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Percolation on nonamenable products at the uniqueness threshold


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by

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ABSTRACT. – Let X and Y be infinite quasi-transitive graphs, such that the automorphism group of X is not amenable. For i.i.d. percolation on the direct product $X \times Y$, we show that the set of retention parameters $p$ where a.s. there is a unique infinite cluster, does not contain its infimum $p_u$. This extends a result of Schonmann, who considered the direct product of a regular tree and $\mathbb{Z}$. © 2000 Éditions scientifiques et médicales Elsevier SAS

Key words: Percolation, Cayley graphs, Amenability

RÉSUMÉ. – Soit X et Y des graphes infinis quasi-transitifs, tels que le groupe d’automorphismes de X n’est pas moyennable. Pour la percolation i.i.d. sur le produit direct $X \times Y$, nous montrons que l’ensemble des paramètres $p$ pour lesquels p.s. il y a un unique amas infini ne contient pas son infimum $p_u$. Cela étend un résultat de Schonmann, qui considérait le produit direct d’un arbre régulier avec $\mathbb{Z}$. © 2000 Éditions scientifiques et médicales Elsevier SAS

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1. INTRODUCTION

Let $X = (V_X, E_X)$ be an infinite, locally finite, connected graph. Say that $X$ is transitive if its automorphism group $\text{Aut}(X)$ has a single orbit in $V_X$; more generally, if $\text{Aut}(X)$ has finitely many orbits in $V_X$, then $X$ is called quasi-transitive. In i.i.d. bond percolation with retention parameter $p \in [0, 1]$ on $X$, each edge is independently assigned the value 1 (open) with probability $p$, and the value 0 (closed) with probability $1 - p$. We write $P^X_p$, or simply $P_p$, for the resulting probability measure on $A_{\text{connected component of open edges}}$ is called a cluster. The critical parameters for percolation on $X$ are

$$p_c(X) = \inf \{ p \in [0, 1] : P^X_p (\exists \text{ an infinite cluster}) = 1 \};$$

$$p_u(X) = \inf \{ p \in [0, 1] : P^X_p (\exists \text{ a unique infinite cluster}) = 1 \}.$$

We now state our result; further background and references will follow.

THEOREM 1.1. — Let $X$ and $Y$ be infinite, locally finite, connected quasi-transitive graphs and suppose that $\text{Aut}(X)$ is not amenable. Then on the direct product graph $X \times Y$,

$$\mathbf{P}_{p_u} (\exists \text{ a unique infinite cluster}) = 0.$$

Remarks. —

• For the definition of amenable groups, see, e.g., [14].

• Theorem 1.1 and its proof may be adapted to site percolation as well.

• In the case where $X$ is a regular tree of degree $d \geq 3$ and $Y = \mathbb{Z}$, Theorem 1.1 is due to Schonmann [20].

• Given two graphs $X = (V_X, E_X)$ and $Y = (V_Y, E_Y)$, the direct product graph $X \times Y$ has vertex set $V_X \times V_Y$; the vertices $(x_1, y_1)$ and $(x_2, y_2)$ in $V_X \times V_Y$ are adjacent in $X \times Y$ iff either $x_1 = x_2$ and $[y_1, y_2] \in E_Y$, or $y_1 = y_2$ and $[x_1, x_2] \in E_X$.

• Our proof of Theorem 1.1 is based on the following ingredients:

  (i) The characterization of $p_u$ in terms of connection probabilities between large balls, due to Schonmann [19]; see Theorem 2.1.

  (ii) The principle that for a (possibly dependent) percolation process, that is invariant under a nonamenable automorphism group, high marginals yield infinite clusters. This principle was proved by Häggström [8] for regular trees; it was extended to graphs with a nonamenable automorphism group by Ben-
jamini, Lyons, Peres and Schramm [2]. (See Theorems 2.2 and 2.3 below.)

(iii) The shadowing method used in Pemantle and Peres [16] to prove that there is no automorphism-invariant measure on spanning trees in any nonamenable direct product $X \times Y$ of the type considered in Theorem 1.1.

The first two ingredients are explained in the next section; (ii) was used in [3] to prove that percolation at level $p_c$ on any nonamenable Cayley graph has no infinite clusters. Section 3 contains the proof of Theorem 1.1, and we will point out there where the shadowing method is used.

