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First occurrence time of a large density fluctuation for a system of independent random walks

by

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ABSTRACT. – We obtain sharp asymptotics for the first time a "macroscopic" density fluctuation occurs in a system of independent simple symmetric random walks on Z^d. Also, we show the convergence of the moments of the rescaled time by establishing tail estimates. © 2000 Éditions scientifiques et médicales Elsevier SAS

Key words: Independent random walks, Exponential approximation, Tail estimates

RÉSUMÉ. – Nous obtenons des estimées asymptotiques de la distribution du premier temps d'apparition d'une fluctuation "macroscopique" en densité pour un système de particules indépendantes sur Z^d. Aussi, nous obtenons la convergence des moments du temps renormalisé en établissant des estimées de queue de sa distribution. © 2000 Éditions scientifiques et médicales Elsevier SAS

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1. INTRODUCTION

We consider a system of independent, simple symmetric random walks on the lattice $\mathbb{Z}^d$. We assume that the system is in equilibrium, with a density of particle $\rho \in (0, +\infty)$. We study the distribution of the first time more than $\rho' n^d$ particles are present in a hypercube of volume n^d with $\rho' > \rho$. Also, we obtain sharp asymptotics in the limit $n \to \infty$.

The problem of finding asymptotics for the first occurrence time of a rare event for a Markov process has a history which traces back to Harris [14]. His results, as well as more recent ones in [8,16,17], are restricted to Markov chains satisfying strong recurrence properties (*Harris recurrence*).

The related problem concerning the exit time from the basin of attraction of a metastable state for certain particle systems [7,9,19–21], has in part stimulated the study of occurrence time of rare events for interacting particle systems. For non-conservative spin-flip dynamics the problem is rather well understood in the case of attractive systems [18] and systems whose equilibrium measure satisfies a logarithmic-Sobolev inequality [2]. In both cases, the fast convergence to equilibrium implies that the system performs several almost independent attempts before reaching the rare event; this guarantees that the distribution of the first occurrence time is close to exponential.

A subtler question is whether the quasi-exponentiality holds for systems possessing a conservation law. Typically, we expect a positive answer, even though the multiplicity of invariant measures prevents ergodicity arguments to be exploited. In fact, in most interesting systems, the time needed to "recover" from a large scale fluctuation is large, but much smaller than the time needed for such fluctuation to occur. It is therefore natural to conjecture that, after rescaling the time according to the larger time scale, the system "looks like" an ergodic process.

So far, no general technique for translating this idea into rigorous mathematics has been developed and analysis has proceeded case by case. The first model to be studied was the zero-range process [11], followed by the symmetric simple exclusion (SSEP) in one dimension [12], the SSEP in any dimension [3] and the contact process [22]. For these models a quite sharp result has been proved. Namely, that there is a constant β_n such that T_n/β_n converges to an exponential variable of mean 1, and β_n is estimated up to a constant.

A key technical argument in most of the cited studies is duality: when we estimate the probability of an event A_t which depends on the state of the system at time t in a finite space region $\Lambda \subset \mathbb{Z}^d$, we need in principle to control the evolution of the entire infinite system; when duality holds, $P(A_t)$ is determined by a finite number of particles which evolve backward in time according to the dual process.

Although we follow the strategy of [12] to prove convergence in distribution, the absence of a simple dual process presented a major difficulty. Moreover, unlike [11,12,3], we show convergence of the moments of T_n/β_n to the moments of an exponential variable of mean 1, in dimension larger than two. For moments convergence, we follow an approach of [1] which relies on an inequality of Varadhan [23].

2. MODEL AND MAIN RESULTS

We consider independent particles evolving as simple random walks on $\mathbb{Z}^d$. Denote by $\eta_t(i)$ the number of particles that occupy the site i at time t. The process $(\eta_t)_{t\geqslant 0}$ is a Markov process with generator

$$Lf(\eta) = \sum_{i\in\mathbb{Z}^d} \frac{1}{2d} \sum_{j:\ |j-i|=1} \eta(i)\left[f(\eta^{i,j}) - f(\eta)\right], \quad \text{for } \eta \in \mathbb{N}^{\mathbb{Z}^d},$$

where $\eta^{i,j}(k) = \eta(k) + \delta_{kj} - \delta_{ki}$. The extremal invariant measures for this process are the product measures, ν_ρ, whose marginals have Poisson distribution with parameter $\rho \geqslant 0$ (see [10]). We denote by P^μ the law of $\eta(t)$ started with μ, and E^μ the corresponding expectation.

For ρ and $\rho' > \rho$ given, we define the event

$$A_n = \left\{\eta \in \mathbb{N}^{\mathbb{Z}^d}:\ \frac{1}{|\Lambda_n|}\eta(\Lambda_n) \geqslant \rho'\right\}$$
$$\text{with } \Lambda_n = \{1,\dots,n\}^d, \text{ and } \eta(\Lambda_n) = \sum_{i\in\Lambda_n}\eta(i).$$

Also, let $T_n = \inf\{t \geqslant 0:\ \eta_t \in A_n\}$, and $[x] = \min\{n \in \mathbb{N}:\ n \geqslant x\}$. Our main results are

THEOREM 1. – *Suppose $0 < \rho < \rho'$. There are positive constants C, c, c' and a sequence $(\beta_n)_{n\geqslant 0}$ such that*

$$\sup_{t\geqslant 0}\left|P^{\nu_\rho}\left(\frac{T_n}{\beta_n} > t\right) - \mathrm{e}^{-t}\right| \leqslant \mathrm{e}^{-Cn^d},$$

where β_n satisfies

$$cn^{d-1}\nu_\rho\big(\eta(\Lambda_n)=\big[|\Lambda_n|\rho'\big]\big)\leqslant\frac{1}{\beta_n}\leqslant c'n^{d-1}\nu_\rho\big(\eta(\Lambda_n)=\big[|\Lambda_n|\rho'\big]\big).$$

THEOREM 2. – *Suppose $d\geqslant 3$. Then, for every $k\geqslant 1$*

$$\lim_{n\to\infty}E^{\nu_\rho}\left(\frac{T_n}{\beta_n}\right)^k=k!$$

The proof of Theorem 1 consists into three main steps: the lower bound for T_n is treated in Section 3, the upper bound in Section 4 and the independence property in Section 5. The proof is then completed in Section 6. The proof of Theorem 2, given in Section 7, relies on uniform estimates on the tails of the distribution of T_n/β_n. In Section 8 we list some problems left open.

3. LOWER BOUND

We define

$$\partial\Lambda_n=\big\{i\in\Lambda_n^c:\ \mathrm{dist}(i,\Lambda_n)=1\big\}.$$

For any $i\in\partial\Lambda_n$ there is exactly one $j\in\Lambda_n$, say j_i, such that $\mathrm{dist}(i,j_i)=1$. For simplicity, we let $l=[\rho' n^d]$, and we will often drop subscripts n or ρ. For $0<s<t$ let $N(s,t)$ be the number of particles that, in the time interval (s,t), enter Λ_n while there are $l-1$ particles in Λ_n. Note that $N(s,t)$ can be written in the form

$$N(s,t)=\sum_{i\in\partial\Lambda_n}\int_s^t Z_{r^-}\,\mathrm{d}J_r^i,\tag{3.1}$$

where J_r^i is the process that counts the number of particles that move from the site i to the site j_i during time $[0,r]$, and

$$Z_r=\chi\big(\eta_r(\Lambda_n)=l-1\big),$$

where $\chi(A)$ is the characteristic function of A.

PROPOSITION 1. – *There is a constant $a>0$ such that for every $t>0$*

$$P^\nu(T_n\geqslant t)\geqslant 1-a|\partial\Lambda_n|\nu\big(\eta(\Lambda_n)=l\big)t-\nu\big(\eta(\Lambda_n)\geqslant l\big).\tag{3.2}$$

Proof. – By using (3.1) and that J_r^i has intensity $\eta_r(i)/2d$, we have

$$E^\nu N(0,t) = \sum_{i\in\partial\Lambda_n} \int_0^t E^\nu\big(Z_s\eta_s(i)\big)\,\mathrm{d}s = \frac{t}{2d}\sum_{i\in\partial\Lambda_n} E^\nu\big(Z\eta(i)\big)$$
$$= \frac{\rho}{2d}|\partial\Lambda_n|\nu\big(\eta(\Lambda_n) = l-1\big)t.$$

Now

$$\{T_n < t\} \subset \{\eta_0(\Lambda_n) \geqslant l\} \cup \{N(0,t) \geqslant 1\}, \qquad \chi\big(N(0,t) \geqslant 1\big) \leqslant N(0,t)$$

and,

$$\frac{\nu(\eta(\Lambda_n) = l-1)}{\nu(\eta(\Lambda_n) = l)} = \frac{[\rho' n^d]}{\rho n^d}$$

so that (3.2) follows at once. □

4. UPPER BOUND

A system of independent random walks can be realized as follows. Suppose that for any site $i \in \mathbb{Z}^d$ there is a sequence $(n_{i,k}(t))_{k\geqslant 1}$ of independent Poisson processes of intensity 1. Processes associated to different sites are independent. We call these processes *clocks*. If t is a jump time for $n_{i,k}$, then we say that the clock of level k at site i rings at time t.

