A deviation inequality for non-reversible Markov processes


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by

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ABSTRACT. – Using the dissipative criterion of Lumer–Philips for the
contraction semigroup, we get in this Note a new deviation inequality
for $\int_0^t V(X_s) \, ds$ by means of the symmetrized Dirichlet form. A more
explicit version is obtained in the case where the logarithmic Sobolev
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Key words: Dirichlet forms, Deviation inequality, Logarithmic Sobolev inequality

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RÉSUMÉ. – Par le critère de dissipativité de Lumer–Philips pour
la contractivité de semigroupes, on obtient une inégalité nouvelle de
déviation pour $\int_0^t V(X_s) \, ds$ via la forme de Dirichlet symmetrisée. Une
expression plus explicite est obtenue dans le cas où l’inégalité de Sobolev
logarithmique est vraie. © 2000 Éditions scientifiques et médicales
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Mots Clés: Forme de Dirichlet, Inégalité de déviation, Inégalité de Sobolev
logarithmique

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1. Let \((\Omega, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, (X_t)_{t \in \mathbb{R}^+}, (P_x)_{x \in E})\) be a conservative càdlàg Markov process with values in a Polish space \(E\), with semigroup of transition probability \((P_t(x, dy))\). We assume that \(\mu\) is a probability measure on \(E\) (equipped with the Borel \(\sigma\)-field \(\mathcal{B}\)), which is invariant and ergodic with respect to \((P_t)\). For any initial measure \(\nu\) on \(E\), write \(P_\nu := \int_E P_x \nu(dx)\).

We denote by \((\mathcal{L}, D_p(\mathcal{L}))\) the generator of \((P_t)\) acting on \(L^p(E, \mu)\) \((D_p(\mathcal{L})\) being its domain in \(L^p\), where \(1 \leq p < +\infty\). The symmetrized Dirichlet form is given by

\[
\mathcal{E}^\sigma(f, g) := \frac{1}{2} \left[\langle -\mathcal{L} f, g \rangle_\mu + \langle -\mathcal{L} g, f \rangle_\mu \right], \quad \forall f, g \in D_2(\mathcal{L}),
\]

where \(\langle \cdot, \cdot \rangle_\mu\) is the usual inner product in \(L^2(E, \mu)\).

Under the assumption below

\[(\mathcal{E}^\sigma, D_2(\mathcal{L}))\text{ is closable}, \quad (H1)\]

its closure \((\mathcal{E}^\sigma, D(\mathcal{E}^\sigma))\) corresponds to a symmetric Markov semigroup \((P_t^\sigma)_{t \geq 0}\) on \(L^2(E, \mu)\).

Given a measurable function \(V : E \to \mathbb{R}\), \(\mu\)-integrable. In this note we are interested to the probability of deviation of the empirical mean \(\frac{1}{t} \int_0^t V(X_s) \, ds\) from its real (or asymptotic) mean \(m := \int_E V \, d\mu := \langle V \rangle_\mu\), i.e.,

\[
P_\nu \left( \left| \frac{1}{t} \int_0^t V(X_s) \, ds - m \right| > r \right).
\]

Introduce

\[
J_V(r) := \inf \left\{ \mathcal{E}^\sigma(f, f) \mid f \in D(\mathcal{E}^\sigma) \cap L^2(|V| \, d\mu), \int f^2 \, d\mu = 1; \right. \left. \int V f^2 \, d\mu = r \right\}
\]

for every \(r \in \mathbb{R}\) \((\text{Convention: } \inf \emptyset := +\infty)\). As is easily seen, \(J_V\) is a convex function on \(\mathbb{R}\). Then \([J_V < +\infty]^0\) (interior) is some interval \((a, b)\) where \(-\infty \leq a \leq b \leq +\infty\).

