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Some remarks on isoperimetry of Gaussian type

by

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ABSTRACT. – We give a martingale proof of Gaussian isoperimetry, which also contains Bobkov's inequality on the two-point space and its extension to non symmetric Bernoulli measures. We derive the equivalence of different forms of Gaussian type isoperimetry. This allows us to prove a sharp form of Bobkov's inequality for the sphere and to get isoperimetric estimates for the unit cube. © 2000 Éditions scientifiques et médicales Elsevier SAS

Key words: Isoperimetry, Gaussian measure

RÉSUMÉ. – Nous donnons une démonstration de l'isopérimétrie gaussienne par une technique de martingale. Cette démonstration fournit aussi l'inégalité de Bobkov sur l'espace à deux points et une généralisation aux mesures de Bernoulli non symétriques. Nous montrons l'équivalence de plusieurs formes d'inégalités isopérimétriques de type gaussien. Ceci nous permet de prouver une inégalité de Bobkov précise pour la sphère et d'obtenir des estimations isopérimétriques

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pour le cube unité.

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1. INTRODUCTION

It is well known that among subsets of the sphere with prescribed volume, spherical caps have minimal boundary measure. Using Poincaré's limit argument, which gives a representation of the Gaussian measure on $\mathbb{R}^n$ as a limit of projections on $\mathbb{R}^n$ of the invariant probability measures on high dimensional spheres, Borell [9] and Sudakov–Tsirel'son [18] proved that half-spaces are solutions to the isoperimetric problem in Gauss space; later, Ehrhard [11] obtained this result by means of symmetrization techniques. In this article, we shall be mainly interested by a different approach due to Bobkov [4], who emphasized a functional version of this Gaussian isoperimetric problem (inequality (2) below); we begin with some notation.

The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $|x|$; if μ is a Borel probability measure on $\mathbb{R}^n$ and if A is an arbitrary Borel subset of $\mathbb{R}^n$, the boundary μ-measure of A is denoted by

$$\mu^+(A) = \liminf_{h\to 0^+} \frac{\mu(A_h) - \mu(A)}{h},$$

where $A_h = \{x \in \mathbb{R}^n;\ d(x, A) \leqslant h\}$ is the h-enlargement of A for the Euclidean distance. Let γ_1 be the standard Gaussian probability measure on $\mathbb{R}$, with density $\varphi(x) = \exp(-x^2/2)/\sqrt{2\pi}$, $x \in \mathbb{R}$, and let $\Phi(x) = \int_{-\infty}^{x} d\gamma_1$. The Gaussian isoperimetric function I is defined for every $t \in [0, 1]$ by

$$I(t) = \varphi \circ \Phi^{-1}(t);$$

this value $I(t)$ represents the minimal Gaussian boundary measure of an interval of Gaussian measure t, which is achieved for sets of the form $(-\infty, a)$ or $(b, +\infty)$ with $a = \Phi^{-1}(t)$ or $b = \Phi^{-1}(1-t)$. We have thus $I(t) = \mathrm{e}^{-a^2/2}/\sqrt{2\pi}$ exactly when $t = \int_{-\infty}^{a} d\gamma_1$. Notice that $I(0) = 0$, and $I(t) = I(1-t)$ for $0 \leqslant t \leqslant 1$; the reader can check that this function I satisfies on $(0, 1)$ the differential relation $I\,I'' = -1$, which will play an important role in the next section (proof of Proposition 1).

The standard estimate for the tail of the Gaussian distribution gives that $I(t) \sim t\sqrt{2\log(1/t)}$ as $t \to 0$.

The isoperimetric inequality for the standard Gaussian measure γ_n on $\mathbb{R}^n$ with density $\exp(-|x|^2/2)/\sqrt{2\pi}$, $x \in \mathbb{R}^n$ can be stated as follows: for every measurable set $A \subset \mathbb{R}^n$, we have

$$\gamma_n^+(A) \geqslant I\big(\gamma_n(A)\big). \tag{1}$$

In other words, if we define $a \in \mathbb{R}$ by the equation $\gamma_n(A) = \int_{-\infty}^{a} d\gamma_1$, then $\gamma_n^+(A) \geqslant \exp(-a^2/2)/\sqrt{2\pi}$. Clearly, this inequality is an equality for affine half-spaces in $\mathbb{R}^n$.