Suppose $\eta_t(i) = r$, i.e., there are r particles in i at time t. We imagine that these particles occupy r different levels. These levels at site i are called *boxes* and can be identified with pairs (i,k), $i \in \mathbb{Z}^d$, $k \geqslant 1$. So the event $\eta_t(i) = r$ can be described by saying that the boxes (i,k), $1 \leqslant k \leqslant r$, are *occupied*, while all other boxes at site i are empty.

We now describe the dynamics. Suppose the clock $n_{i,k}$ rings at time t. If the box (i,k) is not occupied then nothing happens. Otherwise the particle at (i,k) moves to the lowest unoccupied level of a site j, randomly chosen with uniform probability among the $2d$ nearest neighbors; also, the remaining particles in boxes (i,k') for $k' > k$ move back one level. It is easy to check that each particle evolves as a simple random walk, independently of all others.

We define $\hat{N}(s,t)$ to be the number of times the clocks of level 1 in the sites in $\partial\Lambda_n$ ring within time s and t, while there are exactly l particles in Λ_n, i.e.,

$$\hat{N}(s,t) = \sum_{i\in\partial\Lambda} \int_s^t Z_{r^-}\,\mathrm{d}n_{i,1}(r), \quad \text{with } Z_r = \chi\big(\eta_r(\Lambda) = l\big). \tag{4.1}$$

We denote by $1/\gamma_n = 2dn^{d-1}\nu(\eta(\Lambda) = l)$.

PROPOSITION 2. – *There are constants $B, D > 0$ such that, for all $n > 0$, $t > 0$*

$$P^{\nu}(T_n < t\gamma_n) \geqslant \frac{t}{Bt + D}.$$

Proof. – For any $T > 0$, $\{T_n < T\} \supset \{\hat{N}(0,T) \geqslant 1\}$. Thus, Cauchy–Schwarz inequality gives

$$\big(E\hat{N}(0,T)\big)^2 = \big(E\big[\hat{N}(0,T)\chi\{T_n < T\}\big]\big)^2 \leqslant E\big[\hat{N}^2(0,T)\big]P(T_n < T),$$

which implies

$$P(T_n < T) \geqslant \frac{(E\hat{N}(0,T))^2}{E[\hat{N}^2(0,T)]}. \tag{4.2}$$

Now, using (4.1) and the fact that $n_{i,1}$ is a Poisson process of intensity 1, we have

$$E^{\nu}\hat{N}(0,t\gamma_n) = t.$$

Therefore, all we need to show is that there are constants $B, D > 0$ such that

$$E^{\nu}\hat{N}^2(0,t\gamma_n) \leqslant Bt^2 + Dt. \tag{4.3}$$

First note that for any $T > 0$

$$\hat{N}(0,T) = \sum_{i\in\partial\Lambda} \int_0^T Z_{t^-}\,\mathrm{d}M_t^i + \sum_{i\in\partial\Lambda} \int_0^T Z_t\,\mathrm{d}t,$$

where M_t^i is a martingale. Therefore, using the inequality $(a+b)^2 \leqslant 2(a^2+b^2)$, we obtain

$$\hat{N}^2(0,T) \leqslant 2|\partial\Lambda|^2 \int_0^T\int_0^T Z_t Z_s\,\mathrm{d}t\,\mathrm{d}s + 2\sum_{i,j\in\partial\Lambda} \int_0^T Z_{t^-}\,\mathrm{d}M_t^i \int_0^T Z_{t^-}\,\mathrm{d}M_t^j.$$

Thus, by stationarity and Theorem 15 in [6], we get

$$E\hat{N}^2(0,T) \leqslant 4|\partial\Lambda|^2 T \int_0^T E^\nu(Z_0 Z_t)\,\mathrm{d}t + 2|\partial\Lambda|\nu\big(\eta(\Lambda)=l\big)T.$$

Thus, it remains to show that there are constants $B, D > 0$ such that

$$|\partial\Lambda|^2 t\gamma_n \int_0^{t\gamma_n} E^\nu(Z_0 Z_s)\,\mathrm{d}s \leqslant Bt^2 + Dt. \tag{4.4}$$

We denote by $O_k(t)$ the event that exactly k particles among those which were in Λ at time zero, are at time t in Λ^c. We denote by $I_k(t)$ the event that exactly k particles among those which were in Λ^c at time zero, are at time t in Λ. We have

$$\begin{aligned}
&P^\nu\big(\eta_0(\Lambda)=l, \eta_t(\Lambda)=l\big)\\
&\quad= \sum_{k=0}^{l} P^\nu\big(\eta_0(\Lambda)=l,\ O_k(t),\ I_k(t)\big)\\
&\quad= \sum_{k=0}^{l} \nu\big(\eta(\Lambda)=l\big)P^\nu\big(O_k(t)\mid \eta_0(\Lambda)=l\big)P^\nu\big(I_k(t)\big)\\
&\quad= \nu\big(\eta(\Lambda)=l\big)\sum_{k=0}^{l} P^{\mu_l}\big(O_k(t)\big)P^\nu\big(I_k(t)\big),
\end{aligned}$$

where μ_l is the measure corresponding to l independent particles uniformly distributed in Λ. Let q_t be the probability that one particle is outside Λ at time t, if at time 0 it is uniformly distributed in Λ. Then,

$$P^{\mu_l}\big(O_k(t)\big) = \binom{l}{k} q_t^k (1-q_t)^{l-k}.$$

Moreover, by reversibility,

$$\begin{aligned}
P^\nu\big(I_k(t)\big) = P^\nu\big(O_k(t)\big) &= \sum_{m\geqslant k}\binom{m}{k} q_t^k(1-q_t)^{m-k}\,\mathrm{e}^{-n^d\rho}\,\frac{(\rho n^d)^m}{m!}\\
&= \mathrm{e}^{-\rho n^d q_t}\,\frac{(\rho n^d q_t)^k}{k!}.
\end{aligned}$$

Case 1: $t \geqslant n^4$. In this case, for some $M > 0$,

$$1 - q_t \leqslant \frac{Cn^d}{t^{d/2}} \leqslant \frac{1}{Mn^d}.$$

Thus, writing q for q_t, we have

$$\begin{aligned}
&\sum_{k=0}^{l} P^{\mu_l}\big(O_k(t)\big)P^{\nu}\big(I_k(t)\big)\\
&\quad\leqslant \sum_{k=0}^{l}\binom{l}{k}\left(\frac{1}{Mn^d}\right)^{l-k} \mathrm{e}^{-\rho n^d q}\,\frac{(\rho n^d q)^k}{k!}\\
&\quad= \mathrm{e}^{-\rho n^d q}\,\frac{(\rho n^d q)^l}{l!}\sum_{k=0}^{l}\frac{l!}{k!(l-k)!}\,\frac{l!}{(l-k)!}\left(\frac{1}{M\rho q n^{2d}}\right)^k\\
&\quad\leqslant C'\nu\big(\eta(\Lambda)=l\big)\sum_{k=0}^{l}\frac{(l^2/M\rho q n^{2d})^k}{k!}\leqslant C\nu\big(\eta(\Lambda)=l\big),
\end{aligned}$$

where the inequality

$$\mathrm{e}^{-\rho n^d q}\,\frac{(\rho n^d q)^l}{l!}\leqslant C'\nu\big(\eta(\Lambda)=l\big)$$

comes easily from $1-q\leqslant 1/Mn^d$. Therefore, when $t\gamma_n > n^4$

$$|\partial\Lambda|^2 t\gamma_n\int_{n^4}^{t\gamma_n} E^{\nu}(Z_0 Z_t)\,\mathrm{d}t\leqslant C|\partial\Lambda|^2(t\gamma_n)^2\nu\big(\eta(\Lambda)=l\big)^2\leqslant Ct^2. \tag{4.5}$$

Case 2: $t < n^4$. We define

$$a_k=\binom{l}{k}q^k(1-q)^{l-k}\quad\text{and}\quad b_k=\mathrm{e}^{-\rho n^d q}\,\frac{(\rho n^d q)^k}{k!}.$$

