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The asymmetric simple exclusion model with multiple shocks *

by

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ABSTRACT. – We consider the one-dimensional totally asymmetric simple exclusion process with initial product distribution with densities $0 \leqslant \rho_0 < \rho_1 < \cdots < \rho_n \leqslant 1$ in $(-\infty, c_1\varepsilon^{-1})$, $[c_1\varepsilon^{-1}, c_2\varepsilon^{-1}), \ldots,$ $[c_n\varepsilon^{-1}, +\infty)$, respectively. The initial distribution has shocks (discontinuities) at $\varepsilon^{-1}c_k$, $k = 1, \ldots, n$, and we assume that in the corresponding macroscopic Burgers equation the n shocks meet in r^* at time t^*. The microscopic position of the shocks is represented by second class particles whose distribution in the scale $\varepsilon^{-1/2}$ is shown to converge to a function of n independent Gaussian random variables representing the fluctuations of these particles "just before the meeting". We show that the density field at time $\varepsilon^{-1}t^*$, in the scale $\varepsilon^{-1/2}$ and as seen from $\varepsilon^{-1}r^*$ converges weakly to a random measure with piecewise constant density as $\varepsilon \to 0$; the points of discontinuity depend on these limiting Gaussian variables. As a corollary we show that, as $\varepsilon \to 0$, the distribution of the process at site $\varepsilon^{-1}r^* + \varepsilon^{-1/2}a$ at time $\varepsilon^{-1}t^*$ tends to a non-trivial convex combination of the product measures with densities ρ_k, the weights of the combination being explicitly computable. © 2000 Éditions scientifiques et médicales Elsevier SAS

Key words: Asymmetric simple exclusion process, Dynamical phase transition, Shock fluctuations

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1. INTRODUCTION AND RESULTS

It is well known that the hydrodynamical behavior of the one-dimensional asymmetric exclusion process is described by the inviscid Burgers equation

$$\partial_t \rho + \gamma \partial_r \big(\rho(1-\rho)\big) = 0, \tag{1.1}$$

where γ is the mean of the jump distribution. Since Eq. (1.1) develops discontinuities, one has to be careful about the precise statement, but loosely speaking, if (r,t) is a continuity point of $\rho(r,t)$, for a given initial measurable profile $\rho_0(\cdot)$, then at the macroscopic point (r,t) the system is distributed according to the measure $\nu_{\rho(r,t)}$, where ν_ρ is the product Bernoulli measure on $\{0,1\}^{\mathbb{Z}}$ with $\nu_\rho(\eta(x)=1)=\rho$ for all x. This is known as local equilibrium. As it is known, the exact statement involves a space/time change, under Euler scale, and for all these developments we refer to Andjel and Vares [2], Rezakhanlou [11], Landim [8].

The problem with which we are concerned here involves the description of the (microscopic) behavior of the system at certain discontinuity points (r,t) (or shock fronts) of the solution of Eq. (1.1). For example, if $\gamma > 0$ and the initial profile is a step function

$$\rho_0(r) := \alpha\,\mathbf{1}\{r<0\} + \beta\,\mathbf{1}\{r \geqslant 0\},$$

with $0 \leqslant \alpha < \beta \leqslant 1$, the entropy solution of Eq. (1.1) is $\rho(r,t) = \alpha\,\mathbf{1}\{r < vt\} + \beta\,\mathbf{1}\{r \geqslant vt\}$, where $v := \gamma(1-\alpha-\beta)$ is the velocity of the shock front and $\mathbf{1}\{\cdot\}$ is the indicator function of the set $\{\cdot\}$. This description is valid only for continuity points. The investigation of what happens to the system if looked from this shock front was first studied by Wick [13] for a different model, and for the asymmetric simple exclusion in the particular situations $\alpha = 0$ and $\alpha+\beta = 1$ by De Masi et al. [4] and Andjel, Bramson and Liggett [1], respectively. They all proved that at the shock front one sees a fair mixture of ν_α and ν_β. This result has then been extended so as to cover all cases $0 \leqslant \alpha < \beta \leqslant 1$ by Ferrari and Fontes [6], from now on referred as [FF]. [FF] worked with the nearest neighbor asymmetric exclusion process, whose generator is the closure of

$$Lf(\eta) := \sum_{x\in\mathbb{Z}} \sum_{y=x\pm1} p(x,y)\eta(x)\big(1-\eta(y)\big)\big(f(\eta^{x,y}) - f(\eta)\big) \tag{1.2}$$

for f a cylinder function in $\{0,1\}^{\mathbb{Z}}$, with

$$\eta^{x,y}(z) := \begin{cases} \eta(z), & \text{if } z \neq x, y, \\ \eta(y), & \text{if } z = x, \\ \eta(x), & \text{if } z = y, \end{cases}$$

where $p(x, x+1) := p$, $p(x, x-1) := q := 1-p$, with $1/2 < p \leqslant 1$. This process was first studied by Spitzer [12]. Calling $\mu_{\alpha,\beta}$ the product measure on $\{0,1\}^{\mathbb{Z}}$ with site marginals

$$\mu_{\alpha,\beta}(\eta(x)=1) := \begin{cases} \alpha, & \text{if } x < 0, \\ \beta, & \text{if } x \geqslant 0, \end{cases} \tag{1.3}$$

denoting S_t as the semigroup corresponding to the above generator and θ_x as the space shift $\mu\theta_x(f) := \int f(\theta_x\eta)\mu(d\eta)$, with $\theta_x\eta(z) := \eta(x+z)$, [FF] proved that

$$\mu_{\alpha,\beta} S_t \theta_{[vt]} \xrightarrow{\omega^*} \frac{1}{2}(\nu_\alpha + \nu_\beta) \tag{1.4}$$

as $t \to +\infty$, and where $[x]$ denotes the integer part of x. This corresponds to the exact statement made above, under Euler scale, and for the macroscopic point (r, t) in the front line, i.e., $r = vt$. In fact, in the above mentioned references, more detailed analysis is performed, by looking at the microscopic structure of the shock represented by a second class particle. A second class particle jumps to empty sites with the same rates as the other particles, but interchanges positions with the regular particles at the rate holes do. A formal definition using coupling is given in the next section. Calling X_t the position of a second class particle added at the origin, the process as seen from the second class particle $\theta_{X_t}\eta_t$ has distribution asymptotically product to the right and left of the origin with densities α and β respectively, uniformly in time. The velocity of the second class particle is the same as the velocity of the shock in the Burgers equation: $E_{\mu_{\alpha,\beta}} X_t = \gamma(1-\alpha-\beta)t$. [FF] proved that the fluctuations of the position of the particle are Gaussian: calling $\widetilde{X}_t := X_t - vt$,

