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A deviation inequality for non-reversible Markov processes

by

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ABSTRACT. – Using the dissipative criterion of Lumer–Philips for the contraction semigroup, we get in this Note a new deviation inequality for $\int_0^t V(X_s) \, ds$ by means of the symmetrized Dirichlet form. A more explicit version is obtained in the case where the logarithmic Sobolev inequality holds.

Key words: Dirichlet forms, Deviation inequality, Logarithmic Sobolev inequality

AMS classification: 60F10, 60J25

RÉSUMÉ. – Par le critère de dissipativité de Lumer–Philips pour la contractivité de semigroupes, on obtient une inégalité nouvelle de déviation pour $\int_0^t V(X_s) \, ds$ via la forme de Dirichlet symétrisée. Une expression plus explicite est obtenue dans le cas où l’inégalité de Sobolev logarithmique est vraie.

Mots Clés: Forme de Dirichlet, Inégalité de déviation, Inégalité de Sobolev logarithmique

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1. Let \((\Omega, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, (X_t)_{t \in \mathbb{R}^+}, (P_x)_{x \in E})\) be a conservative càdlàg Markov process with values in a Polish space \(E\), with semigroup of transition probability \((P_t(x, dy))\). We assume that \(\mu\) is a probability measure on \(E\) (equipped with the Borel \(\sigma\)-field \(\mathcal{B}\)), which is invariant and ergodic with respect to \((P_t)\). For any initial measure \(\nu\) on \(E\), write \(P_\nu := \int_E P_x \nu(dx)\).

We denote by \((\mathcal{L}, D_p(\mathcal{L}))\) the generator of \((P_t)\) acting on \(L^p(E, \mu)\) \((D_p(\mathcal{L})\) being its domain in \(L^p)\), where \(1 \leq p < +\infty\). The symmetrized Dirichlet form is given by

\[
\mathcal{E}^\sigma(f, g) := \frac{1}{2} \left[ \langle -\mathcal{L} f, g \rangle_{\mu} + \langle -\mathcal{L} g, f \rangle_{\mu} \right], \quad \forall f, g \in D_2(\mathcal{L}),
\]

where \(\langle \cdot, \cdot \rangle_{\mu}\) is the usual inner product in \(L^2(E, \mu)\).

Under the assumption below

\((\mathcal{E}^\sigma, D_2(\mathcal{L}))\) is closable, \hspace{1cm} (H1)

its closure \((\mathcal{E}^\sigma, D(\mathcal{E}^\sigma))\) corresponds to a symmetric Markov semigroup \((P^\sigma_t)_{t \geq 0}\) on \(L^2(E, \mu)\).

Given a measurable function \(V : E \to \mathbb{R}\), \(\mu\)-integrable. In this note we are interested to the probability of deviation of the empirical mean \(\frac{1}{t} \int_0^t V(X_s) \, ds\) from its real (or asymptotic) mean \(m := \int_E V \, d\mu := \langle V \rangle_{\mu}\), i.e.,

\[
P_\nu \left( \left| \frac{1}{t} \int_0^t V(X_s) \, ds - m \right| > r \right).
\]

Introduce

\[
J_V(r) := \inf \left\{ \mathcal{E}^\sigma(f, f) \left| f \in D(\mathcal{E}^\sigma) \cap L^2(|V| \, d\mu), \int f^2 \, d\mu = 1; \right. \right. \left. \left. \text{ and } \int Vf^2 \, d\mu = r \right\}
\]

for every \(r \in \mathbb{R}\) \((Convention: \inf \emptyset := +\infty)\). As is easily seen, \(J_V\) is a convex function on \(\mathbb{R}\). Then \([J_V < +\infty]^0\) (interior) is some interval \((a, b)\) where \(-\infty \leq a \leq b \leq +\infty\).