Using an argument similar to [2], there is a constant $\beta>0$ such that

$$\forall n,\quad \sum_{k=0}^{l} a_k b_k\leqslant \mathrm{e}^{-\beta q n^d}.$$

Indeed, consider two independent random variables X, Y with $P(X=k)=a_k$ and $P(Y=k)=b_k$. Then, for any $\theta\in\mathbb{R}$,

$$\begin{aligned}
\sum_{k=0}^{l} a_k b_k &= P(X=Y)\leqslant E\big\{\mathrm{e}^{\theta(X-Y)}\big\}=E\big(\mathrm{e}^{\theta X}\big)E\big(\mathrm{e}^{-\theta Y}\big)\\
&=\big[\big(\mathrm{e}^{\theta}q+1-q\big)^{\rho'/q}\,\mathrm{e}^{-(1-\mathrm{e}^{-\theta})\rho}\big]^{qn^d}.
\end{aligned}\tag{4.6}$$

By elementary calculus

$$\left(\mathrm{e}^{\theta} q+1-q\right)^{\rho'/q} \mathrm{e}^{-(1-\mathrm{e}^{-\theta})\rho}=\exp\left[(\rho'-\rho)\theta+\mathrm{o}(\theta)\right]$$

uniformly for $q \in [0, 1]$. Thus, for $\theta < 0$ small enough, the expression in (4.6) is bounded by $\mathrm{e}^{-\beta n^d q}$ for some $\beta > 0$. Thus,

$$|\partial\Lambda|^2 t\gamma_n \int_0^{n^4} E^{\nu}(Z_0 Z_t)\,\mathrm{d}t \leqslant Ctn^{d-1} \int_0^{n^4} \mathrm{e}^{-\beta q_t n^d}\,\mathrm{d}t.$$

In the case $d \geqslant 2$, we have

$$q_t \geqslant \frac{1}{kn}\left(1-\mathrm{e}^{-t}\right) \tag{4.7}$$

for some $k > 0$. Indeed, define $\partial_{\mathrm{int}}\Lambda = \{i \in \Lambda\colon \operatorname{dist}(i, \Lambda^c) = 1\}$. Now, $1/n$ is a lower bound for the probability that a particle uniformly distributed in Λ_n, be in $\partial_{\mathrm{int}}\Lambda \subset \Lambda$ at time zero. Consider now a particle that is in $\partial_{\mathrm{int}}\Lambda$ at time zero, and denote by x_t its position at time t. Let τ be the first jump time for this particle. Clearly,

$$P\left(x_{\tau^+} \in \Lambda^c, \tau < t \mid x_0 \in \partial_{\mathrm{int}}\Lambda\right) \geqslant \left(1-\mathrm{e}^{-t}\right)/2d.$$

Moreover, by reflection

$$P\left(x_t \in \Lambda^c \mid x_{\tau^+} \in \Lambda^c, \tau < t\right) \geqslant \frac{1}{2}.$$

It follows that

$$P\left(x_t \in \Lambda^c, \tau < t \mid x_0 \in \partial_{\mathrm{int}}\Lambda\right) \geqslant \left(1-\mathrm{e}^{-t}\right)/4d,$$

and (4.7) follows. Thus, using that for $t \in [0, 1]$, $1-\exp(-t) \geqslant t/2$,

$$\begin{aligned}\int_0^{n^4} \mathrm{e}^{-\beta q_t n^d}\,\mathrm{d}t &= \int_0^{1} \mathrm{e}^{-\beta q_t n^d}\,\mathrm{d}t + \int_1^{n^4} \mathrm{e}^{-\beta q_t n^d}\,\mathrm{d}t\\ &\leqslant \int_0^{1} \mathrm{e}^{-\gamma n^{d-1} t}\,\mathrm{d}t + n^4 \mathrm{e}^{-\lambda n^{d-1}} \leqslant \frac{C}{n^{d-1}}\end{aligned}$$

for suitable $\gamma, \lambda, C > 0$. That completes the upper bound for $d \geqslant 2$.

For the case $d=1$, let $X_t(i)$ be a random walk starting at i. Then,

$$nq_t = \sum_{i=1}^{n} P\big(X_t(i) \notin \Lambda\big) = E\big(\min(|X_t(0)|, n)\big). \tag{4.8}$$

Therefore, using the fact that, for some $c>0$, $E(\min(|X_t(0)|, n)) \geqslant \min(c\sqrt{t}, n)$, we get

$$\begin{aligned}\int_0^{n^4} \mathrm{e}^{-\beta q_t n}\,\mathrm{d}t &\leqslant \int_0^{n^2/c^2} \mathrm{e}^{-\beta c\sqrt{t}}\,\mathrm{d}t + n^4\,\mathrm{e}^{-\beta n}\\ &\leqslant \int_0^{+\infty} \mathrm{e}^{-\beta c\sqrt{t}}\,\mathrm{d}t + n^4\,\mathrm{e}^{-\beta n} \leqslant C\end{aligned}$$

which completes the proof of the upper bound. □

5. THE INDEPENDENCE PROPERTY

PROPOSITION 3. – *For any $\alpha>0$ small enough and $s > \mathrm{e}^{\alpha n^d}/\gamma_n$, there are constants $C_1, C_2 > 0$ such that*

$$\begin{aligned}&\sup_{t>0}\big|P^\nu\big(T_n \geqslant \gamma_n(t+s)\big) - P^\nu(T_n \geqslant \gamma_n t)P^\nu(T_n \geqslant \gamma_n s)\big|\\ &\quad\leqslant (1+s)C_1\,\mathrm{e}^{-C_2 n^d}.\end{aligned}$$

In view of $\{N(0,t)=0\}\Delta\{T_n \geqslant t\} \subset \{\eta_0(\Lambda_N) \geqslant l\}$, it is enough to prove

$$\begin{aligned}&\sup_{t>0}\big|P^\nu\big(N(0,\gamma_n(t+s))=0\big) - P^\nu\big(N(0,\gamma_n t)=0\big)P^\nu\big(N(0,\gamma_n s)=0\big)\big|\\ &\quad\leqslant C_1(1+s)\,\mathrm{e}^{-C_2 n^d}.\end{aligned} \tag{5.1}$$

By using Proposition 1 and the stationarity of the process started with ν, it is easy to see that (5.1) is implied by

$$\begin{aligned}&\sup_{t>0}\big|P^\nu\big(N(0,\gamma_n t) + N\big(\gamma_n t + D_n, \gamma_n(t+s)\big)=0\big)\\ &\qquad - P^\nu\big(N(0,\gamma_n t)=0\big)P^\nu\big(N(D_n,\gamma_n s)=0\big)\big|\\ &\quad\leqslant (1+s)C_1\,\mathrm{e}^{-C_2 n^d},\end{aligned} \tag{5.2}$$

where $D_n = (\gamma_n s)^{1/(d+1)}$. Intuitively, D_n is a time small compared to the average time needed to see the fluctuations, but much longer than the time needed for the particles in the vicinity of Λ_n to "mix".

By using reversibility,

$$P^{\nu}\big(N(0,\gamma_n t)+N\big(\gamma_n t+D_n,\gamma_n(t+s)\big)=0\big)$$
$$=\int \nu(\mathrm{d}\eta)P^{\eta}\big(N(0,\gamma_n t)=0\big)P^{\eta}\big(N(D_n,\gamma_n s)=0\big).$$

Thus, the l.h.s. of (5.2) is bounded above by

$$\int \nu(\mathrm{d}\eta)\nu(d\zeta)\big|P^{\eta}\big(N(D_n,\gamma_n s)=0\big)-P^{\zeta}\big(N(D_n,\gamma_n s)=0\big)\big|. \qquad (5.3)$$

In order to give estimates for (5.3) we couple two systems of independent random walks, $(\eta_t)_{t\geqslant 0}$, $(\xi_t)_{t\geqslant 0}$ driven by the same clocks $n_{i,k}(t)$, $i\in\mathbb{Z}^d$, $k\geqslant 1$. Suppose the clock $n_{i,k}$ rings at time t. If level k is occupied by only one η or a ξ-particle, the "unmatched" particle, say an η-particle, chooses at random one nearest neighbor j, and moves to the lowest level of j containing no η-particle. If level k is occupied by an η and a ξ-particle, then the two "matched" particles move together to the site j, and move to the lowest level containing no matched particles; if there are unmatched particles at j, they have to move up by one level. Finally, if level k is empty, then nothing happen. It is easy to see that both η_t and ξ_t evolve as systems of independent random walks.

