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Forward-backward martingale decomposition and compactness results for additive functionals of stationary ergodic Markov processes

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ABSTRACT. – We extend the forward-backward martingale decomposition of Meyer-Zheng-Lyons's type from the symmetric case to the general stationary situation for the partial sum $S_{\cdot}(f)$ with f satisfying a finite energy condition. As corollaries we obtain easily a maximal inequality and a tightness result related to Donsker's invariance principle, and especially a criterion of *a.s. compactness* related to Strassen's strong invariance principle. © Elsevier, Paris

Key words: forward-backward martingale decomposition, the functional central limit theorem or Donsker's invariance principle, the functional law of iterated logarithm or Strassen's strong invariance principle.

RÉSUMÉ. – On étend la décomposition de martingale progressive-rétrograde du type Meyer-Zheng-Lyons du cas symétrique au cas stationnaire général pour des sommes partielles $S_t(f)$ avec f satisfaisant une condition d'énergie finie. Comme corollaires, on obtient facilement une inégalité maximale associée au principe d'invariance de Donsker et un

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critère de *compacité p.s.* pour le principe d'invariance forte de Strassen.

Mots clés : décomposition de martingale progressive-rétrograde, le principe d'invariance de Donsker, le principe d'invariance forte de Strassen.

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1. INTRODUCTION

1.1. Consider a Markov process $(\Omega, \mathcal{F}, (\mathcal{F}_t), (X_t), (\theta_t), (\mathbb{P}_x)_{x\in E})$ valued in a Polish space E, with transition probability semigroup (P_t) and with an invariant and ergodic probability measure α on $(E, \mathcal{B})$. Here,

- $t \in \mathbf{T}$, $\mathbf{T} = \mathbb{N}$ (discrete time) or $\mathbb{R}^+$ (continuous time); And $(X_t)_{t\in\mathbb{R}^+}$ is *càdlàg* in E in the continuous time case;
- $\theta_s X_t(\omega) := X_t(\theta_s\omega) = X_{t+s}(\omega), \quad \forall s,t \in \mathbf{T}, \omega \in \Omega; \quad \mathcal{B}$ is the Borel σ-field of E;
- the *past* σ-field $\mathcal{F}_t = \sigma(X_s; s \le t)$; $\mathcal{G}_t = \sigma(X_s; s \ge t)$ (the *future* $\sigma-$field); $\mathcal{F} = \mathcal{G}_0$;
- $\mathbb{P}_x(X_0 = x) = 1, \quad \mathbb{E}^x f(X_t) := \mathbb{E}^{\mathbb{P}_x} f(X_t) = \int_E P_t(x,dy)f(y)$;
- $\mathbb{E}(\theta_t\eta \mid \mathcal{F}_t) = \mathbb{E}^{X_t}\eta, \quad \mathbb{P}_\alpha := \int_E \alpha(dx)\mathbb{P}_x - a.s.$ for any bound $\mathcal{G}_0$-measurable η;
- $\alpha P_t(A) := \int_E \alpha(dx)P_t(x,A) = \alpha(A), \quad \forall A \in \mathcal{B}$ (the invariance of α w.r.t. (P_t));
- $\forall f \in b\mathcal{B}$, the space of real bounded $\mathcal{B}$-measurable functions, $P_t f = f, \alpha - a.s. \Longrightarrow f$ is constant $\alpha - a.s.$

The last two points together are equivalent to say that $\mathbb{P} = \mathbb{P}_\alpha$ is $(\theta_t)_{t\in\mathbf{T}}$-invariant and ergodic on $(\Omega, \mathcal{F})$. In this paper we do not distinguish the σ-fields on Ω and their completions w.r.t. $\mathbb{P}$. Throughout this paper, $< \cdot, \cdot >$ and $\|\cdot\|_0$ denote respectively the inner product and norm in $L^2(E,\alpha)$, $\mathbb{E}(\cdot)$ the expectation w.r.t. $\mathbb{P} = \mathbb{P}_\alpha$.

For any $f \in L^2_0(E,\alpha) := \{f \in L^2(\alpha); < f, 1 >= 0\}$, consider the partial sum

$$S_n(f) := \sum_{k=0}^{n-1} f(X_k), \quad \text{if } n \in \mathbf{T} = \mathbb{N} \tag{1.1}$$
$$\text{or} \quad S_t(f) := \int_0^t f(X_s)ds, \quad \text{if } t \in \mathbf{T} = \mathbb{R}^+.$$

The main motivations of this paper are the Donsker (weak) invariance principle or the functional central limit theorem (in abridge: FCLT), and the Strassen strong invariance principle or the functional law of iterated logarithm (in abridge: FLIL) for the partial sum $S_t(f)$ as $t \to +\infty$ or more generally for an additive functional (in abridge: AF) (S_t). By AF, we mean

$$S_0 = 0, \quad S_{s+t} = S_s + \theta_s S_t, \quad \forall s,t \in \mathbf{T}, \quad I\!P - a.s.$$

The main purpose of this paper is to study consequences of the following finite energy condition for $f \in L_0^2(\alpha)$,

$$\text{(1.2D)} \qquad |< f,u >| \le C\sqrt{< (I-P_1)u,u >}, \quad \forall u \in L_0^2(E,\alpha),$$

in the *D*iscrete time case and

$$\text{(1.2C)} \quad |< f,u >| \le C\sqrt{< -\mathcal{L}u,u >}, \quad \forall u \in \mathcal{D} \subset L_0^2(E,\alpha)\bigcap I\!D(\mathcal{L})$$

in the *C*ontinuous time, where $\mathcal{L}$ is the generator of $(P_t)_{t\in I\!R^+}$ in $L^2(E,\alpha)$, $\mathcal{D}$ is an appropriate domain to be specified later, $C \ge 0$ is a constant.

1.2. In the reversible (or symmetric) case (i.e., $P_t^* = P_t$ where P_t^* is the adjoint operator of P_t in $L^2(\alpha)$), Kipnis and Varadhan [KV, 1986] showed that (1.2) is equivalent to the natural *minimal* condition

$$\text{(1.3)} \qquad \lim_{t\to+\infty} \frac{I\!E(S_t(f))^2}{t} := \sigma^2(f) < +\infty,$$

and established the FCLT of $S_t(f)$ under (1.2) or (1.3). Their main tool for the passage from CLT to FCLT is the following maximal inequality ([KV, Lemma 1.4 and Lemma 1.12]

(1.4)

$$I\!P\left(\sup_{t\in I\!D\bigcap[0,T]} |g(X_t)| \ge l\right) \le \frac{3}{l}\sqrt{< g,g > +T < g,Ag >}, \quad \forall l > 0$$

where $I\!D$ is the set of all diadic points $j/2^k$, $A = I - P_1$ or $-\mathcal{L}$ according to $\mathbf{T} = I\!N$ or $I\!R^+$.

The further works extend their result in two directions:

1) the symmetry assumption is relaxed as the *quasi-symmetry or strong sector condition* below:

$$\text{(1.5)} \quad < Au,v > \le K\sqrt{< Au,u >}\cdot\sqrt{< Av,v >}, \quad \forall u,v \in: I\!D(A)$$

where $A = I - P_1$ or $-\mathcal{L}$ according to $\mathbf{T} = I\!N$ or $I\!R^+$.