Define now \(I_V\) as the lower semi-continuous (l.s.c. in short) regularization of \(J_V\). Obviously \(I_V(m) = J_V(m) = 0\) and \(I_V : \mathbb{R} \to [0, +\infty]\) is convex. Then \(I_V\) is non-decreasing on \([m, +\infty)\) and non-increasing on
Notice that when $a < b$, then for any $r \in \mathbb{R}$,

$$I_V(r) =
\begin{cases}
  J_V(r), & \text{if } r \in (a, b); \\
  J_V(a+), & \text{if } r = a; \\
  J_V(b-), & \text{if } r = b; \\
  +\infty, & \text{otherwise.}
\end{cases}
$$

(3)

When our Markov process $(X_t)$ is $\mu$-reversible (or $(P_t)$ is $\mu$-symmetric), Deuschel and Stroock [4, Theorem 5.3.10, p. 210] (1989) proved essentially the following large deviation estimation (where a general level-2 large deviation lower bound is given)

$$
\lim_{t \to \infty} \frac{1}{t} \log P_v \left( \frac{1}{t} \int_0^t V(X_s) \, ds - m > r \right) = -I_V(m + r), \quad \forall r \geq 0,
$$

(4)


In this little note we propose to extend and strengthen (4). Our main observation is

**Theorem 1.** Assume (H1). For any initial measure $\nu$ such that $\nu \ll \mu$ and $\frac{d\nu}{d\mu} \in L^2(\mu)$, we have for all $t > 0$, all $r > 0$,

$$
P_v \left( \frac{1}{t} \int_0^t V(X_s) \, ds - m > r \right) \leq \left\| \frac{d\nu}{d\mu} \right\|_{L^2(\mu)} \cdot \exp \left[ -t \cdot I_V(m + r) \right],
$$

(5)

and

$$
P_v \left( \frac{1}{t} \int_0^t V(X_s) \, ds - m < -r \right) \leq \left\| \frac{d\nu}{d\mu} \right\|_{L^2(\mu)} \cdot \exp \left[ -t \cdot I_V(m - r) \right].
$$

(6)

**Remark 2.** In the symmetric case, the deviation inequality (5) is sharp in its exponent for large time $t$, by (4). The main differences between (4) and (5) are:

(i) The symmetry assumption required in (4) is removed for (5);

(ii) In (5), $t$ and $r$, being arbitrary, are fixed unlike in (4) which is only an asymptotic relation ($t \to +\infty$). Hence (5) is much more stronger and practical.

However in the non-symmetric case, inequality (5) is no longer asymptotically exact. In fact, when the level-2 large deviation principle of Donsker–Varadhan holds and $V$ is bounded, the limit (4) is given by a contraction form of the Donsker–Varadhan entropy functional, which is different from the expression in terms of Dirichlet form. See Deuschel and Stroock [4, Chapter VI] and Ben Arous and Deuschel [1] (1994).
Nevertheless that last large deviation result requires quite restrictive conditions in the non-symmetric case: indeed there exist geometrically ergodic irreducible Markov processes so that the level-1 large deviation principle fails (see Bryc and Smolenski [2] (1993)). While the deviation inequality (5) requires only (H1), which is satisfied in the most part of interesting cases. Moreover (H1) can be removed in case that $V$ is bounded, see Remarks 3(a) below.

2. Proof of Theorem 1. Consider the Feynman–Kac semigroup

$$P_t^V f(x) := E^x f(X_t) \cdot \exp \left( \int_0^t V(X_s) \, ds \right)$$

(7)

where $f \geq 0$ is $\mathcal{B}$-measurable. We shall establish for any $\mu$-integrable function $V : E \to \mathbb{R},$

$$0 < \|P_t^V\|_2 \leq e^{\Lambda(V)}, \quad \forall t \geq 0,$$

(8)

where

$$\|P_t^V\|_2 := \sup \{ \|P_t^V f\|_{L^2(\mu)} : f \geq 0 \text{ and } \langle f^2 \rangle_\mu \leq 1 \}$$

$$= \sup \{ \langle P_t^V f, g \rangle_\mu : f, g \geq 0 \text{ and } \langle f^2 \rangle_\mu \leq 1, \langle g^2 \rangle_\mu \leq 1 \},$$

and

$$\Lambda(V) := \sup \left\{ -\mathcal{E}_V^\sigma(f, f) \big| f \in \mathcal{D}(\mathcal{E}_V^\sigma), \int f^2 \, d\mu = 1 \right\}.$$

