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Combining \textit{m}-dependence with markovness


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Combining $m$-dependence with Markovness

by

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ABSTRACT. – Generally, no stationary sequence of random variables which is Markov of order $n$ but not of order $n - 1$ and $m$-dependent but not $(m - 1)$-dependent exists if the state space of the sequence has small cardinality. We show that to ensure the existence for the Markov sequences of order $n = 1$ the number of attainable states must be at least $m + 2$ and that this bound is tight. Given a small state space such a sequence exists only for special $n$ and $m$. On a two-element state space the smallest possible $n$ and $m$ are shown to be 3 and 2, respectively. This results from our parametric description of all binary $m$-dependent sequences, $m \geq 0$, that are Markov of order 3.

RéSUMÉ. – Si l’espace d’états n’est pas suffisamment riche on ne peut pas construire, pour $n$ et $m$ quelconques, une suite aléatoire stationnaire de Markov d’ordre $n$ et pas d’ordre $n - 1$ qui est dans le même temps $m$-dépendante et pas $(m - 1)$-dépendante. Nous montrons que pour les chaînes de Markov, $n = 1$, l’espace d’états doit avoir au moins $m + 1$ éléments et que ce nombre ne peut pas être amélioré. Pour les suites binaires les plus petits $n$ et $m$ admissibles sont 3 et 2, respectivement. C’est une conséquence


Key words and phrases. Markov chains, $m$-dependence, sequences with memory, stationary sequences, binary sequences, conditional independence structures.

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de notre description paramétrique de toutes les suites binaires stationnaires
m-dépendantes, m ≥ 0, de Markov d’ordre 3. © Elsevier, Paris

1. INTRODUCTION

Let ξ = (ξi; i ≥ 1) be a strictly stationary sequence of random variables
taking values in a finite state space S; speaking about a sequence we will
always assume these properties. The sequence ξ is Markov of order n ≥ 0
if (ξi; 1 ≤ i ≤ k) is conditionally independent of (ξi; i ≥ k + n + 1)
given (ξi; k + 1 ≤ i ≤ k + n) for all k ≥ 1. The sequence ξ is dependent
of order m ≥ 0 if (ξi; 1 ≤ i ≤ k) is unconditionally independent of
(ξi; i ≥ k + m + 1), k ≥ 1. For simplicity, we shorten the expressions
“Markov of order n” and “dependent of order m” to n-Markov and
m-dependent, correspondingly.

The aim of this note is to examine how these two properties interfere
under restrictions on the cardinality of the state space. A more precise
formulation will use the following notion of an index of a sequence. Let nξ
be the smallest nonnegative integer n such that a sequence ξ is n-Markov,
let mξ be the smallest m ≥ 0 such that ξ is m-dependent and let dξ be the
cardinality of the set of states which are attained with positive probabilities.
Thus, we have nξ ≥ 0, mξ ≥ 0 and dξ ≥ 1 with nξ = 0 if and only if
mξ = 0. This expresses the sequence ξ is i.i.d. If ξ is Markov of no order
n ≥ 0 it is reasonable to write nξ = ∞ and similarly with the dependence
and mξ; we shall, however, not deal with these cases at all. The triple
⟨nξ, mξ, dξ⟩ will be called index of ξ.

A natural question asks which triple can be equal to the index of a
sequence ξ. In other words, given a triple of integers ⟨n, m, d⟩ does there
exist a (stationary) sequence ξ such that nξ = n, mξ = m and dξ = d,
i.e. in the nontrivial case n > 0 and m > 0, such that it is n-Markov and
not (n - 1)-Markov, m-dependent and not (m - 1)-dependent and takes
exactly d states with positive probabilities?