2) for more general AFs other than $S_t(f)$, especially in the continuous time case.

We present several of them up to the knowledge of the author.

Goldstein [G, 1995] obtained the FCLT for a general anti-symmetric additive functional of a *symmetric* Markov process.

For the simple exclusion process with an *asymmetric* mean zero probability kernel, Varadhan [Va, 1995] established the central limit theorem (in abridge: CLT) of $S_t(f)$ for all $f \in L_0^2(E, \alpha)$ satisfying (1.2), and proved even the FCLT for some special f related to the movement of a tagged particle, by exploiting the *quasi-symmetry* of this process shown in L. Xu [Xu, 1993].

More recently for general quasi-symmetric Markov processes, Osada and Saitoh [OS, 1995] get the finite dimensional CLT for fairly general additive functional (S_t) under a condition of type (1.2) (see [OS, (1.6)]). And they obtained the corresponding FCLT for rather general additive functionals related to reflected diffusions. In a non published preprint [W2, 1995], Kipnis-Varadhan's maximal inequality (1.4) is established with an extra factor K in its RHS which is the constant in the quasi-symmetry condition (1.5). The proof therein is parallel to the original one of Kipnis-Varadhan.

However whether the LIL and the FLIL hold under (1.2) or (1.3) in the quasi-symmetric case is not treated in these quoted works, for lack of an *a.s.* compactness result. In fact as well known, the maximal inequality of type (1.4) gives (or can be used to give) an *a priori* estimation or a criterion of tightness for the laws $\left\{\mathbb{P}\left(\frac{1}{\sqrt{n}}(S_{nt})_{t\in[0,1]}\right), n \to \infty\right\}$ over $\mathbb{D}[0,1]$ (the space of real *càdlàg* functions on $[0,1]$, equipped with Skorohod topology). But it does not seem to furnish *a priori* estimations or (strong) a.s. compactness about the *a.s.* behavior of $\left\{\frac{1}{\sqrt{2n\log\log n}}S_n, n \to \infty\right\}$, required by the LIL and FLIL.

For the symmetric Markov processes, it is noted in [W1, 1995] that the ingenious forward-backward martingale decomposition of Meyer-Zheng-Lyons (see [MZ, 1984], [LZ, 1988]) gives very directly not only a better maximal inequality than (1.4), but also a strong *a.s.* compactness result required for the LIL and FLIL.

Hence the idea of this paper can be abstracted as one simple point: to extend the forward-backward martingale decomposition of Meyer-Zheng-Lyons to the general stationary case for $S_\cdot(f)$ for those f satisfying (1.2).

1.3. This paper is organized as follows. The next Section is devoted to the discrete time case. In Section 3 we discuss their counterparts in the continuous time case, which is a little more complicated because of the unboundedness of $A = -\mathcal{L}$. Finally we furnish in the Appendix a semi-FLIL for sums of backward martingale differences, required for the a.s. compactness.

2. THE DISCRETE TIME CASE

2.1. Hilbert spaces $\mathcal{H}_1, \mathcal{H}_{-1}$ induced from (1.2)

Let $\mathbf{T} = I\!N$ and write $P = P_1$. Let P^* be the adjoint operator of P in $L^2(E, \alpha)$ and $P^\sigma = (P + P^*)/2$, the symmetrization of P.

LEMMA 2.1. – $P^\sigma = \frac{P+P^*}{2}$ *is* $\alpha-$*ergodic. In particular* $\forall u \in L_0^2(E, \alpha) := \mathcal{H}_0$,

$$< (I - P^\sigma)u, u >= 0 \Longrightarrow u = 0. \tag{2.1}$$

Proof. – By the ergodicity of P w.r.t. α, for any $A, B \in \mathcal{B}$ with $\alpha(A) \wedge \alpha(B) > 0$, $\exists n \geq 0$, $\int_E 1_A P^n 1_B d\alpha > 0$. Then

$$\int_E 1_A (P^\sigma)^n 1_B d\alpha \geq \frac{1}{2^n} \int_E 1_A P^n 1_B d\alpha > 0,$$

where the ergodicity of P^σ follows. ◇

Let

$$I - P^\sigma = \int_{[0,2]} \lambda dE_\lambda = \int_{(0,2]} \lambda dE_\lambda \tag{2.2}$$

be the spectral decomposition of $I - P^\sigma$ on $\mathcal{H}_0$ ($E_0 = 0$ by (2.1)). By (2.1), $I - P^\sigma : \mathcal{H}_0 \to \mathcal{H}_0$ is injective. Then its inverse

$$R_0^\sigma : I\!D(R_0^\sigma)(\subset \mathcal{H}_0) \longmapsto \mathcal{H}_0$$

is a well defined self-adjoint operator with domain $I\!D(R_0^\sigma) = Ran(I - P^\sigma)$ (the range) and

$$R_0^\sigma = (I - P^\sigma)^{-1} = \int_{(0,2]} \frac{1}{\lambda} dE_\lambda \tag{2.3a}$$

$$(2.3b) \qquad D(R_0^\sigma) = \left\{ f \in \mathcal{H}_0; \int_{(0,2]} \frac{1}{\lambda^2} d < E_\lambda f, f > < +\infty \right\}$$

We introduce now two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_{-1}$ inherited from the condition (1.2).

DEFINITION 2.2. – *Let $\mathcal{H}_1$ be the completion of the pre-Hilbert space $(\mathcal{H}_0 = L_0^2(E, \alpha), < \cdot, \cdot >_1)$ where the inner product is given by*

$$(2.4) \qquad < u, v >_1 := < (I - P^\sigma) u, v > = \int_{(0,2]} \lambda d < E_\lambda f, g > .$$

We define $(\mathcal{H}_{-1}, \| \cdot \|_{-1})$ as the dual Hilbert space of $(\mathcal{H}_1, \| \cdot \|_1)$ w.r.t. the canonical dual relation $\mathcal{H}_0' = \mathcal{H}_0$.

LEMMA 2.3. – (a) *$\mathcal{H}_{-1} \subset \mathcal{H}_0$ and the imbedding is continuous.*

(b) *For every $f \in \mathcal{H}_0$, the following properties are equivalent:*

(b.i) $f \in \mathcal{H}_{-1}$;

(b.ii) f verifies (1.2D);

(b.iii) $\int_{(0,2]} \frac{1}{\lambda} d < E_\lambda f, f > < +\infty$;

(b.iv) $\sum_{n=0}^\infty < (P^\sigma)^n f, f >$ is convergent.