(9)

Here

$$\mathcal{D}(\mathcal{E}_V^\sigma) := \mathcal{D}(\mathcal{E}_\sigma) \cap L^2(|V| \, d\mu), \quad \mathcal{E}_V^\sigma(f, f) = \mathcal{E}_\sigma(f, f) - \int V f^2 \, d\mu.$$

Let us see quickly why (8) implies (5), by a very classical argument borrowed from the Cramér theorem [4]. In fact set $P(\lambda) := \Lambda(\lambda V), \forall \lambda \in \mathbb{R}.$ By Chebychev’s inequality, for all $r, t > 0$ fixed,

$$P_\nu \left( \frac{1}{t} \int_0^t V(X_s) \, ds - m > r \right)$$

$$\leq \inf_{\lambda > 0} \exp \left[ -\lambda t (m + r) \right] \cdot E^\nu \exp \left[ \lambda \int_0^t V(X_s) \, ds \right]$$
It remains to identify the exponent in the last term of (10).

Since $A(V) \geq \lambda m$ by the definition (9), $m$ is a sub-differential of $P(\lambda)$ at $\lambda = 0$. Thus for $r > 0$,

$$
\sup_{\lambda > 0} [\lambda(m + r) - P(\lambda)] = \sup_{\lambda \in \mathbb{R}} [\lambda(m + r) - P(\lambda)],
$$

which is the Legendre transformation $P^*(m + r)$ of $P(\lambda)$.

On the other hand, we have by (9)

$$
P(\lambda) = A(V) = \sup\{\lambda z - J_V(z); z \in \mathbb{R}\} = \sup\{\lambda z - I_V(z); z \in \mathbb{R}\}
$$

for all $\lambda \in \mathbb{R}$. Hence the famous Fenchel–Legendre theorem gives us

$$
P^*(m + r) = I_V(m + r).
$$

Substituting those into (10), we get (5).

Applying (5) to $-V$, we get (6).

Consequently to conclude this theorem, it remains to show (8). We divide its proof into three cases.

**Case 1. $-V$ bounded.** In this bounded case $(P_t^V)$ is a strongly continuous semigroup of bounded operators on $L^2(\mu)$, whose generator is exactly $(\mathcal{L} + V; D_2(\mathcal{L} + V) = D_2(\mathcal{L}))$ by the well known Feynman–Kac formula. By the definition (9) of $A(V)$,

$$
\langle (\mathcal{L} + V - A(V))f, f \rangle_\mu \leq 0, \quad \forall f \in D_2(\mathcal{L}).
$$

That means exactly that the generator $\mathcal{L} + V - A(V)$ with domain $D_2(\mathcal{L})$ is a dissipative operator on $L^2(E, \mu)$ in the sense of Lumer and Philips [9, Chapter IX, p. 250]. By the Lumer–Philips Theorem [9, Chapter IX, p. 250], the semigroup $(e^{-tA(V)}P_t^V)$ generated by $\mathcal{L} + V - A(V)$ is
contractive on $L^2(E, \mu)$. In other words,

$$\|e^{-t\Lambda(V)}P_t^V\|_2 \leq 1, \quad \forall t \geq 0,$$

which is exactly (8).

**Case 2.** $V$ upper bounded ($V \leq a$). Considering $V - a$ if necessary, we can assume $V \leq 0$. Take $V_n = \max\{V, -n\}$ for $n \in \mathbb{N}$. We have by the Case 1,

$$\|P_t^V\|_2 \leq \lim_{n \to \infty} \|P_t^{V_n}\|_2 \leq \lim_{n \to \infty} e^{t\Lambda(V_n)} = \exp(t \cdot \inf_{n \geq 1} \Lambda(V_n)). \quad (12)$$