$$\widetilde{X}_t/\sqrt{t} \underset{t\to+\infty}{\xrightarrow{\mathcal{D}}} \mathcal{N}_{\alpha,\beta} \tag{1.5}$$

where $\mathcal{N}_{\alpha,\beta}$ is a centered Gaussian random variable with variance $\gamma(\beta-\alpha)^{-1}(\alpha(1-\alpha)+\beta(1-\beta))$. With this result in hand [FF] proved that, if $-\infty < a < +\infty$, the distribution of the process at time t at the point $vt + a\sqrt{t}$ converges to a mixture of ν_α and ν_β; more precisely, for real a,

$$\mu_{\alpha,\beta} S_t \theta_{[vt+a\sqrt{t}]} \xrightarrow[t\to+\infty]{\omega^*} \nu_\alpha P(\mathcal{N}_{\alpha,\beta} > a) + \nu_\beta P(\mathcal{N}_{\alpha,\beta} \leqslant a) \tag{1.6}$$

which in particular yields $\frac{1}{2}(\nu_\alpha + \nu_\beta)$ for $a = 0$ and interpolates between ν_α and ν_β, as a varies from $-\infty$ to $+\infty$.

Our goal is the consideration of two or more shock fronts and the description of the microscopic behavior of the system at the (macroscopic) time and position of their meeting. To avoid unnecessary technical difficulties, we look the totally asymmetric case: we assume

$$p = 1,\ q = 0 \text{ which implies } \gamma = 1 \text{ in Eq. (1.1).}$$

The extension to $1/2 < p < 1$ will be briefly discussed at the end.

We consider points c_k, densities ρ_k and the existence of a space-time point (r^*, t^*) such that

$$-\infty = c_0 < c_1 < \cdots < c_n < c_{n+1} = \infty, \tag{1.7}$$

$$0 \leqslant \rho_0 < \cdots < \rho_n \leqslant 1, \tag{1.8}$$

$$r^* = c_k + (1 - \rho_{k-1} - \rho_k)t^*, \quad k = 1, \ldots, n. \tag{1.9}$$

With this assumption the entropy solution to Eq. (1.1) with initial data

$$\rho(r) := \sum_{k=0}^{n} \rho_k \mathbf{1}\{c_k \leqslant r < c_{k+1}\}, \tag{1.10}$$

has the property that all n shocks meet at r^* at time t^*. More precisely, the entropy solution is given by

$$\rho(r,t) = \sum_{k=0}^{n} \rho_k \mathbf{1}\{c_k(t) \leqslant r < c_{k+1}(t)\}, \tag{1.11}$$

where

$$c_k(t) = \begin{cases} c_k + (1 - \rho_{k-1} - \rho_k)t, & \text{for } t < t^*, \\ r^* + (1 - \rho_0 - \rho_n)(t - t^*), & \text{for } t \geqslant t^*. \end{cases} \tag{1.12}$$

Notice that after t^* only the extreme densities ρ_0 and ρ_n are seen.

In our discussion we shall need to consider entropy solutions starting with more general increasing step profiles, i.e., not necessarily with all shocks meeting at (t^*, r^*). For this let (ρ_k) satisfy (1.8), $-\infty = b_0 <$

$b_1 < \cdots < b_n < b_{n+1} = +\infty$ and consider the entropy solution to the Burgers equation with initial data

$$\lambda(r) := \sum_{k=0}^{n} \rho_k \mathbf{1}\{b_k \leqslant r < b_{k+1}\}, \tag{1.13}$$

which is given by

$$\lambda(r,t) = \sum_{k=0}^{n} \rho_k \mathbf{1}\{b_k(t) \leqslant r < b_{k+1}(t)\}, \tag{1.14}$$

$$\frac{db_k(t)}{dt} = 1 - \lambda^+(b_k(t),t) - \lambda^-(b_k(t),t), \tag{1.15}$$

$$b_k(0) = b_k, \quad i = 1,\ldots,n,$$

where $\lambda^{\pm}(r,t)$ are the right and left limits of $\lambda(r,t)$: $\lambda^{\pm}(r,t) = \lim_{r'\to 0} \lambda(r \pm r', t)$, with $r' > 0$. In words, $b_1(t) \leqslant \cdots \leqslant b_n(t)$ represent the shock fronts, which move initially as $b_j(t) = b_j + t(1 - \rho_{j-1} - \rho_j)$ for $1 \leqslant j \leqslant n$, until two or more of them meet. When this happens the involved shock fronts coalesce, with the disappearance of all intermediate densities and the front keeps moving with a new velocity given by one minus the sum of the two densities, to its left and to its right, i.e., the two densities which form the shock. This simple description, mainly due to the fact that we are in the one dimensional situation and the initial profile is an increasing step function, allows to define the following map $\psi : \mathbb{R}^n \to \mathbb{R}^n$, $\psi = (\psi_k)$, which will be fundamental in the determination of the distribution of the process at the macroscopic meeting point of the shocks:

Definition of ψ. Given a vector $\underline{x} = (x_1,\ldots,x_n)$ in $\mathbb{R}^n$ let us take a time $t(\underline{x})$ large enough so that defining

$$b_k(\underline{x}) := x_k - t(\underline{x})(1 - \rho_{k-1} - \rho_k), \tag{1.16}$$

then

$$b_1(\underline{x}) < \cdots < b_n(\underline{x}). \tag{1.17}$$

That is, if we consider a family of one shock solutions $\lambda^k(r,t)$, $1 \leqslant k \leqslant n$, starting with

$$\lambda^k(r) := \rho_{k-1} \mathbf{1}\{r < b_k(\underline{x})\} + \rho_k \mathbf{1}\{r \geqslant b_k(\underline{x})\},$$

then x_k is the position of the shock at time $t(\underline{x})$, for the kth equation. What we denote by $\psi(\underline{x})$ is then the position of the shock fronts at time $t(\underline{x})$ in Eqs. (1.14)–(1.15) starting with (1.13) for $(b_k = b_k(\underline{x}))$. It is easy to see that the definition is well posed, i.e., it does not depend on the value of $t(\underline{x})$ provided (1.16) and (1.17) hold.

The coordinates of the vector $\psi(\underline{x}) = (\psi_k(\underline{x}))$ are convex combinations of some x_j. In the particular case of $n = 2$ this becomes

$$\psi_k(x_1, x_2) = \begin{cases} x_k, & \text{if } x_1 \leqslant x_2, \\ x_1 \dfrac{\rho_1 - \rho_0}{\rho_2 - \rho_0} + x_2 \dfrac{\rho_2 - \rho_1}{\rho_2 - \rho_0}, & \text{if } x_1 > x_2, \end{cases} \tag{1.18}$$

for $k = 1, 2$.