Define now \(I_V\) as the lower semi-continuous (l.s.c. in short) regularization of \(J_V\). Obviously \(I_V(m) = J_V(m) = 0\) and \(I_V : \mathbb{R} \to [0, +\infty]\) is convex. Then \(I_V\) is non-decreasing on \([m, +\infty)\) and non-increasing on
(-\infty, m]. Notice that when a < b, then for any \( r \in \mathbb{R} \),

\[
I_V(r) = \begin{cases} 
J_V(r), & \text{if } r \in (a, b); \\
J_V(a+), & \text{if } r = a; \\
J_V(b-), & \text{if } r = b; \\
+\infty, & \text{otherwise.}
\end{cases}
\]  

(3)

When our Markov process \((X_t)\) is \(\mu\)-reversible (or \((P_t)\) is \(\mu\)-symmetric), Deuschel and Stroock [4, Theorem 5.3.10, p. 210] (1989) proved essentially the following large deviation estimation (where a general level-2 large deviation lower bound is given)

\[
\lim_{t \to \infty} \frac{1}{t} \log P_v \left( \frac{1}{t} \int_0^t V(X_s) \, ds - m > r \right) = -I_V(m + r), \quad \forall r \geq 0, \tag{4}
\]

for \(V\) bounded. For general unbounded \(V\), (4) is shown in [7] (1993).

In this little note we propose to extend and strengthen (4). Our main observation is

**Theorem 1.** Assume (H1). For any initial measure \(\nu\) such that \(\nu \ll \mu\) and \(\frac{d\nu}{d\mu} \in L^2(\mu)\), we have for all \(t > 0\), all \(r > 0\),

\[
P_v \left( \frac{1}{t} \int_0^t V(X_s) \, ds - m > r \right) \leq \left\| \frac{d\nu}{d\mu} \right\|_{L^2(\mu)} \cdot \exp[-t \cdot I_V(m + r)], \tag{5}
\]

\[
P_v \left( \frac{1}{t} \int_0^t V(X_s) \, ds - m < -r \right) \leq \left\| \frac{d\nu}{d\mu} \right\|_{L^2(\mu)} \cdot \exp[-t \cdot I_V(m - r)]. \tag{6}
\]

**Remark 2.** In the symmetric case, the deviation inequality (5) is sharp in its exponent for large time \(t\), by (4). The main differences between (4) and (5) are:

(i) The symmetry assumption required in (4) is removed for (5);

(ii) In (5), \(t\) and \(r\), being arbitrary, are fixed unlike in (4) which is only an asymptotic relation \((t \to +\infty)\). Hence (5) is much more stronger and practical.

However in the non-symmetric case, inequality (5) is no longer asymptotically exact. In fact, when the level-2 large deviation principle of Donsker–Varadhan holds and \(V\) is bounded, the limit (4) is given by a contraction form of the Donsker–Varadhan entropy functional, which is different from the expression in terms of Dirichlet form. See Deuschel and Stroock [4, Chapter VI] and Ben Arous and Deuschel [1] (1994).
Nevertheless that last large deviation result requires quite restrictive conditions in the non-symmetric case: indeed there exist geometrically ergodic irreducible Markov processes so that the level-1 large deviation principle fails (see Bryc and Smolenski [2] (1993)). While the deviation inequality (5) requires only (H1), which is satisfied in the most part of interesting cases. Moreover (H1) can be removed in case that $V$ is bounded, see Remarks 3(a) below.

2. Proof of Theorem 1. Consider the Feynman–Kac semigroup

\[ P_t^V f(x) := E^x f(X_t) \cdot \exp \left( \int_0^t V(X_s) \, ds \right) \tag{7} \]

where $f \geq 0$ is $B$-measurable. We shall establish for any $\mu$-integrable function $V : E \to \mathbb{R}$,

\[ 0 < \| P_t^V \|_2 \leq e^{t\Lambda(V)}, \quad \forall t \geq 0, \tag{8} \]

where

\[ \| P_t^V \|_2 := \sup \{ \| P_t^V f \|_{L^2(\mu)} ; f \geq 0 \text{ and } (f^2)_\mu \leq 1 \} \]

\[ = \sup \{ \langle P_t^V f, g \rangle_\mu ; f, g \geq 0 \text{ and } (f^2)_\mu \leq 1, (g^2)_\mu \leq 1 \}, \]

and

\[ \Lambda(V) := \sup \left\{ -\mathcal{E}_V^\sigma(f, f) \big| f \in D(\mathcal{E}_V^\sigma), \int f^2 d\mu = 1 \right\}. \tag{9} \]

