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On the existence of cohomologous continuous cocycles for cocycles with values in some Lie groups

by

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ABSTRACT. – In this paper, we prove that any integrable cocycle defined on a "regular" dynamical system, non-atomic and ergodic, and with values in certain connected Lie groups (among which are the connected, abelian or compact, Lie groups) admits a cohomologous continuous cocycle, which generalizes a result of A.V. Kočergin [4] and D.J. Rudolph [6]. © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Dans ce papier, on montre que tout cocycle intégrable défini sur un système dynamique "régulier", non atomique et ergodique, et à valeurs dans certains groupes de Lie connexes (dont les groupes de Lie connexes, abéliens ou compacts) admet un cocycle continu qui lui est cohomologue, ce qui généralise un résultat de A.V. Kočergin [4] et D.J. Rudolph [6]. © 2000 Éditions scientifiques et médicales Elsevier SAS

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1. INTRODUCTION

Let X be a compact metric space, μ a Borel probability on X and $T: X \to X$ an invertible and bi-measurable transformation that preserves the measure μ. In the following, we also assume that this "regular" dynamical system (X, μ, T) is non-atomic and ergodic.

Let G be a (multiplicative) connected Lie group with identity e and equipped with a Riemannian metric g, invariant under right and left translations. We call d the associated geodesic distance which is also invariant under right and left translations. This condition of invariance, which is realized if and only if $\overline{\mathrm{Ad}(G)}$ is compact, is of course satisfied when G is abelian or compact (see [7]).

We recall that, in this context, a cocycle $\varphi: X \to G$ is a measurable function $\varphi: X \to G$. We denote as usual

$$\varphi^{(n)}(x) := \begin{cases} \varphi(T^{n-1}x) \cdots \varphi(Tx)\varphi(x) & \text{for } n \geqslant 1, \\ e & \text{for } n = 0, \\ \varphi(T^{-n}x)^{-1} \cdots \varphi(T^{-2}x)^{-1}\varphi(T^{-1}x)^{-1} & \text{for } n \leqslant -1. \end{cases}$$

Two cocycles φ_1 and φ_2 are said to be cohomologous if there exists a measurable function $\psi: X \to G$, called the transfer function, such that

$$\varphi_2(x) = \psi(Tx)^{-1}\varphi_1(x)\psi(x), \quad \text{a.e.}$$

Under the previous hypothesis, we shall prove the following theorem.

THEOREM 1.1. – *Let $\varphi: X \to G$ be an integrable cocycle; i.e., such that*

$$\int_X \mathrm{d}(e, \varphi)\,\mathrm{d}\mu < +\infty.$$

Then, there exists a continuous cocycle $\Phi: X \to G$ and a measurable function $\psi: X \to G$ such that

$$\Phi(x) = \psi(Tx)^{-1}\varphi(x)\psi(x), \quad \textit{a.e.}$$

This result generalizes a result of A.V. Kočergin [4] and D.J. Rudolph [6] which deals with the real-valued cocycles and from which we easily deduce a similar result for the cocycles with values in the group $\mathcal{S}^1$ of complex numbers of absolute value 1 or with values in $\mathbb{R}^n$ (see also [2] for a detailed presentation of different results of A.V. Kočergin on regularization, and [1,3,5] for other proof and applications).

Theorem 1.1 is a consequence of the following lemma and theorem which are proved in Section 2 and Section 3, respectively.

Lemma 1.1 is an approximation result of bounded measurable functions with values in G by continuous functions.

LEMMA 1.1. – *For every bounded measurable function $f : X \to G$ and every $\varepsilon > 0$, there exists a continuous function $F : X \to G$ such that*

$$\|\mathrm{d}(e, F)\|_\infty \leqslant \|\mathrm{d}(e, f)\|_\infty \quad \textit{and} \quad \|\mathrm{d}(f, F)\|_1 \leqslant \varepsilon.$$

The theorem below gives as a particular case the existence of a bounded cocycle which is cohomologous to a given integrable cocycle, with control of the transfer function.

