence* $Q = (q_0 = a, q_1, \ldots, q_n = b)$ *of points* $q_i \in \mathbb{Z}^d$. *We call* $\sum_{i=1}^n \|q_i - q_{i-1}\|$ the length *and* $\max_{0\leqslant i\leqslant n-1} \|q_{i+1} - q_i\|$ *the* step size *of* Q. *Note the special case of a path* $Q = (q_0)$. *It is called the* empty path at site q_0 *and we define both its length and step size to be* 0.

DEFINITION 1.14. – *Let* $(\Omega, \mathcal{A}, P, (X_t)_{t\geqslant 0})$ *be a Poisson process with parameter* $\lambda > 0$ *and with values in* $\mathbb{N}^{\mathbb{Z}^d}$. *Let* $T > 0$, $\omega \in \Omega$ *and* $Q = (q_0 = a, q_1, \ldots, q_n = b)$ *a path. We extend* Q *to the infinite sequence* $\widetilde{Q} = (q_0, q_1, \ldots, q_n, q_{n+1} = q_n, \ldots)$ *in which* q_n *is repeated.*

We define a process $(\Omega, \mathcal{A}, P, (Z_t)_{t\in[0,T]})$ *with values in* $\mathbb{N}$ *as follows.*

$$Z : [0, t] \times \Omega \to \mathbb{N}, \qquad (t, \omega) \mapsto Z_t(\omega). \tag{24}$$

If $\omega \in \mathcal{N}_1$ *or it does not jump at* (q_0, t) *for any* $t \in (0, T)$ *we set* $Z_\bullet(\omega) = 0$ *on* $[0, T]$.

Otherwise there is a maximal sequence

$$t_{-1} \stackrel{\text{def}}{=} T > t_0 > t_1 > \cdots > t_{m(\omega)} \tag{25}$$

such that

$$t_i \stackrel{\text{def}}{=} \max\{t \in (0, t_{i-1}) \colon \omega \text{ jumps at } (q_i, t)\} \quad \text{for } 0 \leqslant i \leqslant m(\omega). \tag{26}$$

'Maximal' means that ω *does not jump at* $q_{m(\omega)+1}$ *in the time interval* $(0, t_{m(\omega)})$ *and so the sequence* (25) *cannot be extended.* (*Intuitively one can think that one sits at time* T *at site* q_0 *and, going* backwards *in time, waits for the next jump of* ω *at* q_0 (*which happens at time* t_0), *then jumps (instantly) to* q_1 *and waits* (backwards *in time*) *for the next jump of* ω *at* q_1, *then jumps to* q_2 *etc. After* n *jumps* (*should this occur*) *one does not change the sites any more, but possibly jumps from* q_n *to* q_n. $m(\omega)$ *is the total number of jumps. It is* P*-a.s. finite because* P*-a.a.* ω *have only finitely many jumps at* q_n.)

We set for $t \in [0, T]$*:*

$$\widetilde{Z}_t(\omega) \stackrel{\text{def}}{=} \begin{cases} i & \text{for } t \in [t_i, t_{i-1}), \\ m(\omega) & \text{for } t \in [0, t_{m(\omega)}]. \end{cases} \tag{27}$$

And $Z_\bullet(\omega)$ *is the (uniquely defined) right-continuous function, such that* $Z_\bullet(\omega) = \widetilde{Z}_{T-\bullet}(\omega)$ *everywhere, except possibly where these functions jump. Then* $(\Omega, \mathcal{A}, P, (Z_t)_{t\in[0,T]})$ *is a Poisson process with parameter* λ. (*A precise proof of this uses that the constructed process is 'made of' independent Poisson processes and that these have independent increments.*) *We call it the* Poisson process induced by the path Q.

DEFINITION 1.15. – *In the setting of Definition* 1.14 *we call* Q *a* causal path w.r.t. (t, ω) *if* $Z_T(\omega) \geqslant n$ *and a* maximal causal path w.r.t. (t, ω) *if* $Z_T(\omega) = n$. (*The latter means that* $Q = (q_0, \ldots, q_n)$ *cannot be extended to any causal path* $(q_0, \ldots, q_n, q_{n+1})$.)

We define:

- *Path*(q, n, R) *to be the set of paths that start at* q, *have exactly* n *steps and are of step size at most* R.

- *Path*$(q \to \Lambda)$ *for any* $\emptyset \neq \Lambda \in \mathbb{Z}^d$ *to be the set of paths starting at* q *and ending in* Λ.
- *Path*$_C(t, \omega, q, \Lambda)$ *for* $q \in \Lambda$ *to be the set of causal w.r.t.* (t, ω) *paths* $Q = (q_0 = q, \ldots, q_n)$ *such that*
   1. Q *is maximal causal and* $q_0, \ldots, q_n \in \Lambda$, *or*
   2. $q_0, \ldots, q_{n-1} \in \Lambda$ *and* $q_n \in \Lambda^C$.
- *Path*$_c(t, \omega, q \to \Lambda^C)$ *for* $q \in \Lambda$ *to be the set of causal paths* $(q_0 = q, \ldots, q_n)$ *such that* $q_0, \ldots, q_{n-1} \in \Lambda$ *and* $q_n \in \Lambda^C$. (*So this is the subset of elements in* *Path*$_c(t, \omega, q, \Lambda)$ *for which case* 2. *applies.*)

Remark 1.16. –
1. We have defined the property of being causal for general paths and not related this definition to any kind of interaction. When we study finite range interaction, of range R say, we will consider only causal paths of step size at most R.
2. A term like *inverse causal path* from a to b instead of *causal path* would actually be more appropiate as it corresponds to b affecting a (cf. Definition 2.1) but not necessarily the other way around. However, we prefer the shorter notion.

DEFINITION 1.17 (cf. [3]). – *Let* $(\Omega_1, \mathcal{A}_1)$ *and* $(\Omega_2, \mathcal{A}_2)$ *be measurable spaces. A map* $K : \Omega_1 \times \mathcal{A}_2 \to [0, 1]$ *is called a* Markov kernel from $(\Omega_1, \mathcal{A}_1)$ *to* $(\Omega_2, \mathcal{A}_2)$ *if the two following conditions are satisfied*:

MK1 $\omega_1 \mapsto K(\omega_1, A_2)$ *is* $\mathcal{A}_1$*-measurable for all* $A_2 \in \mathcal{A}_2$.

MK2 $A_2 \mapsto K(\omega_1, A_2)$ *is a probability measure on* $\mathcal{A}_2$ *for all* $\omega_1 \in \Omega_1$.

If $(\Omega_1, \mathcal{A}_1) = (\Omega_2, \mathcal{A}_2)$ *then* K *is called a* Markov kernel on $(\Omega_1, \mathcal{A}_1)$.

Example 1.18. – Let (Y, ϱ_Y) be a metric space and $\mathcal{B}_Y$ its Borel σ-algebra. $\mathcal{C}^0(Y, Y)$ is the space of continuous maps from Y to Y. It has a uniform metric, defined by $\varrho_{\mathcal{C}^0(Y,Y)}(g_1, g_2) = \sup_{\mathbf{y} \in \mathbf{Y}} \varrho_Y(g_1(\mathbf{y}), g_2(\mathbf{y}))$ and the Borel σ-algebra $\mathcal{B}_{\mathcal{C}^0(Y,Y)}$ w.r.t. this metric. Further, let $(\Omega, \mathcal{A}, P)$ be a probability space and

$$S : \Omega \to \mathcal{C}^0(Y, Y), \qquad (28)$$
$$\omega \mapsto S_\omega,$$

a measurable (w.r.t. the σ-algebras $\mathcal{A}$ and $\mathcal{B}_{\mathcal{C}^0(Y,Y)}$) map.

Then

$$K_S(y, Y_1) \stackrel{\text{def}}{=} P\big(\{\omega\colon\ S_\omega(y) \in Y_1\}\big) \qquad (29)$$

for all $y \in Y$, $Y_1 \in \mathcal{B}_Y$, defines a Markov kernel on $(Y, \mathcal{B}_Y)$.

Proof. – To verify MK1 we fix an $Y_1 \in \mathcal{B}_Y$ and show that the map $y \mapsto K_S(y, Y_1)$ is measurable. First we note that S can be seen as a measurable map from $\Omega \times Y$ to Y. We write it as the composite of measurable maps $S \times \mathrm{id}_Y$ and the 'evaluation map':

$$(\omega, y) \mapsto (S_\omega, y) \mapsto S_\omega(y). \qquad (30)$$

The map $S \times \mathrm{id}_Y$ is measurable by assumption and the definition of the product σ-algebra of $\mathcal{C}^0(Y, Y) \times Y$. The evaluation map is continuous (w.r.t. the product topology), hence measurable w.r.t. the Borel σ-algebras. So the composite in (30) is measurable in $\Omega \times M$. It follows that the map $y \mapsto P(\{\omega\colon\ S_\omega(y) \in Y_1\})$ is measurable (cf. Lemma 8.1 on p. 159 in [22]) and so MK1 holds.

Next we show MK2. Consider for fixed $y \in Y$ the composite of measurable maps

$$\omega \mapsto (\omega, y) \mapsto S_\omega(y) \tag{31}$$

that maps Ω to Y. We see that $K(y, \cdot)$ is the image of P w.r.t. this map and so a probability measure which was to be shown. □

DEFINITION 1.19 (cf. [3]). – *Let K be a Markov kernel from $(\Omega_1, \mathcal{A}_1)$ to $(\Omega_2, \mathcal{A}_2)$ and $E^*(\mathcal{A}_i)$ $(i = 1, 2)$ the set of $\mathcal{A}_i$-measurable functions with values in $[0, \infty]$. Then K defines a map from $E^*(\mathcal{A}_2)$ to $E^*(\mathcal{A}_1)$ as follows*:

$$(Kf)(\omega_1) \stackrel{\text{def}}{=} \int_{\Omega_2} K(\omega_1, \mathrm{d}\omega_2) f(\omega_2) \tag{32}$$

for any $f \in E^*(\mathcal{A}_2)$. The notation on the rhs of (32) *means that f is integrated w.r.t. the probabiltiy measure on Ω_2 that is described in Definition* 1.17, MK2.

Example 1.20 (*cf.* [3]). – For the characteristic function χ_{A_2} of an $\mathcal{A}_2$-measurable set A_2 we get

$$K\chi_{A_2}(\omega_1) = K(\omega_1, A_2). \tag{33}$$

Now we consider a special case of Example 1.18.

Example 1.21. – Let $S: Y \to Y$ be a continuous map on (Y, ϱ_Y) and let $(\Omega, \mathcal{A}, P, (X_t)_{t\in I})$ be a counting process with values in $\mathbb{N}$ and $t \in I$.

The map

$$\begin{aligned} S_\omega^t\colon\ & Y \to Y, \\ & y \mapsto S^{X_t(\omega)}(y), \end{aligned} \tag{34}$$

where $S^{X_t(\omega)}$ denotes the $X_t(\omega)$th iterate of S, is well-defined for all $\omega \in \Omega$. Further, $S_\omega(y)$ is measurable w.r.t. (ω, y). In fact, S_ω depends just on $X_t(\omega)$ and so we get a countable, measurable partition of Ω:

$$\Omega = \bigcup_{n\in\mathbb{N}} U(n), \tag{35}$$

$$\text{with } U(n) \stackrel{\text{def}}{=} \{\omega \in \Omega\colon\ X_t(\omega) = n\}. \tag{36}$$

We define a Markov kernel by

$$\begin{aligned} K_S^t(y, Y_1) &\stackrel{\text{def}}{=} P\bigl(\{\omega\colon\ S_\omega^t(y) \in Y_1\}\bigr) \\ &= \sum_{n: S^n(y)\in Y_1} P\bigl(U(n)\bigr) \\ &= \int_\Omega \mathrm{d}P(\omega)\, \chi_{Y_1} \circ S_\omega^t(y) \end{aligned} \tag{37}$$

for $y \in Y$ and $Y_1 \in \mathcal{B}_Y$.

We see that this Markov kernel acts on a measurable function $f : Y \to [0, \infty]$ by

$$\left(K_S^t f\right)(y) = \sum_{n=0}^{\infty} P\left(U(n)\right) f\left(S^n(y)\right). \tag{38}$$

We prepare a generalization of Example 1.21 with a definition and a technical lemma.

DEFINITION 1.22. – *Let $\mathcal{F}$ be the set of finite subsets of $\mathbb{Z}^d$. Consider a fixed $\Lambda \in \mathcal{F} \setminus \{\emptyset\}$. We define $\mathcal{J}$ to be the union of a one-point set $\{j_\infty\}$ and the set of finite sequences $(\Lambda_1, \ldots, \Lambda_n)$ of subsets of Λ. Then $\mathcal{J}$ is countable and we consider it as a measurable space, equipped with the discrete σ-algebra.*

Let $(\Omega, \mathcal{A}, P, (X_t)_{t \in I})$ be a continuous time counting process with values in $\mathbb{N}^\Lambda$ and index-set $I = [0, T)$ or $[0, T]$. We define a map

$$\begin{aligned} \mathbf{j} : \Omega &\to \mathcal{J} \\ \omega &\mapsto \mathbf{j}(\omega). \end{aligned} \tag{39}$$

If $X_\bullet(\omega)$ is non-decreasing, has only finitely many jumps and at most jumps of size 1 then we define $\mathbf{j}(\omega)$ to be the (time-ordered) sequence of jump sets of ω. Otherwise we set $\mathbf{j}(\omega) = j_\infty$. We define for $j \in \mathcal{J}$:

$$U(j) \stackrel{\text{def}}{=} \{\omega\colon \mathbf{j}(\omega) = j\} \tag{40}$$

LEMMA 1.23. – *Let $\Lambda \in \mathcal{F} \setminus \{\emptyset\}$ be fixed and $(\Omega, \mathcal{A}, P, (X_t)_{t \in I})$ a continuous time counting process with index-set $I = [0, T)$ or $I = [0, T]$ and values in $\mathbb{N}^\Lambda$ such that for P-a.a. ω the trajectory $X_\bullet(\omega)$ is non-decreasing, has only finitely many jumps and at most jumps of size* 1. *Then the map* $\mathbf{j}$*, as defined in Definition* 1.22*, is measurable.*

Proof. – We consider the case $I = [0, T]$. The case $I = [0, T)$ is analogous. By assumption $\mathcal{N} = U(j_\infty)$ is measurable and has measure zero. We have to show that $U(j)$ is measurable for any $j = (\Lambda_1, \ldots, \Lambda_n)$. For any $q_1, q_2 \in \Lambda$ and $n_1, n_2 \in \mathbb{N} \setminus \{0\}$ we define $A_1(q_1, n_1, q_2, n_2)$ to be the set of all $\omega \in \Omega \setminus \mathcal{N}$ that have at least n_1 jumps at q_1 and at least n_2 jumps at q_2 and the n_1th jump at q_1 happens at the same time as the n_2th jump at q_2. Similarly, $A_2(q_1, n_1, q_2, n_2)$ is the set of all $\omega \in \Omega \setminus \mathcal{N}$ that have at least n_1 jumps at site q_1 and the n_1th jump at q_1 occurs before the n_2th jump at q_2 (if there is an n_2th jump at q_2 at all – if that is not the case then this second condition is automatically satisfied). We only show the measurability of the sets $A_2(\cdot)$. The proof of the measurability of the sets $A_1(\cdot)$ uses similar arguments.

$$A^{\geqslant}(q_1, n_1, t) \stackrel{\text{def}}{=} \left\{\omega \in \Omega \setminus \mathcal{N}\colon \pi_{q_1} \circ X_t(\omega) \geqslant n_1\right\} \tag{41}$$

is the set of all $\omega \in \Omega \setminus \mathcal{N}$ that have at least n_1 jumps at site q_1 and the n_1th of these jumps happens at the latest at time t.

Analogously,

$$A^{<}(q_2, n_2, t) \stackrel{\text{def}}{=} \left\{\omega \in \Omega \setminus \mathcal{N}\colon \pi_{q_2} \circ X_t(\omega) < n_2\right\} \tag{42}$$

is the set of all $\omega \in \Omega \setminus \mathcal{N}$ with at most $n_2 - 1$ jumps at q_2 in the time interval $[0, t]$.

The sets $A^{\geqslant}(q_1, n_1, t)$ and $A^{<}(q_2, n_2, t)$ are measurable, and so is $A_2(q_1, n_1, q_2, n_2)$ since

$$A_2(q_1, n_1, q_2, n_2) = \bigcup_{t \in [0,T] \cap \mathbb{Q}} \big(A^{\geqslant}(q_1, n_1, t) \cap A^{<}(q_2, n_2, t)\big). \tag{43}$$

Now ω belongs to $U(j)$ if and only if, for all $1 \leqslant k \leqslant n$ and $q_1, q_2 \in \Lambda_k$ and $q_3 \in \Lambda \setminus \Lambda_k$ the following holds:

- If for exactly n_1 indices $1 \leqslant i \leqslant k$ the point q_1 belongs to Λ_i and for exactly n_2 indices $1 \leqslant j \leqslant k$ the point q_2 belongs to Λ_j then $\omega \in A_1(q_1, n_1, q_2, n_2)$.
- If for exactly n_1 indices $1 \leqslant i \leqslant k$ the point q_1 belongs to Λ_i and for exactly $n_3 - 1$ indices $1 \leqslant j < k$ the point q_3 belongs to Λ_j then $\omega \in A_2(q_1, n_1, q_3, n_3)$.
- If for exactly $l \geqslant 0$ indices $1 \leqslant n_1 < n_2 < \cdots < n_l \leqslant n$ a point $q \in \Lambda$ belongs to Λ_{n_i} then $\omega \in \{\tilde{\omega} \in \Omega \setminus \mathcal{N}\colon\ \pi_q \circ X_T(\tilde{\omega}) = l\}$.

We see that $U(j)$ is the intersection of finitely many measurable sets and hence measurable. □

Example 1.24. – We consider a generalization of Example 1.21. Let (Y, ϱ_Y) be a measurable space, $\mathcal{Y}$ its Borel sigma-algebra, Λ a non-empty finite set and $(\Omega, \mathcal{A}, P, (X_t)_{t \in I})$ a counting process with values in $\mathbb{N}^\Lambda$ that has finite expectation and with P-almost surely only jumps of size at most 1. Let $S = (S_{\Lambda_1})_{\Lambda_1 \subset \Lambda}$ be a family of continuous maps on Y^Λ, such that S_{Λ_1} changes at most the Λ_1-coordinates, i.e. if $\mathbf{y}_\Lambda \in Y^\Lambda$ and $q \in \Lambda \setminus \Lambda_1$ we have for the qth coordinate $\pi_q \circ S_{\Lambda_1}(\mathbf{y}_\Lambda) = y_q$.

For $t \in I$ and P-a.a. $\omega \in \Omega$ with $X_t(\omega) \in \mathbb{N}^\Lambda$ we have a finite sequence of jump-sets $\mathbf{j}(\omega) = (\Lambda_1, \ldots, \Lambda_n)$, as defined in Definition 1.22, and it depends measurably on ω, as was shown in Lemma 1.23. We define

$$S_\omega^t : Y^\Lambda \to Y^\Lambda, \tag{44}$$

$$\mathbf{y}_\Lambda \mapsto S_{\mathbf{j}(\omega)}(\mathbf{y}_\Lambda) \stackrel{\text{def}}{=} S_{\Lambda_n} \circ \cdots \circ S_{\Lambda_1}(\mathbf{y}_\Lambda). \tag{45}$$

We get a representation of $K_S^t(\mathbf{y}_\Lambda, Y_1)$, similar to the one in (37):

$$\begin{aligned} K_S^t(\mathbf{y}_\Lambda, Y_1) &= P\big(\{\omega\colon\ S_\omega^t(\mathbf{y}_\Lambda) \in Y_1\}\big) \\ &= \int_\Omega \mathrm{d}P(\omega) \chi_{Y_1} \circ S_\omega^t(\mathbf{y}_\Lambda) \\ &= \sum_{j \in \mathcal{J}\colon\ S_j(\mathbf{y}_\Lambda) \in Y_1} P\big(U(j)\big) \end{aligned} \tag{46}$$

for $\mathbf{y}_\Lambda \in Y^\Lambda$ and $Y_1 \in \bigotimes_{q \in \Lambda} \mathcal{Y}$.

We have seen in Example 1.24 that S_ω^t depends on $\mathbf{j}(\omega)$ only.

As we are interested in spatially extended systems we need some definitions and facts about infinite-dimensional systems.

DEFINITION 1.25. – *S^1 is the one-dimensional sphere. We define it to be isometric as Riemannian manifold to $\mathbb{R}/2\pi\mathbb{Z}$. This defines in particular a metric ϱ_{S^1} on S^1 and also the normalized Lebesgue measure on the* (*completed*) *Borel σ-algebra.*

The diameter of S^1 *is*

$$c_S \overset{\text{def}}{=} \operatorname{diam}_{\varrho_{S^1}}\left(S^1\right) = \pi. \tag{47}$$

(*It seems a bit redundant to introduce the constant* c_S *instead of using* π *in the following. But we indicate that the proofs in Section* 2 *work if* S^1 *is replaced by any compact Riemannian manifold or more general by a bounded metric space with a Borel probability measure. Further, we use the letter '*π*' as notation for projections.*)

We set

$$M \overset{\text{def}}{=} \left(S^1\right)^{\mathbb{Z}^d} \tag{48}$$

and give it the product topology and product Lebesgue measure on the (*completed*) *Borel* σ*-algebra.*

For $\Lambda \subset \mathbb{Z}^d$ we denote by π_Λ the projection on the Λ-coordinates.

Note that the product of the Borel σ-algebras is the same as the Borel σ-algebra for the product space. M is compact and metrizable in the following way:

DEFINITION 1.26. – *Let* $(b(q))_{q\in\mathbb{Z}^d}$ *be a family of positive numbers such that*

$$\lim_{R\to\infty} \sup_{\|q\|\geqslant R} b(q) = 0. \tag{49}$$

Then the metric ϱ_M *on* M*, associated to* $(b(q))_{q\in\mathbb{Z}^d}$*, is defined by*

$$\varrho_M(\mathbf{x},\mathbf{y}) \overset{\text{def}}{=} \sup_{q\in\mathbb{Z}^d} b(q)\varrho_{S^1}(x_q, y_q) \tag{50}$$

for $\mathbf{x}, \mathbf{y} \in M$.

Remark 1.27. –

1. One can easily show that ϱ_M, as defined in Definition 1.26, is in fact a metric and also compatible with the product topology.
2. A sequence $(\mathbf{x}^{(n)})_{n\in\mathbb{N}}$ in M converges w.r.t. the product topology iff it converges w.r.t. each coordinate, i.e. $(x_q^{(n)})_{n\in\mathbb{N}}$ converges for every $q \in \mathbb{Z}^d$. The same holds also for nets $(\mathbf{x}^\Lambda)_{\Lambda\in\mathcal{F}}$.
3. The product topology does not distinguish any particular sites despite the fact that the weights $b(q)$ depend on q. Spatial shifts, like $\mathbf{x} \mapsto \tilde{\mathbf{x}}$ with $\tilde{x}_q = x_{q-r}$ for some $r \in \mathbb{Z}^d$, are homeomorphisms.
4. The space $\mathcal{C}^0(M, M)$ of continuous maps on (M, ϱ_M) is complete w.r.t. the metric defined by

$$\varrho_{\mathcal{C}^0(M,M)}(f, g) \overset{\text{def}}{=} \sup_{\mathbf{x}\in M} \varrho_M\left(f(\mathbf{x}), g(\mathbf{x})\right). \tag{51}$$

   We denote by $\mathcal{B}_{\mathcal{C}^0(M,M)}$ the Borel σ-algebra w.r.t. this metric.

LEMMA 1.28. – *Let* $(\Omega, \mathcal{A})$ *be a measurable space and* $(f^\Lambda)_{\Lambda\in\mathcal{F}\setminus\{\emptyset\}}$ *be a net of measurable maps*

$$f^\Lambda : \Omega \to \mathcal{C}^0(M, M), \\ \omega \mapsto f^\Lambda_\omega, \tag{52}$$

such that for all $\Lambda_1 \in \mathcal{F} \setminus \{\emptyset\}$ *and* $\omega \in \Omega$ *the net* $(\pi_{\Lambda_1} \circ f^\Lambda_\omega)_{\Lambda_1 \subset \Lambda \in \mathcal{F}}$ *converges* (*as* $\Lambda \to \mathbb{Z}^d$) *in* $\mathcal{C}^0(M, (S^1)^{\Lambda_1})$*, say to* $\pi_{\Lambda_1} \circ f_\omega$*.*

Then

$$f_{\omega,q}(\mathbf{x}) \overset{\text{def}}{=} \lim_{\Lambda \to \mathbb{Z}^d} \pi_q \circ f^\Lambda_\omega(\mathbf{x}) \tag{53}$$

defines a measurable map

$$f : \Omega \to \mathcal{C}^0(M, M), \\ \omega \mapsto f_\omega, \tag{54}$$

whose q*th coordinate function is given by* (53)*.*

Proof. – Fix $\omega \in \Omega$, $\mathbf{x} \in M$ and a metric ϱ_M like in Definition 1.26. We show that f_ω is continuous in $\mathbf{x}$. For that let $\varepsilon > 0$ and choose $R_0 \in \mathbb{N}$ such that

$$c_S b(q) < \varepsilon \tag{55}$$

for all q with $\|q\| > R_0$. We note that the qth coordinate function of $\pi_{\Lambda_1} \circ f_\omega \in \mathcal{C}^0(M, (S^1)^{\Lambda_1})$ is the same as the qth coordinate function $f_{\omega,q}$ of f_ω.

By continuity of $\pi_{B_{R_0}(0)} \circ f_\omega$ we can choose a $\delta > 0$ such that for all $\mathbf{y} \in B_\delta(\mathbf{x})$ and all q with $\|q\| \leqslant R_0$:

$$c_S b(q) \varrho_{S^1}\big(f_{\omega,q}(\mathbf{x}), f_{\omega,q}(\mathbf{y})\big) < \varepsilon. \tag{56}$$

From (55) and (56) we conclude that for all $\mathbf{y} \in B_\delta(\mathbf{x})$

$$\varrho_M\big(f_\omega(\mathbf{x}), f_\omega(\mathbf{y})\big) < \varepsilon \tag{57}$$

which was to be shown. Finally f depends measurably on ω because it is pointwise limit of measurable functions with values in a metric space (cf. [22], p. 117, for example). □

Remark 1.29. –

1. Lemma 1.28 is in particular based on the compactness on M w.r.t. the product toplology.
   M is not compact w.r.t. the different metric, defined by
   $$\tilde{\varrho}_M(\mathbf{x}, \mathbf{y}) \overset{\text{def}}{=} \sup_{q \in \Lambda} \varrho_{S^1}(x_q, y_q).$$
   In this case the conclusion from 'local' to 'global' does not hold.
2. As f in (54) is $(\mathcal{A}, \mathcal{B}_{\mathcal{C}^0(M,M)})$-measurable, the map $(\omega, \mathbf{x}) \mapsto f_\omega(\mathbf{x})$ is $(\mathcal{A} \times \mathcal{B}_M, \mathcal{B}_M)$-measurable. We have proved this fact in Example 1.18.

2. Infinite-dimensional systems

In Example 1.24 we used a counting process with values in $\mathbb{N}^\Lambda$ (for finite Λ) and a family of updating-maps on Y^Λ to define Markov kernels on the product Y^Λ. These kernels act on the product space $\mathcal{C}^0(Y^\Lambda)$ of continuous functions (cf. Definition 1.19 and Proposition 2.15). In view of spatially extended systems like coupled map lattices or interacting particle systems we would like to define analogous operators for infinite-dimensional systems ($\Lambda = \mathbb{Z}^d$). As counting process we take the Poisson process $(\Omega, \mathcal{A}, P, (X_t)_{t\geqslant 0})$ with parameter $\lambda > 0$ and values in $\mathbb{N}^{\mathbb{Z}^d}$.