Recently, Bobkov [4] proved a functional version of the Gaussian isoperimetry: for every locally Lipschitz function $f : \mathbb{R}^n \to [0, 1]$, one has

$$I\left(\int_{\mathbb{R}^n} f \, d\gamma_n\right) \leqslant \int_{\mathbb{R}^n} \sqrt{I^2(f) + |\nabla f|^2} \, d\gamma_n. \tag{2}$$

It is easy to see that this inequality implies (1). Bobkov deduces (2) from the following "two-point" isoperimetric inequality: for all $a, b \in [0, 1]$,

$$I\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2}\sqrt{I^2(a) + \left(\frac{b-a}{2}\right)^2} + \frac{1}{2}\sqrt{I^2(b) + \left(\frac{b-a}{2}\right)^2}. \tag{3}$$

Using the remarkable tensorisation properties of this inequality and the central limit theorem, Bobkov shows that (3) implies (2). As it is noticed in [4], inequality (2) for $\mathbb{R}^n$ can also be proved from (1) for $\mathbb{R}^{n+1}$ by choosing $A \subset \mathbb{R}^n \times \mathbb{R}$ to be the subgraph of $\Phi^{-1} \circ f$; actually, this reasoning already appears in Ehrhard's paper [12]: the relation 2.2.1, p. 323 of [12] contains Bobkov's inequality (2); of course the striking point about the paper [4] is that this inequality (2) is obtained there as a consequence of the simple two-point inequality (3).

In Section 2, we extend a Brownian approach to (2) due to Capitaine, Hsu and Ledoux [10]. We get a unified proof of (3) and (2), and an extension of (3) to an isoperimetric inequality for non symmetric Bernoulli measures. The third section contains a proof of the equivalence of different forms of isoperimetry on the Gaussian model. It follows from works by Wang [19] and by Bakry and Ledoux (Theorem 4.1 of [1]) that for any probability measure $d\mu(x) = \mathrm{e}^{-V(x)}\, dx$ on $\mathbb{R}^n$, with $V'' \geqslant \alpha\, Id_{\mathbb{R}^n}$ for some $\alpha \in \mathbb{R}$, and such that $\iint \exp(\varepsilon|x-y|^2)\, d\mu(x)\, d\mu(y) < \infty$ for some $\varepsilon > \sup(0, -\alpha)$, there exists $c > 0$ such that for every Borel set $A \subset \mathbb{R}^n$

$$\mu^+(A) \geqslant c\, I\big(\mu(A)\big). \tag{4}$$

A simple proof of this fact for log-concave probability measures is given by Bobkov [5]: (4) is equivalent to the existence of a number $\varepsilon > 0$ such that $\int \exp(\varepsilon|x|^2)\, d\mu(x) < \infty$ (Herbst condition). Moreover he proves that (4) implies that for all locally Lipschitz functions $f : \mathbb{R}^n \to [0, 1]$,

$$I\Bigg(\int_{\mathbb{R}^n} f\, d\mu\Bigg) \leqslant \int_{\mathbb{R}^n} \bigg(I(f) + \frac{1}{c}|\nabla f|\bigg)\, d\mu. \tag{5}$$

The constants provided by these results are not very good. Sharp ones are given by Bakry and Ledoux [1]: under the hypothesis $V'' \geqslant c^2\, Id_{\mathbb{R}^n}$, one has for $d\mu(x) = \mathrm{e}^{-V(x)}\, dx$ and every f as above

$$I\Bigg(\int_{\mathbb{R}^n} f\, d\mu\Bigg) \leqslant \int_{\mathbb{R}^n} \sqrt{I^2(f) + \frac{1}{c^2}|\nabla f|^2}\, d\mu. \tag{6}$$

Notice that the case $\mu = \gamma_n$ and $c = 1$ gives (2).

It is clear that (6) implies (5), which implies (4). We will show that they are equivalent, with the same constant c. The proof strongly relies on the Gaussian model (2). Then, we give a sharp form of Bobkov's inequality for spheres, using the Gaussian isoperimetric function I. Finally, we improve the isoperimetric estimates of Hadwiger [15] for the unit cube in $\mathbb{R}^n$. In particular, we recover the following result of Hadwiger: among subsets of measure $1/2$ of the unit cube, half-cubes have the smallest boundary measure.

2. BROWNIAN PROOF OF BOBKOV'S INEQUALITIES

In order to simplify the notation we work with real-valued processes and functions. Let $(B_t)_{t\geqslant 0}$ be a standard Brownian motion on $\mathbb{R}$ with natural filtration $(\mathcal{F}_t)_{t\geqslant 0}$ and such that $B_0 = 0$. We assume that all the processes appearing below are adapted with respect to this filtration.