We assume that η_0,ξ_0 are distributed according to the equilibrium measure $\nu\otimes\nu$. We denote by $P^{\nu\otimes\nu}$ the joint law of $(\eta_t,\xi_t)_{t\geqslant 0}$, and by $P^{\eta,\zeta}$ the associated Markov family. Also, N_ζ and N_η denote the counting processes associated with the processes η_u and ζ_u. We now note that

$$\big|P^{\eta}\big(N(D_n,\gamma_n s)=0\big)-P^{\zeta}\big(N(D_n,\gamma_n s)=0\big)\big|$$
$$\leqslant 2P^{\eta,\zeta}\big(N_\eta(D_n,\gamma_n s)\geqslant 1, N_\zeta(D_n,\gamma_n s)=0\big). \qquad (5.4)$$

Let $\{\tau_k,\ k=0,1,\ldots\}$ be the random times at which an η-particle jumps from some bonds of $\partial\Lambda_n$. It may happen that $N_\zeta(D_n,\gamma_n s)=0$ even though there are more than l ζ-particles in Λ_n at time D_n. If we assume that this is not the case, then $N_\zeta(D_n,\gamma_n s)=0$ and $N_\eta(D_n,\gamma_n s)\geqslant 1$, imply that there is a τ_k such that $\eta_{\tau_k^-}$ has l particles in Λ_n, while $\zeta_{\tau_k^-}$ has less than l particles. Thus, the expression in (5.3) is bounded above by

$$2E^{\nu\otimes\nu}\Bigg\{\sum_{k=1}^{\infty}\chi\big(D_n\leqslant\tau_k\leqslant\gamma_n s,\ \eta_{\tau_k^-}(\Lambda_n)=l,$$
$$\exists a\in\Lambda_n\colon\ \zeta_{\tau_k^-}(a)<\eta_{\tau_k^-}(a)\big)\Bigg\}+2\nu\big(\eta(\Lambda_n)\geqslant l\big). \qquad (5.5)$$

By using the same martingale argument used in Proposition 2, the first term in (5.5) can be bounded by

$$\sum_{i\in\partial\Lambda_n}\int_{D_n}^{\gamma_n s} E^{\nu\otimes\nu}\big[\eta_u(i)\chi\big\{\eta_u(\Lambda_n)=l,\ \exists a\in\Lambda_n\colon\ \eta_u(a)>\zeta_u(a)\big\}\big]\,\mathrm{d}u, \tag{5.6}$$

whereas the last term poses no problem. Thus, to prove (5.1), it is enough to prove that (5.6) is bounded by $As\exp(-Cn^d)$. Or, that for any $u\in[D_n,\gamma_n s]$, any $i\in\partial\Lambda_n$,

$$\gamma_n|\partial\Lambda_n|E^{\nu\otimes\nu}\big[\eta_u(i)\chi\{\ldots\}\big]\leqslant A\exp\big(-Cn^d\big). \tag{5.7}$$

Using Cauchy–Schwarz inequality, it is easily seen that it is enough to show

$$\begin{aligned}&P^{\nu\otimes\nu}\big(\eta_u(\Lambda_n)=l,\ \exists a\in\Lambda_n\colon\ \eta_u(a)>\zeta_u(a)\big)\\&\quad\leqslant A\exp\big(-Cn^d\big)\nu\big(\eta(\Lambda_n)=l\big).\end{aligned} \tag{5.8}$$

The next two subsections are devoted to proving (5.8).

5.1. Notations

We let $S=\mathbb{Z}^d\times(\mathbb{N}\setminus\{0\})$ denote the set of boxes. For $x\in S$, its first coordinate, $x(1)$ or x_1, corresponds to a site on the lattice, while the second is a level, for we think that particles on the same site fill different levels. We denote by $X(t,x)$ the position of the box occupied at time t by an η-particle starting on box x (which depends on $\{\eta(s),\xi(s),\ s\leqslant t\}$). Implicitly, we assumed that $\eta(x_1)\geqslant x_2$. Thus, we will always write $\{X(t,x)=y\}$ instead of $\{\eta(x_1)\geqslant x_2$ and $X(t,x)=y\}$. In what follows, for $r>0$ and $i\in\mathbb{Z}^d$, we let $B(i,r)=\{j\in\mathbb{Z}^d\colon\ |i-j|\leqslant r\}$ while, for $x\in S$, we let $B(x,r)=\{y\in S\colon\ |x_1-y_1|\leqslant r\}$.

Since different particles evolve independently, it is clear that for $x,y,x',y',\in S$, $x_1\neq y_1$

$$\begin{aligned}&P\big(X_1(t,x)=x_1',\ X_1(t,y)=y_1'\big)\\&\quad=P\big(X_1(t,x)=x_1'\big)P\big(X_1(t,y)=y_1'\big),\end{aligned} \tag{5.9}$$

where, P stands for $P^{\nu\otimes\nu}$.

We introduce $S_l=\{(x_0,\ldots,x_{l-1})\in S^l\colon\ x_i\neq x_j\}$, and the shorthand notation $X(t,I)\in\Lambda_n$ for $I\in S_l$ to mean $\{X(t,x_0),\ldots,X(t,x_{l-1})\}\in\Lambda_n$

if $I=(x_0,\dots,x_{l-1})$. Thus,

$$\{|\eta_u(\Lambda_n)|=l\}=\bigcup_{I\in S_l}\{X(u,I)\in\Lambda_n\}\cap\{X(u,I^c)\notin\Lambda_n\}. \tag{5.10}$$

If in $X(t,x)$ there is an unmatched η-particle, we say that $X(t,x)$ is a *discrepancy*, and write $\{X(t,x)\in\mathcal{C}_t\}$. An important remark is that the coupling of η_t and ξ_t we choosed does not create discrepancies, i.e., $X(t,x)\in\mathcal{C}_t$ implies $X(u,x)\in\mathcal{C}_u$ for all $u\leqslant t$.

5.2. Estimates

Now, if $\mathcal{A}(I)=\{X(u,I)\in\Lambda_n\}\cap\{X(u,I^c)\notin\Lambda_n\}$, we write

$$\begin{aligned}
&P\big(\eta_u(\Lambda_n)=l,\exists a\in\Lambda_n\colon\ \eta_u(a)>\xi_u(a)\big)\\
&\quad\leqslant\sum_{x_0\in S}P\Big(\bigcup\{\mathcal{A}(I)\colon\ I\in S_l,\ I\ni x_0\},\ X(u,x_0)\in\mathcal{C}_u\Big)\\
&\quad\leqslant\sum_{x_0}\sum_{m=2}^{l}P\Big(\bigcup\{\mathcal{A}(I)\colon\ |I\cap B(x_0,L)|=m,\ I\in S_l,\\
&\qquad\qquad I\ni x_0\}\Big) \qquad (5.11)\\
&\qquad+\sum_{x_0}P\Big(\bigcup\{\mathcal{A}(I)\colon\ |I\cap B(x_0,L)|=1,\ I\in S_l,\ I\ni x_0\},\\
&\qquad\qquad X(u,x_0)\in\mathcal{C}_u\Big) \qquad (5.12)
\end{aligned}$$

where L will be chosen later. We deal with the two summands (5.11) and (5.12) separatly.

Summand (5.11). Consider the events, for $1\leqslant h\leqslant m$

$$\begin{aligned}
A_{x_0,h}&=\{h \text{ particles in } x_0(1) \text{ at time } 0 \text{ end up in } \Lambda_n \text{ at time } u\}\\
&\quad\cap\{\eta(x_0(1))\geqslant x_0(2)\},
\end{aligned}$$

$$\begin{aligned}
B_{L,x_0,h}&=\{h \text{ particles in } B(x_0(1),L)\setminus\{x_0(1)\} \text{ at time } 0 \text{ end up in } \Lambda_n\\
&\quad\text{at time } u\},
\end{aligned}$$

$$\begin{aligned}
C_{L,x_0,m}&=\{l-m \text{ particles in } B(x_0(1),L)^c \text{ at time } 0 \text{ end up in } \Lambda_n\\
&\quad\text{at time } u\}.
\end{aligned}$$

Note that the events $A_{x_0,h}$, $B_{L,x_0,m-h}$ and $C_{L,x_0,m}$ are independent, since they refer to particles that start in disjoint sets of sites. Note that

$$(5.11) \leqslant \sum_{x_0} \sum_{m=2}^{l} \sum_{h=1}^{m} P(A_{x_0,h}) P(B_{L,x_0,m-h}) P(C_{L,x_0,m}).$$