Recall that the set $\mathcal{N}_1$, defined in Definition 1.10, of all $\omega \in \Omega$ such that $X_\bullet(\omega)$ is not nondecreasing, jumps at time 0, has simultaneous jumps or jumps of size greater than one, has P-measure zero. So we have to consider updatings only at single sites. They are given by a family of continuous maps $(S_q)_{q\in\mathbb{Z}^d}$ such that $S_q : M \to M$ changes only the qth coordinate (cf. Example 1.24 for a definition.)

A problem is obviously that the Poisson process, restricted to any finite interval $[0,t]$ of length $t > 0$ is not of finite expectation (cf. Definition 1.1 and Remark 1.9.1). P-a.s. there are infinitely many jumps and it is even impossible to define an order preserving bijection between them and $\mathbb{N}$. However, in Section 2.1 we will show for systems with finite range interaction that for P-a.a. $\omega \in \Omega$, any $q \in \mathbb{Z}^d$ and $t > 0$ the site q is affected in $[0,t]$ (cf. Definition 2.1) by only finitely many sites, so that maps '$\pi_q \circ S^t_\omega$' from M to $(S^1)^{\{q\}}$ and then also 'S^t_ω' from M to M can be defined in a natural way. The proof is based on a percolation argument. Percolation techniques, but different from the ones presented here, were already used by Harris in [15] for proving the existence of certain interacting particle systems of finite range. It follows in particular that $\pi_\Lambda \circ S^t_\omega : M \to (S^1)^\Lambda$ for finite $\Lambda \neq \emptyset$ is the limit (as $\tilde\Lambda \to \mathbb{Z}^d$) of maps that are constructed by using the 'cut offs' $\pi_\Lambda \circ S^t_{\tilde\Lambda,\xi,\omega}$, corresponding to a finite $\tilde\Lambda \supset \Lambda$ and boundary conditions ξ. In fact, this limit also exists and is independent of the boundary conditions for a huge class of infinite range interactions as we will show in Section 2.2. It gives rise to a natural definition of the system. But we also note that for infinite range interaction each site is with positive probability affected by infinitely many other sites. So we cannot use the same definition as for finite range interaction.

In Section 2.3 we define Markov kernels K^t_S for the infinite system S^t and $K^t_{S,\tilde\Lambda}$ for the system $S^t_{\tilde\Lambda}$ that fixes the $\tilde\Lambda^C$-coordinates for a finite $\tilde\Lambda$ (Recall the notation for the complement from Definition 1.11). We show that K^t_S is the weak limit of $K^t_{S,\tilde\Lambda}$ (as $\tilde\Lambda \to \mathbb{Z}^d$), i.e. the corresponding operators on continuous functions converge weakly.

2.1. Finite range interaction

Now we are considering an interaction of range $R \in \mathbb{N}\setminus\{0\}$, i.e. $\pi_q \circ S_q(\mathbf{x})$ depends only on $\mathbf{x}_{B_R(q)}$. (Recall that $B_R(q)$ was defined in (23).)

DEFINITION 2.1. – *Given R as above, $q, \tilde q \in \mathbb{Z}^d$, $T > 0$, $\omega \in \Omega$. We say that $\tilde q$* affects *q* w.r.t. (R,t,ω) *if there is a causal path from q to $\tilde q$ of step size at most R. (Recall that we defined* path *etc. in Definitions* 1.13 *to* 1.15). *If $\emptyset \neq \Lambda \subset \mathbb{Z}^d$ we say that $\tilde q$* affects Λ w.r.t. (R,t,ω) *if $\tilde q$ affects at least one point in Λ w.r.t. (R,t,ω).*

We set

$$\mathrm{Aff}_{(R,t,\omega)}(\Lambda) \overset{\text{def}}{=} \big\{\tilde q \in \mathbb{Z}^d\colon \tilde q \text{ affects } \Lambda \text{ w.r.t. } (R,t,\omega)\big\}, \tag{58}$$

$$\text{and} \quad \Omega_R \overset{\text{def}}{=} \big\{\omega\colon \exists t > 0, q \in \mathbb{Z}^d \text{ such that } |\mathrm{Aff}_{(R,t,\omega)}(q)| = \infty\big\}, \tag{59}$$

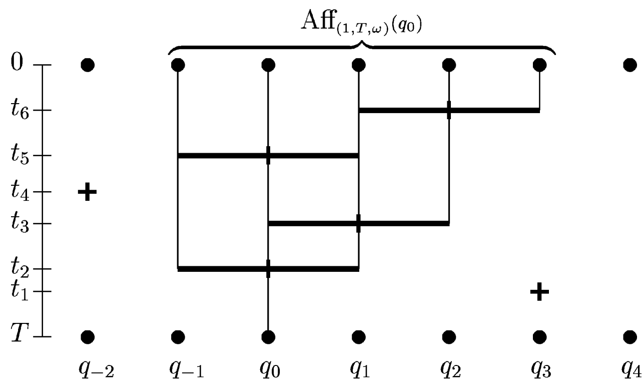

Fig. 1. The history of q_0.

where $|\cdot|$ *denotes the cardinality.*

Fig. 1 is a picture of $\mathrm{Aff}_{(1,T,\omega)}(q_0)$. We consider the finite time-interval $(0, T]$ and nearest neighbour interaction and a particular ω. For each jump we draw a cross at the particular point (q,t). There are jumps at (q_2,t_6), (q_0,t_5), (q_{-2},t_4), (q_1,t_3), (q_0,t_2) and (q_3,t_1) The last jump at q_0 is at time t_2. We draw a thick horizontal line between (q_0,t_2) and (q,t_2) for all nearest neighbours q of q_0 because the updating of q_2 depends also on these sites. So we have to consider the 'histories' of q_0 *and* its nearest neighbours *before* time t_2. Note that $q_3 \in \mathrm{Aff}_{(1,T,\omega)}(q_0)$ and it is updated at time t_1 (and so affected by q_4 for example) but that updating has no influence on q_0 (at time T). We also note that, for example, q_{-1} affects q_0 (w.r.t. $(1,T,\omega)$) but not the other way around. So we have to consider only the time- and space-ordered percolation.

PROPOSITION 2.2. – Ω_R *has* P*-measure zero*:

$$P(\Omega_R) = 0. \tag{60}$$

Proof. – $\mathrm{Aff}_{(R,t,\omega)}(q)$ is increasing in t and so

$$\Omega_R = \bigcup_{t\in\mathbb{N}} \bigcup_{q\in\mathbb{Z}^d} \{\omega\colon |\mathrm{Aff}_{(R,t,\omega)}(q)| = \infty\}. \tag{61}$$

So it is sufficient to show that for fixed $q \in \Lambda$ and $t > 0$ the set $\{\omega\colon |\mathrm{Aff}_{(R,t,\omega)}(q)| = \infty\}$ has P-measure zero. If we set

$$A_N \overset{\text{def}}{=} \{\omega\colon \mathrm{Aff}_{(R,t,\omega)}(q) \not\subset B_N(q)\} \tag{62}$$

it is sufficient to show that

$$\lim_{N\to\infty} P(A_N) = 0. \tag{63}$$

If q is affected by some $\tilde{q} \notin B_N(q)$ w.r.t. (R,t,ω) then there is a maximal causal path of step size at most R from q to $\tilde{q}$ with at least N_0 steps, where N_0 is the smallest integer greater than $\frac{N}{R}$.

Consider any maximal causal path $Q = (q_0 = q, \dots, q_n)$ of step size at most R and with $n \geqslant N_0$. Q is a maximal causal path w.r.t. (t, ω) iff the trajectory of ω w.r.t. the Poisson process induced by Q (cf. Definition 1.14) has exactly n jumps. The probability of this is $p_\lambda(t, n)$ (which was defined in (4).)

We set

$$c_{d,R} \stackrel{\text{def}}{=} |B_R(q)|. \tag{64}$$

(Recall that $B_R(q)$ was defined in (23) and $|\cdot|$ denotes the cardinality.)

Then

$$|\mathrm{Path}(q, n, R)| = c_{d,R}^n \tag{65}$$

because at each step in the path one can choose between $c_{d,R}$ lattice-points.

So we have

$$A_N \subset \bigcup_{n \geqslant N_0} \bigcup_{Q \in \mathrm{Path}(q,n,R)} \{\omega\colon\ Q \text{ is maximal causal w.r.t. } (R, t, \omega)\} \tag{66}$$

and so

$$P(A_N) \leqslant \sum_{n \geqslant N_0} c_{d,R}^n \mathrm{e}^{-\lambda t} \frac{(\lambda t)^n}{n!} \tag{67}$$

$$\leqslant \mathrm{e}^{(c_{d,R}-1)\lambda t} (c_{d,R} \lambda t)^{N_0} \frac{1}{N_0!} \tag{68}$$

which converges to 0 as $N_0 \to \infty$ which was to show. For the last inequality we have used the estimate for the Lagrange remainder in Taylor's formula. □

DEFINITION 2.3. – *Let a finite range interaction* (*i.e. a family of updatings*) *be given by* $(S_q)_{q \in \mathbb{Z}^d}$. *Fix* $\omega \in \Omega \setminus (\Omega_R \cup \mathcal{N}_1)$, $\emptyset \neq \Lambda \subset \tilde{\Lambda} \in \mathcal{F}$, $\xi \in M$ *and* $t > 0$. *Then* ω *has only finitely many jumps in* $\tilde{\Lambda} \times (0, t)$, *say at* $(q_1, t_1), \dots, (q_n, t_n)$ *with* $0 < t_1 < \dots < t_n < t$.

We denote by $\mathbf{x}_{\tilde{\Lambda}} \vee \xi_{\tilde{\Lambda}^C}$ *the point in* M *that has the same* $\tilde{\Lambda}$*-coordinates as* $\mathbf{x}$ *and the same* $\tilde{\Lambda}^C$*-coordinates as* ξ.

We define

$$S_{q,\tilde{\Lambda},\xi} : (S^1)^{\tilde{\Lambda}} \to (S^1)^{\tilde{\Lambda}}, \tag{69}$$
$$S_{q,\tilde{\Lambda},\xi}(\mathbf{x}_{\tilde{\Lambda}}) \stackrel{\text{def}}{=} \pi_{\tilde{\Lambda}} \circ S_q(\mathbf{x}_{\tilde{\Lambda}} \vee \xi_{\tilde{\Lambda}^C}),$$

and

$$\Omega \setminus (\mathcal{N}_1 \cup \Omega_R) \ni \omega \mapsto S^t_{\tilde{\Lambda},\xi,\omega} \in \mathcal{C}^0(M, M), \tag{70}$$
$$S^t_{\tilde{\Lambda},\xi,\omega}(\mathbf{x}) \stackrel{\text{def}}{=} S_{q_n,\tilde{\Lambda},\xi} \circ \dots \circ S_{q_1,\tilde{\Lambda},\xi}(\mathbf{x}_{\tilde{\Lambda}}) \vee \xi_{\tilde{\Lambda}^C}.$$

The maps $S^t_{\tilde{\Lambda},\xi,\omega}$ *are continuous as composites of continuous maps. Furthermore,* $S^t_{\tilde{\Lambda},\xi,\omega}$ *depends only on* $\omega_{\tilde{\Lambda}}$ (*i.e. on* $\pi_{\tilde{\Lambda}} \circ X_\bullet(\omega)$) *and* (70) *gives rise to a countable, measurable partition of* $\Omega \setminus (\mathcal{N}_1 \cup \Omega_R)$: ω *and* $\tilde{\omega}$ *belong to the same set of this partition if they have the same list of jump sites* $(q_1, \dots, q_n)$ (*ordered w.r.t. the jump times*).

Now let $\tilde{\Lambda} \supset \mathrm{Aff}_{(R,t,\omega)}(\Lambda)$ *and* $\xi \in M$ *and define*

$$\pi_\Lambda \circ S^t : \Omega \setminus (\mathcal{N}_1 \cup \Omega_R) \to \mathcal{C}^0\big(M, \big(S^1\big)^\Lambda\big), \tag{71}$$

$$\pi_\Lambda \circ S^t_\omega(\mathbf{x}) \stackrel{\text{def}}{=} \pi_\Lambda \circ S^t_{\tilde{\Lambda},\xi,\omega}(\mathbf{x}_{\tilde{\Lambda}}). \tag{72}$$

The definition does not depend on the choice of $\tilde{\Lambda}$ *or* ξ *because the right-hand side* (*rhs*) *of* (72) *depends, by definition, on the* $\mathrm{Aff}_{(R,t,\omega)}(\Lambda)$*-coordinates of* $\mathbf{x}$ *only.*

Further, the family $(\pi_\Lambda \circ S^t_\omega(\mathbf{x}))_{\Lambda \in \mathcal{F}\setminus\{\emptyset\}}$ *is consistent in the sense that for any* $\emptyset \neq \Lambda_1 \subset \Lambda_2 \in \mathcal{F}$:

$$\pi_{\Lambda_1}\big(\pi_{\Lambda_2} \circ S^t_\omega(x)\big) = \pi_{\Lambda_1} \circ S^t_\omega(x), \tag{73}$$

and so defines a map

$$S^t_\omega : M \to M, \tag{74}$$

$$\big(S^t_\omega(x)\big)_q \stackrel{\text{def}}{=} \pi_q \circ S^t_\omega(x).$$

Finally, we set $S^t_\omega = \mathrm{id}_M$ *for* $\omega \in \Omega_R \cup \mathcal{N}$.

PROPOSITION 2.4. – *The map* S^t_ω*, defined in* (72) *and* (74) *is continuous and depends measurably on* ω.

Proof. – The net $(\tilde{S}^t_{\tilde{\Lambda},\xi,\omega})_{\tilde{\Lambda}\in\mathcal{F}\setminus\{\emptyset\}}$ satisfies the assumptions in Lemma 1.28 and so all statements of Proposition 2.4 follow. □

2.2. Infinite range interaction

We extend our notion of 'S^t_ω' to interactions that are not necessarily of finite range.

Consider a family $(S_q)_{q\in\mathbb{Z}^d}$ of maps $S_q : M \to M$ such that S_q does not change the $\mathbb{Z}^d \setminus \{q\}$-coordinates and $\pi_q \circ S_q : M \to S^1$ is *Lipschitz-continuous w.r.t. all coordinates* and the Lipschitz constants depend only on the relative positions of the sites, i.e. there are constants $w(r)$ for all $r \in \mathbb{Z}^d$ such that for all $q, \tilde{q} \in \mathbb{Z}^d$ and $\mathbf{x}, \mathbf{y} \in M$ with $\mathbf{x}_{\mathbb{Z}^d\setminus\{\tilde{q}\}} = \mathbf{y}_{\mathbb{Z}^d\setminus\{\tilde{q}\}}$ (i.e. $\mathbf{x}$ and $\mathbf{y}$ differ at most in their $\tilde{q}$-coordinates.)

$$\varrho_{S^1}\big(\pi_q \circ S_q(\mathbf{x}), \pi_q \circ S_q(\mathbf{y})\big) \leqslant w(\tilde{q} - q)\varrho_{S^1}(y_{\tilde{q}}, z_{\tilde{q}}). \tag{75}$$

We further assume summability of the Lipschitz-constants, i.e.

$$\sum_{q\in\mathbb{Z}^d} w(q) = c_1 \tag{76}$$

with a positive constant c_1.

We need the following technical lemma.

LEMMA 2.5. – *If* $(w(q))_{q\in\mathbb{Z}^d}$ *is a family of non-negative real numbers satisfying* (76) *then there are families* $(w_1(q))_{q\in\mathbb{Z}^d}$ *and* $(w_2(q))_{q\in\mathbb{Z}^d}$ *of non-negative and positive numbers, respectively, such that*

$$w(q) = w_1(q)w_2(q) \quad \textit{for all } q \in \mathbb{Z}^d, \tag{77}$$

$$\sum_{q\in\mathbb{Z}^d} w_1(q) \leqslant 2c_1 + 1, \tag{78}$$

$$\textit{and} \quad \lim_{R\to\infty} a(R) = 0, \tag{79}$$

where the positive function $a(\cdot)$ *is defined by*

$$a(R) \stackrel{\text{def}}{=} \sup_{\|r_1\|+\cdots+\|r_n\|=R} w_2(r_1)\cdot\dots\cdot w_2(r_n). \tag{80}$$

(*The empty product is defined to be equal to* 1.)

Proof. – We can choose $r_0 = 0 < r_1 < \cdots \in \mathbb{N}$ such that

$$\sum_{\|q\|<r_i} w(q) \geqslant c_1 - 4^{-(i+1)} \quad \text{for } i \geqslant 1. \tag{81}$$

Then we have

$$\sum_{\|q\|<r_1} w(q) \leqslant c_1 \quad \text{and} \quad \sum_{r_i \leqslant \|q\| < r_{i+1}} w(q) \leqslant 4^{-(i+1)} \quad \text{for } i \geqslant 1. \tag{82}$$

We set for $i \geqslant 1$ and $r_{i-1} \leqslant \|q\| < r_i$:

$$w_2(q) \stackrel{\text{def}}{=} 2^{-i}, \tag{83}$$

$$w_1(q) \stackrel{\text{def}}{=} 2^i w(q). \tag{84}$$

Then (77) is obviously satisfied. To prove (78) we use (82) and (84):

$$\sum_{q\in\mathbb{Z}^d} w_1(q) = \sum_{i=0}^{\infty} \sum_{r_i \leqslant \|q\| < r_{i+1}} w_1(q) \leqslant 2c_1 + \sum_{i=1}^{\infty} 2^{-i} = 2c_1 + 1. \tag{85}$$

Now we prove (79). We show by induction (w.r.t. i) that for every $i \geqslant 1$ there is an n_i such that

$$a(R) < 2^{-i} \quad \text{for all } R \geqslant n_i. \tag{86}$$

For $i = 1$ the statement is true with $n_1 = 1$ because $a(R) \leqslant \frac{1}{2}$ for every $R \geqslant 1$ as there is at least one factor on the right-hand-side in (80) and each such factor is at most $\frac{1}{2}$.

Now we assume that the statement holds for i and n_i. We set

$$n_{i+1} \stackrel{\text{def}}{=} r_i + 2n_i. \tag{87}$$

Then every path $(q_0, \dots, q_n)$ of length $R \geqslant n_{i+1}$ has at least one step of size at least r_i (i.e. there is an $1 \leqslant l \leqslant n$ such that $\|q_l - q_{l-1}\| \geqslant r_i$) or it can be divided into two paths both of length at least n_i (i.e. there is an $1 \leqslant l \leqslant m - 1$ such that $\|q_0 - q_1\| + \cdots + \|q_{l-1} + q_l\| \geqslant n_i$ and $\|q_{l+1} - q_l\| + \cdots + \|q_n + q_{n-1}\| \geqslant n_i$). So each product on the right-hand side of (80) has at least one factor less than or equal to $2^{-(i+1)}$ or two factors less than or equal to 2^{-i}. As the other factors are smaller than 1 the product is bounded by $2^{-(i+1)}$ as was to be shown. □

Now we fix (like in Lemma 2.5) a choice of $(w_1(q))_{q\in\mathbb{Z}^d}$ and $(w_2(q))_{q\in\mathbb{Z}^d}$ and so the function a.

DEFINITION 2.6. – *We fix the metric ϱ_M on M by*

$$\varrho_M(\mathbf{x},\mathbf{y}) \stackrel{\text{def}}{=} \sup_{r\in\mathbb{Z}^d} a(\|r\|)\varrho_{S^1}(x_r, y_r). \tag{88}$$

Remark 2.7. – It follows from Remark 1.27.1 and (79) that ϱ_M is a metric and compatible with the product topology.

LEMMA 2.8. – *The maps $S_q : M \to M$ are continuous* (*w.r.t. the product topology on M*).

Proof. – According to Remark 1.27.2 and the uniform choice of the Lipschitz-constants (cf. (75)) we only have to show that the maps $\pi_q \circ S_0 : M \to S^1$ are continuous.

If $q \neq 0$ then the qth coordinate is not changed by S_0 and

$$a(\|q\|)\varrho_{S^1}\big(\pi_q \circ S_0(\mathbf{x}), \pi_q \circ S_0(\mathbf{y})\big) = a(\|q\|)\varrho_{S^1}(x_q, y_q) \leqslant \varrho_M(\mathbf{x},\mathbf{y}). \tag{89}$$

If $q = 0$ we estimate

$$\begin{aligned} a(0)\varrho_{S^1}\big(\pi_0 \circ S_0(\mathbf{x}), \pi_0 \circ S_0(\mathbf{y})\big) &\leqslant a(0)\sum_{r\in\mathbb{Z}^d} w(r)\varrho_{S^1}(x_r, y_r) \\ &\leqslant a(0)\sum_{r\in\mathbb{Z}^d} w(r)\frac{1}{a(\|r\|)}\varrho_M(\mathbf{x},\mathbf{y}) \\ &\leqslant a(0)(2c_1+1)\varrho_M(\mathbf{x},\mathbf{y}), \end{aligned} \tag{90}$$

where we have used (75) for the first, the definition of ϱ_M for the second and (77) for the third inequality. So $\pi_q \circ S_0$ is continuous for all $q \in \mathbb{Z}^d$. □

In the following we estimate the distance (w.r.t. the uniform norm) between $\pi_0 \circ S^t_{\Lambda,\xi,\omega}$ and $\pi_0 \circ S^t_{\Lambda,\tilde{\xi},\omega}$ for different boundary conditions ξ_{Λ^C} and $\tilde{\xi}_{\Lambda^C}$ (that might even depend on the time) at the Λ^C-sites. Conditions (75) and (76) allow us to apply standard estimates for the 'error-growth' for composites of maps. Using the linear nature of the 'Lipschitz-condition' (75), we write the products of sums (over all coordinates, like in (75)) as sums (over paths) of products (corresponding to the particular paths).

We fix $t > 0$, $\Lambda \in \mathcal{F}$ and $\omega \in \Omega \setminus \mathcal{N}_1$. By definition of $\mathcal{N}_1$ (cf. (21)) ω has no jumps at 0, no simultaneous jumps and only finitely many jumps in $\Lambda \times (0,t)$, say at $(q_1,t_1),\ldots,(q_N,t_N)$ with $0 < t_1 < \cdots < t_N < t$. We set $t_0 \stackrel{\text{def}}{=} 0$ and fix arbitrary $\xi = (\xi(t_0),\ldots,\xi(t_N))$, $\tilde{\xi} = (\tilde{\xi}(t_0),\ldots,\tilde{\xi}(t_N)) \in M^{N+1}$ and $\mathbf{x},\mathbf{y} \in M$.

We set $\mathbf{x}(0) \stackrel{\text{def}}{=} \mathbf{x}_\Lambda \vee \xi_{\Lambda^C}(0)$, $\mathbf{y}(0) \stackrel{\text{def}}{=} \mathbf{y}_\Lambda \vee \tilde{\xi}_{\Lambda^C}(0)$ and define for $1 \leqslant i \leqslant N$ recursively:

$$x_q(t_i) \stackrel{\text{def}}{=} \begin{cases} \pi_q \circ S_q(\mathbf{x}(t_{i-1})) & \text{for } q = q_i, \\ x_q(t_{i-1}) & \text{for } q \in \Lambda \setminus \{q_i\}, \\ \xi_q(t_i) & \text{for } q \in \Lambda^C. \end{cases} \tag{91}$$

We define $\mathbf{y}(t_i)$ analogously, using $\mathbf{y}$ and $\tilde{\xi}$ instead of $\mathbf{x}$ and ξ, respectively.

Two points in S^1 can have distance at most $c_S = \operatorname{diam}_{\varrho_{S^1}}(S^1)$. For estimating the distance between $x_q(t_i)$ and $y_q(t_i)$ we define

$$\Delta_q(0) \stackrel{\text{def}}{=} \tilde{\Delta}_q(0) \stackrel{\text{def}}{=} \begin{cases} \varrho_{S^1}(x_q(0), y_q(0)) & \text{for } q \in \Lambda, \\ c_S & \text{for } q \in \Lambda^C, \end{cases} \tag{92}$$

and for $1 \leqslant i \leqslant N$

$$\Delta_q(i) \stackrel{\text{def}}{=} \begin{cases} \sum_{r\in\mathbb{Z}^d} w(r-q)\Delta_r(i-1) & \text{for } q = q_i, \\ \Delta_q(i-1) & \text{for } q \in \Lambda \setminus \{q_i\}, \\ c_S & \text{for } q \in \Lambda^C, \end{cases} \tag{93}$$

$$\tilde{\Delta}_q(i) \stackrel{\text{def}}{=} \begin{cases} \min\{c_S, \sum_{r\in\mathbb{Z}^d} w(r-q)\tilde{\Delta}_r(i-1)\} & \text{for } q = q_i, \\ \tilde{\Delta}_q(i-1) & \text{for } q \in \Lambda \setminus \{q_i\}, \\ c_S & \text{for } q \in \Lambda^C. \end{cases}$$

The functions Δ_q and $\tilde{\Delta}_q$ depend on $\mathbf{x}$, $\mathbf{y}$ and Λ but we do not refer to this in our notation. We have introduced them for estimating the difference between $x_q(t_i)$ and $y_q(t_i)$ (cf. (94)) and so the difference between $x_q(t)$ and $y_q(t)$. This difference depends also on ω and so do the corresponding estimates for Δ_q and $\tilde{\Delta}_q$. In Definition 2.11 we will relate Δ_q and $\tilde{\Delta}_q$ to families of random variables $(Y_\Lambda)_{\Lambda\in\mathcal{F}\setminus\{\emptyset\}}$ and $(\tilde{Y}_\Lambda)_{\Lambda\in\mathcal{F}\setminus\{\emptyset\}}$, respectively. For Δ_q we find a particularly nice expansion (cf. (95)). From this follows the convergence of Y_Λ to zero in expectation (as $\Lambda \to \mathbb{Z}^d$). We will show that $\tilde{Y}_\Lambda$ is bounded by Y_Λ and decreasing and so converges P-almost surely to zero by the Monotone Convergence Theorem (cf. Theorem 2.13).

PROPOSITION 2.9. – *The following holds for* $0 \leqslant i \leqslant N$:

1.

$$\varrho_{S^1}\big(x_q(t_i), y_q(t_i)\big) \leqslant \tilde{\Delta}_q(i) \leqslant \Delta_q(i). \tag{94}$$

2.

$$\Delta_q(i) = \sum_{\substack{(r_0=q,r_1,\dots,r_n) \\ \in \mathrm{Path}_{\mathrm{C}}(t_i,\omega,q,\Lambda)}} w(r_1-r_0)\cdot\dots\cdot w(r_n-r_{n-1})\Delta_{r_n}(0). \tag{95}$$

3. *If in particular* $\mathbf{x}_\Lambda = \mathbf{y}_\Lambda$ *and* $q \in \Lambda$ *then*

$$\Delta_q\big(N(\omega)\big) \leqslant c_S a\big(\operatorname{dist}_{\mathbb{Z}^d}(q, \Lambda^C)\big) \sum_{\substack{(r_0=q,r_1,\dots,r_n) \\ \in \mathrm{Path}_{\mathrm{C}}(t,\omega,q\to\Lambda^C)}} w_1(r_1-r_0)\cdot\dots\cdot w_1(r_n-r_{n-1}), \tag{96}$$

where $N(\omega)$ *is the number of jumps of* ω *in* $\Lambda \times (0,t)$.

Proof. – We prove (94) and (95) by induction w.r.t. i.

$i = 0$: (94) holds by definition of $\Delta_q(0)$ and $\tilde{\Delta}_q(0)$ (cf. (92)). At time 0 no jump has happened and the only summand on the right-hand-side in (95) corresponds to the empty path at site q and so the equality in (95) holds.