Recall that

$$-\Lambda(V_n) = \inf \left\{ \mathcal{E}^\sigma(f, f) - \int V_n f^2 d\mu \mid f \in \mathcal{D}(\mathcal{E}^\sigma) \text{ and } \int f^2 d\mu \leq 1 \right\}$$

$$= \inf \left\{ F_n(f) \mid \int f^2 d\mu \leq 1 \right\},$$

where $F_n: L^2(E, \mu) \to [0, +\infty]$ is given by

$$F_n(f) := \mathcal{E}^\sigma(f, f) - \int V_n f^2 d\mu, \quad \text{if } f \in \mathcal{D}(\mathcal{E}^\sigma), \text{ and } +\infty \text{ else.}$$

By Kato [5, p. 461, Lemma 3.14a] and our assumption (H1), $F_n$ is lower semicontinuous on $L^2(E, \mu)$ with respect to the strong topology, then with respect to the weak topology $\sigma(L^2, L^2)$ (since $F_n$, being the sum of two nonnegative quadratic forms, is convex on $L^2(E, \mu)$). Moreover, since the unit ball $\{ f \in L^2(\mu); \int f^2 d\mu \leq 1 \}$ is compact with respect to $\sigma(L^2, L^2)$, by an elementary analytical lemma (see e.g. [8, Proposition 1.2]),

$$-\inf_{n \geq 1} \Lambda(V_n) = \sup_{n \geq 1} \inf \left\{ F_n(f) \mid \int f^2 d\mu \leq 1 \right\}$$

$$= \inf \left\{ \sup_{n} F_n(f) \mid \int f^2 d\mu \leq 1 \right\} = -\Lambda(V).$$

Substituting it into (12), we get (8) again.

**Case 3.** General case. Take $V_N = \min\{V, N\}$ for $N \in \mathbb{N}$. By the monotone convergence theorem,

$$\|P_t^V\|_2 = \sup \{ \langle P_t^V f, g \rangle \mid f, g \geq 0 \text{ and } \langle f^2 \rangle \mu \leq 1, \langle g^2 \rangle \mu \leq 1 \}$$
\[
\begin{align*}
&= \sup_{N \geq 1} \sup \{ \langle P_t^{V_N} f, g \rangle_{\mu} \mid f, g \geq 0 \text{ and } \langle f^2 \rangle_{\mu} \leq 1, \langle g^2 \rangle_{\mu} \leq 1 \} \\
&\leq \sup_{N \geq 1} e^{t \Lambda(V_N)} = e^{t \Lambda(V)},
\end{align*}
\]
where the third inequality follows from the Case 2, and the last equality follows from the fact that \( D(\mathcal{E}^\sigma) \cap L^\infty(\mu) \) is a form core for all \( \mathcal{E}^\sigma_{V_N}, N \geq 1 \), and for the not necessarily closable quadratic form \( \mathcal{E}^\sigma_V \).

The proof of (8) and then that of Theorem 1 are so finished. \( \square \)

**Remark 3.**

(a) When \( V \) is bounded, it holds that
\[
\| P_t^V \|_2 \leq \exp[t \cdot \Lambda^0(V)]
\]
where
\[
\Lambda^0(V) := \sup \left\{ \int Vf^2 \, d\mu + \langle \mathcal{L} f, f \rangle_{\mu} \mid f \in D_2(\mathcal{L}) \text{ and } \langle f^2 \rangle_{\mu} \leq 1 \right\}
\]
without the assumption (H1) about the closability of \( (\mathcal{E}^\sigma, D_2(\mathcal{L})) \), by the proof in the Case 1 above. As in the proof of (8) \( \Rightarrow \) (5) above, one can deduce from (13) the deviation inequalities (5) and (6) without (H1), but with \( I_V \) substituted by the l.s.c. regularization \( I^0_V \) of
\[
J^0_V(r) := \inf \left\{ \mathcal{E}^\sigma(f, f) \mid f \in D_2(\mathcal{L}), \int f^2 \, d\mu = 1; \int Vf^2 \, d\mu = r \right\}.
\]
When (H1) is satisfied and \( V \) is bounded, \( \Lambda^0(\lambda V) = \Lambda(\lambda V), \forall \lambda \in \mathbb{R} \) (by the fact that \( D_2(\mathcal{L}) \), being a form core of \( \mathcal{E}^\sigma \), is so for \( \mathcal{E}^\sigma_{V} \) because of the boundedness of \( V \)), and then \( I^0_V = I_V \).