Moreover

$$\begin{aligned} P(A_{x_0,h}) &= \sum_{k \geqslant x_0(2)} \mathrm{e}^{-\rho} \frac{\rho^k}{k!} \binom{k}{h} p_u\big(x_0(1), \Lambda_n\big)^h \\ &\leqslant \sum_{k \geqslant x_0(2)} \frac{(2\rho)^k}{k!} p_u\big(x_0(1), \Lambda_n\big) \left(\frac{Cn^d}{u^{d/2}}\right)^{h-1}, \end{aligned}$$

where $p_u(x_0(1), \Lambda_n)$ is the probability that a random walk starting at $x_0(1)$ is in Λ_n at time u. We have used the fact that there is a $C > 0$ such that $p_u(x_0(1), \Lambda_n) \leqslant Cn^d/u^{d/2}$, for all $i \in \mathbb{Z}^d$. Similarly

$$\begin{aligned} P(B_{L,x_0,m-h}) &= \sum_{k \geqslant 0} \mathrm{e}^{-(L^d-1)\rho} \frac{[(L^d-1)\rho]^k}{k!} \binom{k}{m-h} \left(\frac{Cn^d}{u^{d/2}}\right)^{m-h} \\ &\leqslant \left(\frac{L^d \rho C n^d}{u^{d/2}}\right)^{m-h}. \end{aligned}$$

Moreover

$$P(C_{L,x_0,m}) \leqslant \nu\big(\eta(\Lambda_n) \geqslant l-m\big) \leqslant Cn^d \left(\frac{\rho'}{\rho}\right)^m \nu\big(\eta(\Lambda_n) = l\big).$$

Therefore, if L is large enough

$$\begin{aligned} (5.11) &\leqslant \sum_{x_0} \sum_{m=2}^{l} n^d \frac{\rho'}{\rho} \left(\frac{\rho' C n^d L^d}{u^{d/2}}\right)^{m-1} \\ &\quad \times \sum_{k \geqslant x_0(2)} \frac{(2\rho)^k}{k!} p_u\big(x_0(1), \Lambda_n\big) \nu\big(\eta(\Lambda_n) = l\big) \\ &\leqslant C' \sum_{m=2}^{l} n^{2d} \left(\frac{\rho' C n^d L^d}{u^{d/2}}\right)^{m-1} \nu\big(\eta(\Lambda_n) = l\big), \end{aligned}$$

where we used the facts that there is a constant C

$$\sum_{x(2)} \sum_{k \geqslant x_0(2)} \frac{(2\rho)^k}{k!} \leqslant C \quad \text{and} \quad \sum_{x_0(1)} p_u\big(x_0(1), \Lambda_n\big) \leqslant n^d.$$

Finally, recalling that $u \geqslant D_n \geqslant \exp(\frac{\alpha}{d+1} n^d)$, we choose $L = e^{\beta n^d}$ with β small enough, and we get

$$(5.11) \leqslant C'' n^{3d} \frac{L^d}{u^{d/2}} \nu\big(\eta(\Lambda_n) = l\big) \leqslant A\, e^{-Cn^d}\, \nu\big(\eta(\Lambda_n) = l\big)$$

for suitable $A, C > 0$.

Summand (5.12). We show now that $\{X(T, x_0) \in \mathcal{C}_T\}$ is almost independent of $\{X(u, I^c) \notin \Lambda_n\}$ for $T < u$ and I large enough. We introduce the stopping times

$$\tau_0 = \inf\{t\colon\ X(t, x_0) \notin B\big(x_0(1), L/2\big)\},$$
$$\sigma = \inf\{t\colon\ \exists j \in B\big(x_0(1), L\big), \eta_t(j) + \zeta_t(j) \geqslant 2n^d\}.$$

We can think of coupled trajectories $\delta \equiv (\eta_t, \zeta_t)$ as a function of $(\eta_0, \zeta_0, \mathcal{P})$ where $\mathcal{P}$ is the collection of time jumps on each site of $\mathbb{Z}^d$. For simplicity, we let $\delta_t(i) = (\eta_t(i), \zeta_t(i))$.

LEMMA 1. – *Assume we are given $\delta = (\eta_t, \zeta_t)$ and $\gamma = (\eta'_t, \zeta'_t)$ such that $\delta_0(j) = \gamma_0(j)$ for all $j \in B(x_0(1), L)$ and that the jump times in $B(x_0(1), L)$ for both δ and γ are the same. Also, assume that for $i \in B(x_0(1), L/2)$, and $T > 0$, we have $\delta_{T^-}(i) \neq \gamma_{T^-}(i)$. Then there is an integer $m \leqslant |B(x_0(1), L)|$, and a sequence of time jumps in $\mathcal{P}$, $\tau_1 < \tau_2 < \cdots < \tau_m$, such that*

(i) *each τ_k is a jump time for a box (i_k, l_k) which is occupied by either δ or γ.*

(ii) *$i_m \in \partial B(x_0, L)$, $|i_{k+1} - i_k| \leqslant 1$ for $k = 0, \ldots, m$, and $i_0 \equiv i$.*

Proof. – Let $\tau_1 = \inf\{t\colon\ \delta_t(i) \neq \gamma_t(i)\}$. By assumption $\tau_1 \in (0, T)$. Thus, there is i_1, a nearest neighbour of i such that $\delta_{\tau_1^-}(i_1) \neq \gamma_{\tau_1^-}(i_1)$. Note that $\delta_0(i_1) = \gamma_0(i_1)$, and thus one can proceed similarly to build $i_2 \notin \{i, i_1\}$ and $\tau_2 \in (0, \tau_1)$, and by way of induction, one builds easily distinct $\{i, i_1, \ldots, i_k\}$ and $\tau_1 < \tau_2 < \cdots < \tau_k$ as long as $i_k \in B(x_0(1), L)$. This insures that for some $m \leqslant |B(x_0(1), L)|$ we must have $i_m \in \partial B(x_0, L)$. □

Now, we split a δ (i.e. $(\eta_0, \zeta_0, \mathcal{P})$), into $x \equiv (\eta_{B(x_0(1),L)^c}, \zeta_{B(x_0(1),L)^c}, \mathcal{P}_{B(x_0(1),L)^c})$, and $y \equiv (\eta_{B(x_0(1),L)}, \zeta_{B(x_0(1),L)}, \mathcal{P}_{B(x_0(1),L)})$. Let B_T be the event that there is a sequence $\tau_1 < \tau_2 < \cdots < \tau_m$, such that each τ_k is a jump time for an occupied box (i_k, l_k), $i_m \in \partial B(x_0(1), L)$, $|i_{k+1} - i_k| \leqslant 1$ for $k = 0, \ldots, m$, and $i_0 \equiv i$. What Lemma 1 tells us is that for $(x, y) \in$

$\mathcal{A} \equiv B_T^c \cap \{\tau_0 > T\} \cap \{\sigma > T\}$, then,

$$(x, y) \in \{X(T, x_0) \in \mathcal{C}_T\} \Leftrightarrow (\underline{0}, y) \in \{X(T, x_0) \in \mathcal{C}_T\},$$

where we have called $\underline{0}$ the trajectory with no particles in $B(x_0(1), L)^c$ and no time jumps on the site of $B(x_0(1), L)^c$. We call

$$\psi(x, y) \equiv \chi\left((\underline{0}, y) \in \{X(T, x_0) \in \mathcal{C}_T\}\right)$$

Now, we are ready to give an estimate for (5.12). Assuming $T < \Delta_n$:

$$\begin{aligned}
&P\Big(\bigcup\{\mathcal{A}(I):\ |I \cap B(x_0, L)| = 1, I \in S_l, I \ni x_0\}, X(u, x_0) \in \mathcal{C}_u\Big)\\
&\quad = P\Big(\bigcup\{\mathcal{A}(I):\ |I \cap B(x_0, L)| = 1, I \in S_l, I \ni x_0\},\\
&\qquad\qquad X(T, x_0) \in \mathcal{C}_T, \mathcal{A}\Big)\\
&\qquad + P\big(\sigma \leqslant T, X(u, x_0) \in \Lambda_n\big) + P\big(\tau_0 \leqslant T, X(u, x_0) \in \Lambda_n\big)\\
&\qquad + P\big(B_T, \sigma > T, X(u, x_0) \in \Lambda_n\big)\\
&\quad \leqslant P\Big(\bigcup\{X(u, I) \in \Lambda_n, |I| = l - 1, I \subset B(x_0, L)^c\}\Big)\\
&\qquad \times E\big(\psi\chi\big(X(u, x_0) \in \Lambda_n, \tau_0 > T\big)\big)\\
&\qquad + P\big(\sigma \leqslant T, X(u, x_0) \in \Lambda_n\big) + P\big(\tau_0 \leqslant T, X(u, x_0) \in \Lambda_n\big)\\
&\qquad + P\big(B_T, \sigma > T, X(u, x_0) \in \Lambda_n\big)\\
&\quad \leqslant \nu\big(\eta(\Lambda_n) = l - 1\big) P\big(X(T, x_0) \in \mathcal{C}_T, X(u, x_0) \in \Lambda_n\big)\\
&\qquad + 2P\big(\sigma \leqslant T, X(u, x_0) \in \Lambda_n\big) + 2P\big(\tau_0 \leqslant T, X(u, x_0) \in \Lambda_n\big)\\
&\qquad + 2P\big(B_T, \sigma > T, X(u, x_0) \in \Lambda_n\big).
\end{aligned}$$