$i-1 \to i$: (95) holds obviously for i and $q \neq q_i$ as there is no updating at site q and

$$\mathrm{Path}_{\mathrm{C}}(t_i, \omega, q, \Lambda) = \mathrm{Path}_{\mathrm{C}}(t_{i-1}, \omega, q, \Lambda). \tag{97}$$

At site q_i there is a jump at time t_i and so we have

$$\Delta_{q_i}(i) = \sum_{r\in\mathbb{Z}^d} w(r-q_i)\Delta_r(i-1). \tag{98}$$

Using the representation (95) for $\Delta_r(i-1)$ and the fact that every $(q_i, r_1, \ldots, r_n) \in \mathrm{Path}_{\mathrm{C}}(t_i, \omega, q_i, \Lambda)$ can be (uniquely) split into (q_i, r_1) and $(r_1, \ldots, r_n) \in \mathrm{Path}_{\mathrm{C}}(t_{i-1}, \omega, r_1, \Lambda)$, we see that (95) holds for i.

Next we show the first inequality in (94) for i. For $q \in \Lambda^C$ the distances between $x_q(t_i) = \xi_q(t_i)$ and $y_q(t_i) = \tilde{\xi}_q(t_i)$ is bounded by c_S and for $q \in \Lambda \setminus \{q_i\}$ we have $x_q(t_i) = x_q(t_{i-1})$ and $y_q(t_i) = y_q(t_{i-1})$. So in both cases the first inequality in (94) holds.

Now we consider the site q_i where a jump happens at time t_i. Using (75), assumption (94), for $i-1$, and (98), we get

$$\begin{aligned}\varrho_{S^1}\big(x_q(t_i), y_q(t_i)\big) &\leqslant \sum_{r\in\mathbb{Z}^d} w(r-q_i)\varrho_{S^1}\big(x_r(t_{i-1}), y_r(t_{i-1})\big)\\ &\leqslant \sum_{r\in\mathbb{Z}^d} w(r-q_i)\Delta_r(i-1) \leqslant \Delta_{q_i}(i).\end{aligned} \tag{99}$$

So the first inequality in (94) is proved for i. The second follows immediately from (94). So statements 1 and 2 are proved.

Finally, (96) follows from (95): $\Delta_q(0) = 0$ for $q \in \Lambda$. So we only have to sum over paths $(r_0 = q, \ldots, r_n)$ that end in $r_n \in \Lambda^C$.

In particular, if we set $R \stackrel{\text{def}}{=} \|r_n\|$, then

$$\tilde{\Delta}_{r_n}(0) = c_S, \tag{100}$$

$$\mathrm{dist}_{\mathbb{Z}^d}\big(q, \Lambda^C\big) \leqslant R, \tag{101}$$

$$R \leqslant \|r_n - r_{n-1}\| + \cdots + \|r_1 - r_0\|, \tag{102}$$

and so by the choice of w_1, w_2 and a, made before Definition 2.6, we get

$$\begin{aligned}&w(r_1-r_0)\cdot\cdots\cdot w(r_n-r_{n-1})\\ &\quad\leqslant w_1(r_1-r_0)\cdot\cdots\cdot w_1(r_n-r_{n-1})a(R)\\ &\quad\leqslant w_1(r_1-r_0)\cdot\cdots\cdot w_1(r_n-r_{n-1})a\big(\mathrm{dist}_{\mathbb{Z}^d}\big(q,\Lambda^C\big)\big).\end{aligned} \tag{103}$$

Using (95), (100) and (103), we get (96) . □

Remark 2.10. – The summing over causal paths in Proposition 2.9 reflects that the result of an updating depends only on what has happened before.

DEFINITION 2.11. – *We define two families* $(Y_\Lambda)_{\Lambda\in\mathcal{F}\setminus\{\emptyset\}}$ *and* $(\tilde{Y}_\Lambda)_{\Lambda\in\mathcal{F}\setminus\{\emptyset\}}$ *of random variables on* $\Omega \setminus \mathcal{N}_1$. *Let* $\Lambda \in \mathcal{F} \setminus \{\emptyset\}$ *and* $\omega \in \Omega \setminus \mathcal{N}_1$, *say with exactly* $N(\omega)$ *jumps in* $\Lambda \times [0, t]$. *If we choose* $\mathbf{x}, \mathbf{y} \in M$ *with* $\mathbf{x}_\Lambda = \mathbf{y}_\Lambda$ *the value of* $\Delta_0(N(\omega))$ (*as defined by* (92) *and* (94)) *does not depend on* $\mathbf{x}$ *or* $\mathbf{y}$. *We define* $Y_\Lambda(\omega)$ *to be equal to this value*:

$$Y_\Lambda(\omega) \stackrel{\text{def}}{=} \Delta_0\big(N(\omega)\big) \tag{104}$$

$\tilde{Y}_\Lambda$ *is defined analogously, using* $\tilde{\Delta}_0(N(\omega))$ *instead of* $\Delta_0(N(\omega))$.

Remark 2.12. –

1. We remark that Y_Λ depends measurably on ω. In fact there is a countable, measurable partition of $\Omega \setminus \mathcal{N}_1$ such that ω and $\tilde{\omega}$ belong to the same set (of that partition) if the sums for $\Delta_0(t_{N(\omega)})$ and $\Delta_0(t_{N(\tilde{\omega})})$ (cf. (95)) are over the same paths.
2. From (94) we see that

$$\tilde{Y}_\Lambda \leqslant Y_\Lambda. \tag{105}$$

Now we fix $\xi, \mathbf{x} \in M$ and define the map $S^t_{\Lambda,\xi,\omega}$ like in (70).

THEOREM 2.13. –

1. *There is a set $\mathcal{N}_2$ of P-measure zero such that*

$$\lim_{\Lambda \to Z^d} \tilde{Y}_\Lambda = 0 \quad \textit{for } \omega \in \Omega \setminus (\mathcal{N}_1 \cup \mathcal{N}_2). \tag{106}$$

2. *The limit*

$$\pi_0 \circ S^t_\omega \stackrel{\text{def}}{=} \lim_{\Lambda \to \mathbb{Z}^d} \pi_0 \circ S^t_{\Lambda,\xi,\omega} \tag{107}$$

exists in $\mathcal{C}^0(M, S^1)$ for all $\omega \in \Omega \setminus (\mathcal{N}_1 \cup \mathcal{N}_2)$. It is measurable in ω and does not depend on ξ.

3. *There is a set $\mathcal{N} \subset \Omega$ of P-measure zero such that we can define maps*

$$\pi_q \circ S^t_\omega \stackrel{\text{def}}{=} \lim_{q \in \Lambda \to \mathbb{Z}^d} \pi_q \circ S^t_{\Lambda,\xi,\omega} \tag{108}$$

for all $q \in \mathbb{Z}^d$ and $\omega \in \Omega \setminus \mathcal{N}$.
Further, we can define a map $S^t_\omega \in \mathcal{C}^0(M, M)$ by

$$\left(S^t_\omega(\mathbf{x})\right)_q \stackrel{\text{def}}{=} \pi_q \circ S^t_\omega(\mathbf{x}). \tag{109}$$

S^t_ω depends measurably on ω.

Proof. – First we show that

$$\lim_{\Lambda \to \mathbb{Z}^d} E(Y_\Lambda) = 0. \tag{110}$$

We set $R \stackrel{\text{def}}{=} \text{dist}_{\mathbb{Z}^d}(0, \Lambda^C)$. Using (96) we get

$$\begin{aligned} E(Y_\Lambda) &\leqslant \int_\Omega \mathrm{d}P(\omega)\, c_S a(R) \sum_{\substack{(r_0=0,r_1,\ldots,r_n) \\ \in \text{Path}_\text{C}(t,\omega,0\to\Lambda^C)}} w_1(r_1 - r_0)\cdot \cdots \cdot w_1(r_n - r_{n-1}) \\ &= c_S a(R) \sum_{Q \in \text{Path}(0\to\Lambda^C)} w_1(r_1 - r_0)\cdot \cdots \cdot w_1(r_n - r_{n-1}) \\ &\quad \times P\big(\big\{\omega\colon\ Q \in \text{Path}_\text{C}\big(t,\omega,0 \to \Lambda^C\big)\big\}\big) \end{aligned} \tag{111}$$

A path $Q = (q_0 = 0, q_1, \ldots, q_n)$ with $q_n \in \Lambda^C$ is causal w.r.t. (t, ω) (i.e. $Q \in \text{Path}_\text{C}(t, \omega, 0 \to \Lambda^C)$) iff the Poisson process induced by Q has at least n jumps. So we can estimate the probability

$$P\big(\{\omega\colon\ Q\in \mathrm{Path}_{\mathrm{C}}\big(t,\omega,0\to \Lambda^C\big)\}\big)=\sum_{m\geqslant n}\mathrm{e}^{-\lambda t}\frac{(\lambda t)^m}{m!} \tag{112}$$

$$\leqslant \frac{(\lambda t)^n}{n!}. \tag{113}$$

For the last line we have used Taylor's formula, as we did in (68). So we get, using (78),

$$E(Y_\Lambda)\leqslant c_S a(R)\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}\left(\sum_{r\in\mathbb{Z}^d}w_1(r)\right)^n \tag{114}$$

$$\leqslant c_2 a(R) \tag{115}$$

with $c_2=c_S\mathrm{e}^{\lambda t(2c_1+1)}$. (Recall that we consider a fixed t at the moment, so c_2 is a constant.) By (79) we get

$$\lim_{\Lambda\to Z^d}E(Y_\Lambda)=0 \tag{116}$$

and, using (105),

$$\lim_{\Lambda\to Z^d}E(\tilde Y_\Lambda)=0. \tag{117}$$

$\tilde Y_\Lambda(\omega)$ is decreasing for all $\omega\in\Omega\setminus\mathcal{N}_1$: For a fixed ω and $0\in\Lambda_1\subset\Lambda_2\in\mathcal{F}$ we consider the (time-ordered) sequence of jumps $(q_1,t_1),\dots,(q_n,t_n)$ of ω at sites $q_1,\dots,q_2\in\Lambda_1$. It is a subsequence of the sequence of jumps $(\overline{q}_1,\overline{t}_1),\dots,(\overline{q}_m,\overline{t}_m)$ of ω at sites $\overline{q}_1,\dots,\overline{q}_m\in\Lambda_2$. The jumps (q_i,t_i) in the first sequence correspond to jumps $(\overline{q}_{j(i)},\overline{t}_{j(i)})$ in the second one. Then $q_i=\overline{q}_{j(i)}$ and $t_i=\overline{t}_{j(i)}$ but the indices i and $j(i)$ are not the same in general.

We define $\tilde\Delta^1_q(i)$ and $\tilde\Delta^2_q(j)$ as in (92) and (94) for the sets Λ_1 and Λ_2, respectively. We show that

$$\tilde\Delta^1_q(i)\geqslant\tilde\Delta^2_q\big(j(i)\big). \tag{118}$$

If $q\in\Lambda_1^C$ then (118) obviously holds because $\tilde\Delta^1_q(i)=c_S$ is an upper bound for $\tilde\Delta^2_q(j)$. For $q\in\Lambda_1$ we show (118) by induction w.r.t. i.

If $i=0$ then (118) is true by (92). Now assume that (118) holds for all q and a particular $i<n$. For $q\in\Lambda_1\setminus\{q_{i+1}\}$ we have

$$\tilde\Delta^1_q(i+1)=\tilde\Delta^1_q(i)\geqslant\tilde\Delta^2_q\big(j(i)\big)=\tilde\Delta^2_q\big(j(i+1)\big) \tag{119}$$

where the inequality holds by assumption and the equalities by (94). For the site $q=q_{i+1}$ we have by (94)

$$\begin{aligned}\tilde\Delta^1_q(i+1)&=\max\left\{c_S,\sum_{r\in\mathbb{Z}^d}w(r-q)\tilde\Delta^1_r(i)\right\}\\&\geqslant\max\left\{c_S,\sum_{r\in\mathbb{Z}^d}w(r-q)\tilde\Delta^2_r\big(j(i+1)-1\big)\right\}\\&=\tilde\Delta^2_q\big(j(i+1)\big)\end{aligned} \tag{120}$$

which was to be shown. Here we have used that $\tilde{\Delta}_r^1(i) \geqslant \tilde{\Delta}_r^2(j(i+1)-1)$. This follows for $r \in \Lambda_1^C$ from the definition of $\tilde{\Delta}_r^1$ and $\tilde{\Delta}_r^2$ and for $r \in \Lambda_1$ from assumption (118) and the fact that $\tilde{\Delta}_r^2(j(i+1)-1) = \tilde{\Delta}_r^2(j(i))$.

Using the definition of $\tilde{Y}_{\Lambda_1}(\omega)$ and $\tilde{Y}_{\Lambda_2}(\omega)$ (cf. Definition 2.11), we conclude

$$\tilde{Y}_{\Lambda_1}(\omega) \geqslant \tilde{Y}_{\Lambda_2}(\omega) \tag{121}$$

which was to be shown.

We have proved (117) and that $(\tilde{Y}_\Lambda)_{\Lambda \in \mathcal{F} \setminus \{\emptyset\}}$ is decreasing. So we conclude (106), by using the Monotone Convergence Theorem.

Now we prove the second statement in Theorem 2.13, using the first one. First we note that for $\omega \in \Omega \setminus (\mathcal{N}_1 \cup \mathcal{N}_2)$ the map $S^t_{\Lambda,\xi,\omega}$ is continuous since it is the composite of finitely many continuous (cf. Lemma 2.8) updating maps.

For $\Lambda \subset \tilde{\Lambda}$ we have

$$\varrho_{\mathcal{C}^0(M,S^1)}\big(\pi_0 \circ S^t_{\Lambda,\xi,\omega}, \pi_0 \circ S^t_{\tilde{\Lambda},\xi,\omega}\big) \leqslant \tilde{Y}_\Lambda(\omega). \tag{122}$$

So by (106) the net $(\pi_0 \circ S^t_{\Lambda,\xi,\omega})_{\Lambda \in \mathcal{F} \setminus \{\emptyset\}}$ is a Cauchy net with values in $\mathcal{C}^0(M, S^1)$ for $\omega \in \Omega \setminus (\mathcal{N}_1 \cup \mathcal{N}_2)$ and so converges. Furthermore, it is a pointwise limit, i.e. for each particular ω, and so $\pi_0 \circ S^t_\omega$ is measurable in ω. (The last conclusion uses the theorem that the pointwise limit of measurable functions with values in a metric space is measurable. (cf. for example [22], p. 117)).

As mentioned in Remark 1.27.3 there is no distinction of the point 0 by the product topology. So for all $q \in \mathbb{Z}^d$ we can define $\pi_q \circ S^t_\omega$ for all $\omega \in \Omega \setminus \mathcal{N}^q$ where $P(\mathcal{N}^q) = 0$. In the same way we can define for each $\Lambda \in \mathcal{F} \setminus \{\emptyset\}$ and $\omega \in \Omega \setminus \mathcal{N}^\Lambda$ (with $P(\mathcal{N}^\Lambda) = 0$) maps $\pi_\Lambda \circ S^t_\omega \in \mathcal{C}^0(M, (S^1)^\Lambda)$ that depend measurably on ω, and such that $S^t_\omega(\mathbf{x})$ depends measurably on $(\omega, \mathbf{x})$.

The set

$$\mathcal{N} \stackrel{\text{def}}{=} \bigcup_{\Lambda \in \mathcal{F} \setminus \{\emptyset\}} \mathcal{N}^\Lambda \tag{123}$$

has P-measure zero. So by Lemma 1.28 the map S^t_ω is well-defined for $\omega \in \Omega \setminus \mathcal{N}$ and the statements in 3. hold. □

Remark 2.14. –

1. It follows from the proof of Theorem 2.13 that one can define a random dynamical system (cf. [1]), given by the map $[0, \infty) \times \Omega \times M \ni (t, \omega, x) \mapsto S^t_\omega(x)$ and the shifts $\theta(t)$ on Ω such that $X^q_{t_2}(\theta(t_1)\omega) \stackrel{\text{def}}{=} X^q_{t_1+t_2}(\omega) - X^q_{t_1}(\omega)$ where $X^q_t(\omega)$ denotes the number of jumps of ω at site q in the time interval $(0, t]$.
   One can further define for P-a.a. $\omega \in \Omega$ the linear operators '$\circ S^t_\omega$', acting on continuous functions and the corresponding transfer operators '$\mathcal{L}^t_\omega$'. So one has operator-valued random variables. However, in the following we consider only the averaged (w.r.t. ω) operators.
2. We have defined maps $S^t_\omega \in \mathcal{C}^0(M, M)$ for finite range updating in (74) and a special class of infinite range updatings in (109), using (107). Note that the second class does not include the first. It might be interesting to find more general classes

of (single site) updating functions for which the limit in (107) exists, or examples where it does not exist.

2.3. Markov kernels

In Section 1 we defined the Poisson process $(\Omega, \mathcal{A}, P, (X_t)_{t\in[0,T]})$ with parameter λ and values in $\mathbb{N}^\Lambda$, the measure space $(M, \mathcal{B}_M, \mu)$ and the measurable space $(\mathcal{C}^0(M,M), \mathcal{B}_{\mathcal{C}^0(M,M)})$.

We have nets $(S^T_\Lambda)_{\Lambda\in\mathcal{F}\setminus\{\emptyset\}}$ of maps $S^T_\Lambda : \Omega \setminus \mathcal{N} \to \mathcal{C}^0(M,M)$ with limit $S^T \in \mathcal{C}^0(M,M)$, and the following statements hold:

1. S^T_Λ and S^T are $(\mathcal{A}, \mathcal{B}_{\mathcal{C}^0(M,M)})$-measurable.
2. S^T is the pointwise limit of the net $(S^T_\Lambda)_{\Lambda\in\mathcal{F}\setminus\{\emptyset\}}$.
3. For fixed $\mathbf{x}\in M$ the map $S^T_{\Lambda,\bullet}(\mathbf{x}) : \Omega \to M$ is $(\mathcal{A}, \mathcal{B}_M)$-measurable.

More precisely, for finite range interaction (cf. Section 2.1) S^T was defined in (74) and S^T_Λ in (70). (Now we drop the fixed boundary condition ξ and the '$\tilde{}$' in the notation for convenience.) For infinite range interaction (cf. Section 2.2) we define S^T_Λ in the same way as for finite range interaction and the existence of the limit S^T is established in (109). Note that these maps are a priori not defined on a set of P-measure zero. For these exceptional $\omega \in \Omega$ we define S^T_ω and $S^T_{\Lambda,\omega}$ to be equal to the identity on M.

Statement 3. follows from measurability w.r.t. $(\omega, \mathbf{x})$ of $S^T_{\Lambda,\omega}(\mathbf{x})$ (Proposition 2.4 and Remark 1.29.2 for finite range interaction and statement 3. of Theorem 2.13 and Remark 1.29.2 for infinite range interaction), the fact that one-point-sets in M are measurable, and Fubini's Theorem.

Like in Example 1.18 we set

$$K^T_S : M \times \mathcal{B}_M \to [0,1], \qquad K^T_S(\mathbf{x}, A) \stackrel{\text{def}}{=} P\big(\{\omega:\ S^T_\omega(\mathbf{x}) \in A\}\big). \tag{124}$$

The corresponding operator, applied to an $f \in \mathcal{C}^0(M)$, is

$$\big(K^T_S f\big)(\mathbf{x}) = \int_M K^T_S(\mathbf{x}, \mathrm{d}\mathbf{y}) f(\mathbf{y}) \tag{125}$$

$$= \int_\Omega \mathrm{d}P(\omega) f \circ S^T_\omega(\mathbf{x}). \tag{126}$$

(125) is the definition (cf. (32)), and (126) is a consequence of (124).

We define analogously the Markov kernels $K^T_{S,\Lambda}$ and corresponding operators for the Poisson process with values in $\mathbb{N}^\Lambda$.

PROPOSITION 2.15. – *K^T_S and $K^T_{S,\Lambda}$ are bounded linear operators on $\mathcal{C}^0(M)$.*

Proof. – We give the proof for K^T_S. The one for $K^T_{S,\Lambda}$ is analogous. Let $\omega \in \Omega$, $f \in \mathcal{C}^0(M)$ and $(\mathbf{x}^{(n)})_{n\in\mathbb{N}}$ a sequence in M with limit $\mathbf{x}$. Then

$$\lim_{n\to\infty} S^T_\omega\big(\mathbf{x}^{(n)}\big) = S^T_\omega(\mathbf{x}) \tag{127}$$

$$\text{and so} \quad \lim_{n\to\infty} f \circ S^T_\omega\big(\mathbf{x}^{(n)}\big) = f \circ S^T_\omega(\mathbf{x}). \tag{128}$$

Further,

$$\left\| f \circ S_\omega^T \right\|_\infty \leqslant \| f \|_\infty. \tag{129}$$

Using the Dominated Convergence Theorem, we conclude

$$\begin{aligned}\lim_{n\to\infty} \left(K_S^T f\right)\left(\mathbf{x}^{(n)}\right) &= \lim_{n\to\infty} \int_\Omega \mathrm{d}P(\omega)\, f \circ S_\omega^T\left(\mathbf{x}^{(n)}\right) \\ &= \int_\Omega \mathrm{d}P(\omega)\, f \circ S_\omega^T(\mathbf{x}) \\ &= \left(K_S^T f\right)(\mathbf{x}). \end{aligned}\tag{130}$$

So $K_S^T f$ is continuous. Continuity of the operator follows from (126) and (129). □

PROPOSITION 2.16. – *The net* $(K_{S,\Lambda}^T)_{\Lambda\in\mathcal{F}\setminus\{\emptyset\}}$ *converges weakly to* K_S^T *(as* $\Lambda \to \mathbb{Z}^d$*), i.e. for all* $f \in \mathcal{C}^0(M)$*:*

$$\lim_{\Lambda\to\mathbb{Z}^d} K_{S,\Lambda}^T f = K_S^T f. \tag{131}$$

Proof. – We have

$$\left\| K_S^T f - K_{S,\Lambda}^T f \right\|_\infty \leqslant \int_\Omega \mathrm{d}P(\omega) \left\| f \circ S_\omega^T - f \circ S_{\Lambda,\omega}^T \right\|_\infty. \tag{132}$$

Because of condition 2 on p. 26 and (129) the rhs converges to 0 (as $\Lambda \to \mathbb{Z}^d$). □

Remark 2.17. – It follows from Remark 2.14.1 and the homogeneity of Poisson processes w.r.t. time that $(K_S^t)_{t\geqslant 0}$ is a semigroup.

3. Transfer operators

In this section we define transfer operators for the Markov kernels for a special class of updating functions that we have already studied in [13]. First we recall some definitions and notations from [13].

For $\delta > 0$ we denote by A_δ the annulus

$$A_\delta \stackrel{\text{def}}{=} \{z \in \mathbb{C} \mid -\delta \leqslant \ln|z| \leqslant \delta\} \tag{133}$$

and by Γ its positively oriented boundary.

For $\emptyset \neq \Lambda \subset \mathbb{Z}^d$ the normalized Lebesgue measure on $(S^1)^\Lambda$ is denoted by μ^Λ. For finite Λ it is given by

$$\mathrm{d}\mu^\Lambda(\mathbf{z}) = \frac{\mathrm{d}\mathbf{z}}{(2\pi\mathrm{i})^{|\Lambda|}} \frac{1}{\mathbf{z}} \stackrel{\text{def}}{=} \prod_{p\in\Lambda} \frac{\mathrm{d}z_p}{2\pi\mathrm{i}} \frac{1}{z_p}. \tag{134}$$

We also use $\mathrm{d}\mu^\Lambda(\mathbf{z})$ as a shorthand notation for the right-hand side of (134) for $\mathbf{z} \in A_\delta^\Lambda$.

In *Assumption I* (see below) we will fix a $\delta > 0$. For $\Lambda \in \mathcal{F}$ we denote by $\mathcal{H}_\Lambda$ the space of continuous functions on the polyannulus A_δ^Λ that are holomorphic on its interior and write $\|\cdot\|_\Lambda$ for the uniform norm on $\mathcal{H}_\Lambda$. As a function on A_δ^Λ is also a function on $A_\delta^{\mathbb{Z}^d}$ we can drop the index Λ and mean the uniform norm on the infinite-dimensional polyannulus. $\mathcal{H}$ is the vectorspace of all consistent families $\phi = (\phi_\Lambda)_{\Lambda\in\mathcal{F}}$ of functions $\phi_\Lambda \in \mathcal{H}_\Lambda$. Consistency means

$$(\pi_{\Lambda_1}\phi_{\Lambda_2})(\mathbf{z}_{\Lambda_1}) \stackrel{\text{def}}{=} \int\limits_{(S^1)^{\Lambda_2\setminus\Lambda_1}} \mathrm{d}\mu^{\Lambda_2\setminus\Lambda_1}(\mathbf{z}_{\Lambda_2\setminus\Lambda_1})\,\phi(\mathbf{z}_{\Lambda_1}\vee\mathbf{z}_{\Lambda_2\setminus\Lambda_1}) = \phi_{\Lambda_1} \tag{135}$$

for all $\Lambda_1 \subseteq \Lambda_2 \in \mathcal{F}$ and $\mathbf{z}_{\Lambda_1} \in A_\delta^{\Lambda_1}$. (Note that we use the same symbol 'π_Λ' for projections of functions and projections of coordinates, for example from M to $(S^1)^\Lambda$.)

For $0 < \vartheta < 1$ and $\phi \in \mathcal{H}$ we define

$$\|\phi\|_\vartheta = \sup_{\Lambda\in\mathcal{F}} \vartheta^{|\Lambda|}\|\phi_\Lambda\|_\Lambda, \tag{136}$$

$$\|\phi\|_{var} \stackrel{\text{def}}{=} \lim_{\Lambda\to\mathbb{Z}^d} \int\limits_{(S^1)^\Lambda} \mathrm{d}\mu^\Lambda\,|\phi_\Lambda|. \tag{137}$$

We set

$$\mathcal{H}_\vartheta \stackrel{\text{def}}{=} \{\phi\in\mathcal{H}:\ \|\phi\|_\vartheta < \infty\}, \tag{138}$$

$$\mathcal{H}^{bv} \stackrel{\text{def}}{=} \{\phi\in\mathcal{H}:\ \|\phi\|_{var} < \infty\}, \tag{139}$$

$$\mathcal{H}_\vartheta^{bv} \stackrel{\text{def}}{=} \mathcal{H}^{bv}\cap\mathcal{H}_\vartheta. \tag{140}$$

Then $(\mathcal{H}_\vartheta, \|\cdot\|_\vartheta)$ is a Banach space. For $\phi \in \mathcal{H}^{bv}$ and $\psi \in \mathcal{C}^0(M)$ we define

$$\psi_\Lambda(\mathbf{z}_\Lambda) \stackrel{\text{def}}{=} \int\limits_{(S^1)^{\Lambda^C}} \mathrm{d}\mu^{\Lambda^C}(\mathbf{z}_{\Lambda^C})\,\psi(\mathbf{z}_\Lambda\vee\mathbf{z}_{\Lambda^C}), \tag{141}$$

$$\int\limits_M \mathrm{d}\mu\,\psi\phi \stackrel{\text{def}}{=} \lim_{\Lambda\to\mathbb{Z}^d} \int\limits_{(S^1)^\Lambda} \mathrm{d}\mu_\Lambda\,\psi_\Lambda\phi_\Lambda. \tag{142}$$

Finally we recall the definition of a transfer operator: Let $\tilde{\mu}$ be a measure on the (completed) Borel σ-algebra of a metric space $\widetilde{M}$ and $\widetilde{S}:\widetilde{M}\to\widetilde{M}$ be a measurable map that is non-singular w.r.t. μ, i.e. for all measurable $A\in\widetilde{M}$, $\mu(A)=0$ implies $\mu(\widetilde{S}^{-1}(A))=0$. The *Perron–Frobenius operator* (or *transfer operator*) $\mathcal{L}_{\widetilde{S}}$, acting on $L^1(\widetilde{M})$, is defined via the equation

$$\int\limits_{\widetilde{M}} \mathrm{d}\tilde{\mu}\,\psi\circ\widetilde{S}\phi = \int\limits_{\widetilde{M}} \mathrm{d}\tilde{\mu}\,\psi\mathcal{L}_{\widetilde{S}}\phi \tag{143}$$

that must hold for all $\psi \in L^\infty(\widetilde{M})$ and $\phi \in L^1(\widetilde{M})$.