(b) Note also the following (indicated by the referee): the inequality (8) implies not only (5) and (6), but also (with the same argument)
\[
\mathbb{E}^\mu f(X_0)g(X_t)1_{\left[ \frac{1}{2} \int_0^t V(X_s) \, ds - m > r \right]} \leq \| f \|_{L^2(\mu)} \| g \|_{L^2(\mu)} \cdot \exp[-t \cdot I_V(m + r)], \forall r, t > 0.
\]

(c) Applying the Lumer–Philips theorem to \( \mathcal{L} - V \) in \( L^p(\mu) \) with \( 1 \leq p < +\infty \), we get, instead of (8), that for any \( V \) bounded,
\[
\| P_t^V \|_p \leq \exp(t \Lambda_p(V))
\]
where

\[
\Lambda_\rho(V) := \sup \left\{ \int V |f|^\rho \, d\mu + \langle \text{sgn}(f) |f|^{\rho-1}, \mathcal{L}f \rangle_\mu \mid f \in D_\rho(\mathcal{L}) \right\}.
\]

3. In this paragraph we do not require (H1) but we assume the log-Sobolev inequality below: there exists \(C > 0\) such that for all \(f \in D_2(\mathcal{L})\),

\[
\int_E f^2 \log f^2 - \langle f^2 \rangle_\mu \log \langle f^2 \rangle_\mu \leq C (-\mathcal{L} f, f)_\mu. \tag{14}
\]

Consider the log-Laplace transformation of \(V - m\):

\[
H(\lambda) = \log \int_E e^{\lambda V} \, d\mu - \lambda m \tag{15a}
\]

and its Legendre transformation

\[
H^*(r) = \sup \{ \lambda r - H(\lambda) ; \lambda \in \mathbb{R} \}. \tag{15b}
\]

By the classical Cramèr’s theorem [4], \(H^*\) governs the large deviation principle of the i.i.d. sequence of common law \(\mu(V - m \in \cdot)\).

The following result says that the log-Sobolev inequality (14) implies a same type of estimation as in the i.i.d. case.

**Corollary 4.** Assume (14) (not (H1)). Then for any \(V \in L^1(\mu)\),

\[
\frac{1}{t} \log \|P_t^V\|_2 \leq \frac{1}{C} \log \int_E e^{CV} \, d\mu. \tag{16}
\]

In particular for each initial measure \(v \ll \mu\) with \(\frac{dv}{d\mu} \in L^2(\mu)\) and for all \(r > 0, t > 0\)

\[
P_v \left( \frac{1}{t} \int_0^t V(X_s) \, ds - m > r \right) \leq \left\| \frac{dv}{d\mu} \right\|_{L^2(\mu)} \cdot \exp \left( - \frac{t}{C} H^*(r) \right). \tag{17}
\]

*Proof.* The deviation inequality (17) follows from (16) by Chebyshev’s inequality as in Theorem 1. To show the key (16), assume at first that \(V\) is bounded.
By (13) in Remark 3, we have

\[
\frac{1}{t} \log \| P_t^V \|_2 
\leq \sup \left\{ \int V f^2 \, d\mu + \langle \mathcal{L} f, f \rangle_\mu \mid f \in D_2(\mathcal{L}) \quad \text{and} \quad \langle f^2 \rangle_\mu = 1 \right\}
\]

\[
\leq \sup \left\{ \int V f^2 \, d\mu - \frac{1}{C} \int f^2 \log f^2 \, d\mu \mid f \in D_2(\mathcal{L}) \quad \text{and} \quad \langle f^2 \rangle_\mu = 1 \right\} \quad \text{(by (14))}
\]

\[
= \frac{1}{C} \log \int_E e^{CV} \, d\mu,
\]

where the last equality follows from Donsker–Varadhan’s variational formula (see e.g. [8]).