In that case,

$$(2.5) \qquad \begin{aligned} \|f\|_{-1} &= \inf \{C \geq 0; \text{ the condition (1.2D) holds with } C\} \\ &= < R_0^\sigma f, f > = \int_{(0,2]} \frac{1}{\lambda} d < E_\lambda f, f > \\ &= \sum_{n=0}^\infty < (P^\sigma)^n f, f > . \end{aligned}$$

(c) *$D(R_0^\sigma)$ is dense in $(\mathcal{H}_{-1}, \| \cdot \|_{-1})$. $I - P^\sigma$ can be extended as an isomorphism: $\mathcal{H}_1 \to \mathcal{H}_{-1}$, R_0^σ can be extended as an isomorphism $\bar{R}_0^\sigma : \mathcal{H}_{-1} \to \mathcal{H}_1$ and $\bar{R}_0^\sigma$ is the inverse of $I - P^\sigma : \mathcal{H}_1 \to \mathcal{H}_{-1}$.*

Proof. – (a) Since $\mathcal{H}_0 \subset \mathcal{H}_1$ is a continuous and dense imbedding with

$$(2.6a) \qquad \|u\|_1 := \sqrt{< u, u >_1} \leq \sqrt{2 < u, u >} := \sqrt{2} \|u\|_0, \quad \forall u \in \mathcal{H}_0.$$

then $\mathcal{H}_{-1} = H_1' \subset \mathcal{H}_0' = \mathcal{H}_0$ and this imbedding is dense with

$$(2.6b) \qquad \|f\|_0 \leq \sqrt{2} \|f\|_{-1}.$$

(b) The equivalence between (b.i) and (b.ii) follows from the definition of $\mathcal{H}_{-1}$. The all other equivalences as well as (2.5) follow from (2.2), (2.3a,b) and the simple observation below

$$(2.7)\qquad \|f\|_{-1} = \left(\int_{(0,2]} \frac{1}{\lambda} d < E_\lambda f, f >\right)^{1/2} = \sqrt{< R_0^\sigma f, f >} = \|R_0^\sigma f\|_1, \quad \forall f \in \mathbb{D}(R_0^\sigma).$$

(c) They are all obvious by (2.7) and $\|g\|_1 = \|(I - P^\sigma)g\|_{-1}, \quad \forall g \in \mathcal{H}_0$. ◇

Since $I - P, I - P^*$ are injective on $\mathcal{H}_0$, we can define the potential (or Poisson) operators

$$R_0 = (I - P)^{-1} : \mathbb{D}(R_0) = Ran(I - P)(\subset \mathcal{H}_0) \mapsto \mathcal{H}_0$$

$$R_0^* = (I - P^*)^{-1} : \mathbb{D}(R_0^*) = Ran(I - P^*)(\subset \mathcal{H}_0) \mapsto \mathcal{H}_0$$

LEMMA 2.4. – *a)* $\mathbb{D}(R_0) \bigcup \mathbb{D}(R_0^*) \subset \mathcal{H}_{-1}$ *and*

$$(2.8)\qquad \begin{aligned} &\|f\|_{-1} \le \sqrt{2}\|R_0 f\|_0, \quad \forall f \in \mathbb{D}(R_0);\\ &\|f\|_{-1} \le \sqrt{2}\|R_0^* f\|_0, \quad \forall f \in \mathbb{D}(R_0^*)\end{aligned}$$

b) It holds that

$$(2.9)\qquad \begin{aligned} &\|R_0 f\|_1 \le \|f\|_{-1}, \quad \forall f \in \mathbb{D}(R_0);\\ &\|R_0^* f\|_1 \le \|f\|_{-1}, \quad \forall f \in \mathbb{D}(R_0^*).\end{aligned}$$

Proof. – Notice that

$$(2.10a)\qquad \begin{aligned} &2 < u, (I - P)u > - < (I - P)u, (I - P)u >\\ &=< u, u > - < Pu, Pu > \ge 0\end{aligned}$$

$$(2.10b)\qquad \begin{aligned} &2 < u, (I - P^*)u > - < (I - P^*)u, (I - P^*)u >\\ &=< u, u > - < P^*u, P^*u > \ge 0.\end{aligned}$$

Let $f \in \mathbb{D}(R_0)$. For each $u \in \mathcal{H}_0$,

$$\begin{aligned} &< f, u > =< R_0 f, (I - P^*)u >\\ &\le \|R_0 f\|_0 \cdot \|(I - P^*)u\|_0 \le \|R_0 f\|_0 \sqrt{2 < u, (I - P^*)u >}\end{aligned}$$

by (2.10b). Hence $f \in \mathcal{H}_{-1}$ and the first inequality in (2.8) is shown. The first inequality in (2.9) follows from

$$(\|R_0 f\|_1)^2 =< R_0 f, (I-P)R_0 f >=< R_0 f, f >\leq \|R_0 f\|_1 \cdot \|f\|_{-1}.$$

Similarly we get the other parts of (2.8), (2.9) for R_0^*. ◇

2.2. Forward-backward martingale decomposition

We are now ready to show the key

THEOREM 2.5. – *Let* $\mathbf{T} = I\!N$. *There exist three bounded linear mappings*

$$(2.11)\qquad \begin{aligned} &G : \mathcal{H}_{-1} \longrightarrow \mathcal{H}_0 = L_0^2(E,\alpha),\\ &< Gf, Gf >\leq 2(\|f\|_{-1})^2\\ &M_1^{\rightarrow} : \mathcal{H}_{-1} \longrightarrow L^2(\Omega, \mathcal{F}_1, I\!P) \ominus L^2(\Omega, \mathcal{F}_0, I\!P),\\ &I\!E(M_1^{\rightarrow}(f))^2 \leq 2(\|f\|_{-1})^2\\ &M_1^{\leftarrow} : \mathcal{H}_{-1} \longrightarrow L^2(\Omega, \mathcal{G}_0, I\!P) \ominus L^2(\Omega, \mathcal{G}_1, I\!P),\\ &I\!E(M_1^{\leftarrow}(f))^2 \leq 2(\|f\|_{-1})^2 \end{aligned}$$

such that the following forward-backward martingale decomposition holds $I\!P - a.s.$

$$(2.12)\quad 2S_n(f) = M_n^{\rightarrow}(f) + M_n^{\leftarrow}(f) + Gf(X_0) - Gf(X_n), \quad \forall n \in I\!N$$

where

$$M_n^{\rightarrow}(f) = \sum_{k=1}^n \theta_{k-1} M_1^{\rightarrow}(f), \quad M_n^{\leftarrow}(f) = \sum_{k=1}^n \theta_{k-1} M_1^{\leftarrow}(f).$$

Proof. – Assume at first $f \in I\!D(R_0^\sigma)$. Let $g = R_0^\sigma f \in \mathcal{H}_0$. Set

$$(2.13)\qquad m(g) := g(X_1) - Pg(X_0), \quad m^*(g) := g(X_0) - P^* g(X_1).$$

Obviously $I\!E(m(g) \mid \mathcal{F}_0) = 0$, i.e., $m(g) \in L^2(\mathcal{F}_1, I\!P) \ominus L^2(\mathcal{F}_0, I\!P)$. And similarly $I\!E(m^*(g) \mid \mathcal{G}_1) = 0$, i.e., $m^*(g) \in L^2(\mathcal{G}_0, I\!P) \ominus L^2(\mathcal{G}_1, I\!P)$.