PROPOSITION 1. – *Let $(M_t)_{t\geqslant 0}$ and $(N_t)_{t\geqslant 0}$ be $(\mathcal{F}_t)_{t\geqslant 0}$ real-valued martingales with $M_t = M_0 + \int_0^t m_s\, dB_s$, $N_t = N_0 + \int_0^t n_s\, dB_s$, and let $A_t = A_0 + \int_0^t a_s\, ds$ be an increasing process, such that A_t is bounded for* *every $t \geqslant 0$ and $A_0 \geqslant 0$. Assume that $a_t\,|N_t|^2 \geqslant |m_t|^2$ for every $t \geqslant 0$, and that for some $\varepsilon \in (0, 1/2)$, we have $M_t \in [\varepsilon, 1-\varepsilon]$ for every $t \geqslant 0$. Then*

$$\left(\sqrt{I^2(M_t) + A_t\,|N_t|^2}\right)_{t \geqslant 0}$$

is a submartingale.

The result remains true, with essentially the same proof, when M_t is a real martingale and N_t a vector-valued martingale with respect to the n-dimensional Brownian motion $(B_t^{(n)})$. In this case, (m_t) is a vector process, $dM_t = m_t \,.\, dB_t^{(n)}$ is the scalar product in $\mathbb{R}^n$ and (n_t) is a matrix-valued process. The condition above remains $a_t|N_t|^2 \geqslant |m_t|^2$, this time with the Euclidean norm.

Proof. – Let $J : \mathbb{R} \to \mathbb{R}$ be a positive C^2 function, constant outside [0, 1] and such that $J(x) = I(x)$ when $\varepsilon \leqslant x \leqslant 1-\varepsilon$. Let $F(x, y, t) = \sqrt{J^2(x) + ty^2}$. Direct computations give

$$\frac{\partial^2 F}{\partial x^2} = \frac{1}{F^3}\left(tJ'^2(x)y^2 + J^3(x)J''(x) + ty^2 J(x)J''(x)\right),$$
$$\frac{\partial^2 F}{\partial y^2} = \frac{tJ^2(x)}{F^3}, \qquad \frac{\partial^2 F}{\partial x \partial y} = -\frac{tyJ(x)J'(x)}{F^3}.$$

Writing Q_t for the triple (M_t, N_t, A_t), we get by Itô's formula

$$X_t := F(Q_t) = F(Q_0) + \int_0^t \left(\frac{\partial F}{\partial x}(Q)\,dM + \frac{\partial F}{\partial y}(Q)\,dN\right) + \int_0^t \Delta(s)\,ds,$$

with

$$\Delta(s) = \frac{\partial F}{\partial t} a_s + \frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}(Q_s)m_s^2 + 2\frac{\partial^2 F}{\partial x \partial y}(Q_s)m_s n_s + \frac{\partial^2 F}{\partial y^2}(Q_s)n_s^2\right).$$

Since the stochastic integral has a bounded integrand, it is a martingale. Hence (X_t) is a martingale plus $\int_0^t \Delta(s)\,ds$. But since $J(M_t)$ and $I(M_t)$ coincide, we get using the relation $II'' = -1$ and omitting the variables

$$2F^3\Delta = F^2N^2a + \left(I'^2AN^2 - I^2 - AN^2\right)m^2 - 2(II'AN)mn + I^2An^2.$$

Since $N^2a \geqslant m^2$, we have $F^2N^2a \geqslant (I^2 + AN^2)m^2$, hence

$$2F^3\Delta \geqslant A\left(I'^2N^2m^2 - 2II'Nmn + I^2n^2\right) = A(I'Nm - In)^2 \geqslant 0,$$

thus (X_t) is a submartingale. □

The preceding computation is not difficult, but does not really explain why the result was intuitively clear. Given a non-negative semimartingale Z such that $Z + dZ = Z + q\,dB + r\,dt$, Itô's formula shows that $\sqrt{Z}$ is a submartingale precisely when the formal second degree polynomial in the β variable

$$T = Z + q\beta + r\beta^2$$

has a non-positive discriminant $q^2 - 4Zr \leqslant 0$, or in other words when $T \geqslant 0$ for every real value of β. When $Z = I^2(M) + AN^2$, our formal expression is equal to

$$\begin{aligned} T &= \left(I^2 + 2II'm\beta + I'^2m^2\beta^2\right) + II''m^2\beta^2 + aN^2\beta^2 \\ &\quad + A\left(N^2 + 2Nn\beta + n^2\beta^2\right) \\ &\geqslant (I + I'm\beta)^2 + A(N + n\beta)^2 \geqslant 0 \end{aligned}$$

since $II''m^2 + aN^2 = -m^2 + aN^2 \geqslant 0$. The trick is simply that the increase of A_t (multiplied by N_t^2) must compensate the fact that $II'' < 0$. If we were trying to do the same for a different function J, we see that all is needed is that $(JJ'')(M_t)\,m_t^2 + a_tN_t^2 \geqslant 0$ for every t.