2. BACKGROUND

In an infinite tree, clearly $p_u = 1$, and in quasi-transitive amenable graphs, the arguments of Burton and Keane [5] yield that $p_u = p_c$ (see [6]). Examples of transitive graphs where $p_c < p_u < 1$ were provided by Grimmett and Newman [7], Benjamini and Schramm [4] and Lalley [11]. The conjecture stated in [4] that $p_c < p_u$ on any nonamenable Cayley graph, is still open. Benjamini and Schramm also conjectured that on any quasi transitive graph, for all $p > p_u$ there is a unique infinite cluster $P_p$-a.s. This was established by Haggström and Peres [9] under a unimodularity assumption, and by Schonmann [19] in general. The latter paper also contains the following useful expression for $p_u$:

**THEOREM 2.1** ([19]). – Let $X$ be any quasi-transitive graph. Then

$$p_u(X) = \inf \left\{ p : \lim_{R \to \infty} \inf_{x, z \in V_x} P_p(B_R(x) \leftrightarrow B_R(z)) = 1 \right\},$$

(2.1)

**Notation.** – Let $(V, E)$ be a locally finite graph.

- For $K_1, K_2 \subseteq V$, we write $K_1 \leftrightarrow K_2$ for the event that there is an open path from some vertex in $K_1$ to some vertex in $K_2$.
- For $x, z \in V$ and $F \subseteq E$, denote by $\text{dist}(x, z; F)$ the minimal length of a path in $F$ from $x$ to $z$.
- For $x \in V$ and $R > 0$, let $B_R(x) := \{z \in V : \text{dist}(x, z; E) \leq R\}$.

In [10], Theorem 2.1 is used to show that $p_u(\Gamma) \leq p_c(\mathbb{Z}^d)$ for any graph $\Gamma$ which is a direct product of $d$ infinite connected graphs of bounded degree.
Next, we discuss the relation between nonamenability and invariant percolation. Let $X$ be a locally finite graph, and endow the automorphism group $\text{Aut}(X)$ with the topology of pointwise convergence. Then any closed subgroup $G$ of $\text{Aut}(X)$ is locally compact, and the stabilizer $S(x) = S_G(x) := \{ g \in G : gx = x \}$ of any vertex $x$ is compact. We start with a qualitative statement.

**Theorem 2.2** ([2, Theorem 5.1]). - Let $X$ be a locally finite graph and let $G$ be a closed subgroup of $\text{Aut}(X)$. Then $G$ is nonamenable iff there exists a threshold $\eta_G > 0$, such that if a $G$-invariant site percolation $\Lambda$ on $X$ satisfies $P[x \notin \Lambda] < \eta_G$ for all $x \in V_X$, then $\Lambda$ has infinite clusters with positive probability.

The proof of this result in [2] uses a method of Adams and Lyons [1], that does not yield any estimate for the threshold $\eta_G$. Although Theorem 2.2 suffices for the proof of Theorem 1.1, we take this opportunity to complete the discussion of quantitative thresholds from Section 4 of [2]. This avoids the nonconstructive definition of amenability via invariant means, and will also allow us to obtain quantitative bounds on the intrinsic graph metric within the unique percolation cluster for $p > p_u$. (See the second remark in Section 4.)

Say that a subgroup $G$ of $\text{Aut}(X)$ is quasi-transitive if it has finitely many orbits in $V_X$. Let $\mu$ be the left Haar measure on $G$, and denote $\mu_*(v) := \mu[S(v)]$ for $v \in V_X$. For any finite set $K \subset V_X$, denote by $\partial K$ the set of vertices in $V_X \setminus K$ adjacent to $K$, and let $\mu_*(K) := \sum_{x \in K} \mu_*(x)$. Define

$$\kappa_G := \inf \left\{ \frac{\mu_*([\partial K])}{\mu_*(K)} : K \subset V_X \text{ is finite nonempty} \right\}.$$

For $x \in V_X$ and $\omega \subset V_X$, denote by $C(x, \omega)$ the connected component of $x$ in $\omega$ with respect to the edges induced from $E_X$. (This component is empty if $x \notin \omega$.)