We have by (2.10a) and Lemma 2.3

$$(2.14a)\qquad \begin{aligned} I\!E[m(g)]^2 &=< g, g > - < Pg, Pg >\\ &\leq 2 < g, (I-P)g >= 2 < g, g >_1\\ &= 2 < f, f >_{-1} \end{aligned}$$

and similarly by (2.10b) and Lemma 2.3

$$(2.14b)\qquad \begin{aligned} I\!E[m^*(g)]^2 &=< g, g > - < P^* g, P^* g >\\ &\leq 2 < g, (I-P^*)g >= 2 < g, g >_1\\ &= 2 < f, f >_{-1} . \end{aligned}$$

Observe

$$
\begin{aligned}
S_n((I-P)g) &= \sum_{k=1}^{n} (g(X_k) - Pg(X_{k-1})) + g(X_0) - g(X_n) \\
&= \sum_{k=1}^{n} \theta_{k-1} m(g) + g(X_0) - g(X_n)
\end{aligned} \tag{2.15a}
$$

and

$$
\begin{aligned}
&S_n((I-P^*)g) \\
&= \sum_{k=1}^{n} (g(X_{k-1}) - P^*g(X_k)) - P^*g(X_0) + P^*g(X_n) \\
&= \sum_{k=1}^{n} \theta_{k-1} m^*(g) - P^*g(X_0) + P^*g(X_n).
\end{aligned} \tag{2.15b}
$$

Taking the sum of (2.15a) and (2.15b) and noting that $f = (I-P^\sigma)g$, we get

(2.16)

$$
2S_n(f) = \sum_{k=1}^{n} \theta_{k-1} m(g) + \sum_{k=1}^{n} \theta_{k-1} m^*(g) + (I-P^*)g(X_0) - (I-P^*)g(X_n),
$$

which is exactly our forward-backward martingale decomposition for $f \in \mathbb{D}(R_0^\sigma)$.

Remark by (2.10b) that

$$
< (I-P^*)g, (I-P^*)g > \leq 2 < g, (I-P^*)g > = 2 < f, f >_{-1} .
$$

Hence the linear mappings defined by

$$
\begin{aligned}
f &\longrightarrow M_1^{\rightarrow}(f) := m(R_0^\sigma f) \in L^2(\mathcal{F}_1, \mathbb{P}) \ominus L^2(\mathcal{F}_0, \mathbb{P}) \\
f &\longrightarrow M_1^{\leftarrow}(f) := m^*(R_0^\sigma f) \in L^2(\mathcal{G}_0, \mathbb{P}) \ominus L^2(\mathcal{G}_1, \mathbb{P}) \\
f &\longrightarrow Gf := (I-P^*)R_0^\sigma f \in L_0^2(E, \alpha) = \mathcal{H}_0
\end{aligned} \tag{2.17}
$$

for $f \in \mathbb{D}(R_0^\sigma)$ are bounded w.r.t. $\|f\|_{-1}$, and all with norm $\leq \sqrt{2}$.

Therefore they admit all the continuous extension to the whole space $f \in \mathcal{H}_{-1}$, denoted by the same notations. Hence the forward-backward martingale decomposition (2.12) follows from (2.16) by continuous extension. $\diamondsuit$

Remark 2.6. – The forward-backward martingale decomposition (2.12) is the main new point that we incorporate into the studies of (1.2). The decomposition of this type appeared at first in Meyer and Zheng [MZ,1985] and it was developped systematically by Lyons and Zheng [LZ, 1988], for the *symmetric* Markov processes in the continuous time. It should be emphasized that (2.12) is not the exact counterpart of their original decomposition which is done for $g(X_t) - g(X_0)$ instead of $S_t(f)$. This is a key point and we find very difficult to extend their original version to the actual non-symmetric case.

Notice also that this decomposition is not *unique*, because in general there exists $0 \neq \eta \in L^2$, measurable w.r.t. $\sigma(X_0, X_1)$, such that $\mathbb{E}(\eta \mid \mathcal{F}_0) = \mathbb{E}(\eta \mid \mathcal{G}_1) = 0$, and $M_1^{\rightarrow}(f) + \eta, M_1^{\leftarrow}(f) - \eta$ satisfy also (2.12).

2.3. Several corollaries

We present now several compactness results as direct corollaries of (2.12). We begin by a maximal inequality and a criterion of tightness (for laws only).

COROLLARY 2.7. – *For each $f \in \mathcal{H}_{-1}$ or satisfying (1.2D),*

(a) the maximal inequality below holds:

$$\mathbb{E} \sup_{0 \leq k \leq n} |S_k(f)|^2 \leq (24n + 3) < f, f >_{-1}, \quad \forall n \geq 1. \tag{2.18}$$

(b) the family of the laws of

$$t \longrightarrow \frac{1}{\sqrt{n}} S_{[nt]}(f) \in \mathbb{D}([0,1]), \quad n \geq 1$$

on $\mathbb{D}([0,1])$ under $\mathbb{P}$ is precompact for the weak convergence topology and any limit probability measure of this sequence is supported by $C([0,1])$, the space of continuous real functions on $[0,1]$. Here $[x]$ denotes the integer part of $x \geq 0$, $\mathbb{D}([0,1])$ is the space of all càdlàg functions $\eta : [0,1] \to \mathbb{R}$, equipped with the Skorohod topology.

Proof. – (a) It consists to control the three terms appeared in (2.12). At first for the $(\mathcal{F}_n)$-martingale $M_n^{\rightarrow}(f) := \sum_{k=1}^n \theta_{k-1} M_1^{\rightarrow}(f)$, by Doob's maximal inequality and (2.11),

$$\mathbb{E} \max_{k \leq n} (M_k^{\rightarrow}(f))^2 \leq 4\mathbb{E}(M_n^{\rightarrow}(f))^2 = 4n\mathbb{E}(M_1^{\rightarrow}(f))^2 \leq 8n < f, f >_{-1} .$$

For the sum of backward martingale differences $M_n^{\leftarrow}(f) := \sum_{k=1}^n \theta_{k-1} M_1^{\leftarrow}(f)$, note that

$$M_n^{\leftarrow}(f) - M_{n-k}^{\leftarrow}(f), k = 0, 1, \cdots, n$$

is a $(\mathcal{G}_{n-k})_{k=0,1,\cdots,n}$-martingale. Hence by Cauchy-Schwarz and by Doob's maximal inequality and (2.11) again,

$$\begin{aligned} \mathbb{E} \max_{k \le n} (M_k^{\leftarrow}(f))^2 &\le 2\mathbb{E}(M_n^{\leftarrow}(f))^2 + 2\mathbb{E} \max_{k \le n} \left(M_n^{\leftarrow}(f) - M_{n-k}^{\leftarrow}(f)\right)^2 \\ &\le 2\mathbb{E}(M_n^{\leftarrow}(f))^2 + 8\mathbb{E}(M_n^{\leftarrow}(f))^2 \\ &= 10n\mathbb{E}(M_1^{\leftarrow}(f))^2 \le 20n < f, f >_{-1} \end{aligned}$$

Finally by Cauchy-Schwarz and (2.11),

$$\begin{aligned} &\mathbb{E} \max_{k \le n} (Gf(X_k) - Gf(X_0))^2 \\ &\le 2\mathbb{E}[Gf(X_0)]^2 + 2\mathbb{E} \max_{1 \le k \le n} (Gf(X_k))^2 \\ &\le 2\mathbb{E}[Gf(X_0)]^2 + 2\mathbb{E} \sum_{k=1}^n [Gf(X_k)]^2 \\ &= 2(n+1) < Gf, Gf > \le 4(n+1) < f, f >_{-1} . \end{aligned}$$

By the three estimations above, we get by (2.12) that

$$4\mathbb{E} \max_{k \le n} (S_n(f))^2 \le 3 \times [8n + 20n + 4(n+1)] < f, f >_{-1}$$

where (2.18) follows.