We consider a family of product measures μ^ε on $\{0, 1\}^{\mathbb{Z}}$ with profile ρ: the marginal one-site distribution is given by

$$\mu^\varepsilon\big(\eta(x) = 1\big) = \rho(\varepsilon x),$$

where $\rho(\cdot)$ is given in (1.10) and corresponds to the case when all shocks meet at the same point r^* at time t^*. Considering μ^ε as the initial measure, our first goal is to look at the asymptotic distribution of the process at time τ_ε around site $[x_\varepsilon]$ in the scale $\varepsilon^{-1/2}$ as $\varepsilon \to 0$, where

$$\tau_\varepsilon := t^* \varepsilon^{-1}, \qquad x_\varepsilon := r^* \varepsilon^{-1}. \tag{1.19}$$

Let $\mathcal{Y}_0 := -\infty$, $\mathcal{Y}_{n+1} := \infty$ and

$$(\mathcal{Y}_1, \dots, \mathcal{Y}_n) := \psi(\mathcal{X}_1, \dots, \mathcal{X}_n), \tag{1.20}$$

where $(\mathcal{X}_1, \dots, \mathcal{X}_n)$ are independent centered Gaussian random variables with variances $(D_1, \dots, D_n)$ given by

$$D_k := \frac{\rho_{k-1}(1 - \rho_{k-1}) + \rho_k(1 - \rho_k)}{\rho_k - \rho_{k-1}} t^*. \tag{1.21}$$

THEOREM 1.1. – *Let μ^ε $(0 < \varepsilon \leqslant 1)$ be the product measure on $\{0, 1\}^{\mathbb{Z}}$, associated to the profile $\rho(\cdot)$, given by* (1.10). *Let S_t denote the semigroup associated to the totally asymmetric n.n. simple exclusion process, corresponding to the generator given by* (1.2) *with $p = 1$. Then, for $a \in \mathbb{R}$,*

$$\mu^{\varepsilon} S_{t^*\varepsilon^{-1}} \theta_{[r^*\varepsilon^{-1}+a\varepsilon^{-1/2}]} \xrightarrow[\varepsilon\to 0]{\omega^*} \sum_{k=0}^{n} \nu_{\rho_k} P(\mathcal{Y}_k \leqslant a < \mathcal{Y}_{k+1}), \tag{1.22}$$

where $\mathcal{Y}_k$ *are defined in* (1.20).

We shall see that, as in (1.5), $(\mathcal{Y}_1, \ldots, \mathcal{Y}_n)$ represents the limiting fluctuations of the shocks at time $t^*\varepsilon^{-1}$, around $r^*\varepsilon^{-1}$. In the case of only one shock $(n = 1)$ this agrees with (1.6). The crucial difference is that if, say, the kth and $(k+1)$th microscopic shocks meet before $t^*\varepsilon^{-1}$, they coalesce and change the velocity and the intermediate zone of density ρ_k disappears. This explains the weight of ν_{ρ_k} in (1.22): it is the same as the probability that

(a) the kth and $(k+1)$th shocks have not collided yet, and
(b) these shocks are to the left and right of the point we are looking at, respectively.

The coalescing dynamics of the microscopic shocks in the scale $\varepsilon^{-1/2}$ is as in the Burgers equation. Its relation with the n one-shock dynamics—represented by $(\mathcal{X}_k)$—is given by the function ψ.

We turn now to the profile seen from the meeting point and time of the shock fronts, scaled by $\varepsilon^{-1/2}$. For $\varepsilon > 0$ and a local function f, consider the (random) measures on the real line given by

$$\Lambda_\varepsilon(da) := \varepsilon^{1/2} \sum_{x\in\mathbb{Z}} f(\theta_{x+[x_\varepsilon]} \eta_{\tau_\varepsilon}) \delta_{\varepsilon^{1/2}x}(da), \tag{1.23}$$

where x_ε and τ_ε are given by (1.19), $\delta_{\varepsilon^{1/2}x}$ is the Dirac delta measure at $\varepsilon^{1/2}x$, and η_t is the configuration at time t for the initial measure μ^ε .

Let $(\mathcal{Y}_1, \ldots, \mathcal{Y}_n)$ be as in (1.20) and consider the random measure

$$\Lambda(da) := \sum_{k=0}^{n} \nu_{\rho_k}(f)\, \mathbf{1}\{\mathcal{Y}_k \leqslant a < \mathcal{Y}_{k+1}\}\, da. \tag{1.24}$$

THEOREM 1.2. – Λ_ε *converges in law to* Λ *as* $\varepsilon \to 0$, *for the usual weak topology on the space of measures.*

The analysis is based on the well known strategy of identifying microscopically the shocks with second class particles, defined through the so called basic coupling of different versions of the process starting with measures with different uniform densities. From the dependence of their locations on the initial condition, which is a one-shock fact as in [FF], we can ascertain their distribution around the macroscopic meeting place $[x_\varepsilon]$ at the macroscopic meeting time τ_ε; in the scale $\varepsilon^{-1/2}$ this is given by $(\mathcal{X}_1, \dots, \mathcal{X}_n)$. If we look at the positions of the shocks at time $\tau_\varepsilon - \alpha\varepsilon^{-1/2}$, the law of large numbers apply. This is also a one-shock fact (Ferrari [5]). If α is big enough these positions are ordered, with very large probability, and they are represented by the variables $b_k(\mathcal{X}_1, \dots, \mathcal{X}_n)$ as in (1.16), taking $t(\underline{x}) = \alpha\varepsilon^{-1/2}$. From then on, using the invariance of the product measures involved, we can follow their trajectory up to the meeting time through successive applications of the one-shock law of large numbers. The final positions of the shocks in the scale $\varepsilon^{-1/2}$ are given by the variables $\mathcal{Y}_k + [x_\varepsilon]$.

Theorem 1.2 follows from the weak convergence of a suitable function of the second class particles to the variables $\mathcal{Y}_k$, and the asymptotic properties of the measure as seen from the second class particles. We show Theorem 1.1 as a corollary of Theorem 1.2, by using the attractiveness of the process. The proof avoids proving firstly the translation invariance of the weak limits of Theorem 1.1, as in the proof given by [FF] for the one-shock case.