The next theorem combines several results from [2]; we will provide the additional arguments needed below.

**Theorem 2.3.** - Let $X$ be a locally finite graph, and suppose that $G$ is a closed quasi-transitive subgroup of $\text{Aut}(X)$. Choose a complete set $\{v_1, \ldots, v_L\}$ of representatives in $V_X$ of the orbits of $G$. Then

(i) $G$ is nonamenable iff $\kappa_G > 0$. 

(ii) Let $\Lambda$ be a $G$-invariant site percolation on $X$. If $\kappa_G > 0$, then

$$\sum_{i=1}^{L} P[|C(v_i, \Lambda)| < \infty] \leq \sum_{i=1}^{L} \frac{\kappa_G + \deg(v_i)}{\kappa_G} P[v_i \notin \Lambda]. \quad (2.2)$$

Consequently, if

$$\forall x \in V_X, \quad P[x \notin \Lambda] < \frac{\kappa_G}{\kappa_G + \deg(x)}, \quad (2.3)$$

then $\Lambda$ has infinite clusters with positive probability.

(The threshold in (2.3) is sharp for regular trees, see Häggström [8, Theorem 8.1].)

To prove Theorem 2.3, we need the following version of the mass transport principle, obtained from Corollary 3.7 in [2] by setting $a_i \equiv 1$:

**Lemma 2.4.** Let $X$, $G$ and $\{v_1, \ldots, v_L\}$ be as in Theorem 2.3. Suppose that the function $f : V_X \times V_X \to [0, \infty]$ is invariant under the diagonal action of $G$. Then

$$\sum_{i=1}^{L} \sum_{z \in V_X} f(v_i, z) = \sum_{j=1}^{L} \sum_{u \in V_X} f(u, v_j) \frac{\mu_*(u)}{\mu_*(v_j)}. \quad (2.4)$$

**Proof of Theorem 2.3.**

(i) This follows from Theorem 3.9 and Lemma 3.10 in [2].

(ii) Let $v, z \in V_X$ and $\omega \subseteq V_X$. If $v \in \omega$, the component $C(v, \omega)$ is finite, and $z \in \partial C(v, \omega)$, then define

$$f_0(v, z, \omega) = \frac{\mu_*(z)}{\mu_*(\partial C(v, \omega))};$$

otherwise, take $f_0(v, z, \omega) = 0$. For any vertex $v$, clearly

$$\sum_{z \in V_X} f_0(v, z, \omega) = 1_{|0 < |C(v, \omega)| < \infty}. \quad (2.5)$$

Since $v$ can be adjacent to at most $\deg(v)$ components of $\omega$,
The function $f(v, z) := \mathbb{E} f_0(v, z, \Lambda)$ is invariant under the diagonal action of $G$. By (2.4) and (2.5), for any $v \in V_X$ we have
\[ \sum_{z \in V_X} f(v, z) = \mathbb{P}[0 < |C(v, \Lambda)| < \infty] \]
and
\[ \sum_{u \in V_X} f(u, v) \mu_*(u) \leq \frac{\text{deg}(v)}{\kappa_G} \mathbb{P}[v \notin \Lambda]. \]

Taking $v = v_i$ and summing over $i$, we obtain from Lemma 2.4 that
\[ \sum_{i=1}^{I} \mathbb{P}[0 < |C(v_i, \Lambda)| < \infty] \leq \sum_{i=1}^{I} \frac{\text{deg}(v_i)}{\kappa_G} \mathbb{P}[v_i \notin \Lambda]. \quad (2.6) \]

Since $\mathbb{P}[|C(v_i, \Lambda)| < \infty] = \mathbb{P}[0 < |C(v_i, \Lambda)| < \infty] + \mathbb{P}[v_i \notin \Lambda]$, (2.2) follows. Finally, if (2.3) holds, then the right-hand side of (2.2) is less than $L$, so at least one of the probabilities on the left-hand side of (2.2) is less than 1. □

3. PROOF OF NONUNIQUENESS AT $p_u$

We will use the canonical coupling of the percolation processes for all $p$, obtained by equipping the edges of a graph $(V, E)$ with i.i.d. random variables $\{U(e)\}_{e \in E}$, uniform in $[0, 1]$. Denote by $\mathbb{P}$ the resulting product measure on $[0, 1]^E$. For each $p$, the edge set $E(p) := \{ e \in E : U(e) \leq p \}$ has the same distribution as the set of open edges under $\mathbb{P}_p$. Denote by $C(w, p)$ the connected component of a vertex $w$ in the subgraph $(V, E(p))$, and for $W \subseteq V$, write $C(W, p) := \bigcup_{w \in W} C(w, p)$. We need the following easy lemma.