(b) It follows from (2.12) by the following three facts: the classical FCLT for the forward martingale $M_.^{\rightarrow}(f)$; and

$$\text{(2.19)} \quad \frac{1}{n} \max_{k \le n} (Gf(X_k) - Gf(X_0))^2 \longrightarrow 0, \quad \mathbb{P} - a.s. \text{ and in } L^1(\mathbb{P})$$

(by Birkhoff's ergodic theorem); and finally Lemma A.1 in Appendix for $M^{\leftarrow}(f)$. $\diamondsuit$

Remark 2.8. – For each $f \in \mathbb{D}(R_0) \subset \mathcal{H}_{-1}$, we have by (2.15a) with $g = R_0 f, M_n(R_0 f) = \sum_{k=1}^n \theta_{k-1} m(R_0 f)$,

$$\text{(2.20)} \quad \begin{aligned} &\mathbb{E}\left(\max_{1 \le k \le n} |R_0 f(X_k) - R_0 f(X_0)|^2 \right) \\ &\le 2\left[\mathbb{E} \max_{1 \le k \le n} (M_k(R_0 f))^2 + \mathbb{E} \max_{1 \le k \le n} (S_k(f))^2 \right] \\ &\le 2[8n < R_0 f, f > +(24n+3) < f, f >_{-1}] \\ &\le 2 \times (32n+3) < f, f >_{-1} \end{aligned}$$

where we have used $< R_0 f, f >=< R_0 f, R_0 f >_1 \leq < f, f >_{-1}$ (by Lemma 2.4). Up to a numerical factor (2.20) implies, with $g = R_0 f$, Kipnis and Varadhan's maximal inequality (1.4), and (2.20) does not contain the explosing term $< g, g >=< R_0 f, R_0 f >$ as in (1.4) when one would approach to some $f \in \mathcal{H}_{-1}$ but $f \notin I\!D(R_0)$.

As said in the introduction, (2.12) implies not only the maximal inequalities (2.18) and (2.20), but also the *a.s.* compactness related to FLIL, stated in

COROLLARY 2.9. – *Let $f \in \mathcal{H}_{-1}$. With $I\!P$–probability one, the sequence*

$$t \longrightarrow \frac{1}{h(n)} S_{[nt]}(f) \in I\!D([0,1]), \quad n \geq 1$$

is precompact in $I\!D([0,1])$ and all its limit points are contained in $\mathbf{K}(\sqrt{2}||f||_{-1})$, where

$$\mathbf{K}(\sigma) = \left\{ \begin{array}{l} \eta \in C([0,1]; \quad \eta(t) = \int_0^t k(s)ds \text{ where} \\ k \in L^2([0,1], ds) \text{ and } \int_0^1 k^2(s)ds \leq \sigma^2 \end{array} \right\}$$

is the ball of radius $\sigma \geq 0$ in the usual Cameron-Martin subspace, and $h(t) := \sqrt{2t \log\log(t \vee e^2)}$.

Proof. – It follows from (2.12) by the classical FLIL for the forward martingale $M_{\cdot}^{\rightarrow}(f)$ ([HH]), and (2.19) for $Gf(X_n) - Gf(X_0)$, and finally Lemma A.1 for $M^{\leftarrow}(f)$. $\diamondsuit$

3. CONTINUOUS TIME CASE

3.1. General situation

Let $\mathbf{T} = I\!R^+$. The situation is more delicate because of the unboundedness of the generator $\mathcal{L}$ of (P_t) in $L^2(E, \alpha)$. Our first assumption allows us to define the symmetrization $\mathcal{L}^\sigma$ of the generator $\mathcal{L}$.

(H1) *$\mathcal{D}$ is a sub-algebra of $C(E)$ contained in $I\!D(\mathcal{L}) \bigcap I\!D(\mathcal{L}^*)$, so that $(\frac{\mathcal{L}+\mathcal{L}^*}{2}, \mathcal{D})$ is essentially self-adjoint in $L^2(E, \alpha)$.*

Here $C(E)$ is the space of real continuous functions on E, and $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$ in $L^2(E, \alpha)$ (then the generator of (P_t^*) in $L^2(\alpha)$ by [Ka]).

Let $\mathcal{L}^\sigma$ be the closure of $(\frac{\mathcal{L}+\mathcal{L}^*}{2}, \mathcal{D})$, which is self-adjoint by (H1) and definite nonpositive. Let $(\mathcal{E}^\sigma, \mathbb{D}(\mathcal{E}^\sigma))$ be the symmetric form associated to $-\mathcal{L}^\sigma$. It is the closure of

$$\mathcal{E}^\sigma(u,v) = \frac{1}{2}(< -\mathcal{L}u, v > + < u, -\mathcal{L}v >) = < -\frac{\mathcal{L}+\mathcal{L}^*}{2}u, v >, \quad \forall u, v \in \mathcal{D} \tag{3.1}$$

We assume in further

(H2) *for any $u \in \mathbb{D}(\mathcal{E}^\sigma) \bigcap \mathcal{H}_0$, $\mathcal{E}^\sigma(u,u) = 0 \Longrightarrow u = 0$, α-a.s.*

This condition means that $-\mathcal{L}^\sigma : \mathbb{D}(\mathcal{L}^\sigma)(\subset \mathcal{H}_0) \to \mathcal{H}_0$ is injective. Then

$$R_0^\sigma := (-\mathcal{L}^\sigma)^{-1} : Ran(\mathcal{L}^\sigma) = \mathbb{D}(R_0^\sigma)(\subset \mathcal{H}_0) \longrightarrow \mathbb{D}(\mathcal{L}^\sigma) \tag{3.2}$$

is a well defined self-adjoint operator on $\mathcal{H}_0$.

DEFINITION 3.1. – *$\mathcal{H}_1$ is defined as the completion of the pre-Hilbert space $\mathbb{D}(\mathcal{L}^\sigma)$ w.r.t. the inner product $< u, v >_1 := \mathcal{E}^\sigma(u,v)$ or the norm $||u||_1 := \sqrt{\mathcal{E}^\sigma(u,u)}$.*

$\mathcal{H}_{-1}$ is defined as the completion of the pre-Hilbert space $\mathbb{D}(R_0^\sigma)$ w.r.t. the inner product $< f, g >_{-1} = < R_0^\sigma f, g >$ or w.r.t. the norm $||f||_{-1} = \sqrt{< R_0^\sigma f, f >}$.

LEMMA 3.2. – (a) *$\mathcal{L}^\sigma$ can be extended as an isomorphism $\bar{\mathcal{L}}^\sigma : \mathcal{H}_1 \to \mathcal{H}_{-1}$ and R_0^σ can be extended as an isomorphism $\bar{R}_0^\sigma : \mathcal{H}_{-1} \to \mathcal{H}_1$.*

(b) *$\mathcal{H}_{-1}$ is the dual Hilbert space of $\mathcal{H}_1$ and the dual bilinear relation $< \cdot, \cdot >_{-1,1}$ on $\mathcal{H}_{-1} \times \mathcal{H}_1$ is the continuous extension of $< f, u >, \forall (f,u) \in \mathbb{D}(R_0^\sigma) \times \mathbb{D}(\mathcal{L}^\sigma)$.*

(c) *For each $f \in \mathcal{H}_0$, the following properties are equivalent*

(c.i) $f \in \mathcal{H}_{-1}$;

(c.ii) f satisfies (1.2C) with $\mathcal{D}$ given in (H1);

(c.iii) $\int_0^{+\infty} < f, P_t^\sigma f > dt < +\infty$, where $(P_t^\sigma = e^{t\mathcal{L}^\sigma})$ is the semigroup generated by $\mathcal{L}^\sigma$.