Here

\[ D(\mathcal{E}_V^\sigma) := D(\mathcal{E}^\sigma) \cap L^2(|V| d\mu), \quad \mathcal{E}_V^\sigma(f, f) = \mathcal{E}^\sigma(f, f) - \int V f^2 d\mu. \]

Let us see quickly why (8) implies (5), by a very classical argument borrowed from the Cramer theorem [4]. In fact set $P(\lambda) := \Lambda(\lambda V), \forall \lambda \in \mathbb{R}$. By Chebychev’s inequality, for all $r, t > 0$ fixed,

\[ P_{\nu} \left( \frac{1}{t} \int_0^t V(X_s) \, ds - m > r \right) \]

\[ \leq \inf_{\lambda > 0} \exp \left[ -\lambda t (m + r) \right] \cdot E^{\nu} \exp \left[ \lambda \int_0^t V(X_s) \, ds \right] \]
It remains to identify the exponent in the last term of (10). Since $\Lambda(\lambda V) \geq \lambda m$ by the definition (9), $m$ is a sub-differential of $P(\lambda)$ at $\lambda = 0$. Thus for $r > 0$,

$$
\sup_{\lambda > 0} \left[ \lambda (m + r) - P(\lambda) \right] = \sup_{\lambda \in \mathbb{R}} \left[ \lambda (m + r) - P(\lambda) \right],
$$

which is the Legendre transformation $P^*(m + r)$ of $P(\lambda)$.

On the other hand, we have by (9)

$$
P(\lambda) = \Lambda(\lambda V) = \sup \left\{ \lambda z - J_V(z); z \in \mathbb{R} \right\} = \sup \left\{ \lambda z - I_V(z); z \in \mathbb{R} \right\}
$$

for all $\lambda \in \mathbb{R}$. Hence the famous Fenchel–Legendre theorem gives us

$$
P^*(m + r) = I_V(m + r).$

Substituting those into (10), we get (5).

Applying (5) to $-V$, we get (6).

Consequently to conclude this theorem, it remains to show (8). We divide its proof into three cases.

**Case 1.** $-V$ bounded. In this bounded case $(P_t^V)$ is a strongly continuous semigroup of bounded operators on $L^2(\mu)$, whose generator is exactly $(\mathcal{L} + V; D_2(\mathcal{L} + V) = D_2(\mathcal{L}))$ by the well known Feynman–Kac formula. By the definition (9) of $A(V)$,

$$
\langle (\mathcal{L} + V - A(V)) f, f \rangle_\mu \leq 0, \quad \forall f \in D_2(\mathcal{L}). \quad (11)
$$

That means exactly that the generator $\mathcal{L} + V - A(V)$ with domain $D_2(\mathcal{L})$ is a dissipative operator on $L^2(E, \mu)$ in the sense of Lumer and Philips [9, Chapter IX, p. 250]. By the Lumer–Philips Theorem [9, Chapter IX, p. 250], the semigroup $(e^{-tA(V)} P_t^V)$ generated by $\mathcal{L} + V - A(V)$ is
contractive on $L^2(E, \mu)$. In other words,
\[ \|e^{-t\Lambda(V)} P_t^V\|_2 \leq 1, \quad \forall t \geq 0, \]
which is exactly (8).