**Lemma 3.1.** Consider the coupling defined above on a graph $(V, E)$, and fix $p_1 < p_2$ in $[0, 1]$. For any two sets $K, W \subseteq V$ and $M < \infty$, denote by $A_M(K, W; p_1)$ the event that infinitely many vertices in $C(K, p_1)$ are within distance at most $M$ from $C(W, p_1)$. Then
\[ \mathbb{P}[K \leftrightarrow W \text{ in } E(p_2) \mid A_M(K, W; p_1)] = 1. \]

**Proof.** On the event $A_M(K, W; p_1)$, there are infinitely many paths $\{\psi_j\}$ of length at most $M$ from $C(K, p_1)$ to $C(W, p_1)$. Each of these paths intersects at most finitely many of the others, so we can extract an infinite
subcollection \( \{ \psi'_j \} \) of edge-disjoint paths. Thus on \( \mathcal{A}_M(K, W; p_1) \),

\[
P_\psi' \text{ open in } \mathcal{E}(p_2) \mid \mathcal{E}(p_1) \supseteq (p_2 - p_1)^M
\]

for each \( j \), and the assertion follows. \( \square \)

**Proof of Theorem 1.1.** We will show that in \( X \times Y \), if

\[
P_p[\exists \text{ a unique infinite cluster}] = 1,
\]

then \( p > p_u \). Let \( G = \text{Aut}(X) \), and fix a threshold \( \eta_G > 0 \) as in Theorem 2.2. (By Theorem 2.3, we can take \( \eta_G = \kappa_G / (\kappa_G + D_X) \) where \( D_X := \max_{x \in V_X} \deg(x) \).) Denote by \( C_\infty(p) \) the unique infinite cluster in \( \mathcal{E}(p) \), and define

\[
\Gamma_1 = \Gamma_1(r) := \{ v \in V_{X \times Y} : B_r(v) \cap C_\infty(p) \neq \emptyset \}.
\]

By (3.1) and quasi-transitivity of \( X \times Y \), there exists \( r \) such that

\[
\forall v \in V_{X \times Y}, \quad P[v \notin \Gamma_1(r)] < \eta_G / 6.
\]

Next, define

\[
\Gamma_2 = \Gamma_2(r, n) := \{ v \in V_{X \times Y} : \forall v_0, v_1 \in B_{r+1}(v) \cap C_\infty(p), \quad \text{dist} (v_0, v_1; \mathcal{E}(p)) < n \}.
\]

Once \( r \) is chosen, we can find \( n \) such that

\[
\forall v \in V_{X \times Y}, \quad P[v \notin \Gamma_2(r, n)] < \eta_G / 6.
\]

Denote by \( D = D_{X \times Y} \) the maximal degree in \( X \times Y \).

**CLAIM.** Fix \( r, n \) as above. If

\[
p_* > p - \frac{\eta_G}{6D^{r+n}},
\]

then

\[
\lim_{R \to \infty} \inf_{v^1, v^2 \in V_{X \times Y}} P_{p_*} [B_R(v^1) \leftrightarrow B_R(v^2)] = 1.
\]

By Theorem 2.1, the last equation yields that \( p_u \leq p_* \), so the claim implies that

\[
p_u \leq p - \frac{\eta_G}{6D^{r+n}}.
\]
To prove the claim, choose $p_1, p_2$ such that

$$p_1 < p_2 < p_* \quad \text{and} \quad p - p_1 < \frac{\eta_G}{6D^{r+n}}. \quad (3.7)$$

Use the canonical coupling variables $\{U(e)\}$ to define

$$\Gamma_3 = \Gamma_3(r, n, p_1) := \{v \in V_{X \times Y} : U(e) \notin [p_1, p] \text{ for all } e \text{ in } B_{r+n}(v)\}.$$ 