The previous result appears in the more abstract setting of the Wiener space in [10], for the special case of $M_t = \mathbb{E}[f(B_1) \mid \mathcal{F}_t]$, $A_t = t \wedge 1$ and $N_t = \mathbb{E}[\nabla f(B_1) \mid \mathcal{F}_t]$ (the three processes are constant when $t \geqslant 1$), where f is any regular function on $\mathbb{R}^n$ taking values in [0, 1]. As a consequence, Capitaine, Hsu and Ledoux obtain Bobkov's inequality (2) in the equivalent form:

$$\begin{aligned} I\left(\mathbb{E}f(B_1)\right) &= \mathbb{E}\,F(M_0, N_0, 0) \leqslant \mathbb{E}\,F(M_1, N_1, 1) \\ &= \mathbb{E}\sqrt{I^2\left(f(B_1)\right) + \left|\nabla f(B_1)\right|^2}. \end{aligned}$$

Remark. – When $f(s) = \mathbf{1}_{s \geqslant x}$,

$$F(M_t, N_t, t) = \frac{1}{\sqrt{1-t}} \exp -\frac{(x + B_t)^2}{2(1-t)}$$

is a martingale. This corresponds to the equality case in the isoperimetric inequality.

Our next aim is to recover Bobkov's two-point inequality by this submartingale approach. We will use the following stopping times, for $d, e \in \mathbb{R}$

$$T_d = \inf\{t \geqslant 0;\ B_t = d\} \quad \text{and} \quad T_{d,e} = T_d \wedge T_e.$$

PROPOSITION 2. – *Let* $m \in (0,1)$, $d < 0 < e$ *be such that* $[m+d, m+e] \subset (0,1)$. *Then*

$$(Y_t)_{t\geqslant 0} := \left(\sqrt{I^2(m + B_{T_{d,e}\wedge t}) + T_{d,e} \wedge t}\right)_{t\geqslant 0}$$

is a submartingale.

Proof. – Let $\tau = T_{d,e}$. We apply Proposition 1 with $M_t = m + B_{t\wedge\tau}$, $N_t = 1$, and $A_t = t \wedge \tau$; the conditions on m, d, e imply that M_t stays in some interval $[\varepsilon, 1-\varepsilon]$; for $t \leqslant \tau$ we have $m_t = a_t = 1$, and $m_t = a_t = 0$ when $t > \tau$, thus $a_t N_t^2 = a_t \geqslant m_t^2$ for every t. □

We need the following classical facts about the exit time of an interval (see for example [17]); recall that these results are obtained by applying the stopping time theorem to $T_{d,e}$ and to the martingales $(B_t)_{t\geqslant 0}$, $(B_t^2 - t)_{t\geqslant 0}$ and $(B_t^3 - 3tB_t)_{t\geqslant 0}$.

LEMMA 3. – *Let* d, e *be such that* $ed < 0$. *Then, the hitting times of* d *and* e *satisfy*

$$P(T_d < T_e) = \frac{e}{e-d}, \qquad \mathbb{E}\,T_{d,e} = -de,$$
$$\mathbb{E}\,[T_d \mid T_d < T_e] = \frac{d(d-2e)}{3}.$$

Using the preceding results, we may now derive Bobkov's two-point inequality for non-symmetric measures (if we take $p = 1/2$ in the next result, we get inequality (3), the symmetric case):

PROPOSITION 4. – *Let* $a, b, p, q \in [0,1]$, *with* $p + q = 1$. *Then one has*

$$I(pa+qb) \leqslant p\sqrt{I^2(a) + \frac{1-p^2}{3}(a-b)^2} + q\sqrt{I^2(b) + \frac{1-q^2}{3}(a-b)^2}.$$

Proof. – As a consequence of Proposition 2, and with the same notation, we have $I(m) = \mathbb{E}\,Y_0 \leqslant \mathbb{E}\,Y_t$. When t tends to infinity, we get letting $p_{d,e} := P(T_d < T_e)$

$$\begin{aligned} I(m) &\leqslant \mathbb{E}\sqrt{I^2(m + B_{T_{d,e}}) + T_{d,e}} \\ &= p_{e,d}\mathbb{E}\left[\sqrt{I^2(m+e) + T_e} \mid T_e < T_d\right] \\ &\quad + p_{d,e}\mathbb{E}\left[\sqrt{I^2(m+d) + T_d} \mid T_d < T_e\right]. \end{aligned}$$