We present answers only if n = 1 and partially if d = 2 here. In
the second section devoted to the usual Markov chains (1-Markovness)
we prove that a triple ⟨1, m, d⟩ is the index of a sequence if and only
if 1 ≤ m ≤ d - 2. Then we turn our attention entirely to the binary
sequences, S = {0, 1}, and during a technical preparation in the third
section we reveal that every \((n, m, 2)\)-sequence, this is an abbreviation for \(n\)-Markov, \(m\)-dependent and two-element state space, is i.i.d. provided \(n \leq 2\) or \(m \leq 1\). For some years I conjectured this be valid for any \(n\) and \(m\) nonnegative. It is, however, not the case. We will see in Section 4 that all \((3, m, 2)\)-sequences are 3-dependent and therefore the only two candidates for indices of 3-Markov binary sequences are \((3, 2, 2)\) and \((3, 3, 2)\). Both these triples are really indices and, moreover, we provide a complete characterization of the distributions of all \((3, 2, 2)\)-sequences in the fifth section and all \((3, 3, 2)\)-sequences in the sixth section, respectively.

Though Markov chains is an old topic, Markov chains with 1-dependence appeared for the first time in [1] and then in [2],[6] where the focus was on the structure of block-factors. Notes on binary sequences of this type are in [11] and [12]. Our question is akin to the problems around probabilistic conditional independence structures [8]; [4] and [5] settle an unconditional case for sequences of random variables. The latest review of the field is in [7]. It is also worthwhile to mention the paper [9] where Markovness was combined with \(m\)-independence. That means any \(m\) variables of \(\xi\) are mutually independent.

2. MARKOV SEQUENCES OF FIRST ORDER

It is not unexpected that a solution of our problem for 1-Markov sequences will be based on an analysis of transition matrices. Let us remind that a sequence \(\xi\) with the state space \(S = \{1, 2, \ldots, d\}\), \(d = d_\xi\), is 1-Markov if and only if the probability of every event \(\xi_1 = s_1 \cdots \xi_{k+1} = s_{k+1}\), denoted by \([s_1 \ldots s_{k+1}]\), is equal to \([s_1] p_{s_1 s_2} \cdots p_{s_{k} s_{k+1}}\), \(k \geq 1\), where \([s] > 0\) is the probability of \(\xi_1 = s\) and \(p_{s,t}\) is the conditional probability of \(\xi_2 = t\) given \(\xi_1 = s\), \(s, t \in S\). The \((s,t)\)-entry of the \(k\)-th power of the transition matrix \(P = (p_{s,t}; 1 \leq s, t \leq d)\) contains the conditional probability of \(\xi_{k+1} = t\) given \(\xi_1 = s\), \(k \geq 1\).

If the sequence \(\xi\) is, moreover, \(m\)-dependent, \(m \geq 0\), then \(\xi_1\) is independent of \(\xi_{m+2}\) and \(P^{m+1} = Q\) where the matrix \(Q\) has constant columns, \(t\)-th one containing the probability \([t]\), \(t \in S\). It is not difficult to see that, on contrary, this matrix equality implies that \(\xi\) is \(m\)-dependent. In fact, it implies \(\xi_k\) is independent of \(\xi_{k+m+1}\) what together with the conditional independence of \(\xi_k\) and \((\xi_i; i \geq k+m+2)\) given \(\xi_{k+m+1}\) yield \(\xi_k\) is independent of \((\xi_i; i \geq k + m + 1)\). Repeating the same reasoning once again we obtain the desired \(m\)-dependence.

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LEMMA 1. - If a Markov sequence of first order with $d$-element state space, $d \geq 2$, is $m$-dependent, $m \geq 0$, then it is $(d - 2)$-dependent.

This assertion is nontrivial for $m > d - 2$ as it provides reduction of the order of dependence due to “small” state space.

Proof. - Knowing that $P^{m+1} = Q$ we deduce that the spectra of both matrices are equal to $\{0, 1\}$. Since $P$ is primitive (all entries of some of its powers are positive) the number 1 is an eigenvalue of $P$ with algebraic multiplicity one, see [10]. We can write $P = T \mathbf{W} T^{-1}$ where $T$ is a regular matrix and $\mathbf{W}$ is the Jordan canonical form of $P$, see [3]. The matrix $\mathbf{W}$ is block-diagonal. One of the blocks consists of the single eigenvalue 1 and the remaining blocks have zeros on their diagonals and ones on their superdiagonals. The $k$-th power of such a block of size $b \times b$, $b \geq 2$, is a zero matrix once $k \geq b$. Thus, we can conclude that each matrix $\mathbf{W}^k$, $k \geq d - 1$, has only one nonzero entry; obviously it is the eigenvalue 1. Now, if $m + 1 \geq d - 1$ then $\mathbf{W}^{m+1} = \mathbf{W}^{d-1}$ and consequently $P^{d-1} = P^{m+1} = Q$ what means that the examined sequence is $(d - 2)$-dependent.