In that case,

$$\begin{aligned} ||f||_{-1} &= \inf \{C \geq 0;\ (1.2C) \text{ is valid with } C\} \\ &= \int_0^\infty < f, P_t^\sigma f > dt \end{aligned} \tag{3.3}$$

Proof. – Its proof is the same as in the discrete time case, so omitted. ◇

We turn now to check the forward-backward martingale decomposition. At first for each test-function $u \in \mathcal{D} \subset \mathbb{D}(\mathcal{L}) \bigcap \mathbb{D}(\mathcal{L}^*) \bigcap C(E)$,

$$M_t(u) := u(X_t) - u(X_0) + \int_0^t -\mathcal{L}u(X_s)ds \tag{3.4a}$$

is an additive $(\mathcal{F}_t)$-forward *càdlàg* martingale; reversing the time,

$$M_t^*(u) := u(X_0) - u(X_t) + \int_0^t -\mathcal{L}^*u(X_s)ds, \tag{3.4b}$$

is additive and *càdlàg*, such that $(M_T^*(u) - M_{T-t}^*(u), 0 \leq t \leq T)$ is a backward $(\mathcal{G}_{T-t})-$martingale. By Ito's formula,

$$\begin{aligned} \mathbb{E}(M_t(u))^2 &= \mathbb{E}\int_0^t \big(\mathcal{L}(u^2) - 2u\mathcal{L}u\big)(X_s)ds \\ &= 2t < -\mathcal{L}u, u >= 2t < u, u >_1; \\ \mathbb{E}(M_t^*(u))^2 &= \mathbb{E}\int_0^t \big(\mathcal{L}^*(u^2) - 2u\mathcal{L}^*u\big)(X_s)ds \\ &= 2t < -\mathcal{L}^*u, u >= 2t < u, u >_1 . \end{aligned} \tag{3.5}$$

By Doob's maximal inequality, for any $T > 0$ fixed,

$$\mathbb{E}\max_{t\leq T}(M_t(u))^2 \leq 4\mathbb{E}[M_T(u)]^2 = 8T < u, u >_1 \tag{3.6a}$$

and

$$\begin{aligned} &\mathbb{E}\max_{t\leq T}(M_t^*(u))^2 \\ &\leq 2\mathbb{E}[M_T^*(u)]^2 + 2\mathbb{E}\max_{t\leq T}\big(M_{T-t}^*(u) - M_T^*(u)\big)^2 \\ &\leq 10\mathbb{E}[M_T^*(u)]^2 = 20T < u, u >_1 . \end{aligned} \tag{3.6b}$$

Taking the sum of (3.4a) and (3.4b), we get

$$-2\int_0^t \mathcal{L}^\sigma u(X_s)ds = M_t(u) + M_t^*(u), \quad \forall t \geq 0, \quad \mathbb{P} - a.s. \tag{3.7}$$

Let $\mathbb{B}_T$ be the Banach space of all real $(\mathcal{F}_t)-$adapted *càdlàg* processes $X = (X_t)_{t\in[0,T]}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with norm $\|X\|_T = \sqrt{\mathbb{E}\sup_{t\in[0,T]}|X_t|^2}$. As $\mathcal{D}$ is dense in $\mathcal{H}_1$, the linear mappings $u \to M_\cdot(u) \in \mathbb{B}_T$ and $u \to M_\cdot^*(u) \in \mathbb{B}_T$ can be extended to $u \in \mathcal{H}_1$, and (3.4a,b), (3.5), (3.6a,b) still hold.

Now by (H1), for any $f \in \mathbb{D}(R_0^\sigma)$, $u = R_0^\sigma f \in \mathbb{D}(\mathcal{L}^\sigma)$, there are $u_k \in \mathcal{D}, k \geq 1$ so that $u_k \to u$, $\mathcal{L}^\sigma u_k \to \mathcal{L}^\sigma u$ in $\mathcal{H}_0$. By (3.7), we get for any $f \in \mathbb{D}(R_0^\sigma)$ and for any $T > 0$,

$$(3.8) \quad 2\int_0^t f(X_s)ds = M_t(R_0^\sigma f) + M_t^*(R_0^\sigma f), \quad \forall t \in [0,T], \quad \mathbb{P} - a.s.$$

The two sides of (3.8), as element in $(\mathbb{B}_T, ||\cdot||_T)$, are both continuous w.r.t. the norm $||f||_0 + ||f||_{-1}$. Consequently by continuous extension we get

THEOREM 3.3. – *Assume (H1) and (H2). Let $f \in L_0^2(E,\alpha)$ satisfy (1.2C). Then the forward-backward martingale decomposition below holds:*

$$(3.9) \qquad 2\int_0^t f(X_s)ds = M_t^{\rightarrow}(f) + M_t^{\leftarrow}(f), \quad \forall t \geq 0, \quad \mathbb{P} - a.s.,$$

where $M_\cdot^{\rightarrow}(f) := M_\cdot(\bar{R}_0^\sigma f)$ and $M_\cdot^{\leftarrow}(f) := M_\cdot^*(\bar{R}_0^\sigma f)$ are additive and càdlàg in time t, linear and bounded in $f \in \mathcal{H}_0 \bigcap \mathcal{H}_{-1}$, verifying

$$(3.10) \qquad \mathbb{E}(M_t^{\rightarrow}(f))^2 = \mathbb{E}(M_t^{\leftarrow}(f))^2 = 2t < f,f >_{-1} .$$

In particular we have

(a) *The maximal inequality below holds*

$$(3.11) \qquad \mathbb{E} \sup_{0\leq t\leq T} |S_t(f)|^2 \leq 14T < f,f >_{-1} .$$

(b) *The family of the laws of $\left\{\frac{1}{\sqrt{n}}S_{n\cdot}(f) \in C([0,1]),\ n \geq 1\right\}$ under $\mathbb{P}$ is precompact.*

(c) *With $\mathbb{P}$–probability one, the sequence*

$$\left\{t \longrightarrow \frac{1}{h(n)}S_{nt}(f) \in C([0,1]),\ n \geq 1\right\}$$

is precompact in $C([0,1])$ and all its limit points are contained in $\mathbf{K}(\sqrt{2}||f||_{-1})$.

Proof. – The maximal inequality (3.11) follows from (3.6a,b) and (3.9). The compactness criteria (b), (c) follow from (3.9), the discrete time's FCLT and FLIL for forward or backward martingale (Lemma A.1) and the following estimation

$$\frac{1}{n}\left(\sup_{t\leq n}|M_t(\bar{R}_0^\sigma f)-M_{[t]}(\bar{R}_0^\sigma f)|^2+\sup_{t\leq n}|M_t^*(\bar{R}_0^\sigma f)-M_{[t]}^*(\bar{R}_0^\sigma f)|^2\right)\longrightarrow 0$$

both in $L^1(\mathbb{P})$ and $\mathbb{P} - a.s.$ (by Birkhoff ergodic theorem). ◇

Remark 3.4. – Several simple but key properties in the discrete time case do no longer hold automatically in the continuous time case:

1) The assumption (H2) is not always valid in the continuous time case, contrary to (2.1) in the discrete time situation.