By the concavity of the square root function, and using Lemma 3,

$$\begin{aligned}
I(m) &\leqslant P(T_e < T_d)\sqrt{I^2(m+e) + \mathbb{E}[T_e \mid T_e < T_d]} \\
&\quad + P(T_d < T_e)\sqrt{I^2(m+d) + \mathbb{E}[T_d \mid T_d < T_e]} \\
&= \frac{-d}{e-d}\sqrt{I^2(m+e) + \frac{e(e-2d)}{3}} \\
&\quad + \frac{e}{e-d}\sqrt{I^2(m+d) + \frac{-d(2e-d)}{3}}.
\end{aligned}$$

Assume that $a \geqslant b$. To prove the result, we set $m = pa + qb$, $e = a - m = q(a-b)$ and $d = b - m = p(b-a)$. $\square$

Let μ_p be the measure on $\{-1, 1\}$ defined by $\mu(\{1\}) = p$, $\mu(\{-1\}) = q$, and let $\mu_p^n = (\mu_p)^{\otimes n}$. Set $c(1) = (1-p^2)/3$ and $c(-1) = (1-q^2)/3$. For $f : \{-1,1\}^n \to [0,1]$ we consider the following "modulus of gradient":

$$D(f)(x) = \sqrt{\sum_{i=1}^{n} c(x_i)\big[f(x) - f\big(s_i(x)\big)\big]^2},$$

where $x = (x_i)_{i=1}^n \in \{-1,1\}^n$ and $s_i(x) = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n)$. By a classical tensorisation argument (which we recall in the remark after Proposition 5, in a slightly different setting), the previous proposition yields the following isoperimetric inequality on $(\{-1,1\}^n, \mu_p^n)$:

$$I(\mathbb{E}_{\mu_p^n} f) \leqslant \mathbb{E}_{\mu_p^n}\sqrt{I^2(f) + D(f)^2}.$$

This extends Corollary 1 in [4]. Similar isoperimetric inequalities for the Bernoulli measure, involving different moduli of gradient, are derived by Bobkov and Götze in [6].

3. EQUIVALENT FORMS OF GAUSSIAN TYPE ISOPERIMETRY

In order to deal with spheres, we need to generalize slightly our setting. Let (M, g) be a Riemannian manifold; for every subset A of M, we may define as before $A_h = \{x \in M;\, d(x, A) \leqslant h\}$, the h-enlargement of A for the geodesic distance. For any Borel probability measure μ on M, the boundary μ-measure of a Borel subset A of M is again defined by

$$\mu^+(A) = \liminf_{h\to 0^+} \frac{\mu(A_h) - \mu(A)}{h}.$$

PROPOSITION 5. – *Let $c > 0$ and let μ be a Borel probability measure on M, absolutely continuous with respect to the Riemannian volume. Then the following properties are equivalent*:

(i) *For every measurable $A \subset M$, $\mu^+(A) \geqslant c\, I(\mu(A))$.*

(ii) *For every locally Lipschitz function $f : M \to [0, 1]$,*

$$I\left(\int_M f\, d\mu\right) \leqslant \int_M \left(I(f) + \frac{1}{c}|\nabla f|\right) d\mu.$$

(iii) *For every locally Lipschitz function $f : M \to [0, 1]$,*

$$I\left(\int_M f\, d\mu\right) \leqslant \int_M \sqrt{I^2(f) + \frac{1}{c^2}|\nabla f|^2}\, d\mu.$$

Proof. – It is well known that (iii) $\Rightarrow$ (i) (or that (ii) $\Rightarrow$ (i), for the same reason: take $f_\varepsilon(x) = (1 - d(x, A)/\varepsilon)_+$ and let $\varepsilon \to 0$). The implication (i) $\Rightarrow$ (ii) was done in [3] for the Gaussian measure and $c = 1$ but the proof extends to the general case [5]. However, we give here a shorter proof. Assuming (i), the co-area formula yields:

$$\int_M |\nabla f|\, d\mu = \int_0^1 \mu^+(\{f \leqslant t\})\, dt \geqslant c \int_0^1 I\big(\mu(\{f \leqslant t\})\big)\, dt$$

(for this formula, see [13], Theorem 3.2.12, which deals with the Lebesgue measure on $\mathbb{R}^n$ and uses the $n - 1$ dimensional Hausdorff measure of the set $\{f = t\}$ instead of $\mu^+(\{f \leqslant t\})$; the manifold case follows from 3.2.12 with the usual partition of unity argument; for more general situations of this co-area formula, see also [7]). Let ν be the distribution of f with respect to μ. Following [1] we may assume that ν is absolutely continuous with respect to Lebesgue's measure on $\mathbb{R}$ and has a positive density on its support $[a, b] \subset [0, 1]$. For $t \in [0, 1]$, set