THEOREM 1.2. – *Let $\varphi_1, \varphi_2 : X \to G$ be two cocycles such that $\|\mathrm{d}(\varphi_1, \varphi_2)\|_1 < +\infty$.*

For every $C > \|\mathrm{d}(\varphi_1, \varphi_2)\|_1$, there exists a measurable map $\psi : X \to G$ such that, if we put

$$\tilde{\varphi}(x) := \psi(Tx)^{-1}\varphi_1(x)\psi(x), \quad x \in X,$$

then we have:

$$\text{(i)} \quad \|\mathrm{d}(\tilde{\varphi}, \varphi_2)\|_\infty \leqslant C,$$

$$\text{(ii)} \quad \mu(\mathrm{Supp}(\psi)) \leqslant \frac{\|\mathrm{d}(\varphi_1, \varphi_2)\|_1}{C}.$$

We can apply Theorem 1.2 and Lemma 1.1 to obtain Theorem 1.1. Indeed, with these results, we may construct by induction three sequences of functions

$$\varphi_n : X \to G, \quad \text{measurable},$$

$$\psi_n : X \to G, \quad \text{measurable}, \quad n \geqslant 0,$$

$$\Phi_n : X \to G, \quad \text{continuous},$$

such that

- $\varphi_0 = \varphi$, $\Phi_0 = e$, $\varphi_1 = (\psi_0 \circ T)^{-1}\varphi_0\psi_0$ is bounded,
- For $n \geqslant 1$,

$$\|\mathrm{d}(\Phi_n, \varphi_n \Phi_1{}^{-1}\Phi_2{}^{-1} \cdots \Phi_{n-1}{}^{-1})\|_1 \leqslant \frac{1}{2^{2n}},$$

$$\varphi_{n+1} = (\psi_n \circ T)^{-1} \varphi_n \psi_n \quad \text{a.e.},$$

$$\|\mathrm{d}(\Phi_n \cdots \Phi_2 \Phi_1, \varphi_{n+1})\|_\infty \leqslant \frac{1}{2^n} \quad (*),$$

$$\mu(\mathrm{Supp}(\psi_n)) \leqslant \frac{1}{2^n},$$

$$\|\mathrm{d}(e, \Phi_{n+1})\|_\infty \leqslant \frac{1}{2^n}.$$

Thus, since $\sum_{n\geqslant 0} \|\mathrm{d}(e, \Phi_n)\|_\infty < +\infty$, the sequence of products $(\Phi_n \cdots \Phi_2 \Phi_1)_n$ converges uniformly to a continuous function $\Phi : X \to G$, which by $(*)$ implies that the sequence $(\varphi_n)_n$ converges almost everywhere to Φ.

Furthermore, since $\sum_{n\geqslant 0} \mu(\mathrm{Supp}(\psi_n)) < +\infty$, the sequence of products $(\psi_0 \psi_1 \cdots \psi_n)_n$ converges almost everywhere to a measurable function $\Psi : X \to G$.

So, we obtain the identity

$$\Phi(x) = \Psi(Tx)^{-1} \varphi(x) \Psi(x), \quad \text{a.e.},$$

by taking the limit with respect to n in the sequence of expressions

$$\varphi_{n+1}(x) = \big((\psi_0\psi_1 \cdots \psi_n)(Tx)\big)^{-1} \varphi(x) (\psi_0\psi_1 \cdots \psi_n)(x), \quad n \geqslant 0, \quad \text{a.e.}$$

2. PROOF OF LEMMA 1.1

It is sufficient to prove that, for every bounded measurable function $f : X \to G$ and for every $\varepsilon_1, \varepsilon_2 > 0$, there exists a continuous function $F : X \to G$ such that

$$\|\mathrm{d}(e, F)\|_\infty \leqslant \|\mathrm{d}(e, f)\|_\infty \quad \text{and} \quad \mu\{x \in X \mid d(f(x), F(x)) \geqslant \varepsilon_1\} \leqslant \varepsilon_2.$$

Without loss of generality, let us assume that f is not almost everywhere equal to e.

The proof of the statement is divided into three steps.

First step: Case where G is a Euclidian space.

In this case, let us cover the closed ball of centre e and radius $\|\mathrm{d}(e, f)\|_\infty$,

$$\overline{B}(e, \|\mathrm{d}(e, f)\|_\infty),$$

by a finite number of pairwise disjoint measurable sets, $A_1, A_2, \dots, A_k$, such that there exist $c_1, c_2, \dots, c_k$ in G that satisfy

$$\max_{g \in A_i} \mathrm{d}(c_i, g) \leqslant \varepsilon_1.$$

Then we consider a first approximation $\widetilde{F}$ of F as the product of continuous functions f_i, $i = 1, \dots, k$, constructed, by using that μ is Borel and that G is a complete Riemannian manifold, in such a way that

$$f_i(x) = e \quad \text{on } \mathcal{O}_i^c \cup \Big(\bigcup_{j \neq i} \mathcal{K}_j\Big),$$

$$f_i(x) = c_i \quad \text{on } \mathcal{K}_i \quad \text{and} \quad \big\|\mathrm{d}(e, f_i)\big\|_\infty \leqslant \mathrm{d}(e, c_i),$$

where $\mathcal{O}_i$ is an open set that contains $E_i := f^{-1}(A_i)$, $\mathcal{K}_i$ is a compact set that is contained in E_i, and $\mu(\mathcal{O}_i \setminus \mathcal{K}_i) \leqslant \varepsilon_2/k$.