2. SECOND CLASS PARTICLES AND MORE

Ferrari, Kipnis and Saada [7] and Ferrari [5] have shown that in the case that $\mu_{\alpha,\beta}$ is the initial measure, the shock front is well described by X_t, the position of a "second class particle" initially located at the origin. This fact together with the validity of a central limit theorem for X_t as $t \to +\infty$, proven by [FF], are the essentials for the proof that in the totally asymmetric case $\mu_{\alpha,\beta} S_t \theta_{[(1-\alpha-\beta)t]}$ tends to $\frac{1}{2}(\nu_\alpha + \nu_\beta)$. For all this, as well as in the present work, the main tool is coupling. To realize a coupling of several evolutions of the simple exclusion process, corresponding to several initial configurations, is particularly simple through the graphical construction: to each pair of sites $(x, x+1)$, let us associate a Poisson process with rate 1, and at each of its occurrences we put an arrow $x \to x+1$. Construct all such Poisson processes as independent in some space $(\Omega, \mathcal{A}, P)$. Given any realization of arrows ω and a initial configuration η, we may realize an evolution η_t corresponding to L imposing that whenever an arrow $x \to x+1$ appears, if there is a particle at x and no particle at $x+1$, then this particle moves to $x+1$; otherwise, nothing happens. (In the non-totally asymmetric cases, we have arrows $(x, x+1)$ with rate p and $(x, x-1)$ with rate q, all Poisson processes being independent.) Since the probability of two simultaneous arrows is zero this construction makes sense and defines the process of interest. (One could also state the coupling via a suitable generator, cf. Liggett [9] and Chapter VIII of Liggett [10].) From this coupling, the attractiveness property of the dynamics is immediate: if η^0 and η^1 are two initial configurations such that $\eta^0 \leqslant \eta^1$ (i.e., $\eta^0(x) \leqslant \eta^1(x)\ \forall x$) and we write η_t^0, η_t^1 for their evolutions using the same arrows, then $\eta_t^0 \leqslant \eta_t^1$ for each t.

Let us then couple in this way realizations of the exclusion process with random initial configurations, distributed according to ν_{ρ_k}, $k = 0, \ldots, n$. In fact we have a small perturbation since we will add particles at $[\varepsilon^{-1}c_k]$. For example, let $U = (U_x)_{x\in\mathbb{Z}}$ be a family of i.i.d. random variables uniformly distributed on $[0, 1]$, taken as independent of all the Poisson processes of arrows, where, if needed, we enlarge the basic probability space $(\Omega, \mathcal{A}, P)$. Then, for (c_k) and (ρ_k) as in Section 1 define

$$\sigma^k([\varepsilon^{-1}c_j]) = \mathbf{1}\{j \leqslant k\}, \quad j = 1, \ldots, n,$$
$$\sigma^k(x) = \mathbf{1}\{U_x < \rho_k\}, \quad x \in \mathbb{Z} \setminus \{[\varepsilon^{-1}c_1], \ldots, [\varepsilon^{-1}c_n]\}.$$

Using the same graphical construction above described we may consider the simultaneous evolution of all these configurations, which we denote by σ_t^k, $k = 0, \ldots, n$, on the space $(\Omega, \mathcal{A}, P)$. The marginal distribution of σ_t^k is the simple exclusion process under the invariant distribution ν_{ρ_k}.

Consider the configurations on $\{0, 1\}^{\mathbb{Z}}$ given by $\xi_t^0 = \sigma_t^0$ and for $k = 1, \ldots, n$,

$$\xi_t^k(x) = \sigma_t^k(x) - \sigma_t^{k-1}(x). \tag{2.1}$$

It is easy to see that when considering the joint motion of $(\xi_t^1, \ldots, \xi_t^n)$, then for $j < k$ the ξ^j particles have priority over the ξ^k particles: if there is a ξ^j particle at site x, a ξ^k particle at $x + 1$ and an arrow from x to $x + 1$, then they exchange positions. Otherwise, the interaction is the usual exclusion.

Denote X_t^k the position at time t of the ξ^k particle which was at site $[c_k\varepsilon^{-1}]$ at time 0.

The essential tools in [FF] (with $\mu_{\alpha,\beta}$ as the initial measure) include the joint realization of the evolutions (η'_t, Z_t) and $(\sigma_t^0, \xi_t^1, X_t^1)$, where η'_0 is distributed as $\mu_{\alpha,\beta}$ conditioned to have the site 0 occupied by a second class particle, and Z_t describes the position of this single second class particle, while σ_t^k, ξ_t^k and X_t^1 were defined above for $n = 1$, $\rho_0 = \alpha$, $\rho_1 = \beta$.

When $p = 1$ the above coupling can be achieved by

$$\eta'_t(x) = \begin{cases} \sigma_t^0(x), & \text{if } x < X_t^1, \\ \sigma_t^1(x), & \text{if } x \geqslant X_t^1, \end{cases}$$

and $Z_t = X_t^1$. Together with the law of large numbers and the central limit theorem for X_t^1, this coupling is the basic ingredient in [FF]. Still restricting ourselves to the case $p = 1$, a natural extension of this to our evolution η_t starting with μ^ε is the following.

Definition of (Y_t^k). Let us recall that we consider the case $p = 1$, and X_t^k are as just described above. Let $Y_t^0 \equiv -\infty$, $Y_t^{n+1} \equiv \infty$. For $k = 1, \dots, n$, we define $Y_t^{k,i}$ and t_i inductively in $i \geqslant 0$ as follows. Let $t_0 = 0$ and $Y_t^{k,0} = X_t^k$ for all $k = 1, \dots, n$ and $t \geqslant 0$. Having defined t_ℓ and $Y_t^{k,\ell}$ for all $k = 1, \dots, n$ and $t \geqslant t_\ell$, if $t_\ell = \infty$, we stop the inductive procedure; otherwise, let

$$t_{\ell+1} = \inf\{t \geqslant t_\ell\colon Y_t^{i,\ell} = Y_t^{i+1,\ell} + 1\}$$

(with the usual convention that $\inf\emptyset = \infty$). Denote by i_ℓ the index involved. If finite, $t_{\ell+1}$ is the time of the first crossing after time t_ℓ of two particles whose positions at time t_ℓ are $Y_{t_\ell}^{\cdot,\ell}$. At time $t_{\ell+1}$, a $\sigma^{i_\ell-1}|\sigma^{i_\ell+1}$ discrepancy appears at the position $Y_{t_{\ell+1}}^{i_\ell+1}$. Since $p = 1$, from $t_{\ell+1}$ on, $Y_t^{i_\ell,\ell}$ and $Y_t^{i_\ell+1,\ell}$ do not uncross, so we can ignore the ξ^{i_ℓ} particles and consider only the evolution of $(\sigma^i\colon i \neq i_\ell)$. Accordingly, we define, for $t \geqslant t_{\ell+1}$, $Y_t^{i_\ell,\ell+1} = Y_t^{i_\ell+1,\ell+1}$ to be the position of this $\sigma^{i_\ell-1}|\sigma^{i_\ell+1}$ discrepancy at time t. We also make $Y_t^{i,\ell+1} = Y_t^{i,\ell}$ for $i \neq i_\ell, i_\ell + 1$. This concludes the inductive step.

Notice that the inductive steps in the above procedure correspond to instants of crossings of shocks, after each of which we coalesce the crossing shocks into a single one, which is then made to follow a new discrepancy, leaving the remaining shocks untouched.