$N(t)=\nu([0,t])=\mu(\{f\leqslant t\})$. We have to show that

$$\int_0^1 I\big(N(t)\big)\,dt \geqslant I\left(\int_0^1 t\,d\nu(t)\right)-\int_0^1 I(t)\,d\nu(t).$$

Let $k=N^{-1}\circ\Phi:\mathbb{R}\to[a,b]$. We apply the weak functional form of Gaussian isoperimetry to k and we get, since the distribution of k with respect to γ_1 is ν, that

$$\begin{aligned}\int |k'|\,d\gamma_1 &\geqslant I\left(\int k\,d\gamma_1\right)-\int I(k)\,d\gamma_1\\ &= I\left(\int_0^1 t\,d\nu(t)\right)-\int_0^1 I(t)\,d\nu(t).\end{aligned}$$

We know by definition that $\int_{-\infty}^x d\gamma_1=\int_0^{k(x)}d\nu=N(k(x))$ for every x real, hence $\mathrm{e}^{-x^2/2}/\sqrt{2\pi}=I(N(k(x)))$ and we obtain by the change of variables $t=k(x)$

$$\int |k'|\,d\gamma_1=\int_{-\infty}^{+\infty} k'(x)\,\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}\,dx=\int_0^1 I\big(N(t)\big)\,dt.$$

This finishes the proof that (i) $\Rightarrow$ (ii). We show now that (ii) implies (iii). The beginning is similar to [1]. Let ν be the distribution of f, and let N,k be as before. We apply (ii) to $\psi_\varepsilon(f)$, where $\psi_\varepsilon(t)=1$ when $t\leqslant r$, $\psi_\varepsilon(t)=0$ when $t\geqslant r+\varepsilon$ and ψ_ε linear (continuous) in between. Letting $\varepsilon\to 0$ one gets

$$cI\big(N(r)\big)\leqslant\theta(r)N'(r),$$

where $\theta(s)=\mathbb{E}\,[|\nabla f|\mid f=s]$. Differentiating the relation that defines k, we get $N'(k(x))k'(x)=\mathrm{e}^{-x^2/2}/\sqrt{2\pi}=I(N(k(x)))$. Setting $r=k(x)$, the previous inequality becomes $ck'(x)\leqslant\theta(k(x))$. We apply (2) to k:

$$I\left(\int k\,d\gamma_1\right)\leqslant\int\sqrt{I^2(k)+|k'|^2}\,d\gamma_1\leqslant\int\sqrt{I^2(k)+\frac{1}{c^2}\,|\theta(k)|^2}\,d\gamma_1.$$

Since the distribution of k with respect to γ_1 is ν, we may translate the preceding line as

$$
\begin{aligned}
I\left(\int_0^1 t\,d\nu(t)\right) &\leqslant \int_0^1 \sqrt{I^2(t)+\frac{1}{c^2}\theta^2(t)}\,d\nu(t)\\
&= \int_0^1 \sqrt{\left(\mathbb{E}\left[I(f)\mid f=t\right]\right)^2+\left(\mathbb{E}\left[\frac{|\nabla f|}{c}\,\Big|\, f=t\right]\right)^2}\,d\nu(t).
\end{aligned}
$$

The Minkowski inequality $\sqrt{(\mathbb{E}a)^2+(\mathbb{E}b)^2}\leqslant \mathbb{E}\sqrt{a^2+b^2}$ for the conditional expectation gives

$$
\begin{aligned}
I\left(\int_M f\,d\mu\right) &\leqslant \int_0^1 \mathbb{E}\left[\sqrt{I^2(f)+\frac{1}{c^2}|\nabla f|^2}\,\Big|\, f=t\right]d\nu(t)\\
&= \int_M \sqrt{I^2(f)+\frac{1}{c^2}|\nabla f|^2}\,d\mu. \qquad \square
\end{aligned}
$$

Remark. – Among the three equivalent forms of Gaussian isoperimetry, (iii) is formally stronger and can be tensorized: we assume given two probability measures μ_1 and μ_2 on M_1 and M_2; the product space $M_1 \times M_2$ is equipped with the natural Riemannian product metric, for which $|\nabla|^2 = |\nabla_1|^2 + |\nabla_2|^2$; if μ_1 and μ_2 satisfy (iii) with the same constant c, so does $\mu_1 \otimes \mu_2$. Hence, (i) and (ii) can also be tensorized (this does not seem obvious without using (iii)): if μ_1 and μ_2 satisfy (i) with the same constant c, so does $\mu_1 \otimes \mu_2$.