Now we obtain easily that

$$\mu\big\{x \in X \mid \mathrm{d}\big(f(x), \widetilde{F}(x)\big) \geqslant \varepsilon_1\big\} \leqslant \varepsilon_2,$$

and if we set

$$F(x) = \begin{cases} \widetilde{F}(x) & \text{if } x \in \overline{B}\big(e, \|\mathrm{d}(e, f)\|_\infty\big), \\ \dfrac{\widetilde{F}(x)}{\mathrm{d}(e, \widetilde{F}(x))} \big\|\mathrm{d}(e, f)\big\|_\infty & \text{otherwise,} \end{cases}$$

F is as desired.

Second step: There exists a constant η_G such that the conclusions of the lemma are valid for any measurable function $f : X \to G$ such that $\|\mathrm{d}(e, f)\|_\infty \leqslant \eta_G$.

Indeed, since the exponential map is one-to-one and onto, and Lipschitz, from a closed ball $\overline{B}(0, \eta_G)$ of the Lie algebra $\mathcal{G}$ of G (equipped with the scalar product given by the Riemannian metric g) onto the ball $\overline{B}(e, \eta_G)$ of G, this statement is a consequence of the result of the first step applied to $\mathcal{G}$.

Third step: The general case.

Let us choose an integer k such that $\|\mathrm{d}(e, f)\|_\infty / k \leqslant \eta_G$ and let us put $a_i := \varepsilon_i / k$, $i = 1, 2$.

By cutting a minimizing geodesic from e to $f(x)$, $x \in X$, into k pieces of the same length, we may define k measurable functions $f_1, f_2, \dots, f_k$ from X into G such that

$$f = f_1 \cdot f_2 \cdots f_k, \quad \|\mathrm{d}(e, f_i)\|_\infty \leqslant \eta_G, \quad 1 \leqslant i \leqslant k, \quad \text{and}$$
$$\|\mathrm{d}(e, f)\|_\infty = \sum_{i=1}^{k} \|\mathrm{d}(e, f_i)\|_\infty.$$

From the second step, there exists for each i a continuous function F_i such that

$$\|\mathrm{d}(e, F_i)\|_\infty \leqslant \|\mathrm{d}(e, f_i)\|_\infty \quad \text{and} \quad \mu\{x \in X \mid \mathrm{d}(f_i(x), F_i(x)) \geqslant a_1\} \leqslant a_2.$$

Then the product $F = F_1 \cdot F_2 \cdots F_k$ satisfies the inequalities

$$\|\mathrm{d}(e, F)\|_\infty \leqslant \sum_{i=1}^{k} \|\mathrm{d}(e, F_i)\|_\infty \leqslant \sum_{i=1}^{k} \|d(e, f_i)\|_\infty = \|\mathrm{d}(e, f)\|_\infty$$

and

$$\begin{aligned}
&\mu\{x \in X \mid \mathrm{d}(f(x), F(x)) \geqslant \varepsilon_1\} \\
&\quad \leqslant \mu\{x \in X \mid \text{there exists an integer } i \text{ such that } \mathrm{d}(f_i(x), F_i(x)) \geqslant a_1\} \\
&\quad \leqslant \sum_{i=1}^{k} \mu\{x \in X \mid \mathrm{d}(f_i(x), F_i(x)) \geqslant a_1\} \leqslant \varepsilon_2,
\end{aligned}$$

which completes the proof. □

3. PROOF OF THEOREM 1.2

We shall use the following result on Lie groups.