Now for \( V \) unbounded, set \( V_n = \min\{\max\{V, -n\}, n\} \). We have

\[
\| P_t^V \|_2 \leq \liminf_{n \to +\infty} \| P_t^{V_n} \|_2 \leq \lim_{n \to +\infty} \left( \int e^{CV_n} \, d\mu \right)^{t/C} = \left( \int e^{CV} \, d\mu \right)^{t/C}
\]

by the bounded case shown above and the dominated convergence (and Fatou’s lemma if the last integral is infinite). (16) is hence established. \( \square \)

Remark 5. – Ledoux [6] (1999) develops systematically the so called Herbst method which consists to derive deviation inequalities from a log-Sobolev inequality. The strategy consists to apply a log-Sobolev inequality to \( e^{\lambda F} \) to obtain a differential inequation, from which a control on \( E e^{\lambda F} \) is deduced by comparison lemma. Nevertheless for that strategy works here for \( F = \int_0^t V(X_s) \, ds \), we should assume that a log-Sobolev inequality on the path space \( (D([0, t], E), P_t) \) holds, which is in general not the case here.

Even in case that such a path level log-Sobolev inequality holds, it seems that the Herbst method does not give directly better estimation than (17). For instance, let \( (B_t) \) be the Brownian motion on a Riemannian manifold \( E \), with generator \( \Delta/2 \), where \( \Delta \) is the Laplace–Beltrami operator. Assume that the Ricci curvature satisfies \( |Ric_u| \leq K \) for all \( u \in \mathcal{O}(E) \) (the bundle of orthonormal frames on \( E \)). By Capitaine–Hsu–Ledoux [3, (6)], the path level log-Sobolev inequality below holds:

\[
E^x \left( F^2 \log F^2 \right) - E^x F^2 \log E^x F^2 \leq 2e^{Kt} E^x |DF|^2_{H}
\]

(18)
for any \( x \in E \) and \( F : C([0, t]; E) \to \mathbb{R} \) provided that the right side term above is finite, where \( |DF|_H \) is the norm in the Cameron–Martin subspace of the Malliavin derivative \( DF \) on the path space. Now the Herbst method developed in [6, §2.3] yields: if \( |DF|_H^2 \leq \sigma^2, \) \( P_x \)-a.s., then

\[
P_x(F - E^xF > r) \leq \exp\left(-\frac{r^2}{2e^{Kt}\sigma^2}\right).
\]

(19)

Using the notations of [3], we can easily prove that for \( F = \int_0^t V(B_s) \, ds \) with \( \|\nabla V\|_\infty := \sup_{x \in E} |\nabla V(x)| < +\infty \) (where \( |\nabla V(x)| \) is the Riemannian norm of the gradient of \( V \) at \( x \)),

\[
|DF|_H^2 \leq \int_0^t \left( \int_s^t |\nabla V| (B_u) \, du \right)^2 \, ds \leq \|\nabla V\|_\infty^2 \cdot \frac{t^3}{3}, \quad P_x \text{-a.s.}
\]

We then obtain by (19),

\[
P_x\left( \int_0^t V(B_s) \, ds - E^x\int_0^t V(B_s) \, ds > rt \right) \leq \exp\left(-\frac{3r^2}{2te^{Kt}\|\nabla V\|_\infty^2}\right).
\]

(20)

That estimation is quite interesting and sharp for small \( t \), but not so for large \( t \). On the other hand, when \( E \) is compact, the log-Sobolev inequality (14) holds (a well known fact), then (17) is valid and it gives a much better estimation than (20) for large \( t \).

Our approach in Corollary 4 consists to apply log-Sobolev inequality after obtaining the control of \( \|P_t^V\|_2 \) (in Theorem 1), not before, unlike in the Herbst method. One can regard it as another application of log-Sobolev inequality, complementing those amply developed by Ledoux [6].

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