Let us sketch the argument, due to Bobkov, which shows that the inequality in (iii) can be tensorized. Let $f(x_1,x_2)$ be a regular fonction on $M_1 \times M_2$ and let $F(x_1)=\int_{M_2} f(x_1,x_2)\,d\mu_2(x_2)$. For each fixed $x_1 \in M_1$, we may write using (iii) for μ_2 and for the function $x_2 \to f(x_1,x_2)$

$$
I\big(F(x_1)\big) \leqslant \int_{M_2} \sqrt{I^2\big(f(x_1,x_2)\big)+\frac{1}{c^2}\big|\nabla_2 f(x_1,x_2)\big|^2}\,d\mu_2(x_2) =: B_1(x_1).
$$

Next, using (iii) for μ_1 and F,

$$
\begin{aligned}
I\left(\int_{M_1\times M_2} f\,d\mu_1\,d\mu_2\right) &= I\left(\int_{M_1} F(x_1)\,d\mu_1(x_1)\right)\\
&\leqslant \int_{M_1} \sqrt{I^2\big(F(x_1)\big)+\frac{1}{c^2}\big|\nabla_1 F(x_1)\big|^2}\,d\mu_1(x_1)
\end{aligned}
$$

$$=: \int_{M_1} A(x_1)\, d\mu_1(x_1).$$

We bound $I(F(x_1))$ by $B_1(x_1)$, and expressing $\nabla_1 F(x_1)$ as an integral of $\nabla_1 f$ we get

$$\frac{1}{c}\left|\nabla_1 F(x_1)\right| \leqslant \int_{M_2} \frac{1}{c}\left|\nabla_1 f(x_1, x_2)\right| d\mu_2(x_2) =: B_2(x_1);$$

both expressions $B_1(x_1)$ and $B_2(x_1)$ are integrals in the x_2 variable of some functions $b_1(x_1, x_2)$ and $b_2(x_1, x_2)$; by Minkowski, we have for every $x_1 \in M_1$

$$A(x_1) \leqslant \sqrt{B_1^2(x_1) + B_2^2(x_1)} \leqslant \int_{M_2} \sqrt{b_1^2(x_1, x_2) + b_2^2(x_1, x_2)}\, d\mu_2(x_2),$$

and the result follows after integrating on M_1, since $|\nabla f|^2 = |\nabla_1 f|^2 + |\nabla_2 f|^2$.

We shall give two applications of Proposition 5. For the first one, let $rS^n \subset \mathbb{R}^{n+1}$ be the Euclidean sphere of radius $r > 0$, with the Riemannian metric induced by that of $\mathbb{R}^{n+1}$. Let σ_{rS^n} be the uniform probability measure on rS^n. The spherical isoperimetric function is for $a \in [0, 1]$:

$$I_{rS^n}(a) = \inf\left\{\sigma_{rS^n}^+(A),\ \sigma_{rS^n}(A) = a\right\}.$$

Since the infimum is achieved for caps, one can give an analytic expression of it. Notice that with our notations $I_{rS^n} = I_{S^n}/r$. When a is close to 0, it is clear that

$$I_{rS^n}(a) \sim \frac{\alpha_n}{r} a^{\frac{n-1}{n}},$$

for some α_n depending only on the dimension. Furthermore, the Gaussian isoperimetric function satisfies $I(a) \sim a\sqrt{2\log(1/a)}$ as $a \to 0$, so the constant $c_{rS^n} = \inf_{a\in(0,1)} I_{rS^n}(a)/I(a)$ is positive (in fact, the infimum is achieved for $a = 1/2$ [2]). Let $\nabla_\circ$ be the spherical gradient. Using the implication (i) $\Rightarrow$ (iii) of Proposition 5, we obtain:

THEOREM 6. – *Let $f : rS^n \to [0,1]$ be a locally Lipschitz function. The following inequality holds:*

$$I\left(\int f\, d\sigma_{rS^n}\right) \leqslant \int \sqrt{I^2(f) + c_{rS^n}^{-2} |\nabla_\circ f|^2}\, d\sigma_{rS^n}.$$

By construction, this is optimal for caps. Previous versions of this inequality existed, involving the Ricci curvature of S^n, $n \geqslant 2$, instead of c_{rS^n} [1].

We turn now to our second application, the isoperimetry for the unit cube $[0,1]^n$ in $\mathbb{R}^n$. Let $d\lambda_n(x) = \mathbf{1}_{[0,1]^n}(x)\, dx$ be the Lebesgue probability measure on the unit cube. Notice that the definition of λ_n^+ assigns the value 0 to the boundary of the cube: we will be measuring only boundaries inside the open cube.