We are ready to define (Y_t^k). For all $1 \leqslant k \leqslant n$ and $t \geqslant 0$, let $Y_t^k = Y_t^{k,\ell}$, if $t_\ell \leqslant t < t_{\ell+1}$. In case $t_n < \infty$, t_{n+1} is defined as ∞.

We further define

$$\eta_t'(x) = \sum_{k=0}^{n} \sigma_t^k(x)\, \mathbf{1}\{Y_t^k \leqslant x < Y_t^{k+1}\}. \tag{2.2}$$

Notice that some of the indicator functions may vanish. This happens when the corresponding Y particles coalesce before t. Since $p = 1$, η_t' has the same law as the evolution starting from μ^ε conditioned to have the sites $[c_k\varepsilon^{-1}]$ occupied by second class particles with respect to the other particles of η' such that the second class particles of lower labels have priority over the ones with higher labels.

THEOREM 2.1. – *Writing* $\widetilde{X}^k_{\tau_\varepsilon} = X^k_{\tau_\varepsilon} - [x_\varepsilon]$ *and* $\widetilde{Y}^k_{\tau_\varepsilon} = Y^k_{\tau_\varepsilon} - [x_\varepsilon]$, *where* τ_ε *and* x_ε *are given by* (1.19), *we have*:

$$\lim_{\varepsilon\to 0} \varepsilon^{1/2}\big(\widetilde{X}^1_{\tau_\varepsilon}, \dots, \widetilde{X}^n_{\tau_\varepsilon}\big) \stackrel{\mathcal{D}}{=} (\mathcal{X}_1, \dots, \mathcal{X}_n), \tag{2.3}$$

$$\lim_{\varepsilon\to 0} \varepsilon^{1/2}\big(\widetilde{Y}^1_{\tau_\varepsilon}, \dots, \widetilde{Y}^n_{\tau_\varepsilon}\big) \stackrel{\mathcal{D}}{=} (\mathcal{Y}_1, \dots, \mathcal{Y}_n), \tag{2.4}$$

where $(\mathcal{X}_k)$ *are i.i.d. centered Gaussian with variances* D_k *given by* (2.21) *and* $(\mathcal{Y}_1, \dots, \mathcal{Y}_n) = \psi(\mathcal{X}_1, \dots, \mathcal{X}_n)$, *with* ψ *defined in Section* 1.

Before proving Theorem 2.1, let us recall that from Theorem 1.1 of [FF] we have

$$\varepsilon^{1/2}\widetilde{X}^k_{\tau_\varepsilon} \xrightarrow[\varepsilon\to 0]{\mathcal{D}} \mathcal{X}_k, \tag{2.5}$$

with $(\widetilde{X}^k_{\tau_\varepsilon})$ and $(\mathcal{X}_k)$ as in the previous statement. We discuss independence below. The basic idea to show this theorem is to look at the system "just before" τ_ε (in macroscopic time); more precisely: let $\alpha > 0$ and $\tau^\alpha_\varepsilon = \tau_\varepsilon - \alpha\varepsilon^{-1/2}$. We shall make $\varepsilon \to 0$ and then $\alpha \to +\infty$.

Presumably, if ε is small, but α is large enough, with overwhelming probability the second class particles in the n one-shock systems, $(X^1_{\tau^\alpha_\varepsilon}, \dots, X^n_{\tau^\alpha_\varepsilon})$ have not yet crossed each other and so $Y^k_{\tau^\alpha_\varepsilon} = X^k_{\tau^\alpha_\varepsilon}$ for $k = 1, \dots, n$. This is due to the control on the asymptotic behavior of X^k_t, known from [FF] and a reasoning as the one used to define $b_k(\underline{x})$ in (1.17). This allows to compare the system with n independent systems of only one shock each, and we conclude that $(Y^1_{\tau^\alpha_\varepsilon}, \dots, Y^n_{\tau^\alpha_\varepsilon})$ are at asymptotically independent Gaussian distances from their expected values (on the $\varepsilon^{-1/2}$ scale). (Notice that since $p = 1$, then each pair X^i and X^j with $i < j$ cross each other only once.)

Since, furthermore, $(Y^1_{\tau^\alpha_\varepsilon}, \dots, Y^n_{\tau^\alpha_\varepsilon})$ are at distance of order $\varepsilon^{-1/2}$ apart from each other, in the time interval from τ^α_ε to τ_ε (of length $\alpha\varepsilon^{-1/2}$) the fluctuations are negligible (we are in the regime of the hyperbolic scaling, thus of the law of large numbers for $Y^k_\cdot$) and it is as if we have a completely deterministic situation.

Rigorous versions of the facts described on the previous two paragraphs lead straightforwardly to Theorem 2.1. We state and prove them in the paragraphs below.

The first step is to describe the behavior of $(X^1_{\tau^\alpha_\varepsilon}, \dots, X^n_{\tau^\alpha_\varepsilon})$. Since each of the coordinates X^k correspond to the motion of a second class tagged particle in a single shock situation, the essential is their dependence on the initial condition. Indeed, if

$$N^k_{t,\varepsilon}(\sigma^{k-1},\sigma^k)$$
$$\stackrel{\text{def}}{=} \sum_{x=[c_k/\varepsilon]+1}^{[(c_k+t(\rho_k-\rho_{k-1}))/\varepsilon]} \left(1-\sigma^k(x)\right) - \sum_{x=[(c_k-t(\rho_k-\rho_{k-1}))/\varepsilon]+1}^{[c_k/\varepsilon]} \sigma^{k-1}(x)$$

it follows from [FF] that

$$\varepsilon t^{-1} E\left|X^k_{t/\varepsilon} - c_k - \frac{N^k_{t,\varepsilon}(\sigma^{k-1},\sigma^k)}{\rho_k-\rho_{k-1}}\right|^2 \underset{t\to+\infty}{\longrightarrow} 0. \tag{2.6}$$

Remark. – Indeed the result in [FF], for X^k_t, is for uniform product measures off the initial location of the tagged particle. A straightforward conditioning argument shows it to be true for any finite perturbation of that measure. From Theorem 3.1 in (Ferrari [5]), but also from (2.6) it holds

$$\lim_{t\to\infty} \varepsilon X^k_{\varepsilon^{-1}t} = c_k + (1-\rho_{k-1}-\rho_k)t, \quad \text{in probability.} \tag{2.7}$$

Proof of (2.3). – If $t \leqslant t^*$ the random variables $N^k_{t,\varepsilon}$ are independent since by (1.9) they are functions of the initial configuration at disjoint sets of sites. This and the product nature of the initial distribution allows us to use the central limit theorem for sums of Bernoulli random variables to show (2.3). □

To compute the limiting joint probabilities of (2.4) let $\Delta > 0$ and consider the following intervals

$$J_i := \Delta[i-1/2, i+1/2),$$
$$J^\varepsilon_i := \varepsilon^{-1/2} J_i = [\underline{j}_i, \overline{j}_i) \text{ centered at } j_i := i\Delta\varepsilon^{-1/2}, \quad i\in\mathbb{Z}.$$

The following lemma takes care of the case when the $X^k_{\tau_\varepsilon}$ have not still crossed each other.