The Markov kernels for our stochastic systems are analogous to the composition operator '$\circ S$' (with deterministic S), acting on functions.

DEFINITION 3.1. – *We define transfer operator for a Markov kernel K analogously to* (143) *by the equation*

$$\int_{\widetilde{M}} \mathrm{d}\tilde{\mu}\,(K\psi)\phi = \int_{\widetilde{M}} \mathrm{d}\tilde{\mu}\,\psi(\mathcal{L}_K\phi). \tag{144}$$

Remark 3.2. –

1. In the cases we consider, the Markov kernel K_S is given by

$$(K_S\psi)(x) = \int_{\Omega} \mathrm{d}P(\omega)\,\psi \circ S_\omega(x), \tag{145}$$

where Ω is a probability space, S_ω depends measurably on ω and the map $\omega \mapsto \mathcal{L}_{S_\omega}$ is well-defined and integrable. Then

$$\begin{aligned}\int_{\widetilde{M}} \mathrm{d}\tilde{\mu}\,(K\psi)\phi &= \int_{\widetilde{M}} \mathrm{d}\tilde{\mu}(x) \int_{\Omega} \mathrm{d}P(\omega)\,\psi \circ S_\omega(x)\phi(x)\\ &= \int_{\widetilde{M}} \mathrm{d}\tilde{\mu}(x) \int_{\Omega} \mathrm{d}P(\omega)\,\psi(x)(\mathcal{L}_{S_\omega}\phi)(x)\\ &= \int_{\widetilde{M}} \mathrm{d}\tilde{\mu}(x)\,\psi(x) \int_{\Omega} \mathrm{d}P(\omega)\,(\mathcal{L}_{S_\omega}\phi)(x).\end{aligned} \tag{146}$$

So $\mathcal{L}_{K_S}$ is given by

$$(\mathcal{L}_{K_S}\phi)(x) = \int_{\Omega} \mathrm{d}P(\omega)\,(\mathcal{L}_{S_\omega}\phi)(x). \tag{147}$$

2. The operator for the infinite dimensional system that we are going to consider act on elements of $\mathcal{H}_\vartheta$ that do not in general correspond to elements of $L^1(M)$. Recall (see [13]) that $\mathcal{H}_\vartheta^{bv}$ can be identified with a subset of $rca(M)$ (or, in other words, a subset of the Borel measures). So for example in Theorem 3.25 we will show that the equation analogous to (144) holds for $\psi \in \mathcal{C}^0(M)$ (rather than $L^\infty(M)$) and $\phi \in \mathcal{H}_\vartheta^{bv}$.

Now we consider a special class of interactions (cf. [13]), namely a family $(S_\Lambda)_{\Lambda\in\mathcal{F}}$ of maps on M that can be written as

$$\begin{aligned} S_\Lambda &: M \to M,\\ S_\Lambda(\mathbf{z}) &= F_\Lambda \circ T_\Lambda(\mathbf{z}) \vee \mathbf{z}_{\Lambda^C}, \end{aligned} \tag{148}$$

where

$$\begin{aligned} F_\Lambda &: \left(S^1\right)^\Lambda \to \left(S^1\right)^\Lambda,\\ \mathbf{z}_\Lambda &= (z_q)_{q\in\Lambda} \mapsto \left(f_q(z_q)\right)_{q\in\Lambda}, \end{aligned} \tag{149}$$

and

$$T_\Lambda : M \to \left(S^1\right)^\Lambda, \tag{150}$$

$$\left(T_\Lambda(\mathbf{z})\right)_q \stackrel{\text{def}}{=} z_q \exp\left(2\pi \mathrm{i}\varepsilon \sum_{k=1}^{\infty} g_{q,k}(\mathbf{z})\right) \quad \text{for } q \in \Lambda \tag{151}$$

and f_q and $g_{q,k}$ satisfy the following assumptions:

Assumption I. – $F(\mathbf{z}) = (f_q(z_q))_{q\in\mathbb{Z}^d}$ where $f_q : S^1 \to S^1$ are real analytic and expanding (i.e. $f_q' \geqslant \lambda_0 > 1$) maps that extend for some δ_1 holomorphically to the interior of an annulus A_{δ_1}. In Proposition 3.1 and 3.2 of [13] we have shown that the holomorphic extension to a sufficiently thin annulus A_δ is expanding in the sense that the preimage of A_δ w.r.t. f_q lies in the interior of A_δ. We fix such a δ_1. Then for every $q \in \mathbb{Z}^d$ the Perron–Frobenius operator $\mathcal{L}_{f_q}$, acting on $\mathcal{H}_{\{q\}}$, has a simple largest eigenvalue 1 with eigenvector h_q, such that $\pi_\emptyset(h_q) = 1$ and the restriction of h_q to S^1 is positive and it splits into

$$\mathcal{L}_{f_q} = \mathcal{Q}_q + \mathcal{R}_q, \tag{152}$$

where $\mathcal{Q}_q$ is a projection onto $\operatorname{span}(h_q)$. We assume that there are positive constants $\eta < 1$, c_h and c_r such that the following two estimates hold for all $q \in \mathbb{Z}^d$:

$$\|\mathcal{Q}_q\|_{\{q\}} \leqslant c_h, \tag{153}$$

$$\|\mathcal{R}_q^n\|_{\{q\}} \leqslant c_r \eta^n, \tag{154}$$

where $\|\cdot\|_{\{q\}}$ denotes both the uniform norm on $\mathcal{H}_{\{q\}}$ (for this we might have to take δ_1 even smaller) and the induced operator-norm. We note that this holds in particular if f_q does not depend on q.

We further have

$$\mathcal{Q}_q \mathcal{R}_q = \mathcal{R}_q \mathcal{Q}_q = 0. \tag{155}$$

Assumption II. – For all $q \in \mathbb{Z}^d$ and $k \geqslant 1$ each map $g_{q,k}$ extends to a holomorphic map $g_{q,k} : A_{\delta_1}^{B_k(q)} \to \mathbb{C}$ (recall definition (23) of $B_k(q)$) and its sup-norm (of modulus) is exponentially bounded by

$$\|g_{q,k}\|_{A_{\delta_1}^{B_k(q)}} \leqslant c_3 \exp\left(-c_g k^d\right) \tag{156}$$

with $c_3 > 0$ and 'large' $c_g > 0$. (In several statements in Section 3 and 4 a lower bound for c_g will come out of our computations. The idea is always that our estimates work, provided c_g is bigger than a certain constant.)

For $\Lambda \in \mathcal{F} \setminus \{\emptyset\}$ we denote by h_Λ the function

$$h_\Lambda(\mathbf{z}_\Lambda) \stackrel{\text{def}}{=} \prod_{q\in\Lambda} h_q(z_q), \tag{157}$$

where h_q is as in Assumption I. We set $h_\emptyset = 1$ and

$$h_{\mathbb{Z}^d} \stackrel{\text{def}}{=} (h_\Lambda)_{\Lambda\in\mathcal{F}} \in \mathcal{H}. \tag{158}$$

We further define for a fixed $\xi \in M$ and $\Lambda \in \mathcal{F} \setminus \{\emptyset\}$ and $\Lambda_1 \subseteq \Lambda$ the *updating at the* Λ_1*-sites with fixed boundary conditions* ξ_{Λ^C} *outside* Λ (or *cut-off* of S_{Λ_1}):

$$S_{\Lambda_1,\Lambda} : (S^1)^\Lambda \to (S^1)^\Lambda, \qquad \mathbf{z}_\Lambda \mapsto \pi_\Lambda \circ S_{\Lambda_1}(\mathbf{z}_\Lambda \vee \xi_{\Lambda^C}). \tag{159}$$

And for $\mathbf{z}_{\Lambda\setminus\Lambda_1} \in (S^1)^{\Lambda\setminus\Lambda_1}$ we define

$$\pi_{\Lambda_1} \circ S_{\Lambda_1,\Lambda}(\cdot \vee \mathbf{z}_{\Lambda\setminus\Lambda_1}) : (S^1)^{\Lambda_1} \to (S^1)^{\Lambda_1}, \qquad \mathbf{z}_{\Lambda_1} \mapsto \pi_{\Lambda_1} \circ S_{\Lambda_1,\Lambda}(\mathbf{z}_{\Lambda_1} \vee \mathbf{z}_{\Lambda\setminus\Lambda_1}). \tag{160}$$

Remark 3.3. –

1. The map defined in (160) is the cut-off of S w.r.t. Λ_1 and boundary conditions $\mathbf{z}_{\Lambda\setminus\Lambda_1} \vee \xi_\Lambda$. So we can use the special representation in terms of integral kernels for its transfer operator, restricted to $\mathcal{H}_{\Lambda_1}$, for the proposition below.
2. The family $(S_q)_{q\in\mathbb{Z}^d}$, defined by (148), satisfies conditions (75) and (76) as one can see from [13]: The partial derivatives are estimated in the proof of Proposition 3.1 there.

LEMMA 3.4. – *Let* $\Lambda \in \mathcal{F} \setminus \{\emptyset\}$ *be the disjoint union of* Λ_1 *and* Λ_2*. The transfer operator, restricted to* $\mathcal{H}_{\Lambda_1}$*, of the map* $S_{\Lambda_1,\Lambda} : (S^1)^\Lambda \to (S^1)^\Lambda$*, defined in* (159) *has the following representation in terms of integral kernels*:

$$(\mathcal{L}_{S_{\Lambda_1,\Lambda}}\phi)(\mathbf{w}_{\Lambda_1} \vee \mathbf{w}_{\Lambda_2}) = \int_{\Gamma^{\Lambda_1}} \mathrm{d}\mu^{\Lambda_1}(\mathbf{z}_{\Lambda_1})\, \phi(\mathbf{z}_{\Lambda_1} \vee \mathbf{w}_{\Lambda_2}) \prod_{q\in\Lambda_1} \frac{(S_{\Lambda_1,\Lambda}(\mathbf{z}_{\Lambda_1} \vee \mathbf{w}_{\Lambda_2}))_q}{(S_{\Lambda_1,\Lambda}(\mathbf{z}_{\Lambda_1} \vee \mathbf{w}_{\Lambda_2}))_q - w_q} \tag{161}$$

for $\phi \in \mathcal{H}_\Lambda$.

Proof. – Let $\psi \in \mathcal{C}^0((S^1)^\Lambda)$. We use the notation $\phi_{\mathbf{w}_{\Lambda_2}}$ for the function $\mathbf{w}_{\Lambda_1} \mapsto \phi(\mathbf{w}_{\Lambda_1} \vee \mathbf{w}_{\Lambda_2})$.

$$\begin{aligned}
&\int_{(S^1)^\Lambda} \mathrm{d}\mu^\Lambda(\mathbf{w}_\Lambda)\, \psi \circ S_{\Lambda_1,\Lambda}(\mathbf{w}_\Lambda)\phi(\mathbf{w}_\Lambda) \\
&\quad= \int_{(S^1)^{\Lambda_2}} \mathrm{d}\mu^{\Lambda_2}(\mathbf{w}_{\Lambda_2}) \int_{(S^1)^{\Lambda_1}} \mathrm{d}\mu^{\Lambda_1}(\mathbf{w}_{\Lambda_1})\, \psi_{\mathbf{w}_{\Lambda_2}} \circ \pi_{\Lambda_1} \circ S_{\Lambda_1,\Lambda}(\mathbf{w}_{\Lambda_1} \vee \mathbf{w}_{\Lambda_2})\phi_{\mathbf{w}_{\Lambda_2}}(\mathbf{w}_{\Lambda_1}) \\
&\quad= \int_{(S^1)^{\Lambda_2}} \mathrm{d}\mu^{\Lambda_2}(\mathbf{w}_{\Lambda_2}) \int_{(S^1)^{\Lambda_1}} \mathrm{d}\mu^{\Lambda_1}(\mathbf{w}_{\Lambda_1})\, \psi_{\mathbf{w}_{\Lambda_2}}(\mathbf{w}_{\Lambda_1})(\mathcal{L}_{\pi_{\Lambda_1}\circ S_{\Lambda_1,\Lambda}(\cdot\vee\mathbf{w}_{\Lambda_2})}\phi_{\mathbf{w}_{\Lambda_2}})(\mathbf{w}_{\Lambda_1}) \\
&\quad= \int_{(S^1)^\Lambda} \mathrm{d}\mu^\Lambda(\mathbf{w}_\Lambda)\, \psi(\mathbf{w}_\Lambda)(\mathcal{L}_{\pi_{\Lambda_1}\circ S_{\Lambda_1,\Lambda}(\cdot\vee\mathbf{w}_{\Lambda_2})}\phi_{\mathbf{w}_{\Lambda_2}})(\mathbf{w}_{\Lambda_1}).
\end{aligned} \tag{162}$$

Using the representation of the transfer operator for $\pi_{\Lambda_1} \circ S_{\Lambda_1,\Lambda}(\cdot \vee \mathbf{w}_{\Lambda_2})$ that we established in Proposition 3.3 of [13], we obtain the rhs of (162). □

Remark 3.5. –

1. We see in particular that $\mathcal{L}_{S_{\Lambda_1,\Lambda}}$ 'acts on the Λ_1-coordinates' only. There is no integration w.r.t. the Λ_2-coordinates.
   For $q \in \Lambda_1$ we can split the factor
   $$\frac{(S_{\Lambda_1,\Lambda}(\mathbf{z}_{\Lambda_1} \vee \mathbf{w}_{\Lambda_2}))_q}{(S_{\Lambda_1,\Lambda}(\mathbf{z}_{\Lambda_1} \vee \mathbf{w}_{\Lambda_2}))_q - w_q}$$
   $$= h_q(w_q, z_q) + r_q(w_q, z_q) + \sum_{k=1}^{\infty} \beta_{q,k}(w_q, \mathbf{z}_{\Lambda_1} \vee \mathbf{w}_{\Lambda_2} \vee \xi_{\Lambda^c}) \tag{163}$$
   as in [13]. The integral kernels h_q and r_q correspond to the operators $\mathcal{Q}_q$ and $\mathcal{R}_q$, introduced in (152) and $\beta_{q,k}$ to $\mathcal{B}_{q,k}$, say. In addition to (155) we have
   $$\mathcal{Q}_q \circ \mathcal{B}_{q,k} = 0 \tag{164}$$
   for all k.
   For a detailed analysis on composites of operators $\mathcal{Q}_q$, $\mathcal{R}_q$, $\mathcal{B}_{q,k}$ that have value 0 we refer to Section 5 in [13].
2. We have established a representation of the transfer operator also for updatings at more than one point at one time. Such simultaneous updatings happen, for example, in certain discrete time processes with positive probability (cf. [12]).
   As in the systems considered here simultaneous updatings P-almost never happen, we can restrict ourself in the following to the case of updatings at single sites, i.e. $\Lambda_1 = \{q\}$.

DEFINITION 3.6. – *We define for fixed $\Lambda \in \mathcal{F} \setminus \{\emptyset\}$, $\xi \in M$ and a finite sequence $j = (q_1, \ldots, q_n) \in J$ of points in Λ the map*
$$S_{j,\Lambda} : (S^1)^{\Lambda} \to (S^1)^{\Lambda}, \tag{165}$$
$$S_{j,\Lambda} \stackrel{\text{def}}{=} S_{q_n,\Lambda} \circ \cdots \circ S_{q_1,\Lambda}.$$
Here $S_{q,\Lambda}$ is the map for the updating at site q. Recall that in Definition 1.22 *we defined the maps* $\mathbf{j}$. *For all $\omega \in \Omega$ there is a finite sequence $\mathbf{j}(\omega) = (q_1, \ldots, q_n)$ and so*
$$\mathcal{L}_{S_{\mathbf{j}(\omega),\Lambda}} \stackrel{\text{def}}{=} \mathcal{L}_{S_{q_n,\Lambda}} \circ \cdots \circ \mathcal{L}_{S_{q_1,\Lambda}} \tag{166}$$
is well-defined.

Before establishing particular representations of the transfer operators $\mathcal{L}_{\Lambda}^{T}$ and $\pi_{\Lambda} \circ \mathcal{L}^{T}$ we consider some special examples. For these we need the following definition.

DEFINITION 3.7. – *We define*:
$$\mathcal{R}(t) \stackrel{\text{def}}{=} \mathrm{e}^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \mathcal{R}^k \tag{167}$$
$$= \exp\bigl(-\lambda t(\mathrm{id} - \mathcal{R})\bigr). \tag{168}$$
Then we have, using (154),

$$\mathcal{R}(t_1)\mathcal{R}(t_2) = \mathcal{R}(t_1 + t_2), \tag{169}$$

$$\|\mathcal{R}(t)\| \leqslant c_r \mathrm{e}^{-(1-\eta)\lambda t}. \tag{170}$$

Example 3.8. – Consider a single site system, say at site q of a lattice, with an updating map $f : S^1 \mapsto S^1$ that satisfies *Assumption* I. We have for fixed time $T > 0$ and jump rate $\lambda > 0$ a Markov kernel K_f^T, acting on functions $\psi \in \mathcal{C}^0(S^1)$ as in (38). Using that $\mathcal{L}_f^n$ ist the transfer operator of f^n (this is a special case of (166)), we get a transfer operator $\mathcal{L}_f^T$, acting on $\mathcal{H}_{\{q\}}$ (this space is defined on p. 28):

$$\begin{aligned} \mathcal{L}_f^T &= \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda T} \frac{(\lambda T)^n}{n!} \mathcal{L}_f^n \\ &= \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda T} \frac{(\lambda T)^n}{n!} (\mathcal{Q} + \mathcal{R})^n &(171) \\ &= \left(1 - \mathrm{e}^{-\lambda T}\right)\mathcal{Q} + \mathcal{R}(T) &(172) \end{aligned}$$

with $\mathcal{R}(t)$ as in Definition 3.7. Note that we think of the summand $\mathrm{e}^{-\lambda t}\mathrm{id}$, corresponding to $n = 0$, as $\mathrm{e}^{-\lambda t} R^0$. By (155) we have for $n \geqslant 1$ that $(\mathcal{Q} + \mathcal{R})^n = \mathcal{Q} + \mathcal{R}^n$, and so we get (172).

We represent the two summands in (172) diagrammatically in Fig. 2. The operator $(1 - \mathrm{e}^{-\lambda T})\mathcal{Q}$ is represented by a thin vertical line (h-strip) and $\mathcal{R}(T)$ as a thick vertical line (r-strip). Note that the operator $\mathcal{R}(T)$ is a sum of operators, each corresponding to an exponent $0 \leqslant n < \infty$. So the r-strip corresponds to that sum of operators rather than to a particular product $\mathrm{e}^{-\lambda T} \frac{(\lambda T)^n}{n!} \mathcal{R}^n$. An analogous statement holds for the h-strip.

Example 3.9. – Now we consider a small perturbation f_ε of the single site system $f_0 = f$ of Example 3.8, that depends on fixed boundary conditions. For simplicity we split the transfer operator for the single updating into $\mathcal{L}_{f_\varepsilon} = \mathcal{Q} + \mathcal{R} + \mathcal{B}$ where $\mathcal{B}$ is the difference between the operators for the perturbed and the unperturbed system. We note that $\mathcal{B}$ corresponds to the sum $\sum_{k=1}^{\infty} \mathcal{B}_{q,k}$ of operators defined in Remark 3.5.1. It follows from (164) that

$$\mathcal{Q} \circ \mathcal{B} = 0. \tag{173}$$

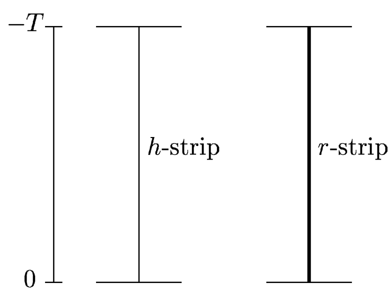

Fig. 2. Single site, unperturbed case: There are only two gum configurations.

For a given number N of updatings we distribute

$$\mathcal{L}_{f_\varepsilon}^N = (\mathcal{Q}+\mathcal{R}+\mathcal{B})^N, \tag{174}$$

using (155) and (173). Let K denote the number of factors $\mathcal{B}$, n_0 the number of factors, either all $\mathcal{Q}$ or all $\mathcal{R}$, before the first factor $\mathcal{B}$ and n_i the number of factors $\mathcal{R}$ after the ith $\mathcal{B}$. So the total number of factors is $N = n_0 + \cdots + n_K + K$.

We get

$$\mathcal{L}_{f_\varepsilon}^T = \sum_{N=0}^{\infty} \mathrm{e}^{-\lambda T} \frac{(\lambda T)^N}{N!} \mathcal{L}_{f_\varepsilon}^N \tag{175}$$

$$= \sum_{N=0}^{\infty} \mathrm{e}^{-\lambda T} \sum_{\substack{n_0,\dots,n_K \geqslant 0 \\ n_0+\cdots+n_K+K=N}} \frac{(\lambda T)^N}{N!} \mathcal{R}^{n_K} \circ \mathcal{B} \circ \cdots \circ \mathcal{B} \circ \mathcal{R}^{n_1} \circ \mathcal{B} \circ (\mathcal{Q}+\mathcal{R})^{n_0} \tag{176}$$

$$= \sum_{K=0}^{\infty} \sum_{\substack{n_0 \geqslant 1 \\ n_1,\dots,n_K \geqslant 0}} \mathrm{e}^{-\lambda T} \frac{(\lambda T)^{n_0+\cdots+n_K+K}}{(n_0+\cdots+n_K+K)!} \mathcal{R}^{n_K} \circ \mathcal{B} \circ \cdots \circ \mathcal{B} \circ \mathcal{R}^{n_1} \circ \mathcal{B} \circ \mathcal{Q} \tag{177}$$

$$+ \sum_{K=0}^{\infty} \sum_{n_0,\dots,n_K \geqslant 0} \mathrm{e}^{-\lambda T} \frac{(\lambda T)^{n_0+\cdots+n_K+K}}{(n_0+\cdots+n_K+K)!} \mathcal{R}^{n_K} \circ \mathcal{B} \circ \cdots \circ \mathcal{B} \circ \mathcal{R}^{n_1} \circ \mathcal{B} \circ \mathcal{R}^{n_0}$$

$$= \sum_{K=0}^{\infty} \int_0^T \lambda\,\mathrm{d}t_1 \int_{t_1}^T \lambda\,\mathrm{d}t_2 \cdots \int_{t_K}^T \lambda\,\mathrm{d}t_K \big(1-\mathrm{e}^{-\lambda t_1}\big) \mathcal{R}(t_K) \circ \mathcal{B} \circ \cdots \circ \mathcal{B} \circ \mathcal{R}(t_1) \circ \mathcal{B} \circ \mathcal{Q}$$

$$+ \sum_{K=0}^{\infty} \int_0^T \lambda\,\mathrm{d}t_1 \int_{t_1}^T \lambda\,\mathrm{d}t_2 \cdots \int_{t_K}^T \lambda\,\mathrm{d}t_K \mathcal{R}(t_n) \circ \mathcal{B} \circ \cdots \circ \mathcal{R}(t_0). \tag{178}$$

For the step from (177) to (178) we have used Lemma 3.10 (s. below). We interpret (178) in the following way. We write the operator $\mathcal{L}_{f_\varepsilon}^T$ as a sum of operators $\mathcal{L}_{\mathcal{C}_g,T}$. For the time being we think of $\mathcal{C}_g$ as a diagram like, for example, in Fig. 3. The vertical axis from top to bottom corresponds to the positively oriented time line. Along this axis we draw K thick horizontal bars, denoted by B, that correspond to the operators $\mathcal{B}$. Between consecutive B's or between the first B and the top or the last B and the bottom or, in the case $K=0$, between the top and the bottom we draw either a thick (r-strip) or a thin (h-strip) line, representing the choice of factors $\mathcal{R}$ or $\mathcal{Q}$ in the product (174), respectively. Note that below a B there must be a thick vertical line. For a fixed T and a K-tuple

$$\mathbf{t} = (t_1,\dots,t_K) \in \{\mathbf{t}\colon\ -T < t_1 < \cdots < t_K < 0\} \tag{179}$$

we think of the top as being fixed at time $-T$, the bottom at 0 and the ith symbol B at t_i. That also fixes the lengths of the particular h- and r-strips. We assign to the triple $(\mathcal{C}_g, T, t)$ an operator $\mathcal{L}_{\mathcal{C}_g,T,\mathbf{t}}$. For example with $\mathcal{C}_g$, T and $\mathbf{t}$ as in Fig. 3, we get

$$\mathcal{L}_{\mathcal{C}_g,T,\mathbf{t}} = \big(1-\mathrm{e}^{-\lambda(t_1+T)}\big) \mathcal{R}(0-t_K) \circ \mathcal{B} \circ \cdots \circ \mathcal{B} \circ \mathcal{R}(t_2-t_1) \circ \mathcal{B} \circ \mathcal{Q}. \tag{180}$$

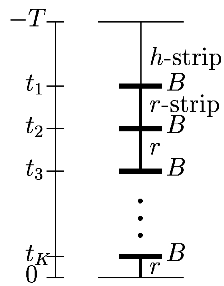

Fig. 3. Single site, perturbed case: example for a gum configuration.

To get $\mathcal{L}_{\mathcal{C}_g}^T$ we integrate (180) over the simplex given by (179) w.r.t. the scaled Lebesgue measure $\lambda^K \,\mathrm{d}t_1 \cdots \mathrm{d}t_K$. The simplex has dimension K, so for each B in $\mathcal{C}_g$ we get one integration. Heuristically, the measure '$\lambda\,\mathrm{d}t_1$' corresponds to the probability that a Poisson process with rate λ jumps in a small time interval. The approach of approximating the continuous time system by discrete time systems is made precise in [12]. We also note the special case $K = 0$ where the simplex degenerates to a single point of measure 1.

Above we have used the following lemma.

LEMMA 3.10. – *Let* $(n_{i,j})_{\substack{0\leqslant i\leqslant K\\ 1\leqslant j\leqslant N}}$ *be a family of non-negative integers and* $\lambda, T > 0$. *Then, with the notation* $t_0 \stackrel{\text{def}}{=} -T$, $t_{K+1} \stackrel{\text{def}}{=} 0$:

$$\mathrm{e}^{-N\lambda T} \frac{(\lambda T)^{n_{0,1}+\cdots+n_{K,N}+K}}{(n_{0,1}+\cdots+n_{K,N}+K)!} \binom{n_{0,1}+\cdots+n_{0,N}}{n_{0,1},\ldots,n_{0,N}} \cdot \cdots \cdot \binom{n_{K,1}+\cdots+n_{K,N}}{n_{K,1},\ldots,n_{K,N}}$$
$$= \int_{-T}^{0} \lambda\,\mathrm{d}t_1 \int_{t_1}^{0} \lambda\,\mathrm{d}t_2 \cdots \int_{t_{K-1}}^{0} \lambda\,\mathrm{d}t_K \prod_{\substack{0\leqslant i\leqslant K\\ 1\leqslant j\leqslant N}} \left(\mathrm{e}^{-\lambda(t_{i+1}-t_i)} \frac{(\lambda(t_{i+1}-t_i))^{n_{i,j}}}{n_{i,j}!} \right). \tag{181}$$

Proof. – We see that the rhs of (181) is equal to

$$\mathrm{e}^{-N\lambda T} \lambda^{n_{0,1}+\cdots+n_{K,N}+K} \prod_{\substack{0\leqslant i\leqslant K\\ 1\leqslant j\leqslant N}} \frac{1}{n_{i,j}!} \int_{-T}^{0} \mathrm{d}t_1 \cdots \int_{-t_{K-1}}^{0} \mathrm{d}t_K \prod_{0\leqslant i\leqslant K} (t_{i+1}-t_i)^{n_{i,1}+\cdots+n_{i,N}}, \tag{182}$$

and so (181) follows by K times applying the identity

$$\int_{\tau}^{0} \mathrm{d}t\,(-t)^n (t-\tau)^m = \frac{n!m!}{(n+m+1)!} (-\tau)^{n+m+1} \tag{183}$$

which can be easily shown by iterated integration by parts. □

Example 3.11. – Now we consider a system with nearest neighbour coupling where only two adjacent sites (1 and 2) are updated and the states at the other sites are fixed.