Thus we are left to show that, for a suitable T,

$$\sum_{x_0} P\big(X(T, x_0) \in \mathcal{C}_T, X(u, x_0) \in \Lambda_n\big) \leqslant A\,\mathrm{e}^{-Cn^d}, \tag{5.13}$$

and that the terms

$$\sum_{x_0} P\big(\sigma \leqslant T, X(u, x_0) \in \Lambda_n\big), \qquad \sum_{x_0} P\big(\tau_0 \leqslant T, X(u, x_0) \in \Lambda_n\big),$$
$$\sum_{x_0} P\big(B_T, \sigma > T, X(u, x_0) \in \Lambda_n\big) \tag{5.14}$$

are all superexponentially small in n^d. Take $T = L^{1/2}$. For (5.13) we have

$$\begin{aligned}
&\sum_{x_0} P\big(X(T,x_0) \in \mathcal{C}_T,\, X(u,x_0) \in \Lambda_n\big) \\
&\quad \leqslant \sum_{x_0} \sum_{y \in \mathbb{Z}^d} \nu\big(\eta\big(x_0(1)\big) \geqslant x_0(2)\big) p_T\big(x_0(1), y\big) p_{u-T}(y, \Lambda_n) \\
&\qquad \times P^\nu\big(\eta_T(y) \neq \xi_T(y)\big) \\
&\quad = P^\nu\big(\eta_T(0) \neq \xi_T(0)\big) \sum_{x_0} \nu\big(\eta\big(x_0(1)\big) \geqslant x_0(2)\big) p_u(x_0, \Lambda_n) \\
&\quad \leqslant C n^d P^\nu\big(\eta_T(0) \neq \xi_T(0)\big).
\end{aligned} \tag{5.15}$$

By [5], Theorem 1, there is A such that

$$P^\nu\big(\eta_T(0) \neq \xi_T(0)\big) < A T^{-1/4},$$

and (5.13) easily follows.

We now consider the terms in (5.14). In all cases, by using Schwarz inequality, the fact that

$$\sum_{x_0} P\big(X(u,x_0) \in \Lambda_n\big) < C n^d,$$

and translation invariance, it is enough to show that the probabilities $P(\sigma \leqslant T)$, $P(\tau_0 \leqslant T)$, $P(B_T, \sigma > T)$ are superexponentially small, when $x_0 = (0,1)$. We treat in the appendix the case of $P(\sigma \leqslant T)$. For $P(\tau_0 \leqslant T)$, we just observe that this quantity is bounded by the probability that a simple random walk starting at the origin is outside $B(0,L)$ at time T, and use standard estimates.

For the term $P(B_T, \sigma > T)$, consider a sequence of times $\tau_1 < \tau_2 < \cdots < \tau_m$ as in the definition of B_T. The sites $i_1, \ldots, i_m$ associated to these times form a path joining $\partial B(0,L)$ to $B(0,L/2)$. Since, under the condition $\sigma > T$ there are never more than $2n^d$ particles in the sites i_j it follows that, for a given path, the probability of having a sequence of jump times as above is bounded by

$$\exp\big(-2n^d T\big) \frac{(2n^d T)^m}{m!}.$$

Moreover the number of paths of length m joining $\partial B(0,L)$ to $B(0,L/2)$ is bounded by $C L^{d-1} (2d)^m$. Therefore

$$
\begin{aligned}
P(B_T, \tau_0 > T) &\leqslant \mathrm{e}^{-2n^d T} \sum_{m \geqslant L/2} C L^{d-1} (2d)^m \frac{(2n^d T)^m}{m!} \\
&\leqslant \left(\frac{C' n^d T}{L/2} \right)^{L/2},
\end{aligned}
$$

which completes the estimate.

6. PROOF OF THEOREM 1

Theorem 1 follows from Propositions 1, 2, 3 as in [12]. For the sake of completeness we give here a sketch of the proof.

Let $(r_n)_{n \geqslant 0}$ be any sequence such that

$$
\forall n, \quad r_n > \max\left(\gamma_n^{-1/2}, \mathrm{e}^{-C_2 n^d/2}\right) \quad \text{and} \quad \lim_{n \to \infty} r_n = 0,
$$

where the constant C_2 is the one that appears in Proposition 3. Define θ_n by $P(T_n > \gamma_n r_n) = \exp(-\theta_n)$. Propositions 1 and 2 imply that

$$
1 - a r_n - \nu\big(\eta(\Lambda_n) \geqslant l\big) \leqslant \mathrm{e}^{-\theta_n} \leqslant 1 - \frac{r_n}{B r_n + D}, \tag{6.1}
$$

where a is the constant appearing in Proposition 1. In particular $\theta_n \to 0$. So, for n large, $\theta_n/2 \leqslant 1 - \mathrm{e}^{-\theta_n} \leqslant \theta_n$, and (6.1) give

$$
(B + D)^{-1} \leqslant (B r_n + D)^{-1} \leqslant \frac{\theta_n}{r_n} \leqslant 2a + 2\frac{\nu(\eta(\Lambda_n) \geqslant l)}{r_n} \leqslant 2a + 1,
$$

where last inequality uses the fact that $r_n > \gamma_n^{-1/2} > \nu(\eta(\Lambda_n) \geqslant l)$ for n large. Set $\alpha_n = \theta_n / r_n$. We need to show that, for $t > 0$ and for some positive constants B, B'

$$
\left| P(T_n \geqslant \gamma_n t) - \mathrm{e}^{-\alpha_n t} \right| \leqslant B\, \mathrm{e}^{-B' n^d}. \tag{6.2}
$$

Note that (6.2) implies Theorem 1, by letting $\beta_n = \gamma_n / \alpha_n$, and observing that by choosing B' small enough we may take $B = 1$ (indeed, the left hand side is always less than 1).

Suppose, first, $t = k r_n$, with k a positive integer. We apply inductively Proposition 3

$$
\left| P(T_n \geqslant \gamma_n t) - \mathrm{e}^{-\alpha_n t} \right| = \left| P(T_n \geqslant k \gamma_n r_n) - \mathrm{e}^{-k\theta_n} \right|
$$

$$
\begin{aligned}
&\leqslant (1+r_n)C_1\,\mathrm{e}^{-C_2 n^d}\left[1+\mathrm{e}^{-\theta_n}+\cdots+\mathrm{e}^{-\theta_n(k-2)}\right]\\
&\leqslant 2C_1\,\mathrm{e}^{-C_2 n^d}\frac{1}{1-\mathrm{e}^{-\theta_n}}\leqslant 4C_1\frac{\mathrm{e}^{-C_2 n^d}}{\theta_n}\\
&=4C_1\frac{\mathrm{e}^{-C_2 n^d}}{r_n\alpha_n},
\end{aligned}
$$

and (6.2) follows. For a general t, one writes $t = kr_n + v_n$ with $0 \leqslant v_n < r_n$, and the argument above is easily modified (see [12], Section 5, for details).

7. PROOF OF THEOREM 2

We begin by defining the *Dirichlet form* associated to the generator L of the system of independent random walks:

$$
\mathcal{D}(f,g) = -\int f Lg\,\mathrm{d}\nu,
$$

whose domain can be obtained by closing the form restricted to bounded local functions. Henceforth, the infimum is always taken over bounded local functions.