**Case 2.** $V$ upper bounded ($V \leq a$). Considering $V - a$ if necessary, we can assume $V \leq 0$. Take $V_n = \max\{V, -n\}$ for $n \in \mathbb{N}$. We have by the Case 1,
\[ \|P_t^V\|_2 \leq \lim_{n \to \infty} \|P_t^{V_n}\|_2 \leq \lim_{n \to \infty} e^{t\Lambda(V_n)} = \exp(t \cdot \inf_{n \geq 1} \Lambda(V_n)). \quad (12) \]
Recall that
\[ -\Lambda(V_n) = \inf\left\{ \mathcal{E}^\sigma(f, f) - \int V_n f^2 d\mu \mid f \in \mathcal{D}(\mathcal{E}^\sigma) \text{ and } \int f^2 d\mu \leq 1 \right\} \]
\[ = \inf\left\{ F_n(f) \mid \int f^2 d\mu \leq 1 \right\}, \]
where $F_n : L^2(E, \mu) \to [0, +\infty]$ is given by
\[ F_n(f) := \mathcal{E}^\sigma(f, f) - \int V_n f^2 d\mu, \quad \text{if } f \in \mathcal{D}(\mathcal{E}^\sigma), \quad \text{and } +\infty \text{ else.} \]

By Kato [5, p. 461, Lemma 3.14a] and our assumption (H1), $F_n$ is lower semicontinuous on $L^2(E, \mu)$ with respect to the strong topology, then with respect to the weak topology $\sigma(L^2, L^2)$ (since $F_n$, being the sum of two nonnegative quadratic forms, is convex on $L^2(E, \mu)$). Moreover, since the unit ball $\{ f \in L^2(\mu) \mid \int f^2 d\mu \leq 1 \}$ is compact with respect to $\sigma(L^2, L^2)$, by an elementary analytical lemma (see e.g. [8, Proposition 1.2]),
\[ -\inf_{n \geq 1} \Lambda(V_n) = \sup_{n \geq 1} \inf\left\{ F_n(f) \mid \int f^2 d\mu \leq 1 \right\} \]
\[ = \inf\left\{ \sup_{n} F_n(f) \mid \int f^2 d\mu \leq 1 \right\} = -\Lambda(V). \]
Substituting it into (12), we get (8) again.

**Case 3.** General case. Take $V_N = \min\{V, N\}$ for $N \in \mathbb{N}$. By the monotone convergence theorem,
\[ \|P_t^V\|_2 = \sup\{ \langle P_t^V f, g \rangle_\mu \mid f, g \geq 0 \text{ and } \langle f^2 \rangle_\mu \leq 1, \langle g^2 \rangle_\mu \leq 1 \} \]
where the third inequality follows from the Case 2, and the last equality follows from the fact that $D(\mathcal{E}(\sigma) \cap L^\infty(\mu))$ is a form core for all $\mathcal{E}_{\sigma N}, N \geq 1,$ and for the not necessarily closable quadratic form $\mathcal{E}_V^\sigma$.

The proof of (8) and then that of Theorem 1 are so finished. \[ \square \]

**Remark 3.** -
(a) When $V$ is bounded, it holds that
\[
\| P_t^V \|_2 \leq \exp[t \cdot A^0(V)]
\]
where
\[
A^0(V) := \sup \left\{ \int V f^2 d\mu + \langle \mathcal{L} f, f \rangle_\mu \mid f \in D_2(\mathcal{L}) \text{ and } \langle f^2 \rangle_\mu \leq 1 \right\}
\]
without the assumption (H1) about the closability of $(\mathcal{E}^\sigma, D_2(\mathcal{L}))$, by the proof in the Case 1 above. As in the proof of $(8) \Rightarrow (5)$ above, one can deduce from (13) the deviation inequalities (5) and (6) without (H1), but with $I_V$ substituted by the l.s.c. regularization $I^0_V$ of
\[
J^0_V(r) := \inf \left\{ \mathcal{E}^\sigma(f, f) \mid f \in D_2(\mathcal{L}), \int f^2 d\mu = 1; \int V f^2 d\mu = r \right\}.
\]
When (H1) is satisfied and $V$ is bounded, $A^0(\lambda V) = A(\lambda V), \forall \lambda \in \mathbb{R}$ (by the fact that $D_2(\mathcal{L})$, being a form core of $\mathcal{E}^\sigma$, is so for $\mathcal{E}^\sigma_{\lambda V}$ because of the boundedness of $V$), and then $I^0_J = I_V$.