LEMMA 2.2. – *For integers $i(1) < \cdots < i(n)$,*

$$\lim_{\varepsilon\to 0} P\left(\widetilde{Y}^k_{\tau_\varepsilon} \in J^\varepsilon_{i(k)},\ k=1,\ldots,n\right) = \prod_k P(\mathcal{X}_k \in J_{i(k)}), \tag{2.8}$$

where $(\mathcal{X}_k)_k$ are as in Theorem 2.1.

Proof. – It follows from the definition of Y^k_t and (2.3). □

Now we study the case of unordered $X^k_{\tau_\varepsilon}$. We consider first the case $n=2$. Take arbitrary integers $i(1) > i(2)$. For $0<\delta<1/2$ let

$$J_i^{\varepsilon,\delta} := \left(i - \frac{1}{2} + \delta, i + \frac{1}{2} - \delta\right)\Delta\varepsilon^{-1/2} = \left(\underline{J}_i + \delta\Delta\varepsilon^{-1/2}, \overline{J}_i - \delta\Delta\varepsilon^{-1/2}\right).$$

The interval $J_i^{\varepsilon,\delta}$ is strictly contained in J_i^ε. Let

$$I^{\varepsilon,\delta}_{i(1),i(2)} = \left(\psi_1\bigl(i(1),i(2)\bigr) - 1 - \delta, \psi_1\bigl(i(1),i(2)\bigr) + 1 + \delta\right)\Delta\varepsilon^{-1/2},$$

where

$$\psi_1\bigl(i(1),i(2)\bigr) = \psi_2\bigl(i(1),i(2)\bigr) = i(1)\,\frac{\rho_1-\rho_0}{\rho_2-\rho_0} + i(2)\,\frac{\rho_2-\rho_1}{\rho_2-\rho_0}. \tag{2.9}$$

This corresponds to the coordinates of the vector defined in (1.18) because $i(1) > i(2)$.

LEMMA 2.3. – *For integers $i(1) > i(2)$, and for all fixed Δ,*

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} P\bigl(\widetilde{X}^k_{\tau_\varepsilon} \in J^{\varepsilon,\delta}_{i(k)},\ k=1,2\bigr)$$
$$= \lim_{\delta\to 0}\lim_{\varepsilon\to 0} P\bigl(\widetilde{X}^k_{\tau_\varepsilon} \in J^{\varepsilon,\delta}_{i(k)},\ \widetilde{Y}^k_{\tau_\varepsilon} \in I^{\varepsilon,\delta}_{i(1),i(2)},\ k=1,2\bigr). \tag{2.10}$$

Proof. – For $I, J \subset \mathbb{R}$ we say

$$I < J \quad \text{if } x<y \text{ for all } x\in I \text{ and } y \in J.$$

Considering, as before, $\tau_\varepsilon^\alpha = \tau_\varepsilon - \alpha\varepsilon^{-1/2}$, let us define the intervals

$$J^\varepsilon_{i(k),\alpha} = \bigl(\underline{J}^\alpha_{i(k)}, \overline{J}^\alpha_{i(k)}\bigr) = \theta_{-\alpha\varepsilon^{-1/2}(1-\rho_{k-1}-\rho_k)+x_\varepsilon} J^\varepsilon_{i(k)},$$
$$J^{\varepsilon,\delta}_{i(k),\alpha} = \bigl(\underline{J}^\alpha_{i(k)} + \delta\Delta\varepsilon^{-1/2}, \overline{J}^\alpha_{i(k)} - \delta\Delta\varepsilon^{-1/2}\bigr) = \theta_{-\alpha\varepsilon^{-1/2}(1-\rho_{k-1}-\rho_k)+x_\varepsilon} J^{\varepsilon,\delta}_{i(k)},$$

which we get at time τ_ε^α by translating $\theta_{x_\varepsilon} J^\varepsilon_{i(k)}$ and $\theta_{x_\varepsilon} J^{\varepsilon,\delta}_{i(k)}$ at velocity $-(1-\rho_{k-1}-\rho_k)$ backwards from time τ_ε to time τ_ε^α. Let α big enough such that

$$J^\varepsilon_{i(1),\alpha} < J^\varepsilon_{i(2),\alpha}.$$

From the one-shock law of large numbers for X_t and the definition of Y^k_t (which coincide with the X^k_t for $t<t_1$ in that defintion), we have:

$$\lim_{\varepsilon\to 0} P\bigl(Y^k_{\tau^\alpha_\varepsilon} \in J^\varepsilon_{i(k),\alpha},\ k=1,2\bigr) = \lim_{\varepsilon\to 0} P\bigl(X^k_{\tau^\alpha_\varepsilon} \in J^\varepsilon_{i(k),\alpha},\ k=1,2\bigr)$$
$$= \lim_{\varepsilon\to 0} P\bigl(\widetilde{X}^k_{\tau_\varepsilon} \in J^\varepsilon_{i(k)},\ k=1,2\bigr).$$

This gives the localization of the Y particles at time τ_ε^α. The idea is that those particles will follow their respective characteristics and thus meet during the time interval $(\tau_\varepsilon^\alpha, \tau_\varepsilon)$, where they coalesce and follow the new characteristic, thus ending in $x_\varepsilon + I_{i(1),i(2)}^{\varepsilon,\delta}$ at time τ_ε. Since we are in the same scale for time and space, the result will follow from the law of large numbers.

To make this rigorous, at time τ_ε^α put $\sigma^0|\sigma^1$ discrepancies (second class particles) at $[\underline{J}_{i(1)}^\alpha + \delta\Delta\varepsilon^{-1/2}] + 1$ and $[\overline{J}_{i(1)}^\alpha - \delta\Delta\varepsilon^{-1/2}]$ and $\sigma^1|\sigma^2$ discrepancies (third class particles) at $[\underline{J}_{i(2)}^\alpha + \delta\Delta\varepsilon^{-1/2}] + 1$ and $[\overline{J}_{i(2)}^\alpha - \delta\Delta\varepsilon^{-1/2}]$. Label their positions $\underline{Y}_t^1$, $\overline{Y}_t^1$, $\underline{Y}_t^2$ and $\overline{Y}_t^2$, respectively, for $t \geqslant \tau_\varepsilon^\alpha$.