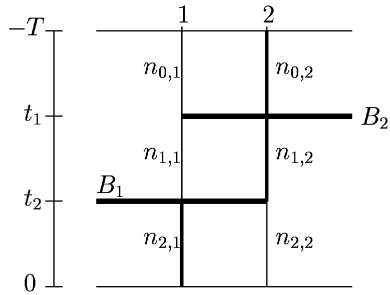

Fig. 4. Two sites, example for a gum configurations.

This time an expansion as in Example 3.9 gives rise to summing operators $\mathcal{L}_{\mathcal{C}_g,T}$ where the $\mathcal{C}_g$ correspond to diagrams like in Fig. 4.

We have to consider all finite ordered sequences of symbols B_1 and B_2.

In the example of Fig. 4 we have the sequence (B_2, B_1). For each B_i we draw a thick horizontal bar, centered in the column corresponding to the ith site and connecting to all sites (columns) on which the operator $\mathcal{B}_i$ depends. We draw the sequence of B_i's 'downwards'. As we consider only nearest neighbour updatings here, the B_i have width 2.

Then we can choose at sites 1 and 2 between h- and r-strips. Note that in Fig. 4 at site 2 an h-strip follows an r-strip. This is possible because they are 'separated' by B_1 and so the corresponding operator is not necessarily 0. (In Section 5.3 of [13] we list combinations of operators that lead to value 0.) Now we consider the particular $\mathcal{C}_g$ and T shown in Fig. 4. Let $n_{0,1}$ denote the number of jumps at site 1 before the jumps that corresponds to $\mathrm{id}_1 \otimes \mathcal{B}_2$ etc. The definition of $n_{i,j}$ in general is analogous. So the total number of jumps is $N = n_{0,1} + \cdots + n_{2,2} + 2$. The first two factors in (184) give the probability that a particular sequence of exactly N jumps occurs. The first binomial coefficient counts the combinations (different sequences) of $n_{0,1}$ jumps at site 1 and $n_{0,2}$ jumps at site 2, the others are explained analogously.

We get the operator:

$$\begin{aligned}
\mathcal{L}_{\mathcal{C}_g,T} = {} & \sum_{\substack{n_{0,1},n_{1,1},n_{2,2}\geqslant 1\\ n_{0,2},n_{1,2},n_{2,1}\geqslant 0}} \mathrm{e}^{-2\lambda T} \frac{(\lambda T)^{n_{0,1}+\cdots+n_{2,2}+2}}{(n_{0,1}+\cdots+n_{2,2}+2)!} \binom{n_{0,1}+n_{0,2}}{n_{0,1}, n_{0,2}} \binom{n_{1,1}+n_{1,2}}{n_{1,1}, n_{1,2}} \\
& \binom{n_{2,1}+n_{2,2}}{n_{2,1}, n_{2,2}} \mathcal{R}_1^{n_{2,1}} \otimes \mathcal{Q}_2 \circ \mathcal{B}_1 \otimes \mathrm{id}_2 \circ \mathcal{Q}_1 \otimes \mathcal{R}_2^{n_{1,2}} \circ \mathrm{id}_1 \otimes \mathcal{B}_2 \circ \mathcal{Q}_1 \otimes \mathcal{R}_2^{n_{0,2}} \\
= {} & \int_{-T}^{0} \lambda\,\mathrm{d}t_1 \int_{t_1}^{0} \lambda\,\mathrm{d}t_2 \left(\sum_{n_{2,1}\geqslant 0} \mathrm{e}^{-\lambda|t_2|} \frac{(\lambda|t_2|)^{n_{2,1}}}{n_{2,1}!} \mathcal{R}_1^{n_{2,1}} \right) \\
& \otimes \left(\sum_{n_{2,2}\geqslant 1} \mathrm{e}^{-\lambda|t_2|} \frac{(\lambda|t_2|)^{n_{2,2}}}{n_{2,2}!} \mathcal{Q}_2 \right) \circ \mathcal{B}_1 \otimes \mathrm{id}_2 \\
& \circ \left(\sum_{n_{1,1}\geqslant 1} \mathrm{e}^{-\lambda(t_2-t_1)} \frac{(\lambda(t_2-t_1))^{n_{1,1}}}{n_{1,1}!} \mathcal{Q}_1 \right)
\end{aligned}$$

$$\otimes \left(\sum_{n_{1,2} \geqslant 0} \mathrm{e}^{-\lambda(t_2-t_1)} \frac{(\lambda(t_2-t_1))^{n_{1,2}}}{n_{1,2}!} \mathcal{R}_2^{n_{1,2}} \right) \circ \mathrm{id}_1 \otimes \mathcal{B}_2$$

$$\circ \left(\sum_{n_{0,1} \geqslant 1} \mathrm{e}^{-\lambda(t_1+T)} \frac{(\lambda(t_1+T))^{n_{0,1}}}{n_{0,1}!} \mathcal{Q}_1 \right)$$

$$\otimes \left(\sum_{n_{0,2} \geqslant 0} \mathrm{e}^{-\lambda(t_1+T)} \frac{(\lambda(t_1+T)^{n_{0,2}}}{n_{0,2}!} \mathcal{R}_2^{n_{0,2}} \right)$$

$$= \int_{-T}^{0} \lambda \, \mathrm{d}t_1 \int_{t_1}^{0} \lambda \, \mathrm{d}t_2 \big(1-\mathrm{e}^{-\lambda|t_2|}\big)\big(1-\mathrm{e}^{-\lambda(t_2-t_1)}\big)\big(1-\mathrm{e}^{-\lambda(t_1+T)}\big) \mathcal{R}_1(|t_2|) \otimes \mathcal{Q}_1$$

$$\circ \mathcal{B}_1 \otimes \mathrm{id}_2 \circ \mathcal{Q}_1 \otimes \mathcal{R}_2(t_2-t_1) \circ \mathrm{id}_1 \otimes \mathcal{B}_2 \circ \mathcal{Q}_1 \otimes \mathcal{R}_2(t_1+T). \tag{184}$$

Note that the operator '$\otimes$' here has higher precedence than '$\circ$', so, for example, $A_1 \otimes A_2 \circ A_3 \otimes A_4$ is understood as $(A_1 \otimes A_2) \circ (A_3 \otimes A_4)$.

Remark 3.12. – In these introductory examples we have seen that our transfer operators can be represented as a sum of particular transfer operators $\mathcal{L}_{\mathcal{C}_g,T}$ each of whose corresponds to a certain diagram $\mathcal{C}_g$ and the time T. We will call such a $\mathcal{C}_g$ a *gum configuration* (see Definition 3.16). The B_i correspond to particular sets of sites in the lattice (The corresponding integral operator $\mathcal{B}_i$ takes these sites into account) and the sequence $(\ldots, B_2, B_1)$ reflects a fixed temporal order. Heuristically, we think of the vertical (corresponding to the time coordinate) distances between the B_i as as being not yet fixed. The (vertical) h- and r-strips of the gum configuration are flexible.

Further, each $\mathcal{L}_{\mathcal{C}_g,T}$ can be written as an operator-valued integral where the variable $\mathbf{t}$ of integration is interpreted as time vector and the integrand $\mathcal{L}_{\mathcal{C}_g,T,\mathbf{t}}$ corresponds to a *specific gum configuration* (see Definition 3.19) that can be thought of as the gum configuration $\mathcal{C}_g$ whose vertical coordinates are specified by T and $\mathbf{t}$.

Now we establish in a formal way a diagrammatic representation of operators $\pi_{\Lambda_1} \circ \mathcal{L}_{S,\Lambda}^T$ and $\pi_{\Lambda_1} \circ \mathcal{L}_S^T$.

For that we need some technical definitions and notation. Some of them are taken from [13]. Note that we also use some standard terminology from elementary graph theory here that we assume to be known to the reader.

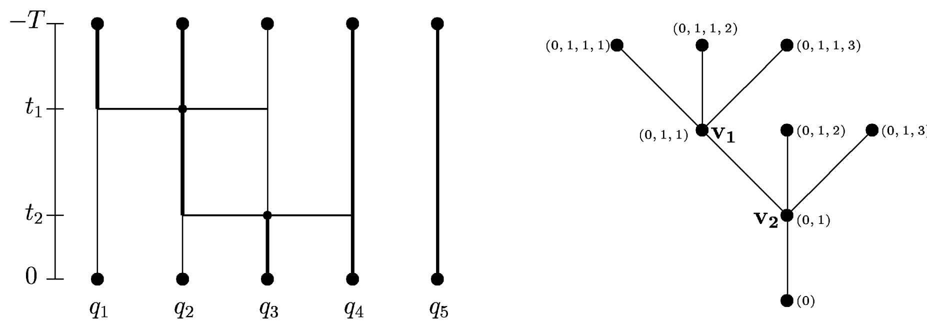

Fig. 5. Specific gum configuration and its labelled tree.

DEFINITION 3.13 (cf. [13]). – *We define the* distance *between two vertices* v_1 *and* v_2 *of a connected graph to be the smallest number* l *such that there is a sequence* $(w_0 = v_1, w_1, \ldots, w_l = v_2)$ *of vertices and for all* $1 \leqslant i \leqslant l$ *the vertices* w_i *and* w_{i-1} *have a common edge. Two vertices of distance 1 from each other are called* neighbours.

A labelled tree *is a tree graph whose vertices are labelled in the following way* (*see also the rhs of Fig.* 5 *for an example*): *The root has label* (0). *If the root has* n *neighbours then these are labelled by* $(0,1), \ldots, (0,n)$, *respectively. In general, any vertex* v *of distance* l *from the root has a label* $(0, s_1, \ldots, s_l)$. *If such a* v *has* n *neighbours each of whose has distance* $l+1$ *from the root then these neighbours are labelled by* $(0, s_1, \ldots, s_l, 1), \ldots, (0, s_1, \ldots, s_l, n)$, *respectively.*

We call a vertex of distance $l \geqslant 1$ *from the root a leaf if it has no neighbour of distance* $l+1$ *from the root. All vertices that are not the root or a leaf are called* branchings.

For $k \geqslant 1$, *we denote by a* k-branching *a branching, say of distance* l *to the root, that has exactly* $v(k)$ *neighbours of distance* $l+1$ *to the root, where* $v(k) \stackrel{\text{def}}{=} |B_k(0)|$ *and* $B_k(0)$ *is as defined in* (23). *In this case we kall* k *the* degree of the branching.

In the following we consider only labelled trees, each of whose branchings is of degree k (*for some* k *depending on the particular branching*). *The number of branchings of degree* k *is denoted by* $n_{\beta,k}$. *We collect these numbers in the parameter* $n_\beta \stackrel{\text{def}}{=} (n_{\beta,1}, n_{\beta,2}, \ldots)$.

A labelled tree that has exactly K *vertices of distance* 1 *from the root and exactly* $n_{\beta,k}$ k*-branchings is called a* labelled tree with parameters K *and* n_β.

(*Below and also in* [13] *a on-to-one-correspondence between the* k*-branchings of a configuration and the* k*-branchings of the corresponding labelled tree is specified. That explains why we are only interested in labelled trees with those particular branchings.*)

The (*non-reflexive*) linear order $\prec_v$ *on the set of labels, and so on the set of vertices (of a labelled tree), is generated by the set of all relations of the form* $(0, s_1, \ldots, s_l) \prec_v (0, s_1, \ldots, s_l, i)$ *or* $(0, s_1, \ldots, s_l, i, \ldots) \prec_v (0, s_1, \ldots, s_l, j, \ldots)$ (*for* $i < j$).

We say that a linear order $\prec_b$ *on the set of branchings* (*of a labelled tree*) *is* compatible with the labelling *if for any two branchings the following implication holds*:

$$\text{label}(\mathbf{v_1}) = (0, s_1, \ldots, s_l) \textit{ and } \text{label}(\mathbf{v_2}) = (0, s_1, \ldots, s_l, i) \quad \Rightarrow \quad \mathbf{v_2} \prec_b \mathbf{v_1}. \tag{185}$$

Finally, we introduce a linear order $\prec$ on $\mathbb{Z}^d$:

$$(k_1, \ldots, k_d) \prec (\tilde{k}_1, \ldots, \tilde{k}_d) \quad \textit{if } k_i < \tilde{k}_i \textit{ for the lowest index } i \textit{ such that } k_i \neq \tilde{k}_i. \tag{186}$$

DEFINITION 3.14. – A *gum tree* τ_g *with parameters* $n_\beta = (n_{\beta,1}, n_{\beta,2}, \ldots)$ *and* $\Lambda_2 \in \mathcal{F} \setminus \{\emptyset\}$ *is given by the following data*:

1. *A labelled tree* τ *with parameters* n_β *and* $|\Lambda_2|$ (*as defined in Definition* 3.13).
2. *A linear order* $\prec_b$ *on the set of branchings of* τ *that is compatible with the labelling.*
3. *A map* pin *from the set of vertices* (*except the root*) *of* τ *to* $\mathbb{Z}^d$ *that satisfies the following conditions*:
   (a) *The restriction of* pin *to the set of vertices, that are labelled by* $(0,1), \ldots, (0, |\Lambda_2|)$ (*We denote the restriction of* pin *to this set by* pin_0.), *is an order-*

preserving bijection onto Λ_2, *i.e. for any two such vertices* $\mathbf{v}$ *and* $\tilde{\mathbf{v}}$

$$\text{label}(\mathbf{v}) \prec_v \text{label}(\tilde{\mathbf{v}}) \quad \Rightarrow \quad \text{pin}(\mathbf{v}) \prec \text{pin}(\tilde{\mathbf{v}}). \tag{187}$$

(b) *If* $\mathbf{v}$ *with* $\text{label}(\mathbf{v}) = s = (s_1, \ldots, s_m)$ *is a* k*-branching and* $\text{pin}(\mathbf{v}) = q \in \mathbb{Z}^d$ *then the restriction of* pin *to the set of vertices with labels* $(s, 1), \ldots, (s, v(k))$ (*We denote the restriction of* pin *to this set by* $\text{pin}_{\mathbf{v}}$.) *is an order-preserving bijection onto* $B_k(q) \subset \mathbb{Z}^d$.

(c) *If* $\text{label}(\mathbf{v_2}) = (0, i)$ *then there is no* k*-branching* $\mathbf{v_3}$ *such that both* $\mathbf{v_2} \prec_b \mathbf{v_3}$ and $\text{pin}(\mathbf{v_2}) \in B_k(\text{pin}(\mathbf{v_3}))$.
Similarly, if $\text{label}(\mathbf{v_2}) = (0, s_1, \ldots, s_l, i)$ *is a branching and* $\text{label}(\mathbf{v_1}) = (0, s_1, \ldots, s_l)$ *then there is no* k*-branching* $\mathbf{v_3}$ *such that both* $\mathbf{v_2} \prec_b \mathbf{v_3} \prec_b \mathbf{v_1}$ and $\text{pin}(\mathbf{v_3}) \in B_k(\text{pin}(\mathbf{v_3}))$.

Remark 3.15. –

1. Note that for each choice of $\Lambda_2 \subset \Lambda_1$ and a labelled tree with parameters n_β and $|\Lambda_2|$ the map pin is automatically fixed (by the first two conditions on pin in Definition 3.14). Then it depends on the third condition (on pin) if the given set Λ_2, gum tree τ and order $\prec_\text{b}$ can be assigned to a (unique) gum configuration.
2. Condition 3(c) on the map pin will be justified in the proof of Proposition 3.23 where we assign to the product of operators a (unique by condition 3(c)) gum configuration and hence a gum tree. Also note that in Definition 3.20 we will define operators for a given gum configuration so we will use assignments between operators and diagrammatic data in both directions.

DEFINITION 3.16. – *A* gum configuration $\mathcal{C}_g$ on Λ ending in $\Lambda_\mathbf{1}$ *is given by the following data*:

1. *A gum tree* τ_g *with parameters* n_β *and* Λ_2 *such that* $\Lambda_2 \subseteq \Lambda_1$. *The corresponding tree has branchings* $\mathbf{v_1} \prec_b \cdots \prec_b \mathbf{v_n}$, *say, with branching-degrees* $b_1, \ldots, b_n$, *respectively.*
We denote the gum tree of a gum configuration $\mathcal{C}_g$ *by* $\tau_g(\mathcal{C}_g)$, *the corresponding tree by* $\tau(\mathcal{C}_g)$ *and its branching parameter by* $n_\beta(\mathcal{C}_g)$. *So the number of branchings is* $|n_\beta(\mathcal{C}_g)| \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} n_{\beta,k}(\mathcal{C}_g)$.
2. *For each* $1 \leqslant i \leqslant n$ *there are maps*

$$u_i : B_{b_i}\big(\text{pin}(\mathbf{v_i})\big) \cap \Lambda \to \{0, 1\}, \tag{188}$$

$$d_i : B_{b_i}\big(\text{pin}(\mathbf{v_i})\big) \cap \Lambda \to \{0, 1\} \tag{189}$$

such that

(a) *If* $q \in B_{b_i}(\text{pin}(\mathbf{v_i})) \cap \Lambda$ *and* j *is the smallest number greater than* i *such that* $q \in B_{b_j}(\text{pin}(\mathbf{v_j}))$ *(if such a* j *exists at all) then* $d_i(q) = u_j(q)$.

(b) *For every* $1 \leqslant i \leqslant n$

$$d_i\big(\text{pin}(\mathbf{v_i})\big) = 1. \tag{190}$$

(c) *If* $q \in B_{b_i}(\text{pin}(\mathbf{v_i})) \cap (\Lambda \setminus \Lambda_1)$ *and there is no* $j > i$ *such that* $q \in B_{b_j}(q_j)$ *then* $d_i(q_i) = 0$.

(*We will see later that the maps* u_i *define from a vertex upwards going* h-strips (*if* $u_i = 0$) *or* r-strips (*if* $u_i = 1$). *Similarly, the maps* d_i *determine downwards going strips. For a strip between two vertices it should be well-defined if it is an* h*-strip or an* r*-strip. Hence we impose condition* (a). *Condition* (b) *says that a strip that goes downwards from a branching must be an* r*-strip.*)

3. *A map* long *from* $\Lambda \setminus \bigcup_{i=1}^{n} B_{b_i}(\mathrm{pin}(\mathbf{v_i}))$ *to* $\{0, 1\}$ *such that*

$$\mathrm{long}(q) = 0 \quad \textit{if } q \notin \Lambda_1. \tag{191}$$

DEFINITION 3.17. – *We define in analogy to Definition* 5.2 *in* [13]

$$\widetilde{\Lambda}(\mathcal{C}_g) \stackrel{\mathrm{def}}{=} \bigcup_{i=1}^{n} B_{b_i}\bigl(\mathrm{pin}(\mathbf{v_i})\bigr), \tag{192}$$

$$\Lambda_r(\mathcal{C}_g) \stackrel{\mathrm{def}}{=} \bigl\{q \in \Lambda \setminus \widetilde{\Lambda}(\mathcal{C}_g) \colon \mathrm{long}(q) = 1\bigr\}, \tag{193}$$

$$\Lambda(\mathcal{C}_g) \stackrel{\mathrm{def}}{=} \widetilde{\Lambda}(\mathcal{C}_g) \cup \Lambda_r(\mathcal{C}_g). \tag{194}$$

We introduce the following notation:

- *In the situation of* 2(a) *the point* q *is the image* (*w.r.t.* pin) *of the vertices* $\mathrm{pin}_{\mathbf{v_i}}^{-1}(q)$ *and* $\mathrm{pin}_{\mathbf{v_j}}^{-1}(q)$. *We say that* $\mathcal{C}_g$ *has an* h-strip (r-strip) *from* $\mathrm{pin}_{\mathbf{v_i}}^{-1}(q)$ *to* $\mathrm{pin}_{\mathbf{v_j}}^{-1}(q)$ *if* $d_i(q) = 0$ ($d_i(q) = 1$). (*We note that we do not distinguish the order of the vertices in this notation*: *A strip from* $\mathbf{v}$ *to* $\tilde{\mathbf{v}}$ *is the same as a strip from* $\tilde{\mathbf{v}}$ *to* $\mathbf{v}$.)
- *If* $q \in B_{b_i}(\mathrm{pin}(\mathbf{v_i})) \cap \Lambda$ *and* $\mathbf{v} = \mathrm{pin}_{\mathbf{v_i}}^{-1}(q)$ *and there is no* $j > i$ *such that* $q \in B_{b_j}(\mathrm{pin}(\mathbf{v_j}))$ *and if* $d_i(q) = 0$ ($d_i(q) = 1$) *we say that* $\mathcal{C}_g$ *has an* h-strip (r-strip) from $\mathbf{v}$ to the bottom.
- *If* $q \in B_{b_i}(\mathrm{pin}(\mathbf{v_i})) \cap \Lambda$ *and* $\mathbf{v} = \mathrm{pin}_{\mathbf{v_i}}^{-1}(q)$ *and there is no number* $j < i$ *such that* $q \in B_{b_j}(\mathrm{pin}(\mathbf{v_j}))$ *and if* $u_i(q) = 0$ ($u_i(q) = 1$) *we say that* $\mathcal{C}_g$ *has an* h-strip (r-strip) from $\mathbf{v}$ to the top.
- *In the situation of* 2(b) *we call the corresponding* r*-strip an* apex-r-strip.
- *If* $q \in \Lambda \setminus \widetilde{\Lambda}(\mathcal{C}_g)$ *and* $\mathrm{long}(q) = 0$ ($\mathrm{long}(q) = 1$) *then we say that* $\mathcal{C}_g$ *has a* long h-strip (long r-strip) *at* q. *So* $\Lambda_r(\mathcal{C}_g) \subset \Lambda_1$ *is the set of* q *where* $\mathcal{C}_g$ *has long* r*-strips.*
- *If* $\mathcal{C}_g$ *has an* r*-strip to the top or a long* r*-strip we say that* $\mathcal{C}_g$ reaches the top.

We denote by $\mathrm{Conf}_{\mathcal{G}}(\Lambda, \Lambda_1)$ *the set of all gum configurations on* Λ *ending in* Λ_1.

DEFINITION 3.18. – *Let* $\mathcal{C}_g$ *be a gum configuration on* Λ *ending in* Λ_1 *with branchings* $\mathbf{v_1} \prec_b \ldots \prec_b \mathbf{v_n}$ *of branching-orders* $b_1, \ldots, b_n$, *respectively, and let* $T \in (0, \infty]$. *Then we define*

$$\mathrm{Simplex}(\mathcal{C}_g, T) \stackrel{\mathrm{def}}{=} \bigl\{(t_1, \ldots, t_n) \colon -T < t_1 < \cdots < t_n < 0\bigr\}. \tag{195}$$

$\mathrm{Simplex}(\mathcal{C}_g, T)$ *is an open subset of* $\mathbb{R}^n$ *and so carries the induced Lebesgue measure.*

For the special case $n_\beta(\mathcal{C}_g) = 0$ *we define* $\mathrm{Simplex}(\mathcal{C}_g, T)$ *to be a single point having measure* 1.

DEFINITION 3.19. – *For* $\mathcal{C}_g \in \mathrm{Conf}_{\mathcal{G}}(\Lambda, \Lambda_1)$, $T \in (0, \infty]$ *and* $\mathbf{t} \in \mathrm{Simplex}(\mathcal{C}_g, T)$ *we call the triple* $(\mathcal{C}_g, T, \mathbf{t})$ *a specific gum configuration.*

Specific gum configurations can be viewed graphically: *The vertices are placed in* $\mathbb{Z}^d \times [-T, 0]$ *and the strips are 'spanned' between vertices, the top* $(t = -T)$ *and the bottom* $(t = 0)$:

- *We assign to each vertex* $\mathbf{v}$ *in* $\tau(\mathcal{C}_g)$ *a point in* $\mathbb{Z}^d \times [-T, 0]$ *in the following way. If* $\mathbf{v_i}$ *is a branching of degree* b_i, $q \in B_{b_i}(\mathrm{pin}(\mathbf{v_i}))$ *and* $\mathbf{v} = \mathrm{pin}_{\mathbf{v_i}}^{-1}(q)$ *then* $\mathbf{v}$ *has time-coordinate* t_i. *In particular* $\mathbf{v_i}$ *has time-coordinate* t_i. *As further* $\mathrm{pin}(\mathbf{v}) = q$ *we assign* $\mathbf{v}$ *to* (q, t_i).
  Let for the following two vertices $\mathbf{v}$ *and* $\tilde{\mathbf{v}}$ *be assigned to* (q, t) *and* $(q, \tilde{t})$, *respectively.*
- *If* $\mathcal{C}_g$ *has an* h*-strip (*r*-strip) from* $\mathbf{v}$ *to* $\tilde{\mathbf{v}}$ *we say that* $(\mathcal{C}_g, T, \mathbf{t})$ *has a* maximal h-strip (maximal r-°*strip*) *from* (q, t) *to* $(q, \tilde{t})$. *We define its length to be* $|t - \tilde{t}|$.
- *If* $\mathcal{C}_g$ *has an* h*-strip* (r*-strip*) *from* $\mathbf{v}$ *to the bottom* (*this has time-coordinate* 0.) *we say that* $(\mathcal{C}_g, T, \mathbf{t})$ *has a* maximal h-strip (maximal r-strip) *from* (q, t) *to* $(q, 0)$. *Its length is* $|t|$.
- *If* $\mathcal{C}_g$ *has an* h*-strip* (r*-strip*) *from* $\mathbf{v}$ *to the top* (*this has time-coordinate* $-T$.) *we say that* $(\mathcal{C}_g, T, \mathbf{t})$ *has a* maximal h-strip (maximal r-strip) *from* (q, t) *to* $(q, -T)$. *Its length is* $T - |t|$. (*Note that for* $T = \infty$ *this length is* ∞.)
- *If* $\mathcal{C}_g$ *has a long* h*-strip* (*long* r*-strip*) *at* q *we say that* $(\mathcal{C}_g, T, \mathbf{t})$ *has a* long h-strip (long r-strip) *at* q. *Its length is* T. (*Long* h*-strips* (*long* r*-strips*) *are also considered as maximal strips.*)

If $(\mathcal{C}_g, T, \mathbf{t})$ *has a maximal* h*-strip* (r*-strip*) *from* $(q, \tilde{t}_1)$ *to* $(q, \tilde{t}_4)$ *and* $\tilde{t}_1 \leqslant \tilde{t}_2 < \tilde{t}_3 \leqslant \tilde{t}_4$ *then we say that* $(\mathcal{C}_g, T, \mathbf{t})$ *has an* h*-strip* (r*-strip*) *from* $(q, \tilde{t}_2)$ *to* $(q, \tilde{t}_3)$ (*or from* $(q, \tilde{t}_3)$ *to* $(q, \tilde{t}_2)$).

For a branching $\mathbf{v_i}$ *and a* $q \in B_{b_i}(\mathrm{pin}(\mathbf{v_i}))$ *we call the maximal* h*-strip* (*if any*) *from* (q, t_i) *to* (q, t) *with* $t_i < t$ $(t_i > t)$ *a* downwards going (upwards going) h-strip associated to the branching. (*Note that in our pictures the positively oriented time-axis goes downwards.*) *The notation for* r*-strips is analogous.*

$(\mathcal{C}_g, T, \mathbf{t})$ *must have a downwards going* r*-strip at the points* $(\mathrm{pin}(\mathbf{v_i}), t_i)$ *because of condition* 2(b). *We call it an apex-*r*-strip.*

An h*-strip* (r*-strip*) *in* $(\mathcal{C}_g, T, \mathbf{t})$ goes to the bottom (to the top) *if the corresponding* h*-strip* (r*-strip*) *in* $\mathcal{C}_g$ *goes to the bottom* (*to the top*).