(b) Note also the following (indicated by the referee): the inequality (8) implies not only (5) and (6), but also (with the same argument)
\[
\mathbb{E}^\mu f(X_0)g(X_t) \left[ \frac{1}{t} \int_0^t V(x_s) ds - m > r \right] 
\leq \| f \|_{L^2(\mu)} \| g \|_{L^2(\mu)} \cdot \exp \left[ -t \cdot I_V(m + r) \right], \forall r, t > 0.
\]
(c) Applying the Lumer–Philips theorem to $\mathcal{L} - V$ in $L^p(\mu)$ with $1 \leq p < +\infty,$ we get, instead of (8), that for any $V$ bounded,
\[
\| P_t^V \|_p \leq \exp(t A_p(V))
\]
where

\[ \Lambda_p(V) := \sup \left\{ \int V |f|^p \, d\mu + \langle \text{sgn}(f) |f|^{p-1}, \mathcal{L}f \rangle_{\mu} \mid f \in D_p(\mathcal{L}), \langle |f|^p \rangle_{\mu} = 1 \right\}. \]

3. In this paragraph we do not require (H1) but we assume the log-Sobolev inequality below: there exists \( C > 0 \) such that for all \( f \in D_2(\mathcal{L}) \),

\[ \int_E f^2 \log f^2 - \langle f^2 \rangle_{\mu} \log \langle f^2 \rangle_{\mu} \leq C \langle -\mathcal{L}f, f \rangle_{\mu}. \]  

(14)

Consider the log-Laplace transformation of \( V - m \):

\[ H(\lambda) = \log \int_E e^{\lambda V} \, d\mu - \lambda m \]  

(15a)

and its Legendre transformation

\[ H^*(r) = \sup\{\lambda r - H(\lambda); \lambda \in \mathbb{R}\}. \]  

(15b)

By the classical Cramèr’s theorem [4], \( H^* \) governs the large deviation principle of the i.i.d. sequence of common law \( \mu(V - m \in \cdot) \).

The following result says that the log-Sobolev inequality (14) implies a same type of estimation as in the i.i.d. case.

**Corollary 4.**—Assume (14) (not (H1)). Then for any \( V \in L^1(\mu) \),

\[ \frac{1}{t} \log \|P_t^V\|_2 \leq \frac{1}{C} \log \int_E e^{CV} \, d\mu. \]  

(16)

In particular for each initial measure \( \nu \ll \mu \) with \( \frac{d\nu}{d\mu} \in L^2(\mu) \) and for all \( r > 0, t > 0 \)

\[ \mathbb{P}_\nu \left( \frac{1}{t} \int_0^t V(X_s) \, ds - m > r \right) \leq \left\| \frac{d\nu}{d\mu} \right\|_{L^2(\mu)} \cdot \exp \left( -\frac{t}{C} H^*(r) \right). \]  

(17)

*Proof.*—The deviation inequality (17) follows from (16) by Chebychev’s inequality as in Theorem 1. To show the key (16), assume at first that \( V \) is bounded.
By (13) in Remark 3, we have

\[
\frac{1}{t} \log \left\| P_t^V \right\|_2 \\
\leq \sup \left\{ \int V f^2 \, d\mu + (\mathcal{L} f, f)_{\mu} \mid f \in \mathbf{D}_2(\mathcal{L}) \text{ and } (f^2)_{\mu} = 1 \right\} \\
\leq \sup \left\{ \int V f^2 \, d\mu - \frac{1}{C} \int f^2 \log f^2 \, d\mu \mid f \in \mathbf{D}_2(\mathcal{L}) \\
\text{ and } (f^2)_{\mu} = 1 \right\} \quad \text{(by (14))} \\
= \frac{1}{C} \log \int_E e^{CV} \, d\mu,
\]

where the last equality follows from Donsker–Varadhan’s variational formula (see e.g. [8]).