Since $D^{r+n}$ bounds the number of edges in a ball of radius $r+n$ in $X \times Y$, (3.7) gives

$$\forall v \in V_{X \times Y}, \quad P[v \notin \Gamma_3(r, n, p_1)] < \eta_G/6.$$

Let $\Gamma_\circ := \Gamma_1(r) \cap \Gamma_2(r, n) \cap \Gamma_3(r, n, p_1)$, and note that $P[(x, y) \notin \Gamma_\circ] < \eta_G/2$ for any $(x, y) \in V_{X \times Y}$. The “shadowing method” which is the key to our argument, is based on defining a site percolation on $X$ that requires “good behavior” simultaneously in two levels, $X \times \{y_0\}$ and $X \times \{y_1\}$. Fix $y_0, y_1 \in V_Y$, and consider

$$\Lambda := \{x \in V_X : (x, y_0) \in \Gamma_\circ \text{ and } (x, y_1) \in \Gamma_\circ\}.$$ 

$\Lambda$ is a $G$-invariant site percolation on $X$, with $P[x \notin \Lambda] < \eta_G$ for every vertex $x$. Thus

$$P[\Lambda \text{ has an infinite component}] > 0, \quad (3.8)$$

by Theorem 2.2. Since the event in (3.8) is $G$-invariant and determined by the i.i.d. variables in the canonical coupling, it must have probability 1. (The action of $G$ on $X$ has infinite orbits, whence the induced action on the random field $\{U_x\}_{x \in E_X}$ is ergodic.)

Our next task is to verify that for any infinite path with vertices $\{x_j\}_{j \geq 1}$ in $\Lambda$, its lift $\xi_0 := \{(x_j, y_0)\}_{j \geq 1}$ to $X \times \{y_0\}$, is “shadowed” by an infinite path with edges in $\mathcal{E}(p_1)$, that remains a bounded distance from $\xi_0$. Indeed, the ball $B_r(x_j, y_0)$ contains a point $\nu_j^0$ in $C_\infty(p)$ by the definition of $\Gamma_1$, and there is a path in $\mathcal{E}(p_1)$ from $\nu_j^0$ to $\nu_{j+1}^0$ by the definitions of $\Gamma_2$ and $\Gamma_3$. Concatenating these finite paths gives an infinite path with edges in $\mathcal{E}(p_1)$, that intersects $B_r(x_j, y_0)$ for each $j \geq 1$. Similarly, there is an infinite path with edges in $\mathcal{E}(p_1)$, that intersects $B_r(x_j, y_1)$ for each $j \geq 1$. 
Therefore, Lemma 3.1 with \( M = 2r + \text{dist}(y_0, y_1; E_Y) \) implies that for any \( x_1 \in V_X \),

\[
P[B_r(x_1, y_0) \leftrightarrow B_r(x_1, y_1) \text{ in } \mathcal{E}(p_2) \mid C(x_1, A) \text{ is infinite}] = 1. \tag{3.9}
\]

Let \( \varepsilon > 0 \). Since the event in (3.8) has probability 1, there exists \( R_0 \) such that for all \( x \in V_X \),

\[
P[B_{R_0}(x) \text{ intersects an infinite component of } A] > 1 - \varepsilon. \tag{3.10}
\]

Let \( R = R_0 + r \). By (3.9), (3.10) and the triangle inequality,

\[
P[B_R(x, y_0) \leftrightarrow B_R(x, y_1) \text{ in } \mathcal{E}(p_2)] > 1 - \varepsilon. \tag{3.11}
\]

Finally, consider two arbitrary vertices \( v^1 = (x^1, y^1) \) and \( v^2 = (x^2, y^2) \) in \( V_{X \times Y} \). For \( y \in V_Y \), let

\[
H_y := \{ B_R(x^1, y^1) \leftrightarrow B_R(x^1, y) \text{ and } B_R(x^2, y^2) \leftrightarrow B_R(x^2, y) \text{ in } \mathcal{E}(p_2) \}.
\]

By (3.11), \( P[H_y] > 1 - 2\varepsilon \) for any \( y \in V_Y \). Consequently,

\[
P[H_y \text{ for infinitely many } y] > 1 - 2\varepsilon. \tag{3.12}
\]