LEMMA 3.1. – *Let C be a positive constant and k a non-negative integer.*

There exists a measurable map

$$\Pi : \left\{(\alpha_0, \alpha_1, \ldots, \alpha_k, \beta_0, \beta_1, \ldots, \beta_k) \in G^{2(k+1)} \mid \sum_{i=0}^{k} \mathrm{d}(\alpha_i, \beta_i) \leqslant (k+1)C\right\}$$
$$\to \{g \in G \mid \mathrm{d}(e, g) \leqslant C\}^{k+1},$$

$$(\alpha_0, \alpha_1, \ldots, \alpha_k, \beta_0, \beta_1, \ldots, \beta_k) \mapsto (\xi_0, \xi_1, \ldots, \xi_k)$$

such that

$$\alpha_k \cdots \alpha_1 \alpha_0 = \beta_k \xi_k \cdots \beta_1 \xi_1 \beta_0 \xi_0.$$

Proof. – Since the distance d is invariant under right and left translations, we have

$$\mathrm{d}\big(\beta_0^{-1}\beta_1^{-1}\cdots\beta_k^{-1}\alpha_k\cdots\alpha_1\alpha_0, e\big) \leqslant \sum_{i=0}^{k}\mathrm{d}(\alpha_i,\beta_i) \leqslant (k+1)C.$$

Thus, since G is a complete Riemannian manifold (being connected and homogeneous), we can choose ξ_0, in a measurable way, on a minimizing geodesic between e and

$$\beta_0^{-1}\beta_1^{-1}\cdots\beta_k^{-1}\alpha_k\cdots\alpha_1\alpha_0$$

such that

$$\mathrm{d}(e,\xi_0) \leqslant C \quad \text{and} \quad \mathrm{d}\big(\beta_0^{-1}\beta_1^{-1}\cdots\beta_k^{-1}\alpha_k\cdots\alpha_1\alpha_0, \xi_0\big) \leqslant kC.$$

Assume now that $\xi_0, \xi_1, \ldots, \xi_p$, $0 \leqslant p < k$, have been chosen satisfying

$$\mathrm{d}(e,\xi_i) \leqslant C, \quad 0 \leqslant i \leqslant p,$$

and

$$\mathrm{d}(\alpha_k\cdots\alpha_1\alpha_0, \beta_k\cdots\beta_{p+1}\beta_p\xi_p\cdots\beta_1\xi_1\beta_0\xi_0) \leqslant (k-p)C.$$

Then we may choose ξ_{p+1} on a minimizing geodesic from e to

$$\beta_{p+1}^{-1}\cdots\beta_k^{-1}\alpha_k\cdots\alpha_1\alpha_0\xi_0^{-1}\beta_0^{-1}\cdots\xi_p^{-1}\beta_p^{-1}$$

in such a way that

$$\mathrm{d}(e,\xi_{p+1}) \leqslant C$$

and

$$\mathrm{d}\big(\beta_{p+1}^{-1}\cdots\beta_k^{-1}\alpha_k\cdots\alpha_1\alpha_0\xi_0^{-1}\beta_0^{-1}\cdots\xi_p^{-1}\beta_p^{-1}, \xi_{p+1}\big),$$

which completes the construction by induction. □

Proof of Theorem 1.2. – Let us put $f(x) := \mathrm{d}(\varphi_1(x), \varphi_2(x))$, $x \in X$. By the pointwise ergodic theorem, we have, for almost all x in X,

$$\lim_{m\to+\infty}\frac{1}{m}\sum_{i=-m}^{-1} f(T^i x) = \lim_{l\to+\infty}\frac{1}{l}\sum_{i=0}^{l-1} f(T^i x) = \|f\|_1 < C.$$

For these x, the set $\mathcal{N}(x)$ of positive integers k such that

$$\sum_{i=-k}^{p} f(T^i x) \geqslant (p+k+1)C, \quad \text{for } p=-k,-k+1,\ldots,-1,$$

is empty or bounded.

So we can put

$$k(x) := \begin{cases} \max \mathcal{N}(x) & \text{if } \mathcal{N}(x) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

and consider $l(x)$ the smallest non-negative integer l such that

$$\sum_{i=-k(x)}^{l} f(T^i x) < (l+k(x)+1)C.$$

With this construction, we have

(1) if $l(x) > 0$ then $k(Tx) = k(x)+1$,

(2) if $l(x) = 0$ then $k(Tx) = 0$,

(3) X is equal to the disjoint union

$$\bigcup_{\{x \mid l(x)=0\}} \{T^{-k(x)}x, \ldots, T^{-1}x, x\},$$

up to a set of measure zero,

(4) the sets

$$A_{k,l} := \{x \in X \mid k(x)=k,\ l(x)=l\},$$
$$k=1,2,\ldots;\ l=0,1,2,\ldots,$$

form a partition of X, up to a set of measure zero, and $T^{-k}A_{k,l} = A_{0,l+k}$.