Let $\tau_\varepsilon' = \tau_\varepsilon'' + \delta\varepsilon^{-1/2}$, where τ_ε'' is defined by $E\underline{Y}_{\tau_\varepsilon''}^1 = E\overline{Y}_{\tau_\varepsilon''}^2$. At time τ_ε', which will be smaller than τ_ε for ε sufficiently small, put $\sigma^0|\sigma^2$ discrepancies at

$$[\overline{J}_{i(1)}^\alpha + (1-\rho_0-\rho_1)(\tau_\varepsilon' - \tau_\varepsilon^\alpha)] \quad \text{and} \quad [\underline{J}_{i(2)}^\alpha + (1-\rho_1-\rho_2)(\tau_\varepsilon' - \tau_\varepsilon^\alpha)].$$

Label their positions $\underline{Y}_t$ and $\overline{Y}_t$, respectively, for $t \geqslant \tau_\varepsilon'$.

Since $p = 1$, we easily check the facts that for $t \geqslant \tau_\varepsilon^\alpha$,

$$\begin{aligned}&\{Y_{\tau_\varepsilon^\alpha}^k \in J_{i(k),\alpha}^{\varepsilon,\delta},\ k = 1,2\}\\ &\quad = \{X_{\tau_\varepsilon^\alpha}^k \in J_{i(k),\alpha}^{\varepsilon,\delta},\ k = 1,2\} \subseteq \{\underline{Y}_t^k \leqslant X_t^k \leqslant \overline{Y}_t^k\}\end{aligned}$$

which shows in particular that for all $t \geqslant \tau_\varepsilon'$:

$$\{Y_{\tau_\varepsilon^\alpha}^k \in J_{i(k),\alpha}^{\varepsilon,\delta},\ k = 1,2\} \subseteq \{X_t^2 \leqslant X_t^1\} \cup \{\underline{Y}_{\tau_\varepsilon'}^1 \leqslant \overline{Y}_{\tau_\varepsilon'}^2\} \tag{2.11}$$

and

$$\{X_t^2 \leqslant X_t^1\} \subseteq \{X_t^2 \leqslant Y_t^k \leqslant X_t^1,\ k = 1,2\}. \tag{2.12}$$

Now, a simple geometric argument relying on the laws of large numbers for $\underline{Y}_t^k$, $\overline{Y}_t^k$, X_t^k (cf. Theorem 3.1 in [FF] and remark following Eqs. (2.6)), (2.11) and (2.12) proves that

$$\begin{aligned}&\lim_{\varepsilon\to 0} P(Y_{\tau_\varepsilon^\alpha}^k \in J_{i(k),\alpha}^{\varepsilon,\delta},\ k = 1,2)\\ &\quad = \lim_{\varepsilon\to 0} P(X_{\tau_\varepsilon^\alpha}^k \in J_{i(k),\alpha}^{\varepsilon,\delta},\ \underline{Y}_{\tau_\varepsilon'} \leqslant Y_{\tau_\varepsilon'}^k \leqslant \overline{Y}_{\tau_\varepsilon'},\ k = 1,2)\end{aligned} \tag{2.13}$$

plus $O(\delta)$, as soon as $\delta > 0$ is small enough. $O(\delta)$ comes from the probability of the event on the right of the union on the right hand side of Eq. (2.11) and the law of large numbers for $\underline{Y}_t^1$ and $\overline{Y}_t^2$.

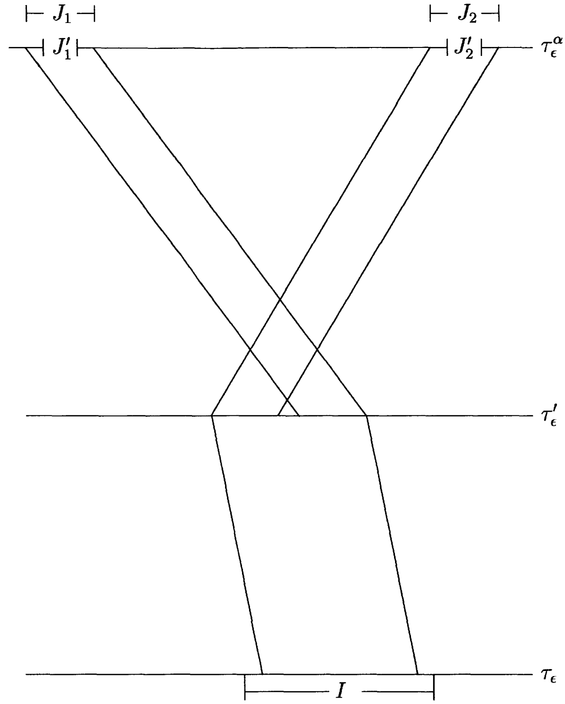

Fig. 1. $J_1 = J^{\varepsilon}_{i(1),\alpha}$, $J_2 = J^{\varepsilon}_{i(2),\alpha}$, $J_1' = J^{\varepsilon,\delta}_{i(1),\alpha}$, $J_2' = J^{\varepsilon,\delta}_{i(2),\alpha}$, $I = I^{\varepsilon,\delta}_{i(1),i(2)}$.

The result now follows from an application of the laws of large numbers for $\underline{Y}_{L\theta}$, $\overline{Y}_t$ and the fact $\underline{Y}_t \leqslant Y_t \leqslant \overline{Y}_t$ for $t \geqslant \tau_\varepsilon'$. (See Fig. 1.) □

The same argument with a more complicated notation shows the above extends to the case of general n, summarized below.

LEMMA 2.4. – *For distinct integers $i(1), \ldots, i(n)$, and for all fixed Δ,*

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} P\big(\widetilde{X}^k_{\tau_\varepsilon} \in J^{\varepsilon,\delta}_{i(k)},\ k = 1, \ldots, n\big)$$
$$= \lim_{\delta\to 0}\lim_{\varepsilon\to 0} P\big(\widetilde{X}^k_{\tau_\varepsilon} \in J^{\varepsilon,\delta}_{i(k)},\ \widetilde{Y}^k_{\tau_\varepsilon} \in I^{\varepsilon,\delta}_{j(k)},\ k = 1, \ldots, n\big), \tag{2.14}$$

where $j(k) = \psi_k(i(1), \ldots, i(n))$.