On this event, the sets \( C(B_R(v^1), p_2) \) and \( C(B_R(v^2), p_2) \) come infinitely often within distance \( \text{dist}(x^1, x^2; E_X) + 2R \) from each other. As \( p_* > p_2 \), we obtain from Lemma 3.1 and (3.12) that

\[
P[B_R(v^1) \leftrightarrow B_R(v^2) \text{ in } \mathcal{E}(p_*)] > 1 - 2\varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, we have established (3.5) and the claim. This implies (3.6) and the theorem. \( \square \)

4. CONCLUDING REMARKS

- **Nonamenability and isoperimetric inequalities.** Say that an infinite graph \( X \) is nonamenable if

\[
\inf \left\{ \frac{|\partial K|}{|K|} : K \subset V_X \text{ is finite nonempty} \right\} > 0. \tag{4.1}
\]

In Theorem 1.1 we assumed that the group \( \text{Aut}(X) \) is nonamenable. Could this assumption be replaced by the weaker assumption that
the graph $X$ is nonamenable? (These assumptions are equivalent if $\text{Aut}(X)$ is quasi-transitive and unimodular, see Salvatori [17].)

- **Intrinsic distance within the infinite cluster.** In the setup of Theorem 1.1, denote by $D$ the maximal degree in $X \times Y$. For $p > p_u = p_u(X \times Y)$, choose $r = r(p)$ and $n = n(p)$ to satisfy (3.2) and (3.3). Then (3.4) implies that

$$D^{r+n} > \frac{\eta G}{6(p - p_u)},$$

(4.2)

If $p_u > p_c$ then $\sup_{p>p_u} r(p) < \infty$, so (4.2) yields a bound on the distribution of the intrinsic distance between vertices in the unique infinite cluster.

- **Kazhdan groups.** Lyons and Schramm [13] proved that $p_u < 1$ for Cayley graphs of Kazhdan groups. The present author observed that their argument can be modified to prove nonuniqueness at $p_u$ on these graphs; see [13].

- **Planar graphs.** Benjamini and Schramm (unpublished) showed that for i.i.d. percolation on a planar nonamenable transitive graph, there is a unique infinite cluster for $p = p_u$. (As noted by the referee, for Cayley graphs of cocompact Fuchsian groups of genus at least 2, this can be inferred from [11].) It is an open problem to find a geometric characterization of nonamenable transitive graphs that satisfy uniqueness at $p_u$.

- **Minimal spanning forests and $p_u$.** The impetus for this note was a suggestion by I. Benjamini and O. Schramm, that uniqueness for i.i.d. percolation at $p = p_u$ on a transitive graph $X$, should be closely related to connectedness of the “free minimal spanning forest” (FMSF) on $X$; this is a random subgraph $(V_X, F)$ of $X$, obtained by labeling the edges in $E_X$ by i.i.d. uniform variables, and removing any edge that has the highest label in a cycle. Indeed, Schramm (personal communication) has recently observed that connectedness of the FMSF implies uniqueness at $p_u$; the converse fails for certain free products, but it is open whether it holds for transitive graphs that satisfy $p_c < p_u < 1$.

- **The contact process.** Let $T_d$ be a regular tree of degree $d \geq 3$. Pemantle [15] considered the contact process on $T_d$ with infection rate $\lambda$. He showed that if $d \geq 4$, then the critical parameter for global survival, $\lambda_1(T_d)$, is strictly smaller than the critical parameter for local survival, $\lambda_2(T_d)$; the result was extended to $T_3$ by Liggett [12]. Zhang [21] showed that the contact process on $T_d$ does not survive
locally at the parameter $\lambda_2(T_d)$, and that for larger values of $\lambda$, the so called “complete convergence theorem” holds. The proof by Schonmann [20] of nonuniqueness for percolation at level $p_u$ on $T \times \mathbb{Z}$, was motivated by these results of Zhang and alternative proofs of them in Salzano and Schonmann [18]. Can the proof of Theorem 1.1 be adapted to show that for any graph $X$ with $\text{Aut}(X)$ nonamenable, the contact process does not survive locally at the parameter $\lambda_2(X)$?

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