Now for \( V \) unbounded, set \( V_n = \min\{\max\{V, -n\}, n\} \). We have

\[
\left\| P_t^V \right\|_2 \leq \liminf_{n \to +\infty} \left\| P_t^{V_n} \right\|_2 \leq \lim_{n \to \infty} \left( \int e^{CV_n} \, d\mu \right)^{1/C} = \left( \int e^{CV} \, d\mu \right)^{1/C}
\]

by the bounded case shown above and the dominated convergence (and Fatou’s lemma if the last integral is infinite). (16) is hence established.

Remark 5. – Ledoux [6] (1999) develops systematically the so-called Herbst method which consists to derive deviation inequalities from a log-Sobolev inequality. The strategy consists to apply a log-Sobolev inequality to \( e^{\lambda F} \) to obtain a differential inequality, from which a control on \( Ee^{\lambda F} \) is deduced by comparison lemma. Nevertheless for that strategy works here for \( F = \int_0^t V(X_s) \, ds \), we should assume that a log-Sobolev inequality on the path space \( (\mathbf{D}([0, t], E), P_t) \) holds, which is in general not the case here.

Even in case that such a path level log-Sobolev inequality holds, it seems that the Herbst method does not give directly better estimation than (17). For instance, let \( (B_t) \) be the Brownian motion on a Riemannian manifold \( E \), with generator \( \Delta/2 \), where \( \Delta \) is the Laplace–Beltrami operator. Assume that the Ricci curvature satisfies \( |\text{Ric}_u| \leq K \) for all \( u \in O(E) \) (the bundle of orthonormal frames on \( E \)). By Capitaine–Hsu–Ledoux [3, (6)], the path level log-Sobolev inequality below holds:

\[
E^x (F^2 \log F^2) - E^x F^2 \log E^x F^2 \leq 2e^{Kt} |DF|^2_{H} \quad (18)
\]
for any $x \in E$ and $F : C([0, t]; E) \to \mathbb{R}$ provided that the right side term above is finite, where $|DF|_H$ is the norm in the Cameron–Martin subspace of the Malliavin derivative $DF$ on the path space. Now the Herbst method developed in [6, §2.3] yields: if $|DF|_H^2 \leq \sigma^2$, $P_x$-a.s., then

$$P_x(F - \mathbb{E}^xF > r) \leq \exp \left( -\frac{r^2}{2e^{Kt}\sigma^2} \right). \quad (19)$$

Using the notations of [3], we can easily prove that for $F = \int_0^t V(B_s) \, ds$ with $\|\nabla V\|_\infty := \sup_{x \in E} |\nabla V(x)| < +\infty$ (where $|\nabla V(x)|$ is the Riemannian norm of the gradient of $V$ at $x$),

$$|DF|_H^2 \leq \int_0^t \left( \int_s^t |\nabla V|(B_u) \, du \right)^2 \, ds \leq \|\nabla V\|_\infty^2 \cdot \frac{t^3}{3}, \quad P_x$-a.s.$

We then obtain by (19),

$$P_x \left( \int_0^t V(B_s) \, ds - \mathbb{E}^x \int_0^t V(B_s) \, ds > rt \right) \leq \exp \left( -\frac{3r^2}{2te^{Kt}\|\nabla V\|_\infty^2} \right). \quad (20)$$

That estimation is quite interesting and sharp for small $t$, but not so for large $t$. On the other hand, when $E$ is compact, the log-Sobolev inequality (14) holds (a well known fact), then (17) is valid and it gives a much better estimation than (20) for large $t$.

Our approach in Corollary 4 consists to apply log-Sobolev inequality after obtaining the control of $\|P_tV\|_2$ (in Theorem 1), not before, unlike in the Herbst method. One can regard it as another application of log-Sobolev inequality, complementing those amply developed by Ledoux [6].

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