COROLLARY 1. - Every $(n, m, d)$-sequence is, $d \geq 2$, dependent of order $(d^{n} - n - 1)$.

Proof. - If $\xi$ fulfils the assumptions, $n > 0$, we consider the sequence $\eta = (\eta_i; i \geq 1)$ of the random variables $\eta_i = (\xi_i, \ldots, \xi_{i+n-1})$. This sequence is obviously 1-Markov, $(n+m-1)$-dependent and $d_{\eta} \leq d_{\xi} \leq d^{n}$. By Lemma 1 it is $(d_\eta - 2)$-dependent what implies that $\xi$ is dependent of order $(d^{n} - 2 - (n - 1))$.

PROPOSITION 1. - A triple of integers $\langle 1, m, d \rangle$ is the index of a sequence if and only if $1 \leq m \leq d - 2$.

Proof. - The necessity of the presented condition is a consequence of the previous lemma and the sufficiency will be approved below by a construction of the desired sequences.

Let $x_1, \ldots, x_d$ be an arbitrary orthonormal base of the Euclidean space $\mathcal{R}^d$ such that $x_1$ has all coordinates equal to $d^{-1/2}$. These vectors are taken as rows and their transpositions, columns, are obtained by using the superindex $T$. For example, $Q = x_1^T x_1$ is a doubly stochastic matrix. If we set $U_k = \sum_{j=2}^{k} x_j^T x_{j+1}$ for $1 \leq k \leq d - 1$ then the powers of these matrices are $U_k^\ell = \sum_{j=2}^{k+1-\ell} x_j^T x_{j+\ell}$, $1 \leq \ell \leq k$. This fact can be obtained by a simple induction argument. Note that $U_k^\ell$, $1 \leq k \leq d - 1$, are zero matrices and that for $\ell < k$ the matrix $U_k^\ell$ is nonzero owing to $x_2^T U_k^\ell = x_{2+\ell}$. In addition, $Q U_k^\ell = U_k^\ell Q$ are zero matrices for $1 \leq \ell \leq k \leq d - 1$, too.
Now, let us have a triple $(1, m, d)$ and $1 \leq m \leq d - 2$. The 1-Markov sequence $\xi$ on a $d$-element state space with the transition matrix $P = Q + \varepsilon U_{m+1}$, $\varepsilon \neq 0$ sufficiently small, and the uniform initial distribution is stationary because $x_1 P = x_1$. In addition, $P^k = Q + \varepsilon^k U_{m+1}$, $1 \leq k \leq m + 1$, what enables to conclude that $\xi$ is $m$-dependent but not $(m - 1)$-dependent, i.e. $n_{\xi} = 1$, $m_{\xi} = m$ and $d_{\xi} = d$.

\section{3. BINARY SEQUENCES: PRELIMINARIES}

From now on we fix the state space as $S = \{0, 1\}$. States $s_k, \ldots, s_\ell$ from $S$, $k \leq \ell$, will be concatenated into words and the word $s_k s_{k+1} \cdots s_\ell$ will be shortened to $s^\ell$. The symbol $S^k$ denotes the set of all words made of letters from $S$ which have the length $k$, $k \geq 0$, e.g. $s^1_1 \in S^k$ and $s^0_1 \in S^0$ is the empty word.