2) Lemma 2.3.(a) and (2.6a,b) are no longer valid in general. For example, $\mathcal{H}_{-1}$ contains much more elements than those in $\mathcal{H}_{-1} \bigcap \mathcal{H}_0$ (such as Revuz measures).

3) Lemma 2.4.(a) does not hold in general.

Remark 3.5. – Let $\phi \in \mathcal{H}_{-1} \backslash \mathcal{H}_0$ and define

$$S.(\phi) := \lim_k S.(f_k), \quad \text{in } \mathbb{B}_T \tag{3.12}$$

where $f_k \in \mathcal{H}_{-1} \bigcap \mathcal{H}_0$ and $f_k \to \phi$ in $\mathcal{H}_{-1}$. This is well defined by (3.11). By continuous extension $S.(\phi)$ satisfies still the forward-backward martingale decomposition (3.9) and consequently all claims in Th.3.3.

3.2. The quasi-symmetric case

In this paragraph we assume the sector condition (1.5) and our Markov process is Hunt. We show now that (1.5) can be used to substitute the assumptions (H1) and (H2) above for Theorem 3.3. We begin by the construction of $\mathcal{L}^\sigma$.

Let $(\mathcal{E}, \mathbb{D}(\mathcal{E}))$ be the closure of the sectorial form $(\mathcal{E}(u,v) = <-\mathcal{L}u, v>$, $\forall u, v \in \mathbb{D}(\mathcal{L}))$. Let $(\mathcal{E}^\sigma, \mathbb{D}(\mathcal{E}^\sigma))$ be the closure of $(\mathcal{E}^\sigma(u,v) = \frac{1}{2} < -\mathcal{L}u, v> + <u, -\mathcal{L}v>, \forall u, v \in \mathbb{D}(\mathcal{L}))$, and $-\mathcal{L}^\sigma$ be the corresponding definite nonnegative self-adjoint operator. It is known that $\mathbb{D}(\mathcal{L})$ and $\mathbb{D}(\mathcal{L}^*)$ are form core of $(\mathcal{E}^\sigma, \mathbb{D}(\mathcal{E}^\sigma))$ and $\mathbb{D}(\mathcal{E}^\sigma) = \mathbb{D}(\mathcal{E})$ (see [Ka, Ch. VI]).

Secondary let us verify (H2). In fact for any $f \in \mathbb{D}(\mathcal{E}^\sigma) \bigcap \mathcal{H}_0$ with $\mathcal{E}^\sigma(f,f) = 0$, then $f \in \mathbb{D}(\mathcal{E})$ and by (1.5),

$$|\mathcal{E}(f,u)| \le K\sqrt{\mathcal{E}(f,f)} \cdot \sqrt{\mathcal{E}(u,u)} = 0, \quad \forall u \in \mathbb{D}(\mathcal{E}).$$

It follows that $f \in \mathbb{D}(\mathcal{L})$ and $\mathcal{L}f = 0$. By the assumed ergodicity of (P_t), $f = 0, \alpha - a.s.$

Hence $R_0^\sigma = (-\mathcal{L}^\sigma)^{-1}$ is well defined on $\mathcal{H}_0$. Define the Hilbert spaces $(\mathcal{H}_1, <\cdot,\cdot>_1, \|\cdot\|_1)$ and $(\mathcal{H}_{-1}, <\cdot,\cdot>_{-1}, \|\cdot\|_{-1})$ as in Definition 3.1. All claims in Lemma 3.2 remain valid with $\mathcal{D} = \mathbb{D}(\mathcal{L})$ in (1.2C).

We can now state the following result whose proof is given in Appendix.

THEOREM 3.6. – *Let* $\mathbf{T} = I\!R^+$, *assume (1.5) and* (X_t) *is a Hunt process. For* $f \in \mathcal{H}_0 \bigcap \mathcal{H}_{-1}$ *(or equivalently (1.2C) with* $\mathcal{D} = I\!D(\mathcal{L})$ *), the forward-backward martingale decomposition* $[(3.9)+(3.10)]$ *still holds. In particular all claims in Theorem 3.3. remain valid.*

Remark 3.7. – Under the sector condition (1.5), we can prove (the detail is left to the reader) in the discrete and continuous time cases both,

(i) $I\!D(R_0)$ is a dense subset in $\mathcal{H}_{-1}$, and $R_0 := A^{-1} : (I\!D(R_0), \|\cdot\|_{-1}) \longrightarrow \mathcal{H}_1$ is a contraction, where $A = I - P$ or $-\mathcal{L}$.

(ii) equivalence between

(a) $f \in \mathcal{H}_{-1}$ *or equivalently (1.2D or C)*;

(b) $\limsup_{t\to\infty} \frac{1}{t} I\!E(S_t(f))^2 < +\infty$;

(c) $\sigma^2(f) = \lim_{t\to\infty} \frac{1}{t} I\!E(S_t(f))^2 \in I\!R$ exists;

(d) $\liminf_{\epsilon\to 0} < R_\epsilon f, f >< +\infty$, where $R_\epsilon = (\epsilon + A)^{-1}$.

Remark 3.8. – By (3.12) in Remark 3.5, Th.3.6 remain valid for $S_.(\phi)$ for any $\phi \in \mathcal{H}_{-1}$. Hence the general finite dimensional CLT in [Va, 1995] and [OS, 1995] for $S_.(\phi)$ becomes the FCLT.

Moreover by the *a.s.* compactness in Corollary 2.9 and Th.3.3.(c) and the property (i) above in Remark 3.7, and by an approximation of f by $(f_k) \subset I\!D(R_0)$ w.r.t. the norm $\|\cdot\|_{-1}$, we get the FLIL below:

THEOREM 3.9. – *In the discrete and continuous time cases both, assume (1.5). For every* $f \in \mathcal{H}_{-1}$, *with probability one the sequence* $\{S_{n\cdot}(f)/h(n); n \geq 1\}$ *is precompact in* $I\!D([0,1])$ *and the set of its limit points is exactly* $\mathbf{K}(\sigma(f))$ *where* $\sigma(f)$ *is given in Remark 3.7.(ii.c).*

APPENDIX

A.1. Semi-FLIL for sums of backward martingale differences

The following lemma were used in Corollary 2.7 and 2.8. and Th. 3.3.

LEMMA A.1. – *Let*

$$m_1 \in L^2(\mathcal{G}_0) \ominus L^2(\mathcal{G}_1).$$

Then $M_n := \sum_{k=1}^n \theta_{k-1} m_1$ *satisfies the FCLT, and* $I\!P - a.s.$

$$\text{(A.1)} \qquad \left\{\frac{1}{h(n)} M_{[n\cdot]}, n \geq 1\right\}$$

is precompact and the set of its limit points $\subset \mathbf{K}(\sigma)$ *where* $\sigma^2 = I\!E(m_1)^2$ *and* $\mathbf{K}(\sigma)$ *is given in Corollary 2.9.*

Proof. – For $a : \mathbb{N} \longrightarrow \mathbb{R}$, the notation a^C with an added exposant C is used to denote the polygone function defined on $\mathbb{R}^+$, connecting (n, a_n), i.e., $a_t^C = a_n + (t-n)(a_{n+1} - a_n), \forall t \in [n, n+1], n \in \mathbb{N}$.