LEMMA 2. – *For all $t > 0$ we have*

$$
\begin{aligned}
&\log P^{\nu}(T_n > \beta_n t)\\
&\quad\leqslant -t\beta_n \inf\left\{\mathcal{D}(f,f)\colon \int f^2\,\mathrm{d}\nu = 1,\ f\equiv 0 \textit{ on } A_n\right\}.
\end{aligned}
\tag{7.1}
$$

Proof. – The proof is based on standard functional analytic arguments, thus we only sketch it. Let $(\eta_t)_{t\geqslant 0}$ be the system of independent random walks in the stationary measure ν, and define the *killed* process

$$
\hat{\eta}_t = \begin{cases} \eta_t & \text{if } T_n > t,\\ D & \text{otherwise,}\end{cases}
$$

where D is a "cemetery state" not belonging to $\mathbb{N}^{\mathbb{Z}^d}$. It is easily shown that $(\hat{\eta}_t)_{t\geqslant 0}$ is a Markov process on $A_n^c \cup \{D\}$. Moreover, its generator $\hat{L}$ is such that if $f:\mathbb{N}^{\mathbb{Z}^d}\to\mathbb{R}$ is local and $f(\xi) = 0\ \forall \xi \in A_n$ (so that f can be identified with a function on $A_n^c\cup\{D\}$), then $\hat{L}f = \chi(A_n^c)Lf$. In particular, $\hat{L}$ is self-adjoint on the Hilbert space $\mathcal{H}_n = \{f \in L^2(\nu)\colon f \equiv 0$ on $A_n\}$. Thus, letting $f = \chi(A_n^c)$, we have

$$P(T_n > \beta_n t) = E^{\nu}\big(f(\hat{\eta}_{\beta_n t})\big) = \int \nu(\mathrm{d}\xi) f(\xi)\big(\mathrm{e}^{t\beta_n \hat{L}} f\big)(\xi)$$

$$= \int_0^{+\infty} \mathrm{e}^{-t\beta_n x} \mu_f(\mathrm{d}x) \leqslant \mathrm{e}^{-t\beta_n \lambda} \|f\|^2_{L^2(\nu)} \leqslant \mathrm{e}^{-t\beta_n \lambda}, \quad (7.2)$$

where μ_f is the spectral measure associated to $\hat{L}$ and f, and λ is the infimum of the spectrum of $-\hat{L}$, i.e.

$$\lambda = \inf\left\{ -\int g\hat{L}g\,\mathrm{d}\nu\colon\ g \in \mathcal{H}_n,\ \|g\|_2 = 1 \right\}$$

$$= \inf\left\{ -\int gLg\,\mathrm{d}\nu\colon\ g \in L^2(\nu),\ \|g\|_2 = 1,\ g \equiv 0 \text{ on } A_n \right\}. \qquad \square$$

By Lemma 2, Theorem 2 follows from Theorem 1 and the following result.

PROPOSITION 4. – *For every $n \geqslant 1$ there exists a constant $c > 0$ such that*

$$\inf\left\{ \mathcal{D}(f, f)\colon \int f^2\,\mathrm{d}\nu = 1,\ f \equiv 0 \textit{ on } A_n \right\} \geqslant \frac{c}{n^{2d-1}}. \quad (7.3)$$

The proof of Proposition 4 is divided into three lemmas. The first, Lemma 3 is from [23].

LEMMA 3. – *Let $d \geqslant 3$. There is a constant $c > 0$ such that for every $u \in l^2(\mathbb{Z}^d)$ there is a sequence $(x_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}^d$ such that $x_0 = 0$, $\forall i\ |x_{i+1} - x_i| = 1$, $\lim_{i\to\infty} x_i = \infty$ and*

$$\sum_{i=0}^{+\infty} |u(x_i)| \leqslant c\sqrt{\sum_{j\in\mathbb{Z}^d} u^2(i)}.$$

Let f be a real valued function on $\mathbb{N}^{\mathbb{Z}^d}$. For $i \neq j \in \mathbb{Z}^d$ we let $(\sigma_i f)(\eta) = f(\sigma_i \eta)$ and $(T_{i,j} f)(\eta) = f(T_{i,j}\eta)$ where, for $\eta \in \mathbb{N}^{\mathbb{Z}^d}$,

$$(\sigma_i\eta)(k) = \begin{cases} \eta(k) & \text{if } k \neq i, \\ \eta(k) + 1 & \text{if } k = i, \end{cases}$$

$$(T_{i,j}\eta)(k) = \begin{cases} \eta(k) & \text{if } k \neq i, j \text{ or } \eta(i) = 0, \\ \eta(k) - 1 & \text{if } k = i \text{ and } \eta(i) > 0, \\ \eta(k) + 1 & \text{if } k = j \text{ and } \eta(i) > 0. \end{cases}$$

In the following lemma we extend to independent random walks an inequality which for SSEP can be found in [23].

LEMMA 4. – *There is a constant $c > 0$ such that for all local functions f*

$$\mathcal{D}(f,f) \equiv \sum_i \sum_{j:\,|j-i|=1} \int \eta(i)\big[T_{i,j}f(\eta) - f(\eta)\big]^2 \nu(\mathrm{d}\eta)$$
$$\geqslant c \int \big[\sigma_0 f(\eta) - f(\eta)\big]^2 \nu(\mathrm{d}\eta). \tag{7.4}$$

Proof. – Let f be a given local function and let $i \in \mathbb{Z}^d$ be outside the range of f, i.e., $f(\eta)$ does not depend on $\eta(i)$. It follows that

$$\sigma_0 f(\eta) \cdot \chi\big(\eta(i) > 0\big) = T_{i,0} f(\eta) \cdot \chi\big(\eta(i) > 0\big). \tag{7.5}$$

Now let $x_0, x_1, \ldots, x_k$ be elements of $\mathbb{Z}^d$ such that $x_0 = 0$, $x_k = i$, $|x_{j+1} - x_j| = 1$. Clearly:

$$T_{i,0} = T_{x_1,x_0} \circ \cdots \circ T_{x_{k-1},x_{k-2}} \circ T_{x_k,x_{k-1}}. \tag{7.6}$$

We also define a norm $\|\cdot\|$ on local functions by

$$\|g\| = \sqrt{\int_{\eta(i)>0} g^2(\eta)\nu(\mathrm{d}\eta)}. \tag{7.7}$$

Using (7.5)–(7.7) and the fact that ν is a product measure, we get

$$\begin{aligned}
&\int \big[\sigma_0 f(\eta) - f(\eta)\big]^2 \nu(\mathrm{d}\eta) \\
&\quad = \frac{1}{\nu(\eta(i) > 0)} \int_{\eta(i)>0} [T_{i,0} f - f]^2 \nu(\mathrm{d}\eta) \\
&\quad = \frac{1}{1 - \mathrm{e}^{-\rho}} \|T_{i,0} f - f\|^2 \\
&\quad = \frac{1}{1 - \mathrm{e}^{-\rho}} \bigg\| \sum_{j=1}^{k-1} \big(T_{x_j,x_{j-1}} \circ \cdots \circ T_{x_k,x_{k-1}} \\
&\qquad - T_{x_{j+1},x_j} \circ \cdots \circ T_{x_k,x_{k-1}}\big) f + \big(T_{x_k,x_{k-1}} - I\big) f \bigg\|^2 \\
&\quad \leqslant \frac{1}{1 - \mathrm{e}^{-\rho}} \left[\sum_{j=1}^{k} \big\| (T_{x_j,x_{j-1}} - I)(T_{i,x_j} f) \big\| \right]^2.
\end{aligned} \tag{7.8}$$

Note that

$$
\begin{aligned}
&\left\|(T_{x_j,x_{j-1}} - I)(T_{i,x_j} f)\right\|^2 \\
&= \left\|(T_{x_j,x_{j-1}} - I)(\sigma_{x_j} f)\right\|^2 \\
&= \int\limits_{\eta(x_j)>0} \left[(T_{x_j,x_{j-1}} - I) f(\eta)\right]^2 \frac{d\nu \circ \sigma_{x_j}^{-1}}{\mathrm{d}\nu}(\eta)\nu(\mathrm{d}\eta) \\
&= \int \frac{\eta(x_j)}{\rho}\left[(T_{x_j,x_{j-1}} - I) f(\eta)\right]^2 \nu(d\eta) \\
&\leqslant \frac{1}{\rho} \sum_{l:\ |l-x_j|=1} \int \eta(x_j)\left[(T_{x_j,l} - I) f(\eta)\right]^2 \nu(\mathrm{d}\eta),
\end{aligned}
\tag{7.9}
$$

where we have used

$$
\chi\left(\eta(x_j) > 0\right) \frac{d\nu \circ \sigma_{x_j}^{-1}}{d\nu}(\eta) = \frac{\eta(x_j)}{\rho}.
$$