Lemma 3.1 allows us to construct a measurable map $\tilde{\varphi}: X \to G$ such that:

(a) for $x \in X$, $\mathrm{d}(\tilde{\varphi}(x), \varphi_2(x)) \leqslant C$,

(b) for $x \in X$ such that $l(x) = 0$,

$$\tilde{\varphi}(x)\tilde{\varphi}(T^{-1}x)\cdots\tilde{\varphi}(T^{-k(x)}x) = \varphi_1(x)\varphi_1(T^{-1}x)\cdots\varphi_1(T^{-k(x)}x).$$

(Choose, for $\tilde{\varphi}(T^i x)$, $-k(x) \leqslant i \leqslant 0$, x such that $l(x)=0$, the product of the $(k(x)+i+1)$th coordinate of

$$\Pi\big(\varphi_1(T^{-k(x)}x),\dots,\varphi_1(T^{-1}x),\varphi_1(x),\varphi_2(T^{-k(x)}x),\dots,\\ \varphi_2(T^{-1}x),\varphi_2(x)\big)$$

by $\varphi_2(T^i x)$.)

Let us show now that the function $\psi : X \to G$ defined by

$$\psi(x) := \begin{cases} \varphi_1(T^{-1}x)\varphi_1(T^{-2}x)\cdots\varphi_1(T^{-k(x)}x) & \text{if } k(x) > 0,\\ \quad \times\,\tilde{\varphi}(T^{-k(x)}x)^{-1}\cdots\tilde{\varphi}(T^{-2}x)^{-1}\tilde{\varphi}(T^{-1}x)^{-1} & \\ e & \text{if } k(x) = 0, \end{cases}$$

satisfies the conclusions of the theorem.

- Firstly, let us prove the identity

$$\tilde{\varphi}(x) = \psi(Tx)^{-1}\varphi_1(x)\psi(x), \quad \text{a.e.}$$

We distinguish two cases.
First case: $l(x) > 0$. Then $k(Tx) = k(x) + 1$ and so

$$\begin{aligned}\psi(Tx) &= \varphi_1(T^{-1}(Tx))\varphi_1(T^{-2}(Tx))\cdots\varphi_1(T^{-k(x)-1}(Tx))\\ &\quad\times\tilde{\varphi}(T^{-k(x)-1}(Tx))^{-1}\cdots\tilde{\varphi}(T^{-2}(Tx))^{-1}\tilde{\varphi}(T^{-1}(Tx))^{-1}\\ &= \varphi_1(x)\psi(x)\tilde{\varphi}(x)^{-1}.\end{aligned}$$

Second case: $l(x) = 0$. Then $k(Tx) = 0$ and thus, by (b), we have

$$\psi(Tx) = e = \varphi_1(x)\psi(x)\tilde{\varphi}(x)^{-1}.$$

- Moreover, we have by (a),

$$\|\mathrm{d}(\tilde{\varphi},\varphi_2)\|_\infty \leqslant C.$$

- Finally, the measure of the support of ψ is less than $\|f\|_1/C$, as $\{x \in X \mid \psi(Tx) \neq e\} \subset [l \neq 0]$ and

$$\begin{aligned}\|f\|_1 &\geqslant \int_{[l\neq 0]} f(x)\,\mathrm{d}\mu(x) = \sum_{l=1}^{+\infty}\int_{\{x\mid l(x)=l\}} f(x)\,\mathrm{d}\mu(x)\\ &= \sum_{l=1}^{+\infty}\sum_{k=0}^{+\infty}\int_{A_{k,l}} f(x)\,\mathrm{d}\mu(x) = \sum_{l=1}^{+\infty}\sum_{i=0}^{l-1}\int_{T^iA_{0,l}} f(x)\,\mathrm{d}\mu(x)\\ &= \sum_{l=1}^{+\infty}\int_{A_{0,l}}\sum_{i=0}^{l-1} f(T^ix)\,\mathrm{d}\mu(x) \geqslant \sum_{l=1}^{+\infty}\int_{A_{0,l}}\sum_{i=0}^{l-1} C\,\mathrm{d}\mu(x)\\ &= C\mu([l\neq 0]). \qquad \square\end{aligned}$$

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Note added in proof

The conclusion of Theorem 1.1 has been recently obtained for compact connected metric groups by E. Lindenstrauss (*Ergodic Theory and Dynamical Systems* 19 (4) (1999) 1063–1076).

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