A sequence $\xi$ is $n$-Markov, $n \geq 0$, if and only if for all $k > 0$ and $s^1_{i+n+k} \in S^{n+k}$ the probability $[s^1_{i+n+k}]$ of the event $\xi_1 = s_1 \cdot \xi_{n+k} = s_{n+k}$ can be factorized as follows

$$[s^1_{i+n+k}] = [s^n_1] \prod_{j=1}^{k} (s^j_{i+n}).$$

In this formula the numbers $(s^j_{i+n})$, conditional probabilities, are defined by the equalities $[s^n_{i+1}] = [s^n_1] (s^1_{i+n})$. If $[s^n_1] = 0$ the choice of $(s^1_{i+n})$ is arbitrary and will not affect our next computations.

An $n$-Markov sequence is $m$-dependent, $m \geq 0$, if and only if for all $s^n_1$, $s^1_{n+m+1} \in S^n$ the following equality

$$0 = \square_{n,m} [s^n_1, s^1_{n+m+1} = [s^n_1] \left[ s^{1+n}_{n+m+1} - \sum_{s^{i+n}_{i+m+1} \in S^n} \prod_{j=1}^{n+m} (s^j_{i+n}) \right]$$

takes place. That means $(\xi_{k+i}; 1 \leq i \leq n)$ is independent of $(\xi_{k+n+m+i}; 1 \leq i \leq n)$, $k \geq 0$, and this fact implies $m$-dependence in a similar way as was done above with $n = 1$. We shall also need the symbol $\square_{n,m} [s^n_1, s^1_{n+m+1})$ denoting the difference in parentheses. Sometimes an argument in $\square_{n,m} [s^n_1, \cdot)$ is omitted to work with a function on $S^n$.

An $n$-Markov sequence, $n \geq 1$, is $(n - 1)$-Markov if and only if

$$0 = \triangle_{n-1} (s^2) = [0 s^n_0] [1 s^n_1] - [0 s^n_0] [1 s^n_0] [0 s^n_0]$$

for all $s^n_0 \in S^{n-1}$. The equalities express that the variable $\xi_k$ is independent of $\xi_{k+n}$ given $(\xi_{k+i}; 1 \leq i \leq n - 1)$. We shall also need the symbol
\[ \Delta_{n-1}^*(s^n_2) \] denoting the above difference with the brackets replaced by parentheses. Thus, \( \Delta_{n-1}^*(s^n_2) = 0 \cdot s^n_2 \cdot 1 \cdot s^n_2 \) \( \Delta_{n-1}^*(s^n_2) \).

Beside the foregoing basic observations we want to summarize and label some other useful facts concerning \((n, m, 2)\)-sequences, \(n, m \geq 1\), \((k \geq 0)\)

1. \( \square_{n, m-1}(s^n_2 0, \cdot) + \square_{n, m-1}(s^n_2 1, \cdot) = 0 \)
2. \( [0 s^n_2] \square_{n, k}(1 s^n_2 \cdot) = [1 s^n_2] \square_{n, k}(0 s^n_2 \cdot) \) if \( \Delta_{n-1}(s^n_2) = 0 \)
3. \( \square^*_{n, m-1}(s^n_2 0, \cdot) = \square^*_{n, m-1}(s^n_2 1, \cdot) = 0 \) if \( \Delta_{n-1}(s^n_2) \neq 0 \)
4. \( \square_{n, m-1}(\cdot, 0 s^n_2) = \square_{n, m-1}(\cdot, 1 s^n_2) = 0 \) if \( \Delta_{n-1}(s^n_2) \neq 0 \)

Some comments are in order. The expression in 1. equals the sum of \( \square_{n,m}(0 s^n_2 \cdot) \) and \( \square_{n,m}(1 s^n_2 \cdot) \). The validity of 2. is clear if \( [0 s^n_2] \) or \( [1 s^n_2] \) is zero; if they are both positive we use \( (0 s^{n+1}_2) = (1 s^{n+1}_2) \). To see 3. we write

\[ 0 = \square_{n,m}(s^n_2 \cdot) = \sum_{t \in S} [s^n_1 t] \square^*_{n, m-1}(s^n_2 t, \cdot) \]

and match these equalities into pairs corresponding to \( 0 s^n_2 \) and \( 1 s^n_2 \); the assumption \( \Delta_{n-1}(s^n_2) \neq 0 \) means that the determinant of two equations in a pair is nonzero. Finally, the validity of 4. is obtained similarly from

\[ 0 = \square_{n,m}(\cdot, s^{n+m+n}_n) = \sum_{t \in S} (s^{2n+m}_n \cdot) \square_{n, m-1}(\cdot, t s^{2n+m-1}_n) \).