Now, using (7.8), (7.9) and Lemma 3, choosing suitably the sequence $(x_j)_{j\geqslant 0}$, we get

$$
\begin{aligned}
&\int \left[\sigma_0 f(\eta) - f(\eta)\right]^2 \nu(d\eta) \\
&\quad \leqslant \frac{c}{\rho(1-\mathrm{e}^{-\rho})} \sum_i \sum_{j:\ |j-i|=1} \int \eta(i)\left[T_{i,j} f(\eta) - f(\eta)\right]^2 \nu(\mathrm{d}\eta). \qquad \square
\end{aligned}
$$

As a consequence of Lemma 4, we can write

$$
\mathcal{D}(f,f) \geqslant \frac{c}{n^d} \sum_{i\in\Lambda_n} \int \left[\sigma_i f(\eta) - f(\eta)\right]^2 \nu(\mathrm{d}\eta). \tag{7.10}
$$

Now, let

$$
\bar{\mathcal{D}}(f,f) = \sum_{i\in\Lambda_n} \int \left[\sigma_i f(\eta) - f(\eta)\right]^2 \nu(\mathrm{d}\eta).
$$

Consider the σ-field $\Gamma_{\Lambda_n} = \sigma\{\eta(i):\ i \in \Lambda_n\}$, and define

$$
\tilde{f} = \sqrt{E^\nu(f^2 \mid \Gamma_{\Lambda_n})}.
$$

It is easily seen that $\sigma_i \tilde{f} = \sqrt{E^\nu(\sigma_i f^2 | \Gamma_{\Lambda_n})}$ for $i \in \Lambda_n$. Thus, by using Cauchy–Schwarz inequality, one checks that

$$\bar{\mathcal{D}}(f, f) \geqslant \bar{\mathcal{D}}(\tilde{f}, \tilde{f}). \tag{7.11}$$

It follows from (7.10), (7.11) and the fact that $f \equiv 0$ on A_n implies $\tilde{f} \equiv 0$ on A_n:

$$\begin{aligned} &\inf\left\{\mathcal{D}(f, f)\colon \int f^2 d\nu = 1,\ f \equiv 0 \text{ on } A_n\right\} \\ &\quad \geqslant \frac{c}{n^d} \inf\left\{\bar{\mathcal{D}}(f, f)\colon \int f^2 d\nu = 1,\ f \equiv 0 \text{ on } A_n,\ f : \mathbb{N}^{\Lambda_n} \to \mathbb{R}\right\}. \end{aligned}$$

Therefore the following lemma completes the proof of Theorem 2.

LEMMA 5. –

$$\inf\left\{\bar{\mathcal{D}}(f, f)\colon \int f^2 d\nu = 1,\ f \equiv 0 \textit{ on } A_n,\ f : \mathbb{N}^{\Lambda_n} \to \mathbb{R}\right\} \geqslant \frac{c}{n^{d-1}\beta_n}.$$

Proof. – Note first that the above statement is equivalent to

$$\inf\left\{\bar{\mathcal{D}}(f, f)\colon \int f^2 d\nu = 1,\ f \equiv 0 \text{ on } A_n,\ f : \mathbb{N}^{\Lambda_n} \to \mathbb{R}\right\} \geqslant c\nu(A_n)$$

for some $c > 0$.

Now note that $\bar{\mathcal{D}}$ is the Dirichlet form of N independent copies of the $\mathbb{N}$-valued Markov process generated by the operator

$$Mf(n) = \rho f(n+1) + nf(n-1) - (\rho + n) f(n).$$

Note that M is reversible with respect to the Poisson measure of density ρ (that we still denote by ν). Now, M has a spectral gap in $L^2(\nu)$. This is equivalent to show that

$$-\langle f, Mf\rangle \equiv \sum_n \nu(n)\left[f(n+1) - f(n)\right]^2 \geqslant c\,\mathrm{Var}_\nu(f) \tag{7.12}$$

for some $c > 0$, where $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\nu)$, and $\mathrm{Var}_\nu(f)$ is the variance of f under ν. It is shown in [4] that (7.12) holds with $c = \rho$.

Now, the spectral gap of N independent copies of a given process is the same as the one of a single copy. It follows that

$$\bar{\mathcal{D}}(f, f) \geqslant c\,\mathrm{Var}_\nu(f)$$

for all $f : \mathbb{N}^{\Lambda_n} \to \mathbb{R}$, where ν is now the product Poisson measure on $\mathbb{N}^{\Lambda_n}$. Therefore, to complete the proof of the lemma, it is enough to show that

$$\mathrm{Var}_\nu(f) \geqslant \nu(A_n)$$

for all f such that $\int f^2 \,\mathrm{d}\nu = 1$ and $f \equiv 0$ on A_n. Indeed, by Cauchy–Schwarz inequality:

$$\left(\int f \,\mathrm{d}\nu\right) = \left(\int f 1(A_n^c) \,\mathrm{d}\nu\right)^2 \leqslant \nu(A_n^c) = 1 - \nu(A_n),$$

and therefore

$$\mathrm{Var}_\nu(f) = 1 - \left(\int f \,\mathrm{d}\nu\right)^2 \geqslant \nu(A_n). \qquad \square$$

8. OPEN PROBLEMS

The following problems, related to the ones considered in this paper, are still unsolved.

(1) We have not succeded in determining the *exact* asymptotics for β_n. This amounts to compute the limit

$$\lim_{n \to +\infty} n^{d-1} \nu\big(\eta(\Lambda_n) = l\big) \beta_n.$$

We have not even shown that this limit exists.

(2) In Section 7 we have shown that, for $d \geqslant 3$,

$$P(T_n \geqslant \beta_n t) \leqslant \exp\left[-\frac{c}{n^{2d-1}} t\right]$$

for some $c > 0$ independent of n. It is natural to ask whether a uniform tail estimate of the form

$$P(T_n \geqslant \beta_n t) \leqslant \mathrm{e}^{-ct}$$

holds for some $c > 0$.

(3) The convergence of the moments of T_n in dimension $d \leqslant 2$ remains an open problem. For the symmetric simple exclusion process one of us [1] has worked out the case $d = 2$.

(4) A related problem is the study of quasi-stationary measures. Suppose $n \geqslant 1$ is arbitrary but fixed. A probability measure μ_n on $\mathbb{N}^{\mathbb{Z}^d}$ is said to be a *quasi stationary measure* if, for all $A \subset \mathbb{N}^{\mathbb{Z}^d}$ measurable and $t > 0$

$$P^{\mu_n}(\eta_t \in A \mid T_n > t) = \mu_n(A).$$

For existence results on quasi stationary measures see Keilath [15] for finite Markov chains and [13] for certain Markov chains with countable state space.
Besides the existence of quasi stationary measures for the model studied in this paper, we would like to know if for each $\rho > 0$ there is a quasi stationary measure μ_n such that $\mu_n \ll \nu_\rho$, and if $\mu_n \to \nu_\rho$ weakly as $n \to +\infty$.

APPENDIX

LEMMA 6. – *Let σ be the stopping time defined in Section 5, i.e.,*

$$\sigma = \inf\{t\colon |\delta_t(i)| > 2n^d \text{ for some } i \in B(0, L)\}.$$

Then there are $a, b > 0$ such that

$$P(\sigma \leqslant T) \leqslant a\,\mathrm{e}^{-bn^d \log n^d}$$

for $T = L^{1/2} = \mathrm{e}^{\gamma n^d}$, where $\gamma > 0$ is a given sufficiently small positive number.

Proof. – First note that

$$P(\sigma \leqslant T) \leqslant 2L^d P\big(\inf\{t \geqslant 0\colon \eta_t(0) \geqslant n^d\} \leqslant T\big). \tag{8.1}$$

It is easy to see that there is a constant $\alpha > 0$ such that

$$\nu\big(\eta(0) > n^d\big) \leqslant \mathrm{e}^{-\alpha n^d \log n^d}.$$

Moreover, as in Proposition 1, define

$$N_t = \sum_{i \in \partial\{0\}} \int_0^t \chi\big(\eta_s(0) = n^d - 1\big)\,\mathrm{d}J_s^i,$$

where J_t^i counts the number of particles that move from i to 0. Thus

$$\{\inf\{t \geqslant 0\colon\ \eta_t(0) \geqslant n^d\} \leqslant T\} \subset \{\eta_0(0) > n^d\} \cup \{N_T \geqslant 1\}$$

which yields

$$\begin{aligned} &P\big(\inf\{t \geqslant 0\colon\ \eta_t(0) \geqslant n^d\} \leqslant T\big) \\ &\quad \leqslant \nu\big(\eta(0) \geqslant n^d\big) + EN_t = \nu\big(\eta(0) \geqslant n^d\big) + 2dT\nu\big(\eta(0) = n^d - 1\big) \end{aligned}$$

and the conclusion follows. □

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