THEOREM 7. – *Let A be a Borel subset of $[0,1]^n$, then*

$$\lambda_n^+(A) \geqslant \sqrt{2\pi}\, I\big(\lambda_n(A)\big).$$

Proof. – It is clear that for any non-empty set $A \subset \mathbb{R}$ and for every $\varepsilon > 0$, one has

$$\lambda_1(A_\varepsilon) \geqslant \inf\big(1, \lambda_1(A) + \varepsilon\big).$$

So $\lambda_1^+(A) \geqslant 1$ if $\lambda_1(A) \in (0,1)$. Since $\max I = 1/\sqrt{2\pi}$, one has for every measurable subset A

$$\lambda_1^+(A) \geqslant \sqrt{2\pi}\, I\big(\lambda_1(A)\big).$$

By the remark following Proposition 5, this property also holds for the product measure λ_n. □

This yields that among subsets of measure $1/2$, half-cubes of the form $H = [0, 1/2] \times [0,1]^{n-1}$ have the smallest boundary measure. Indeed, for $\lambda_n(A) = 1/2$, one has $\lambda_n^+(A) \geqslant \sqrt{2\pi}\, I(1/2) = 1 = \lambda_n^+(H)$. This was already implied by [15], where it is proved that $\lambda_n^+(A) \geqslant 4\lambda_n(A)(1 - \lambda_n(A))$. Nevertheless, our estimate is always better when $\lambda_n(A) \neq 1/2$ (and of course $0 < \lambda_n(A) < 1$). In fact, up to a multiplicative constant, it is the optimal estimate valid for all $n \geqslant 1$. To see this, consider the sets

$$A_{n,t} = \left\{ x \in [0,1]^n;\ \sum_{i=1}^n \left(x_i - \frac{1}{2}\right) \leqslant \frac{\sqrt{n}}{2\sqrt{3}}\, t \right\}.$$

It is clear that the enlargement $(A_{n,t})_\varepsilon$ is equal to $A_{n,t+2\sqrt{3}\varepsilon}$. By the central limit theorem, $\lambda_n(A_{n,t})$ tends to $\Phi(t)$ when n tends to infinity. So for every t, $\lambda_n^+(A_{n,t}) \sim 2\sqrt{3}\, I(\lambda_n(A_{n,t}))$ for n large.

Remark. – A similar result holds for the flat torus $\mathbb{T}_n = (\mathbb{R}/\mathbb{Z})^n = (\frac{1}{2\pi} S^1)^n$, where $\frac{1}{2\pi} S^1$ denotes a circle of length one. The isoperimetric function of $\mathbb{T}_1$ is also constant on $(0,1)$, hence by the same method, the product of a half circle and $\mathbb{T}_{n-1}$ is solution of the isoperimetric problem among sets of measure $1/2$. Extensions of these results will appear in [2].

Actually, our result for $[0,1]^n$ is easy to obtain by a known and simple transportation argument: the Gaussian distribution function Φ sends γ_1 to the uniform probability measure λ_1 on $[0,1]$, and Φ is a Lipschitz map with constant $1/\sqrt{2\pi}$; tensoring this one-dimensional information, we obtain a Lipschitz map with constant $1/\sqrt{2\pi}$ from $\mathbb{R}^n$ to $[0,1]^n$ that sends γ_n to λ_n, and it is then easy to transfer the relevant Gaussian estimates to $[0,1]^n$.

4. LOGARITHMIC SOBOLEV INEQUALITIES

We learned from W. Beckner (see [16]) that the Gaussian logarithmic Sobolev inequality of Gross [14] is a limit case of Bobkov's inequality (2): taking $f = \varepsilon g$ in (2) for a bounded function g and letting ε tend to 0 yields

$$\int g \log g \, d\gamma_n - \left(\int g \, d\gamma_n\right) \log\left(\int g \, d\gamma_n\right) \leqslant \frac{1}{2} \int \frac{|\nabla g|^2}{g} \, d\gamma_n,$$

because $I(\varepsilon) \sim \varepsilon\sqrt{2\log(1/\varepsilon)}$. All the statements of this note have a log-Sobolev version which can be proved by this limit process (or directly by taking $F(x,y,t) = -x\log x + ty^2/2$; see [10] for the log-Sobolev versions of the results of Section 2). The two-point inequality is, for $a, b, p, q > 0$ such that $p + q = 1$

$$\begin{aligned} &p\,a\log a + q\,b\log b - (pa+qb)\log(pa+qb) \\ &\quad\leqslant \frac{pq}{6}\left(\frac{1+p}{a} + \frac{1+q}{b}\right)(a-b)^2. \end{aligned}$$

This inequality is rather good when p is close to $1/2$, but for small p the inequality of [8] is better.

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