Proof of (2.4). – Follows from Lemma 2.4 and (2.3). □

3. PROOF OF THEOREMS 1.1 AND 1.2

Proof of Theorem 1.2. – Let us start by noticing that Theorem 1.2 was stated in terms of the evolution η_t, which is not exactly the η'_t we are considering, for which initially the sites $[c_k\varepsilon^{-1}]$, $k = 1, \dots, n$, are occupied by second class particles. On the other side, it is immediate to see that coupling η_t and η'_t in the usual way, one concludes that the eventual discrepancies, at most n, behave as second class particles. They might even annihilate one another, but in any case diffuse as $\varepsilon^{-1/2}$. This shows that Theorem 1.2 will be proven (i.e., for η_t) once we check the analogous for our perturbed process η'_t. For this, it is sufficient, according to Daley and Vere-Jones [3], Proposition 9.1.VII, to show that for every bounded continuous function Φ with compact support $\int \Phi\, d\Lambda_\varepsilon$ converges weakly to $\int \Phi\, d\Lambda$ as $\varepsilon \to 0$. The former integral is

$$\varepsilon^{1/2}\sum_{x\in\mathbb{Z}} f\left(\theta_{x+[x_\varepsilon]}\eta'_{\tau_\varepsilon}\right)\Phi\left(\varepsilon^{1/2}x\right). \tag{3.1}$$

Let $M \geqslant 0$ be an integer such that the cylinder function f depends only on the coordinates in $\{-M, \dots, M\}$. Then, we use (2.2) to condition on the (standardized) locations of the shocks at time τ_ε, and decompose (3.1) as

$$\begin{aligned}&\sum_{k=0}^{n}\varepsilon^{1/2}\sum_{x\in\mathbb{Z}} f\left(\theta_{x+[x_\varepsilon]}\sigma^k_{\tau_\varepsilon}\right)\Phi\left(\varepsilon^{1/2}x\right)\\ &\qquad\times \mathbf{1}\left\{\widehat{Y}^k_\varepsilon \leqslant \varepsilon^{1/2}(x-M) < \varepsilon^{1/2}(x+M) < \widehat{Y}^{k+1}_\varepsilon\right\} + Z_\varepsilon,\end{aligned} \tag{3.2}$$

where $\widehat{Y}^k_\varepsilon = \varepsilon^{1/2}(Y^k_{\tau_\varepsilon} - [x_\varepsilon])$ and the random variable Z_ε satisfies

$$\begin{aligned}E\left(|Z_\varepsilon|\right) \leqslant \|f\|_\infty &\sum_{k=0}^{n}\varepsilon^{1/2}\sum_{x\in\mathbb{Z}}\Phi\left(\varepsilon^{1/2}x\right)\\ &\times P\left(\widehat{Y}^k_\varepsilon \in \left(\varepsilon^{1/2}(x-M), \varepsilon^{1/2}(x+M)\right)\right).\end{aligned}$$

By Theorem 2.1, $(\widehat{Y}^1_\varepsilon, \dots, \widehat{Y}^n_\varepsilon)$ converges weakly to $\psi(\mathcal{X}_1, \dots, \mathcal{X}_n)$ as $\varepsilon \to 0$, and this implies at once that $\lim_{\varepsilon\to 0} E(|Z_\varepsilon|) = 0$.

The result follows from this. To see it, notice that, for $\varepsilon > 0$, the expression in (3.2) is a function of $(\sigma^k_{\tau_\varepsilon})$ and $(\widehat{Y}^k_\varepsilon)$, which we denote $F_\varepsilon(\sigma^1_{\tau_\varepsilon}, \dots, \sigma^n_{\tau_\varepsilon}, \widehat{Y}^1_\varepsilon, \dots, \widehat{Y}^n_\varepsilon)$. By the product structure of $\sigma^k_{\tau_\varepsilon}$ and the law of large numbers, for all $(y_1, \dots, y_n)$

$$F_\varepsilon\left(\sigma^1_{\tau_\varepsilon}, \dots, \sigma^n_{\tau_\varepsilon}, y_1, \dots, y_n\right) \xrightarrow[\varepsilon\to 0]{} F(y_1, \dots, y_n) := \int \Phi\, d\Lambda_{(y_1,\dots,y_n)}$$

almost surely, where $\Lambda_{(y_1,\dots,y_n)}$ is defined as in (1.24). Both F and ψ are continuous in $(y_1,\dots,y_n)$. Also, $\{F_\varepsilon, \varepsilon > 0\}$ is equicontinuous as a family of functions of $(y_1,\dots,y_n)$. This almost sure convergence and continuity properties of F_ε, F and ψ plus the weak convergence of $(\widehat{Y}_\varepsilon^k)$ yield the weak convergence claimed in the statement of the theorem by standard arguments. We leave the details to the reader. □

Proof of Theorem 1.1. – It suffices to check that

$$\lim_{\varepsilon\to 0} Ef\left(\theta_{[r^*\varepsilon^{-1}+a\varepsilon^{-1/2}]}\eta_{t^*\varepsilon^{-1}}\right) \xrightarrow[\varepsilon\to 0]{\omega^*} \sum_{k=0}^{n} \nu_{\rho_k} f P(\mathcal{Y}_k \leqslant r < \mathcal{Y}_{k+1}),$$

where $\mathcal{Y}_k$ are as in the statement of Theorem 1.1, η_t represents the process starting with μ^ε and f is a cylinder function, increasing (for the usual coordinatewise order). Now, to recover the above expression out of Theorem 1.2 we may take the expectation $E\int \Phi\, d\Lambda_\varepsilon$ where the test function Φ defined by $\Phi(w) = \frac{1}{u}\mathbf{1}\{a \leqslant w < a+u\}$, with $u > 0$ will give an upper bound for $Ef(\theta_{[x_\varepsilon+a\varepsilon^{-1/2}]}\eta_{\tau_\varepsilon})$. Similarly by using Φ defined by $\Phi(w) = \frac{1}{u}\mathbf{1}\{a-u \leqslant w < a\}$, with $u > 0$, we get a lower bound. (The fact that these particular test functions have two points of discontinuity is not a problem, due to the continuity of limit measure Λ.) Here we are using the attractiveness of the system and the monotonicity of the initial profile to get $\mu^\varepsilon S(t) f \leqslant \theta_1 \mu^\varepsilon S(t) f$, for increasing continuous f. Letting u tend to zero both terms will converge to the desired expression. □

Remark 3.5. – A similar analysis can be employed to determine that the (microscopic) measure seen from $([x_\varepsilon + a\varepsilon^{-1/2}], \tau_\varepsilon + s\varepsilon^{-1/2})$, with $a, s \in \mathbb{R}$ fixed, converges as $\varepsilon \to 0$ to a mixture of ν_{ρ_k}. For that, as in the definition of ψ, let

$$\psi_k^s(\underline{x}) = \begin{cases} x_k + s(1-\rho_{k-1}-\rho_k), & \text{if } s \leqslant -t(\underline{x}),\\ b_k\big(\underline{x}, t(\underline{x}) + s\big), & \text{if } s > -t(\underline{x}). \end{cases}$$

This is the function that enters in the corresponding statement.

Remark 3.6. – In this paper we have treated in detail only the totally asymmetric case, where $p = 1$. It is not hard, but technically more cumbersome, to extend the analysis to $1/2 < p < 1$. One can define η_t' in the same way. The extra difficulty comes from the fact that now the shocks can *un*cross each other. However, since the gaussian fluctuations and the law of large numbers remain valid, essentially the same analysis applies, with the appropriate change on parameters. Details are left to the reader.

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