We define

$$\tilde{c}(\mathcal{C}_g, T, \mathbf{t}) \stackrel{\text{def}}{=} \prod_H \bigl(1 - \exp\bigl(\lambda \mathrm{length}(H)\bigr)\bigr), \tag{196}$$

where the product is over all maximal h*-strips* H *that do not end in* $(\Lambda \setminus \Lambda_1) \times \{0\}$.

We draw in the specific gum configuration in Fig. 5 thick horizontal lines for branchings and thin or thick vertical lines for h-strips or r-strips, respectively. There are two branchings of degree 1, at (q_2, t_1) and at (q_3, t_2). The specific gum configuration has, for example, a long r-strip at site q_5, an r-strip from (q_1, t_1) to the top and an h-strip from (q_1, t_1) to the bottom.

Note that the vertices in the labelled gum tree (except the root) are assigned to points in $\mathbb{Z}^d$ (in this example $d = 1$) by the map pin. For example $\mathrm{pin}(\mathbf{v_1}) = q_2$.

DEFINITION 3.20. – *We denote by $\mathcal{R}_q(t)$ the operator as defined in Definition 3.7, acting on the qth coordinate. For $\mathcal{C}_g \in \mathrm{Conf}_{\mathcal{G}}(\Lambda, \Lambda_1)$ with $|n_\beta(\mathcal{C})| \geqslant 1$ branchings at $\mathbf{v_1} \prec_b \cdots \prec_b \mathbf{v_n}$ of degree $b_1, \ldots, b_n$, respectively, we set $t_0 \stackrel{\text{def}}{=} -T$, $t_{n+1} \stackrel{\text{def}}{=} 0$ and define*:

$$\mathrm{Op}_1(i, \mathcal{C}_g, T, \mathbf{t}) \stackrel{\text{def}}{=} \bigotimes_{q \in \Lambda_{\mathcal{Q}}(i, \mathcal{C}_g)} \mathcal{Q}_q \bigotimes_{q \in \Lambda_{\mathcal{R}}(i, \mathcal{C}_g)} \mathcal{R}_q(t_{i+1} - t_i), \tag{197}$$

$$\mathrm{Op}_2(i, k) \stackrel{\text{def}}{=} \mathcal{B}_{\mathrm{pin}(\mathbf{v_i}), k} \bigotimes_{q \in \Lambda \setminus \{\mathrm{pin}(\mathbf{v_i})\}} \mathrm{id}_q, \tag{198}$$

$$\begin{aligned} \mathcal{L}^T_{\mathcal{C}_g, \mathbf{t}} \stackrel{\text{def}}{=} \tilde{c}(\mathcal{C}_g, T, \mathbf{t}) \mathrm{Op}_1(n, \mathcal{C}_g, T, \mathbf{t}) \circ \mathrm{Op}_2(n, b_n) \circ \cdots \\ \circ \mathrm{Op}_1(1, \mathcal{C}_g, T, \mathbf{t}) \circ \mathrm{Op}_2(1, b_1) \circ \mathrm{Op}_1(0, \mathcal{C}_g, T, \mathbf{t}), \end{aligned} \tag{199}$$

$$\text{and} \quad \mathcal{L}^T_{\mathcal{C}_g} \stackrel{\text{def}}{=} \int\limits_{\mathrm{Simplex}(\mathcal{C}_g, T)} \lambda^{|n_\beta(\mathcal{C}_g)|} \, \mathrm{d}\mathbf{t} \, \mathcal{L}^T_{\mathcal{C}_g, \mathbf{t}}, \tag{200}$$

where $\Lambda_{\mathcal{Q}}(i, \mathcal{C}_g)$ is the set of $q \in \Lambda$ such that $(\mathcal{C}_g, T, \mathbf{t})$ has an h-strip from (q, t_i) to (q, t_{i+1}) and $\Lambda_{\mathcal{R}}(i, \mathcal{C}_g)$ is the set of $q \in \mathbb{Z}^d$ such that $(\mathcal{C}_g, T, \mathbf{t})$ has an r-strip from (q, t_i) to (q, t_{i+1}).

If $n_\beta(\mathcal{C}_g) = 0$ we simply set

$$\mathcal{L}^T_{\mathcal{C}_g, \mathbf{t}} \stackrel{\text{def}}{=} \bigotimes_{q:\ \mathrm{long}(q)=0} \mathcal{Q}_q \bigotimes_{q:\ \mathrm{long}(q)=1} \mathcal{R}_q, \tag{201}$$

$$\mathcal{L}^T_{\mathcal{C}_g} \stackrel{\text{def}}{=} \mathcal{L}^T_{\mathcal{C}_g, \mathbf{t}}. \tag{202}$$

Finally we set

$$\pi_{\Lambda_1} \circ \mathcal{L}^T_{S, \Lambda} \stackrel{\text{def}}{=} \sum_{\mathcal{C}_g \in \mathrm{Conf}_{\mathcal{G}}(\Lambda, \Lambda_1)} \pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g}. \tag{203}$$

Remark 3.21. –

1. If H is a maximal h-strip from time t_i to time t_j with $1 \leqslant i < j \leqslant n+1$ then $\mathrm{length}(H) = |t_i - t_j|$ and so the factor $1 - \exp(-\lambda|t_i - t_j|)$ does not depend on T. However, in the case $i = 0$, i.e. $t_i = -T$, the factor $1 - \exp(-\lambda(T - |t_j|))$ depends on T. For $T = \infty$ this is equal to 1.
2. From (196), (197), (198) and (199) we see that the map $\mathbf{t} \mapsto \pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g, \mathbf{t}}$, defined on $\mathrm{Simplex}(\mathcal{C}_g, T)$, is uniformly continuous (because all factors are uniformly continuous w.r.t. $\mathbf{t}$), hence integrable if $T < \infty$. We will see in the next proposition that the integral also exists in the case $T = \infty$. So (200) is well-defined.
3. We see that if $(\mathcal{C}_g, \infty, \mathbf{t})$ has an r-strip going to the top then $\mathcal{L}^\infty_{\mathcal{C}_g, \mathbf{t}} = 0$.

For the following proposition recall that the parameters ε and c_g were introduced in (151) and (156), respectively. The coupling of the interaction between different sites is ‘small’ if ε is ‘small’ and a ‘large’ c_g means ‘strong’ exponential decay of the interaction (w.r.t. spatial distance).

PROPOSITION 3.22. – *There are constants* $0 < \tilde{\vartheta} < \vartheta < 1$ *and a* $c_4 > 0$ *such that for sufficiently small* $\varepsilon > 0$, *large* c_g, *all* $T > 0$, $\Lambda_1 \subset \Lambda \in \mathcal{F} \setminus \{\emptyset\}$ *and* $\phi \in \mathcal{H}_\vartheta$

$$\tilde{\vartheta}^{|\Lambda_1|} \sum_{\mathcal{C}_g \in \mathrm{Conf}_g(\Lambda,\Lambda_1)} \int_{\mathrm{Simplex}(\mathcal{C}_g,T)} \lambda^{|n_\beta(\mathcal{C}_g)|}\, \mathrm{d}\mathbf{t}\, \big\|\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g} \phi_\Lambda\big\| \leqslant c_4 \|\phi\|_\vartheta. \tag{204}$$

For sufficiently large T *this also holds for suitably chosen* $\tilde{\vartheta} = \vartheta$.

Proof. – First we estimate for each $\mathcal{C}_g \in \mathrm{Confg}(\Lambda, \Lambda_1)$ and $\mathbf{t} \in \mathrm{Simplex}(\mathcal{C}_g, T)$ the norm $\|\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} \phi_\Lambda\|_{\Lambda_1}$. For that we follow the proof of (57) and (58) in [13]. The operators $\mathcal{Q}_q$, $\mathcal{R}_q(t_{i+1} - t_i)$ and $\mathcal{B}_{q,k}$ in the representation (199) of the operator $\Lambda_1 \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}}$ as well as the projection operator π_{Λ_1} can be represented by integral operators (see comment after (163) and cf. [13].) Note that by (167), $\mathcal{R}_q(t_{i+1} - t_i)$ is the sum of integral operators.

Using the integral representation of the function $\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} \phi_\Lambda \in \mathcal{H}_{\Lambda_1}$, we proceed as follows:

1. We perform the integration corresponding to all maximal r-strips of the specific gum configuration $(\mathcal{C}_g, T, \mathbf{t})$. In the estimate (205) an r-strip R gives rise to a factor $c_r \exp(-(1-\eta)\lambda \mathrm{length}(R))$ (see (170).)
2. For each maximal h-strip that does not end in $(\Lambda \setminus \Lambda_1) \times \{0\}$ we perform the integration $(1 - \exp(\lambda \mathrm{length}(H)))h_q \cdot \mu_q$ (i.e. integration w.r.t. the q-coordinate and multiplication by a scalar factor of h_q.) That leads to a factor $c_h(1 - \exp(\lambda \mathrm{length}(H)))$ (see (153).)
3. For all maximal h-strips ending in $(\Lambda \setminus \Lambda_1) \times \{0\}$ we perform the integration corresponding to the projection π_{Λ_1} which leads to a factor 1 in the estimate.
4. For each operator $\mathcal{B}_{q,k}$ we estimate the contribution of its integral kernel from above by $\tilde{c}_3 \varepsilon \exp(-c_g k^d)$. That estimate is derived from (156) in the same way as (55) in [13] is proved. In particular, the constant $\tilde{c}_3$ is a product of the constant c_3 in (156) and constants depending on the geometry of the annulus defined in (133) and with parameter δ as fixed in *Assumption* I.
5. The integral operator $\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}}$ acts on the function $\phi_\Lambda \in \mathcal{H}_\Lambda$. However, we only have to estimate the norm of $\phi_{\Lambda(\mathcal{C}_g)}$ as ϕ_Λ is simply integrated w.r.t. the $(\Lambda \setminus \Lambda(\mathcal{C}_g))$-coordinates, i.e. at least w.r.t. these, possibly also w.r.t. others. To see that, note that the application of the projection operator π_{Λ_1} or $\mathcal{Q}_q$ mean integration w.r.t. the $(\Lambda \setminus \Lambda_1)$- or the q-coordinates, respectively.

By $\tilde{n}_h$ we denote the number of maximal h-strips that have spatial coordinate in $\Lambda(\mathcal{C}_g) \cup \Lambda_1$ (for the other h-strips there is simply an integration to do, giving rise to a factor 1 in the estimates) and by $\tilde{n}_r$ the number of maximal r-strips.

Then we get the estimates

$$\begin{aligned}\big\|\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} \phi_\Lambda\big\|_{\Lambda_1} &\leqslant (c_3\varepsilon)^{|n_\beta|} \exp\left(-c_g \sum_{k=1}^{\infty} k^d n_{\beta,k}\right) c_h^{\tilde{n}_h} c_r^{\tilde{n}_r} \\ &\quad \times \prod_R \exp\big(-(1-\eta)\lambda \mathrm{length}(R)\big) \tilde{c}(\mathcal{C}_g, T, \mathbf{t}) \|\phi_{\Lambda(\mathcal{C}_g)}\|_{\Lambda(\mathcal{C}_g)},\end{aligned} \tag{205}$$

where the product is over all maximal r-strips R of $(\mathcal{C}_g, T, \mathbf{t})$ and $\tilde{c}(\mathcal{C}_g, T, \mathbf{t})$ is as defined in (196), and

$$\|\phi_{\Lambda(\mathcal{C}_g)}\|_{\Lambda(\mathcal{C}_g)} \leqslant \vartheta^{-|\Lambda_r|-\sum_{k=1}^{\infty}(3k)^d n_{\beta,k}} \|\phi\|_\vartheta \leqslant \vartheta^{-|\Lambda_r|} \prod_{k=1}^{\infty} \vartheta^{-(3k)^d n_{\beta,k}} \|\phi\|_\vartheta. \tag{206}$$

Now we consider a labelled tree τ with parameters $n_{\beta,k}$ and K, a set $\Lambda_2 \subset \Lambda_1$ with $|\Lambda_2| = K$ and the set $A(\tau, \Lambda_2)$ of all $\mathcal{C}_g \in \mathrm{Conf}_g(\Lambda, \Lambda_1)$ whose labelled tree is τ and whose gum tree has parameter Λ_2. Note that there can be different linear orders on the branchings of τ. We want to estimate

$$\tilde{\vartheta}^{|\Lambda_1|} \sum_{\mathcal{C}_g \in A(\tau,\Lambda_2)} \int_{\mathrm{Simplex}(\mathcal{C}_g,T)} \lambda^{|n_\beta(\mathcal{C}_g)|} \,\mathrm{d}\mathbf{t} \left\|\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} \phi_\Lambda\right\| \tag{207}$$

and consider this expression as integral over the union of all sets $\mathrm{Simplex}(\mathcal{C}_g, T)$.

We change the variables of integration: Let the branchings of τ denote by $\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_{|\mathbf{n}_\beta|}$. A given $\mathcal{C}_g \in A(\tau, \Lambda_2)$ has an ordered set of branchings $\mathbf{v}_1 \prec_{\mathrm{b}} \cdots \prec_{\mathrm{b}} \mathbf{v}_{|\mathbf{n}_\beta|}$, so that $\mathbf{v}_\mathbf{i} = \tilde{\mathbf{v}}_{\mathbf{j}(\mathbf{i})}$, where j is a permutation on the set of indices.

Further, for given $T > 0$ and $\mathbf{t} \in \mathrm{Simplex}(\mathcal{C}_g, T)$ the time-coordinate t_i corresponds to the branching $\mathbf{v}_\mathbf{i}$ whereas we denote by $\tilde{t}_1, \ldots, \tilde{t}_{|n_\beta|}$ the lengths of the apex-r-strips that correspond to the branching $\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_{|\mathbf{n}_\beta|}$, respectively. In particular, the $\tilde{t}_i$ are bounded by T.

For each $\mathbf{t} = (t_1, \ldots, t_{|n_\beta|}) \in \bigcup_{\mathcal{C}_g \in A(\tau)} \mathrm{Simplex}(\mathcal{C}_g, T)$ there is a unique $\tilde{\mathbf{t}} = (\tilde{t}_1, \ldots, \tilde{t}_{|n_\beta|})$. The images of the different simplices $\mathrm{Simplex}(\mathcal{C}_g, T)$ w.r.t. this map are disjoint subsets of $[0, T]^n$. Further, the change of variables from $\mathbf{t}$ to $\tilde{\mathbf{t}}$ is linear and has a determinant of modulus 1. We see that by doing the transformation in several steps: $\tilde{t}_{j(1)}$ is given by a linear equation

$$\tilde{t}_{j(1)} = \mathrm{Lin}_1(t_2, \ldots, t_n) - t_1 \tag{208}$$

and $\tilde{t}_{j(2)}$ by

$$\tilde{t}_{j(2)} = \mathrm{Lin}_2(\tilde{t}_{j(1)}, t_3, \ldots, t_n) - t_2 \tag{209}$$

etc. and the statement about the determinant follows. So we can estimate in (207) the term '$\sum_{\mathcal{C}_g \in A(\tau,A_2)} \int_{\mathrm{Simplex}(\mathcal{C}_g,T)} \mathrm{d}\mathbf{t} \ldots$' by '$\int_{[0,T]^{|n_\beta(\tau)|}} \mathrm{d}\tilde{\mathbf{t}} \ldots$' and so in the estimate of (204) we replace '$\sum_{\mathcal{C}_g} \int_{\mathrm{Simplex}(\mathcal{C}_g,T)} \mathrm{d}\mathbf{t} \ldots$' by '$\sum_{\Lambda_2,\tau} \int_{[0,T]^{|n_\beta(\tau)|}} \mathrm{d}\tilde{\mathbf{t}} \ldots$' where the sum is over all $\Lambda_2 \subseteq \Lambda_1$ and labelled trees τ with parameters $|\Lambda_2|$ and n_β.

Next we want to estimate in the last sum the contribution corresponding to all labelled trees with parameter K (and arbitrary n_β): For a fixed $0 \leqslant K \leqslant |\Lambda_1|$ there are exactly $\binom{|\Lambda_1|}{K}$ subsets $\Lambda_2 \subset \Lambda_1$ with $|\Lambda_2| = K$. By Lemma 8.2.(2) of [13], for fixed Λ_2 and n_β with $|n_\beta| \geqslant |\Lambda_2|$, the number of labelled trees with parameter $|\Lambda_2|$ and n_β is bounded from above by $4^{|\Lambda_2|} \prod_{k=1}^{\infty} (\exp(\tilde{c}_d k^d))^{n_{\beta,k}}$, where $\tilde{c}_d$ is a constant depending only on the dimension d of the lattice.

Now we consider a given a set Λ_2, a labelled tree τ and a choice of $\tilde{\mathbf{t}}$. There is at most one gum tree τ_g, having tree τ, such that a specified gum configuration with parameter Λ_2 and gum tree τ_g exists and the $\tilde{t}_i$ are the lengths of apex-r-chains. (Then the order on the branches is determined by the $\tilde{t}_i$.)

If such a gum tree exists then the (specific) gum configuration is uniquely defined by the choice of up- and downwards going h- and r-strips and long h- and r-strips.

For each choice of an h-strip or r-strip we get a factor $c_h(1 - \exp(\lambda \text{length}(H)))$ or $c_r \exp(-(1-\eta)\lambda \text{length}(R))$, respectively, as mentioned at the beginning of this proof. So for each branching we can estimate the contribution of constant factors c_h and c_r of all possible choices from above by a factor $\exp(c_{12}k^d)$. This factor will be compensated for by the factor $\exp(-c_g k^d)$ that, as mentioned at the beginning of the proof, from the estimate for the operator $\mathcal{B}_{q,k}$ corresponding to the branching. For that the constant c_g has to be sufficiently large.

There are not more than $|\Lambda_1| - |\Lambda_2|$ sites for which we can choose between long h-strips and long r-strips. A long r-strip gives rise to a factor $c_r \exp(-(1-\eta)\lambda T)$, and a long h-strip to a factor at most c_h.

Gum configurations $\mathcal{C}_g$ without branchings (i.e. $n_\beta(\mathcal{C}_g) = 0$) can only have long r-chains (that must end in Λ_1) or long h-chains. This case corresponds to the summand for $K = 0$ in (211).

We remark that the sum '$\sum_{n=0}^{\infty}$' in (211) also includes the estimate for this special case $K = 0$. Then the gum configurations have no branchings and so $n_\beta = 0$. The sum '$\sum_{n=0}^{\infty}$' should then be replaced by a factor 1 (to avoid confusion). However, this sum is at least 1 and so the estimate is correct.

We estimate the left-hand side (lhs) of (204):

$$\tilde{\vartheta}^{|\Lambda_1|} \sum_{\mathcal{C}_g \in \text{Confg}(\Lambda,\Lambda_1)} \int_{\text{Simplex}(\mathcal{C}_g,T)} \lambda^{|n_\beta(\mathcal{C}_g)|} \, \mathrm{d}\mathbf{t} \left\| \pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} \phi_\Lambda \right\| \tag{210}$$

$$\leqslant \tilde{\vartheta}^{|\Lambda_1|} \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} \left(c_h + \vartheta^{-1} c_r \exp\bigl(-(1-\eta)\lambda T\bigr)\right)^{|\Lambda_1|-K} 4^K \tag{211}$$

$$\times \sum_{n=K}^{\infty} \left(\varepsilon\lambda \sum_{k=1}^{\infty} \exp\bigl(-c_g k^d\bigr) \exp\bigl((\tilde{c}_d + c_{12})k^d\bigr) c^{k^d} \vartheta^{-ck^d} \times \int_0^T \mathrm{d}t \, c_r \exp\bigl(-(1-\eta)\lambda T\bigr) \right)^n \|\phi\|_\vartheta$$

$$\leqslant c_4 \left(\tilde{\vartheta} c_h + \frac{\tilde{\vartheta}}{\vartheta} c_r \exp\bigl(-(1-\eta)\lambda T\bigr) + \tilde{\vartheta}\varepsilon_1 \right)^{|\Lambda_1|} \|\phi\|_\vartheta$$

with $\lim_{\varepsilon\to 0} \varepsilon_1 = 0$. So there are $0 < \tilde{\vartheta} < \vartheta$ such that for sufficiently small ε

$$\tilde{\vartheta} c_h + \frac{\tilde{\vartheta}}{\vartheta} c_r \exp\bigl(-(1-\eta)T\bigr) + \tilde{\vartheta}\varepsilon_1 < 1 \tag{212}$$

and so (204) holds uniformly in Λ_1 and Λ. For sufficiently large T we can choose $\tilde{\vartheta} = \vartheta$ such that (212) holds. So (204) is proved. □

PROPOSITION 3.23. – *$\mathcal{L}^T_{S,\Lambda} \stackrel{\text{def}}{=} \pi_\Lambda \circ \mathcal{L}^T_{S,\Lambda}$ is the transfer operator, restricted to $\mathcal{H}_\Lambda$, for $K^T_{S,\Lambda}$, i.e.*

$$\int_{(S^1)^\Lambda} \mathrm{d}\mu \big(K^T_{S,\Lambda}\psi_\Lambda\big)\phi_\Lambda = \int_{(S^1)^\Lambda} \mathrm{d}\mu\, \psi_\Lambda \big(\mathcal{L}^T_{S,\Lambda}\phi_\Lambda\big). \tag{213}$$

For all $\psi \in \mathcal{C}^0((S^1)^\Lambda)$ and $\phi_\Lambda \in \mathcal{H}_\Lambda$ the operator $\pi_{\Lambda_1} \circ \mathcal{L}^T_{S,\Lambda}$ is the composite of π_{Λ_1} and $\mathcal{L}^T_{S,\Lambda}$ for any $\Lambda_1 \subset \Lambda$.

Proof. – The first claim is a special case of the last statement. Note that $\pi_{\Lambda_1} \circ \mathcal{L}^T_{S,\Lambda}$ is a priori the operator defined in (203). Now we prove that it is actually the composite of π_{Λ_1} and the transfer operator, restricted to $\mathcal{H}_\Lambda$, for $K^T_{S,\Lambda}$. The convergence of the following expressions follows from Proposition 3.22.

We consider all ordered finite sequences of jump-sites in Λ. A particular sequence $(q_1, \ldots, q_N)$ occurs in a fixed time interval $(-T, 0)$ with probability $\mathrm{e}^{-|\Lambda|\lambda T} \frac{(\lambda T)^N}{N!}$ because this is the probability of having exactly N jumps in total, divided by the number $|\Lambda|^N$ of different sequences of length k (which have all the same probability).

The sequence corresponds to a map $S_{q_N} \circ \cdots \circ S_{q_1}$ (cf. Definition 3.6) and so by (166), to a transfer operator

$$\pi_{\Lambda_1} \circ \mathcal{L}_{S_{q_N}} \circ \cdots \circ \mathcal{L}_{S_{q_1}}. \tag{214}$$

So the composite of π_{Λ_1} and the transfer operator for $K^T_{S,\Lambda}$ is equal to the following sum over all (possibly empty) sequences:

$$\sum_{(q_1,\ldots,q_N)} \mathrm{e}^{-|\Lambda|\lambda T} \frac{(\lambda T)^N}{N!} \pi_{\Lambda_1} \circ \mathcal{L}_{S_{q_N}} \circ \cdots \circ \mathcal{L}_{S_{q_1}} \tag{215}$$

because this is equal to the rhs of (147). The probability space Ω is partitioned into countably many sets, each corresponding to a particular sequence of jump sites. So we can write the integral here as a weighted sum.

The factors in (214) can be split

$$\mathcal{L}_{S_q} = \left(\mathcal{Q}_q + \mathcal{R}_q + \sum_{l=1}^{\infty} \mathcal{B}_{q,l}\right) \otimes \mathrm{id}_{\Lambda\setminus\{q\}}. \tag{216}$$

Expanding the product in (214), we get a sum of operators. Recall the rules (155) and (164), and also that we have, with μ_q denoting the integration w.r.t. the normalized Lebesgue measure $\mu^{\{q\}}$ (as defined in (134)),

$$\mu_q \circ \mathcal{R}_q = 0, \tag{217}$$
$$\mu_q \circ \mathcal{B}_{q,l} = 0,$$

for all $q \in \Lambda$ and $l \geqslant 1$ (as $\mathcal{Q}_q = h_q \cdot \mu_q$). So some of the summands in the expansion are zero, namely if $\mathcal{Q}_q$ is followed by $\mathcal{R}_q$, or $\mathcal{R}_q$ or $\mathcal{B}_q$ are followed by $\mathcal{Q}_q$ or a projection π_{Λ_1} with $q \notin \Lambda_1$. 'Following' here means that there is in between no $\mathcal{B}_{q,l}$ with $\|\tilde{q} - q\| \leqslant l$. In the following we rule out these combinations. (Compare this to the notion of non-zero configurations in [13].)

Now we represent each summand

$$\pi_{\Lambda_1} \circ A_N \circ \cdots \circ A_1 \tag{218}$$

in the expansion of (214) which is the composite of operators π_{Λ_1}, $\mathcal{B}_q \otimes \mathrm{id}_{\Lambda\setminus\{q\}}$, $\mathcal{R}_q \otimes \mathrm{id}_{\Lambda\setminus\{q\}}$ and $\mathcal{Q}_q \otimes \mathrm{id}_{\Lambda\setminus\{q\}}$ with variable $q \in \Lambda$, by a gum configuration as follows:

We define the obvious order of the factors in (218) such that A_1 comes before A_2 etc. Each $\mathcal{B}_{q,l}$ corresponds to a l-branching (which is assigned by the map pin to q) and the order of the operators $\mathcal{B}_{q,l}$ defines the linear order of the branchings. The other factors $\mathcal{Q}_q$ and $\mathcal{R}_q$ determine the h- and r-strips in the following way:

1. *Strips between two vertices:* Let $\mathbf{v}_i \prec_\mathrm{b} \mathbf{v}_j$ be two branchings of degree b_i and b_j, respectively, and $q \in B_{b_i}(\mathrm{pin}(\mathbf{v}_i)) \cap B_{b_j}(\mathrm{pin}(\mathbf{v}_j)) \cap \Lambda$ such that there is no other branching $\mathbf{v}_k$, of degree b_k, say, with $\mathbf{v}_i \prec_\mathrm{b} \mathbf{v}_k \prec_\mathrm{b} \mathbf{v}_j$ and $q \in B_{b_k}(\mathrm{pin}(\mathbf{v}_k))$. Then $\mathcal{C}_g$ has an h-strip between $\mathrm{pin}_{\mathbf{v}_i}^{-1}(q)$ and $\mathrm{pin}_{\mathbf{v}_j}^{-1}(q)$ if there is a factor $\mathcal{Q}_q \otimes \mathrm{id}_{\Lambda\setminus\{q\}}$ in (218) between the two factors corresponding to $\mathbf{v}_i$ and $\mathbf{v}_j$. Otherwise $\mathcal{C}_g$ has an r-strip there (even if there is no factor $\mathcal{R}_q \otimes \mathrm{id}_{\Lambda\setminus\{q\}}$.)
2. *Strips from a vertex and the top:* Let $\mathbf{v}_j$ be a b_j-branching and $q \in B_{b_j}(\mathrm{pin}(\mathbf{v}_j))$. Assume there is no b_i-branching $\mathbf{v}_i$ with $\mathbf{v}_i \prec_\mathrm{b} \mathbf{v}_j$ and $q \in B_{b_i}(\mathrm{pin}(\mathbf{v}_i))$. Then $\mathcal{C}_g$ has an h-strip from $\mathrm{pin}_{\mathbf{v}_j}^{-1}(q)$ to the top if there is a factor $\mathcal{Q}_q \otimes \mathrm{id}_{\Lambda\setminus\{q\}}$ in (218) before the factor corresponding to $\mathbf{v}_j$. Otherwise $\mathcal{C}_g$ has an r-strip there.
3. *Strips from a vertex and the bottom:* Let $\mathbf{v}_i$ be a b_i-branching and $q \in B_{b_i}(\mathrm{pin}(\mathbf{v}_i))$. Assume there is no b_j-branching $\mathbf{v}_j$ with $\mathbf{v}_i \prec_\mathrm{b} \mathbf{v}_j$ and $q \in B_{b_j}(\mathrm{pin}(\mathbf{v}_j))$. Then $\mathcal{C}_g$ has an h-strip from $\mathrm{pin}_{\mathbf{v}_i}^{-1}(q)$ to the bottom if there is a factor $\mathcal{Q}_q \otimes \mathrm{id}_{\Lambda\setminus\{q\}}$ in (218) after the factor corresponding to $\mathbf{v}_j$ or if $q \in \Lambda \setminus \Lambda_1$. Otherwise $\mathcal{C}_g$ has an r-strip there.
4. *Long strips:* Let $q \in \Lambda$ and assume that there is no b_i-branching $\mathbf{v}_i$ with $q \in B_{b_i}(\mathrm{pin}(\mathbf{v}_i))$. Then $\mathcal{C}_g$ has a long h-strip at site q if there is a factor $\mathcal{Q}_q \otimes \mathrm{id}_{\Lambda\setminus\{q\}}$ in (218) or if $q \in \Lambda \setminus \Lambda_1$. Otherwise $\mathcal{C}_g$ has a long r-strip there.