The FCLT is easy: note that $(M_k^{(n)} = M_n - M_{n-k}, \quad k = 0, \cdots, n)$ is a $(\mathcal{G}_{n-k})_{0 \le k \le n}$-martingale. By the classical FCLT for martingales (see [HH]),

$$\frac{1}{\sqrt{n}}(M_{n\cdot}^{(n),C}) \longrightarrow \sigma W.$$

in law on $C[0,1]$. Consequently

$$\frac{1}{\sqrt{n}} M_{n\cdot}^C = \frac{1}{\sqrt{n}}\left(M_n^{(n)} - M_{n-n\cdot}^{(n),C}\right) \longrightarrow \sigma(W_1 - W_{1-\cdot}),$$

in law on $C[0,1]$. But $(W_1 - W_{1-t})_{t \in [0,1]}$ is still a Brownian Motion, the FCLT follows.

We translate the semi-FLIL in (A.1) (the full FLIL is an equality instead of $\subset$) as

$$\text{(A.2)} \qquad dist\left(\frac{1}{h(n)} M_{n\cdot}^C, \mathbf{K}(\sigma)\right) \longrightarrow 0, \text{ in } C[0,1], \quad \mathbb{P} - a.s.$$

Unlike the FCLT, (A.2) is far from to be a direct consequence of the classical FLIL for martingale and it is relied on two results.

At first we have proved in [W1] by means of a variant of Skorohod's representation that

$$\text{(A.3)} \qquad \limsup_{n \to \infty} \frac{1}{h(n)} \sup_{k \le n} |M_k| \le 2\sigma = 2\sqrt{\mathbb{E}(m_1)^2}.$$

Having this *a priori* estimation and using an approximation procedure if necessary, we can assume that m_1 is bounded.

In that bounded case, Dembo [De, 1996] proves that $\mathbb{P}\left(\frac{1}{h(n)} M_{n\cdot}^{(n),C} \in \bullet\right)$ satisfies the large deviation principle with speed $\frac{h^2(n)}{n} = 2 \log\log n$ and with rate function

$$I(\gamma) = \frac{1}{2\sigma^2} \int_0^1 \gamma'(t)^2 dt,$$
$$\text{if } \gamma \in \text{ the Cameron-Martin space and } +\infty \text{ otherwise}$$

on $C[0,1]$ w.r.t. the uniform convergence topology. Then by contraction principle,

$$\mathbb{P}\left(\frac{1}{h(n)} M_{n\cdot}^C \in A\right), \quad A \subset C[0,1] \text{ (Borel subset)}$$

satisfies the same large deviation principle.

Finally as well explained in [DS], this large deviation principle implies (A.2). $\diamondsuit$

A.2. Proof of Theorem 3.6

We prove the forward-backward decomposition (3.9) in the actual context, which without (H1) requires Fukushima's decomposition from the Dirichlet form theory under (1.5) (see [MR, p. 180, Th. 2.5]):

for every $\mathcal{E}-$quasi continuous $g \in I\!D(\mathcal{E}^\sigma)$, $\forall t \geq 0$,

$$(A.4) \qquad g(X_t) - g(X_0) = M_t(g) + N_t(g) = -M_t^*(g) - N_t^*(g), \quad I\!P - a.s.$$

where $(M_t(g))$ (resp. $(M_T^*(g) - M_{T-t}^*(g))$) is a $(\mathcal{F}_t)$ (resp. $(\mathcal{G}_{T-t})$) martingale with

$$(A.5) \qquad I\!E[M_t(g)]^2 = I\!E[M_t^*(g)]^2 = 2t\mathcal{E}^\sigma(g,g), \quad \forall t \geq 0,$$

and $N(g), N^*(g)$ are continuous additive functional of *zero enery* so that

$$(A.6) \qquad \begin{aligned} N_t(g) &= \int_0^t \mathcal{L}g(X_s)ds, \quad \forall g \in I\!D(\mathcal{L}); \\ N_t^*(v) &= \int_0^t \mathcal{L}^*v(X_s)ds, \quad \forall v \in I\!D(\mathcal{L}^*). \end{aligned}$$

By (A.4), we get $\forall t \geq 0$, $I\!P - a.s.$

$$(A.7) \qquad -N_t(g) - N_t^*(g) = M_t(g) + M_t^*(g).$$

Now we prove that $\forall g \in I\!D(\mathcal{L}^\sigma)$, $I\!P - a.s.$,

$$(A.8) \qquad N_t(g) + N_t^*(g) = 2\int_0^t \mathcal{L}^\sigma g(X_s)ds,$$

for each $t \geq 0$ fixed (then for all $t \geq 0$ by the continuity of the two sides of (A.8)). Note that $[(A.7) + (A.8)]$ gives (3.8) which leads easily to (3.9) by (A.5).

To prove (A.8) for $t \geq 0$ fixed, we take $g_k \in I\!D(\mathcal{L}), g_k^* \in I\!D(\mathcal{L}^*)$ such that $g_k \to g, g_k^* \to g$ in $I\!D(\mathcal{E}^\sigma)$ as $k \to \infty$ (possible as $I\!D(\mathcal{L}), I\!D(\mathcal{L}^*)$ are form core of $I\!D(\mathcal{E}^\sigma)$). It is well known that

$$(A.9) \qquad S_t(\mathcal{L}g_k + \mathcal{L}^*g_k^*) = N_t(g_k) + N_t^*(g_k^*) \longrightarrow N_t(g) + N_t^*(g) \quad \text{in } L^2(I\!P).$$

Notice also $\forall u \in I\!D(\mathcal{E})$,

$$(A.10) \qquad < \mathcal{L}g_k + \mathcal{L}^*g_k^*, u > = -\mathcal{E}(g_k, u) - \mathcal{E}(u, g_k^*) \longrightarrow -2\mathcal{E}^\sigma(g,u) = 2 < \mathcal{L}^\sigma g, u > .$$

Now for (A.8), it remains to show for $\phi_k = \mathcal{L}g_k + \mathcal{L}^* g_k^*$,

$$\text{(A.11)} \qquad I\!\!E S_t(\phi_k) \prod_{i=0}^{m} u_i(X_{t_i}) \longrightarrow I\!\!E 2 S_t(\mathcal{L}^\sigma g) \prod_{i=0}^{m} u_i(X_{t_i})$$

as $k \to \infty$, where $0 = t_0 < t_1 < \cdots < t_m = t$, $u_i \in b\mathcal{B}$ are arbitrary. This can be done by recurrence on $m \geq 0$ and here we treat only the case $m = 1$.

$$\begin{aligned} \text{the LHS of (A.11)} &=< u_0, \int_0^t P_s(\phi_k P_{t-s} u_1) ds > \\ &= \int_0^t < P_s^* u_0 \cdot P_{t-s} u_1, \phi_k > ds. \end{aligned}$$

For any $0 < s < t$, $P_s^* u_0, P_{t-s} u_1 \in I\!\!D(\mathcal{E}^\sigma)$ (a consequence of (1.5)) and they are bounded, then their product $P_s^* u_0 \cdot P_{t-s} u_1 \in I\!\!D(\mathcal{E}^\sigma)$. Therefore by (A.10) and the dominated convergence, we get (A.11) for $m = 1$. $\diamondsuit$

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