The assignment of a summand in the sequence (218) of operators to a gum configuration is not injective, as we have already seen in the simple Example 3.8. Now we consider a fixed gum configuration $\mathcal{C}_g$ on Λ ending in Λ_1 and all sequences corresponding to it. We assume $\mathcal{C}_g$ to have at least one branching. The case of no branching is treated in a similar but easier way. To keep the notation simpler, let $\Lambda = \{1, \ldots, |\Lambda|\}$. Any such sequence has the factors $\mathcal{B}_{q_i,l_i}$ $(i = 1, \ldots, K)$ that correspond $1-1$ and order preserving to the K branchings of $\mathcal{C}_g$ as described above. Between two consecutive factors $\mathcal{B}_{q_i,l_i}$ and $\mathcal{B}_{q_{i+1},l_{i+1}}$ there can be factors $\mathcal{Q}_q \otimes \mathrm{id}_{\Lambda\setminus\{q\}}$ (factors $\mathcal{R}_q \otimes \mathrm{id}_{\Lambda\setminus\{q\}}$) if for any $\mathbf{t} \in \mathrm{Simplex}(\mathcal{C}_g, T)$ the specific gum configuration $(\mathcal{C}_g, T, \mathbf{t})$ has an h-strip (an r-strip) from (q, t_i) to (q, t_{i+1}). Similarly, the option of having such factors before $\mathcal{B}_{q_1,l_1}$ or after $\mathcal{B}_{q_K,l_K}$ depends on the h-strips and r-strips of $\mathcal{C}_g$ in the obvious way. Further, such factors belong to particular maximal r-strips or h-strips in the obvous way.

Let us denote the number of factors $\mathcal{Q}_q \otimes \mathrm{id}_{\Lambda\setminus\{q\}}$ or $\mathcal{R}_q \otimes \mathrm{id}_{\Lambda\setminus\{q\}}$ before $\mathcal{B}_{q_1,l_1}$ by $n_{0,q}$ and the number of such factors after $\mathcal{B}_{q_i,l_i}$ by $n_{i,q}$.

Note that for every maximal h-strip, at site q,say, that does not go to the bottom at a site in $\Lambda \setminus \Lambda_1$, there must be at least one factor $Q_q \otimes \mathrm{id}_{\Lambda\setminus\{q\}}$ because of rules 1, 3 and 4 on p. 47. We denote this condition on the family $(n_{i,q})_{\substack{0\leqslant i\leqslant K\\ q\in\Lambda}}$ of numbers by *condition A*. In total there are

$$N = \sum_{q\in\Lambda}\sum_{i=0}^{K} n_{i,q} + K \tag{219}$$

factors that correspond to a particular sequence of N jump sites and such a sequence occurs with probability $\mathrm{e}^{-|\Lambda|\lambda T}\frac{(\lambda T)^N}{N!}$. The $\sum_{q\in\Lambda} n_{0,q}$ factors before $\mathcal{B}_{q_1,l_1}$ can occur in any of the $\binom{n_{0,1}+\cdots+n_{0,|\Lambda|}}{n_{0,1},\ldots,n_{0,|\Lambda|}}$ different orders, that all have the same probability.

So if we sum over all (products of) operators that correspond to $\mathcal{C}_g$, weighted with the probability that the corresponding sequence of jumps (in the underlying Poisson process) occurs, we get

$$\begin{aligned}
&\sum_{\substack{(n_{i,q})\\ \text{condition A}}} \mathrm{e}^{-|\Lambda|\lambda T}\frac{(\lambda T)^N}{N!}\binom{n_{0,1}+\ldots+n_{0,|\Lambda|}}{n_{0,1},\ldots,n_{0,|\Lambda|}}\cdots\binom{n_{K,1}+\cdots+n_{K,|\Lambda|}}{n_{K,1},\ldots,n_{K,|\Lambda|}} \qquad (220)\\
&\quad\times A_{K,1}^{n_{K,1}}\otimes\cdots\otimes A_{K,|\Lambda|}^{K,n_{|\Lambda|}}\circ\mathcal{B}_{q_K,l_K}\otimes\mathrm{id}_{\Lambda\setminus\{q_K\}}\circ\cdots\circ\mathcal{B}_{q_1,l_1}\otimes\mathrm{id}_{\Lambda\setminus\{q_1\}}\\
&\quad\circ A_{0,1}^{n_{0,1}}\otimes\cdots\otimes A_{0,|\Lambda|}^{n_{0,|\Lambda|}}\\
&=\sum_{\substack{(n_{i,q})\\ \text{condition A}}}\int_{\mathrm{Simplex}(\mathcal{C}_g,T)}\lambda^K\,\mathrm{d}\mathbf{t}\left(\mathrm{e}^{-\lambda(-t_K)}\frac{(\lambda(-t_K))^{n_{K,1}}}{n_{K,1}!}A_{K,1}^{n_{K,1}}\right)\otimes\cdots\\
&\quad\otimes\left(\mathrm{e}^{-\lambda(-t_K)}\frac{(\lambda(-t_K))^{n_{K,|\Lambda|}}}{n_{K,|\Lambda|}!}A_{K,|\Lambda|}^{n_{K,|\Lambda|}}\right)\circ\mathcal{B}_{q_K,l_K}\otimes\mathrm{id}_{\Lambda\setminus\{q_K\}}\circ\cdots\circ\mathcal{B}_{q_1,l_1}\otimes\mathrm{id}_{\Lambda\setminus\{q_1\}}\\
&\quad\circ\left(\mathrm{e}^{-\lambda(t_1+T)}\frac{(\lambda(t_1+T))^{n_{0,1}}}{n_{0,1}!}A_{0,1}^{n_{0,1}}\right)\otimes\cdots\otimes\left(\mathrm{e}^{-\lambda(t_1+T)}\frac{(\lambda(t_1+T))^{n_{0,|\Lambda|}}}{n_{0,|\Lambda|}!}A_{0,|\Lambda|}^{n_{0,|\Lambda|}}\right)\\
&=\int_{\mathrm{Simplex}(\mathcal{C}_g,T)}\lambda^K\,\mathrm{d}\mathbf{t}\prod_H\left(1-\mathrm{e}^{-\lambda\,\mathrm{length}(H)}\right)\left(\sum_{n_{K,1}\geqslant 0}\mathrm{e}^{-\lambda(-t_K)}\frac{(\lambda(-t_K))^{n_{K,1}}}{n_{K,1}!}A_{K,1}^{n_{K,1}}\right)\otimes\cdots\\
&\quad\otimes\left(\sum_{n_{K,|\Lambda|}\geqslant 0}\mathrm{e}^{-\lambda(-t_K)}\frac{(\lambda(-t_K))^{n_{K,|\Lambda|}}}{n_{K,|\Lambda|}!}A_{K,|\Lambda|}^{n_{K,|\Lambda|}}\right)\\
&\quad\circ\mathcal{B}_{q_K,l_K}\otimes\mathrm{id}_{\Lambda\setminus\{q_K\}}\circ\cdots\circ\mathcal{B}_{q_1,l_1}\otimes\mathrm{id}_{\Lambda\setminus\{q_1\}}\circ\left(\sum_{n_{0,1}\geqslant 0}\mathrm{e}^{-\lambda(t_1+T)}\frac{(\lambda(t_1+T))^{n_{0,1}}}{n_{0,1}!}A_{0,1}^{n_{0,1}}\right)\\
&\quad\otimes\cdots\otimes\left(\sum_{n_{0,|\Lambda|}\geqslant 0}\mathrm{e}^{-\lambda(t_1+T)}\frac{(\lambda(t_1+T))^{n_{0,|\Lambda|}}}{n_{0,|\Lambda|}!}A_{0,|\Lambda|}^{n_{0,|\Lambda|}}\right)\\
&=\mathcal{L}_{\mathcal{C}_g}^T. \qquad (221)
\end{aligned}$$

In (220) each operator $A_{k,q}$ stands for either $\mathcal{Q}_q$ or $\mathcal{R}_q$.

We have interchanged summation with integration and multiplication and applied the following computations to each set of factors belonging to the same maximal h-strip that does not end in $(\Lambda \setminus \Lambda_1) \times \{0\}$. For notational simplicity, let the maximal h-strip under consideration cover the time intervals $(t_0, t_1), \ldots, (t_{l-1}, t_l)$. So its length is $t_l - t_0$. And let the operators $\mathcal{Q}_q$ have exponents $n_1, \ldots, n_l$:

$$\sum_{\substack{n_0,\ldots,n_l \geqslant 0 \\ n_0+\ldots+n_l \geqslant 1}} \prod_{i=1}^{l} \left(e^{-\lambda(t_i - t_{i-1})} \frac{(\lambda(t_i - t_{i-1}))^{n_i}}{n_i!} \mathcal{Q}_q \right) \tag{222}$$

$$= \prod_{i=1}^{l} \left(\sum_{n_i=0}^{\infty} e^{-\lambda(t_i - t_{i-1})} \frac{(\lambda(t_i - t_{i-1}))^{n_i}}{n_i!} \mathcal{Q}_q \right) - \prod_{i=1}^{l} e^{-\lambda(t_i - t_{i-1})} \mathcal{Q}_q$$

$$= \left(1 - e^{-\lambda(t_l - t_0)}\right) \mathcal{Q}_q. \tag{223}$$

This explains the appearance of the factor (196) in (221). Recall that each operator $A_{i,q}$ in (221) stands for either $\mathcal{Q}_q$ or $\mathcal{R}_q$. So we can replace

$$\sum_{n_{i,q} \geqslant 0} e^{-\lambda(t_i - t_{i-1})} \frac{(\lambda(t_i - t_{i-1}))^{n_{i,q}}}{n_{i,q}!} \mathcal{Q}_q^{n_{i,q}} = \mathcal{Q}_q, \tag{224}$$

$$\sum_{n_{i,q} \geqslant 0} e^{-\lambda(t_i - t_{i-1})} \frac{(\lambda(t_i - t_{i-1}))^{n_{i,q}}}{n_{i,q}!} \mathcal{R}_q^{n_{i,q}} = \mathcal{R}_q(t_i - t_{i-1}) \tag{225}$$

and get (221).

So we have seen that (215) is equal to $\pi_{\Lambda_1} \circ \mathcal{L}^T_{S,\Lambda}$, as defined in (203). □

For the representation of the transfer operator for the infinite dimensional system we need the following definition.

DEFINITION 3.24. – *Let* $\Lambda_1, \Lambda_2 \subseteq \Lambda \in \mathcal{F} \setminus \{\emptyset\}$ *and* $\mathcal{C}_g \in \mathrm{Conf}_g(\Lambda, \Lambda_1)$. *We say that* $\mathcal{C}_g$ lies in Λ_2 *if* $\Lambda(\mathcal{C}_g) \cup \Lambda_1 \subseteq \Lambda_2$. (*Recall that* $\Lambda(\mathcal{C}_g)$ *was defined in* (194).) *Let both* $\mathcal{C}_g \in \mathrm{Conf}_g(\Lambda, \Lambda_1)$ *and* $\widetilde{\mathcal{C}}_g \in \mathrm{Conf}_g(\widetilde{\Lambda}, \Lambda_1)$ *lie in* $\Lambda \cap \widetilde{\Lambda}$. *If further* $\mathcal{C}_g$ *and* $\widetilde{\mathcal{C}}_g$ *have the same gum tree with the same linear order and if they have the same r-strips then we say that* $\mathcal{C}_g$ *is* equivalent *to* $\widetilde{\mathcal{C}}_g$. *Then we have defined an equivalence relation and further, for* $\mathcal{C}_g$ *equivalent to* $\widetilde{\mathcal{C}}_g$, *we have*:

$$\mathrm{Simplex}(\mathcal{C}_g, T) = \mathrm{Simplex}(\widetilde{\mathcal{C}}_g, T) \quad \textit{for all } T \in (0, \infty], \tag{226}$$

$$\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} \circ \pi_\Lambda = \pi_{\Lambda_1} \circ \mathcal{L}^T_{\widetilde{\mathcal{C}}_g,\mathbf{t}} \circ \pi_{\widetilde{\Lambda}} \quad \textit{for all } \mathbf{t} \in \mathrm{Simplex}(\mathcal{C}_g, T) \tag{227}$$

$$\textit{and} \quad \pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g} \circ \pi_\Lambda = \pi_{\Lambda_1} \circ \mathcal{L}^T_{\widetilde{\mathcal{C}}_g} \circ \pi_{\widetilde{\Lambda}}. \tag{228}$$

(227) *and* (228) *say that the operators in* $L(\mathcal{H}_\vartheta, \mathcal{H}_{\Lambda_1})$ *are the same. We define by* $\mathrm{Conf}_g(\mathbb{Z}^d, \Lambda_1)$ *the set of equivalence classes. Because of* (226) *and* (227) *the simplices and operators for each equivalent class can be defined as being equal to the corresponding object for any representative.*

We will write $\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}}$ *instead of* $\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g,\mathbf{t}} \circ \pi_\Lambda$ *and* $\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g}$ *instead of* $\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g} \circ \pi_\Lambda$ *for the operators from* $\mathcal{H}_\vartheta$ *to* $\mathcal{H}_{\Lambda_1}$.

THEOREM 3.25. –
1. *There are* $0<\tilde\vartheta<\vartheta<1$ *such that for sufficiently small* ε, *large* c_g *and every* $T\in[0,\infty]$ *we can define an operator* $\mathcal{L}_S^T$ *from* $\mathcal{H}_\vartheta$ *to* $\mathcal{H}_{\tilde\vartheta}$ *by*

$$\pi_{\Lambda_1}\circ\mathcal{L}_S^T\phi=\sum_{\mathcal{C}_g\in\mathrm{Conf}_g(\mathbb{Z}^d,\Lambda_1)}\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_g}^T\phi. \tag{229}$$

There is a $T_0>0$ *such that for* $T\geqslant T_0$ *the operator* $\mathcal{L}_S^T$ *maps* $\mathcal{H}_\vartheta$ *into* $\mathcal{H}_\vartheta$. $\mathcal{L}_S^T$ *is the transfer operator, restricted to* $\mathcal{H}_\vartheta^{bv}$, *for the kernel* K_S^T, *i.e.*

$$\int_M \mathrm{d}\mu\,(K_S^T\psi)\phi=\int_M \mathrm{d}\mu\,\psi\,(\mathcal{L}_S^T\phi) \tag{230}$$

for all $\psi\in\mathcal{C}^0(M)$ *and* $\phi\in\mathcal{H}_\vartheta^{bv}$.
2. *The family* $(\mathcal{L}_S^T)_{T\geqslant 0}$ *in* $L(\mathcal{H}_\vartheta)$ *converges exponentially fast to* $\mathcal{L}_S^\infty$:

$$\|\mathcal{L}_S^\infty-\mathcal{L}_S^T\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\tilde\vartheta})}\leqslant c_5\mathrm{e}^{-c_6T} \tag{231}$$

for some positive constants c_5, c_6. *For sufficiently large* T *estimate* (231) *holds also in the norm of* $L(\mathcal{H}_\vartheta)$. *So among the probability measures corresponding to elements in* $\mathcal{H}_\vartheta$ *there is a unique* K_S^T*-invariant probability measure* ν^* *on* M, *say corresponding to* $\nu\in\mathcal{H}_\vartheta$. *The operator* $\mathcal{L}_S^\infty$ *is a projection onto* spanν:

$$\mathcal{L}_S^\infty\phi=\mu(\phi)\nu. \tag{232}$$

Proof. – The infinite sum on the rhs of (229) converges as the prove of estimate (204) applies literally to the case $\Lambda=\mathbb{Z}^d$. Next we want to show that $\pi_{\Lambda_1}\circ\mathcal{L}_S^T$ is the limit of $\pi_{\Lambda_1}\circ\mathcal{L}_{S,\Lambda}^T$ (as $\Lambda\to\mathbb{Z}^d$). The difference between these two operators is due to configurations $\mathcal{C}_g$ in $\mathrm{Confg}(\Lambda,\Lambda_1)$ or in $\mathrm{Confg}(\mathbb{Z}^d,\Lambda_1)$ with $\Lambda(\mathcal{C}_g)\not\subset\Lambda$. For these we can split in estimate (205) the factor that arises from the decay of interaction in the following way (which is the same as the splitting (110) in [13]).

$$\exp\left(-c_g\sum_{k=1}^\infty k^d n_{\beta,k}\right)\leqslant\exp\left(-\tilde c_g\sum_{k=1}^\infty k^d n_{\beta,k}\right)\exp\bigl(-\xi\,\mathrm{dist}(\Lambda_1,\Lambda^C)\bigr) \tag{233}$$

with a suitably chosen $\xi>0$ such that $\tilde c_g=c_g-\xi>0$. (Note that we can choose ξ so small that the estimates, formerly done with c_g work with $\tilde c_g$ instead as well.) So we can estimate

$$\begin{aligned}
&\tilde\vartheta^{|\Lambda_1|}\|\pi_{\Lambda_1}\circ\mathcal{L}_{S,\Lambda}^T-\pi_{\Lambda_1}\circ\mathcal{L}_S^T\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\Lambda_1})}\\
&\quad\leqslant 2\sum_{\substack{\mathcal{C}_g\in\mathrm{Conf}(\mathbb{Z}^d,\Lambda_1),\\ \Lambda(\mathcal{C}_g)\not\subset\Lambda}}\tilde\vartheta^{|\Lambda_1|}\|\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_g}^T\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\Lambda_1})}\\
&\quad\leqslant c_7\exp\bigl(-\xi\,\mathrm{dist}(\Lambda_1,\Lambda^C)\bigr).
\end{aligned} \tag{234}$$

Next we show (230) for the special case that ψ depends only on the Λ_1-coordinates, using (213):

$$\begin{aligned}\int_M \mathrm{d}\mu(\mathbf{z})\big(K_S^T\psi\big)(\mathbf{z})\phi(\mathbf{z}) &= \lim_{\Lambda\to\mathbb{Z}^d}\int_M \mathrm{d}\mu^\Lambda(\mathbf{z}_\Lambda)\big(K_{S,\Lambda}^T\psi\big)(\mathbf{z}_\Lambda)\phi_\Lambda(\mathbf{z}_\Lambda)\\ &= \lim_{\Lambda\to\mathbb{Z}^d}\int_{(S^1)^\Lambda} \mathrm{d}\mu^\Lambda(\mathbf{z}_\Lambda)\,\psi(\mathbf{z}_\Lambda)\big(\mathcal{L}_{S,\Lambda}^T\phi_\Lambda\big)(\mathbf{z}_\Lambda)\\ &= \lim_{\Lambda\to\mathbb{Z}^d}\int_{(S^1)^{\Lambda_1}} \mathrm{d}\mu^{\Lambda_1}(\mathbf{z}_\Lambda)\,\psi(\mathbf{z}_{\Lambda_1})\big(\pi_{\Lambda_1}\circ\mathcal{L}_s^T\phi_\Lambda\big)(\mathbf{z}_{\Lambda_1})\\ &= \int_M \mathrm{d}\mu(\mathbf{z})\,\psi(\mathbf{z})\big(\mathcal{L}_S^T\phi\big)(\mathbf{z}). \end{aligned}\tag{235}$$

We conclude (230) for general $\psi\in\mathcal{C}^0(M)$ by approximating it by ψ_{Λ_1} (cf. (141)), depending only on the Λ_1-coordinates and using continuity w.r.t. ψ of both sides of (230). So 1. is proved.

Next we show (231). We note that for $\Lambda_1=\emptyset$ the lhs (236) in the following estimate is equal to zero as both transfer operators preserve the Lebesgue integral (μ is a 'left eigenvector' with eigenvalue 1.) So we only have to consider the case $|\Lambda_1|\geqslant 1$.

$$\tilde{\vartheta}^{|\Lambda_1|}\big\|\pi_{\Lambda_1}\circ\mathcal{L}_S^\infty-\pi_{\Lambda_1}\circ\mathcal{L}_S^T\big\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\Lambda_1})}\tag{236}$$

$$\leqslant\tilde{\vartheta}^{|\Lambda_1|}\sum_{\mathcal{C}_g\in\mathrm{Conf}(\mathbb{Z}^d,\Lambda_1)}\big\|\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_g}^\infty-\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_g}^T\big\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\Lambda_1})}$$

$$\leqslant\tilde{\vartheta}^{|\Lambda_1|}\big\|\mathcal{Q}_{\Lambda_1}\circ\pi_{\Lambda_1}-\big(1-\mathrm{e}^{-\lambda T}\big)^{|\Lambda_1|}\mathcal{Q}_{\Lambda_1}\circ\pi_{\Lambda_1}\big\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\Lambda_1})}\tag{237}$$

$$+\,\tilde{\vartheta}^{|\Lambda_1|}\sum_{\substack{\mathcal{C}_g\in\mathrm{Confg}(\mathbb{Z}^d,\Lambda_1),\\ \mathcal{C}_g\text{reaches the top}}}\big\|\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_g}^T\big\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\Lambda_1})}\tag{238}$$

$$+\,\tilde{\vartheta}^{|\Lambda_1|}\sum_{\substack{\mathcal{C}_g\in\mathrm{Confg}(\mathbb{Z}^d,\Lambda_1),\\ \mathcal{C}_g\text{does not}\\ \text{reach the top},\\ |n_\beta(\mathcal{C}_g)|\geqslant 1}}\int_{\mathrm{Simplex}(\mathcal{C}_g,\frac{T}{2})}\lambda^{|n_\beta(\mathcal{C}_g)|}\,\mathrm{d}\mathbf{t}$$

$$\times\big\|\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_g,\mathbf{t}}^\infty-\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_g,\mathbf{t}}^T\big\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\Lambda_1})}\tag{239}$$

$$+\,\tilde{\vartheta}^{|\Lambda_1|}\sum_{\substack{\mathcal{C}_g\in\mathrm{Confg}(\mathbb{Z}^d,\Lambda_1),\\ \mathcal{C}_g\text{does not reach the top},\\ |n_\beta(\mathcal{C}_g)|\geqslant 1}}\int_{\substack{\mathrm{Simplex}(\mathcal{C}_g,\infty)\\ \setminus\mathrm{Simplex}(\mathcal{C}_g,\frac{T}{2})}}\lambda^{|n_\beta(\mathcal{C}_g)|}\,\mathrm{d}\mathbf{t}$$

$$\times\big\|\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_g,\mathbf{t}}^\infty\big\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\Lambda_1})}\tag{240}$$

$$+\,\tilde{\vartheta}^{|\Lambda_1|}\sum_{\substack{\mathcal{C}_g\in\mathrm{Confg}(\mathbb{Z}^d,\Lambda_1),\\ \mathcal{C}_g\text{does not reach the top},\\ |n_\beta(\mathcal{C}_g)|\geqslant 1}}\int_{\substack{\mathrm{Simplex}(\mathcal{C}_g,T)\\ \setminus\mathrm{Simplex}(\mathcal{C}_g,\frac{T}{2})}}\lambda^{|n_\beta(\mathcal{C}_g)|}\,\mathrm{d}\mathbf{t}$$

$$\times\big\|\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_g,\mathbf{t}}^T\big\|_{L(\mathcal{H}_\vartheta,\mathcal{H}_{\Lambda_1})}.\tag{241}$$

We have distinguished between the following classes of gum configurations. The first summand (237) corresponds to the operator $\pi_{\Lambda_1} \circ \mathcal{L}^{\infty}_{\mathcal{C}_g} - \pi_{\Lambda_1} \circ \mathcal{L}^{T}_{\mathcal{C}_g}$ where $\mathcal{C}_g$ is the gum configuration that has only long h-strips (no branchings or r-strips). The second summand (238) takes all $\mathcal{C}_g$ into account that reach the top. So all specified configurations $(\mathcal{C}_g, T, \mathbf{t})$ have an r-strip ending at time $-T$. All $(\mathcal{C}_g, \infty, \mathbf{t})$ have an infinitely long r-strip and so the corresponding operator is zero (cf. Remark 3.21.3) and does not appear in (238). The last three summands, (239), (240) and (241), correspond to $\mathcal{C}_g$ that do not reach the top and do not consist only of h-strips. That implies that it has at least one branching and the corresponding domains of integration, $\operatorname{Simplex}(\mathcal{C}_g, \infty)$ and $\operatorname{Simplex}(\mathcal{C}_g, T)$, are not degenerated to a point. We divide them into $\operatorname{Simplex}(\mathcal{C}_g, \frac{T}{2})$ and the particular complements. The reason for this will become clear when we do the estimates. In (239) we integrate the norm of the operator difference $\pi_{\Lambda_1} \circ \mathcal{L}^{\infty}_{\mathcal{C}_g} - \pi_{\Lambda_1} \circ \mathcal{L}^{T}_{\mathcal{C}_g}$ over $\operatorname{Simplex}(\mathcal{C}_g, \frac{T}{2})$ and in (240) and (241) we integrate the norms of the two operators separately over the particular complement sets.

Now we estimate each summand: The first summand (237) is estimated by

$$\tilde{\vartheta}^{|\Lambda_1|} c_h^{|\Lambda_1|}\left(1 - \left(1 - \mathrm{e}^{-\lambda T}\right)^{|\Lambda_1|}\right) \leqslant (\tilde{\vartheta} c_h)^{|\Lambda_1|} \sum_{k=1}^{|\Lambda_1|} \binom{|\Lambda_1|}{k} \left(\mathrm{e}^{-\lambda T}\right)^k$$
$$\leqslant \left(\tilde{\vartheta} c_h \left(1 + \mathrm{e}^{-\frac{1}{2}\lambda T}\right)\right)^{|\Lambda_1|} \mathrm{e}^{-\frac{1}{2}\lambda T} \leqslant \mathrm{e}^{-\frac{1}{2}\lambda T}, \qquad (242)$$

where the last inequality holds if $\tilde{\vartheta}$ is chosen sufficiently small.

For estimating the last summand (241) we note that for $\mathbf{t} \in \operatorname{Simplex}(\mathcal{C}_g, T) \setminus \operatorname{Simplex}(\mathcal{C}_g, \frac{T}{2})$ the sum of the lengths of all r-strips of $(\mathcal{C}_g, T, \mathbf{t})$ is at least $\frac{T}{2}$. (This is because $(\mathcal{C}_g, T, \mathbf{t})$ has a branching, say at time t_i with $|t_i| \geqslant \frac{T}{2}$ and there must be a sequence of apex-r-strips whose lengths add up to at least $\frac{T}{2}$.) So if we split in the estimate (170) for each maximal r-chain, of length t say, the rhs

$$\|\mathcal{R}(t)\| \leqslant c_r \mathrm{e}^{-(1-\eta)\lambda t} = c_r \mathrm{e}^{-\frac{1-\eta}{2}\lambda t} \mathrm{e}^{-\frac{1-\eta}{2}\lambda t} \qquad (243)$$

we can extract the second factor $\exp(-\frac{1-\eta}{2}\lambda t)$. Their product is bounded from above by $\exp(-\frac{1-\eta}{2}\lambda \frac{T}{2})$. We assume that the coupling parameters ε (small) and c_g (large) are such that our analysis still holds with the (in the sum) remaining factors $\exp(-\frac{1-\eta}{2}\lambda \operatorname{length}(R))$ for each maximal r-chain R.

We get

$$\tilde{\vartheta}^{|\Lambda_1|} \sum_{\substack{\mathcal{C}_g \in \operatorname{Confg}(\mathbb{Z}^d, \Lambda_1), \\ \mathcal{C}_g \text{ does not reach the top}, \\ |n_\beta(\mathcal{C}_g)| \geqslant 1}} \int_{\substack{\operatorname{Simplex}(\mathcal{C}_g, T) \\ \setminus \operatorname{Simplex}(\mathcal{C}_g, \frac{T}{2})}} \lambda^{|n_\beta(\mathcal{C}_g)|} \, \mathrm{d}\mathbf{t} \left\|\pi_{\Lambda_1} \circ \mathcal{L}^{T}_{\mathcal{C}_g, \mathbf{t}}\right\|_{L(\mathcal{H}_\vartheta, \mathcal{H}_{\Lambda_1})}$$
$$\leqslant c_8 \exp\left(-\lambda \frac{1-\eta}{2} \frac{T}{2}\right). \qquad (244)$$

Similarly, we can estimate the second (238) and the fourth (240) summand:

$$\tilde{\vartheta}^{|\Lambda_1|} \sum_{\substack{\mathcal{C}_g \in \operatorname{Confg}(\mathbb{Z}^d, \Lambda_1), \\ \mathcal{C}_g \text{ reaches the top}}} \left\|\pi_{\Lambda_1} \circ \mathcal{L}^{T}_{\mathcal{C}_g}\right\|_{L(\mathcal{H}_\vartheta, \mathcal{H}_{\Lambda_1})} \leqslant c_9 \exp\left(-\frac{1-\eta}{2}\lambda T\right), \qquad (245)$$

$$\tilde{\vartheta}^{|\Lambda_1|} \sum_{\mathcal{C}_g \in \mathrm{Confg}(\mathbb{Z}^d, \Lambda_1)} \int_{\substack{\mathrm{Simplex}(\mathcal{C}_g, \infty) \\ \setminus \mathrm{Simplex}(\mathcal{C}_g, \frac{T}{2})}} \lambda^{|n_\beta(\mathcal{C}_g)|} \,\mathrm{d}\mathbf{t} \left\| \pi_{\Lambda_1} \circ \mathcal{L}^{\infty}_{\mathcal{C}_g, \mathbf{t}} \right\|_{L(\mathcal{H}_\vartheta, \mathcal{H}_{\Lambda_1})}$$
$$\leqslant c_{10} \exp\left(-\frac{1-\eta}{2} \lambda \frac{T}{2} \right). \tag{246}$$

We estimate the third summand (239) we use Lemma 3.26 (see below). For that we note that for $\mathcal{C}_g \in \mathrm{Confg}(\mathbb{Z}^d, \Lambda_1)$ and $\mathbf{t} \in \mathrm{Simplex}(\mathcal{C}_g, \frac{T}{2})$ the operators $\pi_{\Lambda_1} \circ \mathcal{L}^{\infty}_{\mathcal{C}_g, \mathbf{t}}$ and $\pi_{\Lambda_1} \circ \mathcal{L}^{T}_{\mathcal{C}_g, \mathbf{t}}$ can both be written as a product of numbers $(1 - \mathrm{e}^{-\mathrm{length}(H)})$ and operators $\mathcal{Q}_q \otimes \mathrm{id}_{\Lambda \setminus \{q\}}$, corresponding to maximal h-strips, $\mathcal{R}_q \otimes \mathrm{id}_{\Lambda \setminus \{q\}}$, corresponding to maximal r-strips, and $\mathcal{B}_q \otimes \mathrm{id}_{\Lambda \setminus \{q\}}$, corresponding to branchings. They have the same structure in the sense that these factors are in $1-1$-correspondence and the quantitative difference is only due to h-strips going to the top or long h-strips in Λ_1 as we can see from representation (199) for $\mathcal{L}^{\infty}_{\mathcal{C}_g, \mathbf{t}}$ and $\mathcal{L}^{T}_{\mathcal{C}_g, \mathbf{t}}$ and also from Remark 3.21.1. So they differ only in the constants $\tilde{c}(\mathcal{C}_g, \infty, \mathbf{t})$ and $\tilde{c}(\mathcal{C}_g, T, \mathbf{t})$. More precisely, an h-strip in $\mathcal{C}_g$ that goes to the top and therefore corresponds to an h-strip in $(\mathcal{C}_g, T, \mathbf{t})$, say from (q, t_i) to $(q, -T)$, and so gives rise to a factor $1 - \exp(-\lambda(T - |t_i|))$ (note that $|t_i| < \frac{T}{2}$) whilst the corresponding h-strip in $(\mathcal{C}_g, \infty, \mathbf{t})$ ends at time $-\infty$ and gives rise to a factor 1. Similarly a long h-strip of $\mathcal{C}_g$ in Λ_1 gives rise to factors $1 - \exp(-\lambda T)$ and 1, respectively. In both cases the difference between the scalar factors (for each h-strip to the top) is bounded by

$$\delta^2 = \exp\left(-\frac{\lambda}{2} T \right). \tag{247}$$

The number of h-strips to the top is bounded by $\sum_{k=1}^{\infty} 3^d n_{\beta,k} k^d$ and the number of long h-strips at sites in Λ_1 by $|\Lambda_1| - K$ (where $n_{\beta,k}$ and K are the parameters of the labelled tree of $\mathcal{C}_g$.)

So we estimate

$$\tilde{\vartheta}^{|\Lambda_1|} \left\| \pi_{\Lambda_1} \circ \mathcal{L}^{\infty}_{\mathcal{C}_g, \mathbf{t}} \phi - \pi_{\Lambda_1} \circ \mathcal{L}^{T}_{\mathcal{C}_g, \mathbf{t}} \phi \right\|_{\Lambda_1}$$
$$\leqslant \delta \prod_{k=1}^{\infty} (1+\delta)^{3^d n_{\beta,k} k^d} (1+\delta)^{|\Lambda_1| - K} (c_3 \varepsilon)^{|n_\beta|} \exp\left(-c_g \sum_{k=1}^{\infty} k^d n_{\beta,k} \right) c_h^{\tilde{n}_h} c_r^{\tilde{n}_r}$$
$$\times \prod_R \exp\bigl(-(1-\eta) \lambda \mathrm{length}(R) \bigr) \tilde{c}(\mathcal{C}_g, T, \mathbf{t}) \|\phi_{\Lambda(\mathcal{C}_g)}\|_{\Lambda(\mathcal{C}_g)}. \tag{248}$$

The factor $(1+\delta)^{3^d n_{\beta,k} k^d}$ and the factor $\vartheta^{-(3k)^d n_{\beta,k}}$ that we get from the estimate (206) of $\|\phi_{\Lambda(\mathcal{C}_g)}\|_{\Lambda(\mathcal{C}_g)}$ are compensated for by $\exp(-c_g k^d n_{\beta,k})$ 'in the usual way'. If ε is sufficiently small and c_g large we can estimate

$$\tilde{\vartheta}^{|\Lambda_1|} \sum_{\substack{\mathcal{C}_g \in \mathrm{Confg}(\mathbb{Z}^d, \Lambda_1), \\ \mathcal{C}_g \text{ does not reach the top}, \\ |n_\beta(\mathcal{C}_g)| \geqslant 1}} \int_{\mathrm{Simplex}(\mathcal{C}_g, \frac{T}{2})} \lambda^{|n_\beta(\mathcal{C}_g)|} \,\mathrm{d}\mathbf{t} \left\| \pi_{\Lambda_1} \circ \mathcal{L}^{\infty}_{\mathcal{C}_g, \mathbf{t}} - \pi_{\Lambda_1} \circ \mathcal{L}^{T}_{\mathcal{C}_g, \mathbf{t}} \right\|_{L(\mathcal{H}_\vartheta, \mathcal{H}_{\Lambda_1})}$$
$$\leqslant c_{11} \mathrm{e}^{-\frac{\lambda}{4} T}. \tag{249}$$

From (242), (245), (249), (246), and (244) we conclude (231) with $c_6 = \frac{1-\eta}{4}\lambda$ and c_5 sufficiently large.

For any $\phi \in \mathcal{H}_\vartheta$ and any $\Lambda \in \mathcal{F}$ we have

$$\pi_\Lambda \circ \mathcal{L}_S^\infty \phi = \sum_{\substack{\mathcal{C}_g \in \mathrm{Confg}(\mathbb{Z}^d,\Lambda),\\ \mathcal{C}_g \text{does not reach the top}}} \pi_\Lambda \circ \mathcal{L}_{\mathcal{C}_g}^\infty \phi$$

$$= \sum_{\substack{\mathcal{C}_g \in \mathrm{Confg}(\mathbb{Z}^d,\Lambda),\\ \mathcal{C}_g \text{does not reach the top}}} \big(\pi_\Lambda \circ \mathcal{L}_{\mathcal{C}_g}^\infty h_{\mathbb{Z}^d}\big).\mu(\phi). \tag{250}$$

The sum in (250) is a priori over all $\mathcal{C}_g \in \mathrm{Conf}_g(\mathbb{Z}^d, \Lambda)$ but, as we have seen before, if $\mathcal{C}_g$ reaches the top the corresponding operator $\pi_\Lambda \circ \mathcal{L}_{\mathcal{C}_g}^\infty$ is zero. If $\mathcal{C}_g$ does not reach the top there are only h-strips going to the top ($-\infty$) and $\pi_\Lambda \circ \mathcal{L}_{\mathcal{C}_g}^\infty$ is a projection onto $\mathrm{span}(h_{\mathbb{Z}^d})$.

We set $\nu_\Lambda \stackrel{\text{def}}{=} \pi_\Lambda \circ \mathcal{L}_{\mathcal{C}_g}^\infty h_{\mathbb{Z}^d}$ and this defines $\nu = (\nu_\Lambda)_{\Lambda\in\mathcal{F}}$. Note that the transfer operator $\mathcal{L}_S^\infty$ preserves the Lebesgue integral and so $\nu_\emptyset = 1$, i.e. ν corresponds to a probability measure. □

In the proof of Theorem 3.25 we have used the following lemma.

LEMMA 3.26. – *Let $A_1, \ldots, A_n, \tilde{A}_1, \ldots, \tilde{A}_n$ be operators on the same Banach space, $0 < \delta < 1$ and $a_1, \ldots, a_n$ positive numbers such that:*

$$\|A_i\| \leqslant a_i \quad \textit{for all } 1 \leqslant i \leqslant n \tag{251}$$

$$\textit{and} \quad \|A_i - \tilde{A}_i\| \leqslant \delta^2 a_i. \tag{252}$$

Then

$$\|A_1 \circ \cdots \circ A_n - \tilde{A}_1 \circ \cdots \circ \tilde{A}_n\| \leqslant \delta(1+\delta)^n a_1 \cdot \cdots \cdot a_n. \tag{253}$$

Proof. – From (252) we get

$$\|\tilde{A}_i\| \leqslant \big(1+\delta^2\big)a_i. \tag{254}$$

So we get via ‘telescope expansion’:

$$\begin{aligned}
&\|A_1 \circ \cdots \circ A_n - \tilde{A}_1 \circ \cdots \circ \tilde{A}_n\| \\
&\quad\leqslant \|A_1 \circ \cdots \circ A_n - \tilde{A}_1 \circ A_2 \circ \cdots \circ A_n\| + \cdots \\
&\qquad + \|\tilde{A}_1 \circ \cdots \circ \tilde{A}_{n-1} \circ A_n - \tilde{A}_1 \circ \cdots \circ \tilde{A}_n\| \\
&\quad\leqslant \delta^2\big(1 + \big(1+\delta^2\big) + \cdots + \big(1+\delta^2\big)^{n-1}\big)a_1 \cdot \cdots \cdot a_n \\
&\quad= \big(\big(1+\delta^2\big)^n - 1\big)a_1 \cdot \cdots \cdot a_n \\
&\quad= \sum_{k=1}^n \binom{n}{k} \delta^{2k} a_1 \cdot \cdots \cdot a_n \leqslant \delta \sum_{k=1}^n \binom{n}{k} \delta^k a_1 \cdot \cdots \cdot a_n \\
&\quad\leqslant \delta(1+\delta)^n a_1 \cdot \cdots \cdot a_n
\end{aligned} \tag{255}$$

and the lemma is proved. □

Remark 3.27. – Analogously to Proposition 6.1.3 in [13], one can also prove a semigroup-like property of the family $(\mathcal{L}_S^t)_{t\geqslant 0}$, using Remark 2.17 and the diagrammatic representation of the operators.

4. Decay of correlations

In the following theorem which is completely analogous to Theorem 7.1 in [13], we state the mixing properties for the invariant probability measure ν^* in terms of the weighted norms.

THEOREM 4.1. – *For sufficiently small ϑ, $\tilde{\vartheta}$, ε and big c_g there is a $\kappa \in (0,1)$ and positive constants c_{12}, c_{13}, c_{14} and c_{15} such that for all finite disjoint $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$ and $\psi \in \mathcal{H}_{\Lambda_2}$ the following holds*:

$$\|\nu_{\Lambda_1\cup\Lambda_2} - \nu_{\Lambda_1}\nu_{\Lambda_2}\|_{\Lambda_1\cup\Lambda_2} \leqslant c_{12}\vartheta^{-|\Lambda_1\cup\Lambda_2|}\kappa^{\operatorname{dist}(\Lambda_1,\Lambda_2)}, \tag{256}$$

$$\|\pi_{\Lambda_1}(\psi\nu) - \nu^*(\psi)\nu_{\Lambda_1}\|_{\Lambda_1} \leqslant c_{13}\vartheta^{-|\Lambda_1\cup\Lambda_2|}\|\psi\|_{\Lambda_2}\kappa^{\operatorname{dist}(\Lambda_1,\Lambda_2)}, \tag{257}$$

$$\begin{aligned}\left\|\pi_{\Lambda_1}\circ\mathcal{L}_S^T(\psi\nu) - \nu^*(\psi)\nu_{\Lambda_1}\right\|_{\Lambda_1} \leqslant{}& c_{14}\vartheta^{-|\Lambda_2|}\tilde{\vartheta}^{-|\Lambda_1|}\|\psi\|_{\Lambda_2}\kappa^{\operatorname{dist}(\Lambda_1,\Lambda_2)}\\ &\times\exp(-c_{15}T)\end{aligned} \tag{258}$$

for every $T > 0$.

Proof. – For a gum configuration $\mathcal{C}_g$ we define in analogy to (109) in [13]

$$b(\mathcal{C}_g) \overset{\text{def}}{=} \sum_{k=1}^{\infty} k n_{\beta,k}(\mathcal{C}_g). \tag{259}$$

In the following we split gum configurations $\mathcal{C}_g \in \operatorname{Conf}(\mathbb{Z}^d, \Lambda_1\cup\Lambda_2)$ with $b(\mathcal{C}_g) \leqslant \frac{1}{2}\operatorname{dist}(\Lambda_1,\Lambda_2)$ into $\mathcal{C}_g = \mathcal{C}_g^1 \cup \mathcal{C}_g^2$ with $\mathcal{C}_g^1 \in \operatorname{Conf}(\mathbb{Z}^d,\Lambda_1)$, $\mathcal{C}_g^2 \in \operatorname{Conf}(\mathbb{Z}^d,\Lambda_2)$ and $\Lambda(\mathcal{C}_g^1)\cap\Lambda(\mathcal{C}_g^2) = \emptyset$.

We write, using (232) and the notation of (158):

$$\begin{aligned}\nu_{\Lambda_1\cup\Lambda_2} ={}& \sum_{\substack{\mathcal{C}_g=\mathcal{C}_g^1\cup\mathcal{C}_g^2\in\operatorname{Confg}(\mathbb{Z}^d,\Lambda_1\cup\Lambda_2),\\ b(\mathcal{C}_g)\leqslant\frac{1}{2}\operatorname{dist}(\Lambda_1,\Lambda_2)}} \left(\pi_{\Lambda_1}\circ\mathcal{L}^T_{\mathcal{C}_g^1}h_{\mathbb{Z}^d}\right)\left(\pi_{\Lambda_2}\circ\mathcal{L}^T_{\mathcal{C}_g^2}h_{\mathbb{Z}^d}\right)\\ &+ \sum_{\substack{\mathcal{C}_g\in\operatorname{Confg}(\mathbb{Z}^d,\Lambda_1\cup\Lambda_2),\\ b(\mathcal{C}_g)>\frac{1}{2}\operatorname{dist}(\Lambda_1,\Lambda_2)}} \pi_{\Lambda_1\cup\Lambda_2}\circ\mathcal{L}^T_{\mathcal{C}_g}h_{\mathbb{Z}^d}.\end{aligned} \tag{260}$$

In estimating the norm of the second summand in (260) we can take out from the estimate for $\|\pi_{\Lambda_1}\circ\mathcal{L}^{\infty}_{\mathcal{C}_g}h_{\mathbb{Z}^d}\|$ a factor

$$\exp\left(-\xi\frac{1}{2}\operatorname{dist}(\Lambda_1,\Lambda_2)\right) = \kappa^{\operatorname{dist}(\Lambda_1,\Lambda_2)} \tag{261}$$

like in (233) such that we get

$$\Bigg\| \sum_{\substack{\mathcal{C}_g \in \mathrm{Confg}(\mathbb{Z}^d, \Lambda_1 \cup \Lambda_2),\\ b(\mathcal{C}_g) > \frac{1}{2}\mathrm{dist}(\Lambda_1, \Lambda_2)}} \pi_{\Lambda_1 \cup \Lambda_2} \circ \mathcal{L}^T_{\mathcal{C}_g} h_{\mathbb{Z}^d} \Bigg\| \leqslant c_{16} \vartheta^{|\Lambda_1 \cup \Lambda_2|} \kappa^{\mathrm{dist}(\Lambda_1, \Lambda_2)}. \tag{262}$$

We write the first summand in (260) as

$$\begin{aligned} &\sum_{\substack{\mathcal{C}_g = \mathcal{C}_g^1 \cup \mathcal{C}_g^2 \in \mathrm{Confg}(\mathbb{Z}^d, \Lambda_1 \cup \Lambda_2),\\ b(\mathcal{C}_g) \leqslant \frac{1}{2}\mathrm{dist}(\Lambda_1, \Lambda_2)}} \big(\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g^1} h_{\mathbb{Z}^d}\big)\big(\pi_{\Lambda_2} \circ \mathcal{L}^T_{\mathcal{C}_g^2} h_{\mathbb{Z}^d}\big) \qquad (263)\\ &= \nu_{\Lambda_1} \nu_{\Lambda_2} - \sum_{\substack{\mathcal{C}_g^1 \mathrm{Confg}(\mathbb{Z}^d, \Lambda_1),\\ \mathcal{C}_g^2 \in \mathrm{Confg}(\mathbb{Z}^d, \Lambda_2),\\ b(\mathcal{C}_g^1) + b(\mathcal{C}_g^2) > \frac{1}{2}\mathrm{dist}(\Lambda_1, \Lambda_2)}} \big(\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g^1} h_{\mathbb{Z}^d}\big)\big(\pi_{\Lambda_2} \circ \mathcal{L}^T_{\mathcal{C}_g^2} h_{\mathbb{Z}^d}\big) \end{aligned}$$

and estimate

$$\Bigg\| \sum_{\substack{\mathcal{C}_g^1 \mathrm{Confg}(\mathbb{Z}^d, \Lambda_1),\\ \mathcal{C}_g^2 \in \mathrm{Confg}(\mathbb{Z}^d, \Lambda_2),\\ b(\mathcal{C}_g^1) + b(\mathcal{C}_g^2) > \frac{1}{2}\mathrm{dist}(\Lambda_1, \Lambda_2)}} \big(\pi_{\Lambda_1} \circ \mathcal{L}^T_{\mathcal{C}_g^1} h_{\mathbb{Z}^d}\big)\big(\pi_{\Lambda_2} \circ \mathcal{L}^T_{\mathcal{C}_g^2} h_{\mathbb{Z}^d}\big) \Bigg\| \leqslant c_{17} \vartheta^{|\Lambda_1 \cup \Lambda_2|} \kappa^{\mathrm{dist}(\Lambda_1, \Lambda_2)}. \tag{264}$$

From (262), (263) and (264) we conclude (256). The proof of (257), using (256), is the same as in [13].

To prove (258) we set $\phi = \psi\nu - \nu(\psi)\nu$. So

$$\pi_{\Lambda_1} \circ \mathcal{L}^T_S \phi = \pi_{\Lambda_1} \circ \mathcal{L}^T_S(\psi\nu) - \nu(\psi)\nu_{\Lambda_1} \tag{265}$$

and in particular

$$\mathcal{L}^\infty_S \phi = 0. \tag{266}$$

We estimate (265), analogously to (129) in [13], using the finer estimate

$$\|\phi_{\Lambda(\mathcal{C})}\|_{\Lambda(\mathcal{C})} \leqslant c_{13} \vartheta^{-|\Lambda_2|} \|\psi\|_{\Lambda_2} \vartheta^{-|\Lambda_r(\mathcal{C})| - \sum_{k=1}^\infty (3k)^d n_{\beta,k}} \kappa^{\mathrm{dist}(\Lambda_1, \Lambda_2) - \sum_{k=1}^\infty k n_{\beta,k}} \tag{267}$$

that we get from (257). For each $\mathcal{C}_g$ we get a ‘good’ factor $\kappa^{\mathrm{dist}(\Lambda_1, \Lambda_2)}$ that we can take out of the sum (over gum configurations), and a ‘bad’ factor $\kappa^{-\sum_{k=1}^\infty k n_{\beta,k}}$. The latter is compensated for in the usual way by the factor $\exp(-c_g \sum_{k=1}^\infty k^d n_{\beta,k})$, provided that c_g is sufficiently large.

Using (266) and (267), we get with the same argument as for the proof of (231):

$$\tilde{\vartheta}^{|\Lambda_1|} \big\|\pi_{\Lambda_1} \circ \mathcal{L}^T_S \phi\big\| \leqslant c_{18} \vartheta^{-|\Lambda_2|} \|\psi\| \kappa^{\mathrm{dist}(\Lambda_1, \Lambda_2)} \exp(-c_{15} T) \tag{268}$$

and (258) follows. □

For our last theorem we need some definitions.

DEFINITION 4.2. – *Every* $\tau = (\tau_1, \ldots, \tau_d) \in \mathbb{Z}^d$ *defines a shift on lattice* $\mathbb{Z}^d$ *by* $(\alpha_1, \ldots, \alpha_d) + (\tau_1, \ldots, \tau_d) = (\alpha_1 + \tau_1, \ldots, \alpha_d + \tau_d)$, *and so* a shift on $(S^1)^{\mathbb{Z}^d}$ *by* $(\tau(\mathbf{x}))_\alpha \stackrel{\text{def}}{=} x_{\alpha+\tau}$ *for* $\mathbf{x} \in (S^1)^{\mathbb{Z}^d}$. *The size* $m(\tau)$ of the shift τ *is* $m(\tau) \stackrel{\text{def}}{=} |\tau_1| + \ldots + |\tau_d|$.

We further define a shift on functions $\psi \in \mathcal{C}((S^1)^{\mathbb{Z}^d})$ *by* $(\psi \circ \tau)(\mathbf{x}) \stackrel{\text{def}}{=} \psi(\tau(\mathbf{x}))$.

The family of maps $(f_q)_{q\in\mathbb{Z}^d}$, *introduced in* (149), *is called* translation-invariant *if these maps are all the same, i.e.* $f_q = f$ *for some* f *and all* $q \in \mathbb{Z}^d$.

The family $(g_{q,k})_{q\in\mathbb{Z}^d}$, *introduced in* (151), *is called* translation-invariant *if* $g_{q,k}(\tau(\mathbf{z})) = g_{q-\tau,k}(\mathbf{z})$ *for all* $q, \tau \in \mathbb{Z}^d$ *and* $\mathbf{z} \in ((S^1)^{\mathbb{Z}^d})$.

If $(f_q)_{q\in\mathbb{Z}^d}$ *and also the families* $(g_{q,k})_{q\in\mathbb{Z}^d}$ *are translation-invariant then we say that the* system *is* translation-invariant.

Remark 4.3. – In case of a translation-invariant system we also have translation-invariance of the action of Markov-kernels on functions: $K_S^T(\psi \circ \tau) = (K_S^T(\psi)) \circ \tau$ for all $\psi \in \mathcal{C}((S^1)^{\mathbb{Z}^d})$ and $\tau \in \mathbb{Z}^d$.

We can state the mixing properties of ν^* w.r.t. spatio-temporal shifts in terms of correlation functions for observables $\psi_1, \psi_2 \in \mathcal{C}^0(M)$ like in Theorem 2.2 of [13] and, using Theorem 4.1, prove them in exactly the same way.

THEOREM 4.4. – *For sufficiently small* ϑ, ε *and large* c_g *there is a* $\kappa \in (0,1)$ *such that for all nonempty* $\Lambda_1, \Lambda_2 \in \mathcal{F}$ *the following holds with the constant* $c(\Lambda_1, \Lambda_2, \kappa) \stackrel{\text{def}}{=} \kappa^{-\max\{\|p-q\|: p\in\Lambda_1, q\in\Lambda_2\}}$ *and some positive constants* c_{19}, c_{20}:

1. *If* $\psi_1 \in \mathcal{C}((S^1)^{\Lambda_1})$ *and* $\psi_2 \in \mathcal{C}((S^1)^{\Lambda_2})$ *then*

$$\left| \int_M \mathrm{d}\nu^* \psi_1 \psi_2 - \left(\int_M \mathrm{d}\nu^* \psi_1 \right) \left(\int_M \mathrm{d}\nu^* \psi_2 \right) \right| \leqslant c_{19} \vartheta^{-|\Lambda_1|-|\Lambda_2|} \|\psi_1\|_\infty \|\psi_2\|_\infty \kappa^{\operatorname{dist}(\Lambda_1,\Lambda_2)}. \tag{269}$$

2. *If* $\psi_1 \in \mathcal{C}((S^1)^{\Lambda_1})$ *and* $\psi_2 \in \mathcal{H} \cap \mathcal{C}((S^1)^{\Lambda_2})$ *then*

$$\left| \int_M \mathrm{d}\nu^* K_S^T(\psi_1 \circ \tau)\psi_2 - \left(\int_M \mathrm{d}\nu^* \psi_1 \circ \tau \right) \left(\int_M \mathrm{d}\nu^* \psi_2 \right) \right| \leqslant c(\Lambda_1, \Lambda_2, \kappa) c_{20}^{|\Lambda_1|+|\Lambda_2|} \|\psi_1\|_\infty \|\psi_2\|_{\Lambda_2} \kappa^{m(\tau)} \exp(-c_{21} T). \tag{270}$$

3. *If the system is translation-invariant and* $\psi_1, \psi_2 \in \mathcal{C}(M)$ *then*

$$\lim_{\max\{m(\tau),T\}\to\infty} \int_M \mathrm{d}\nu^* K_S^T(\psi_1 \circ \tau)\psi_2 = \left(\int_M \mathrm{d}\nu^* \psi_1 \right) \left(\int_M \mathrm{d}\nu^* \psi_2 \right). \quad \square \tag{271}$$

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