**Lemma 2.** - If \( \xi \) is a \((n, m, 2)\)-sequence where \( n \leq 2 \) or \( m \leq 1 \) then \( \xi \) is i.i.d.

**Proof.** - Using Corollary 1 we can restrict ourselves to \( m = 1 \) and \( n \geq 2 \). We shall demonstrate by contradiction that \( \Delta_{n-1}^* \) is identically zero and then apply the induction argument.

Let \( \Delta_{n-1}(t^n_2) \neq 0 \) for some \( t^n_2 \in S^{n-1} \). By fact 3. we have

\[ [s^{2n+1}_{n+2}] = \prod_{j=2}^{n+1} (s^j_{n+1}) \]

as soon as the word \( s^{2n+1}_{2n} \) begins with \( t^n_2 \). We multiply both sides by \([s^{2n}_{n+1}]\)

and sum over \( s^{2n+1}_{2n} \) what gives

\[ [s^{2n}_{n+1}] [s^{2n}_{n+2}] = [s^{2n}_{n+1}] \prod_{j=2}^{n} (s^j_{n+1}). \]

The conclusion is \([s^{2n}_{n+1}] [s^{2n}_{n+2}] = [s^{2n}_{n+1}] [s^{2n+1}_{n+1}] \) for all \( s^{2n+1}_{n+1} \in S^{n+1} \) contradicting the assumption \( \Delta_{n-1}(t^n_2) \neq 0 \).
4. BINARY 3-MARKOV SEQUENCES

From Corollary 1 we know that every \((3, m, 2)\)-sequence is 4-dependent. We will see in a moment that it is even 3-dependent. By Lemma 2 the only nontrivial \(m\)'s to be examined are then 2 and 3. The aim of this section is to prove some auxiliary results about these two cases.

**Lemma 3.** – For every \((3, m, 2)\)-sequence \(\Delta_2(01) = 0\) or \(\Delta_2(10) = 0\).

**Proof.** – Let us suppose that both \(\Delta_2(01)\) and \(\Delta_2(10)\) are nonzero. By fact 3. we deduce

\[
0 = \square^*_{3,m-1}(01t, \cdot) = \square^*_{3,m-1}(10t, \cdot), \quad t \in S.
\]

If \(\Delta_2(00) = 0\) then by fact 2.

\[
0 = [000] \square_{3,m-1}(100, \cdot) = [100] \square_{3,m-1}(000, \cdot)
\]

and since \([100] \neq 0\) (otherwise \(\Delta_2(10) = 0\)) we employ 1. to obtain

\[
0 = \square_{3,m-1}(00t, \cdot), \quad t \in S.
\]

If \(\Delta_2(00) \neq 0\) we have this equality immediately by 3. The same reasoning applies symmetrically to \(\Delta_2(11)\). Thus, we see that the sequence is \((m - 1)\)-dependent. By induction, it is i.i.d., a contradiction.

**Lemma 4.** – A \((3, m, 2)\)-sequence is i.i.d. if and only if both numbers \(\Delta_2(01)\) and \(\Delta_2(10)\) are equal to zero.

**Proof.** – One implication is trivial. If \(\Delta_2(01) = \Delta_2(10) = 0\) and both \(\Delta_2(00)\) and \(\Delta_2(11)\) are nonzero we recall 3., 2. and 1. and, similarly as in the proof above, keep lowering of the order of dependence. Hence, by symmetry, let \(\Delta_2(00) = 0\). Then \(\Delta_2(11) = 0\) would imply the sequence is 2-Markov and by Lemma 2 also i.i.d.

From \(\Delta_2(11) \neq 0\) we deduce \(\square_{3,m-1}(s1t, \cdot) = 0\) for \(s, t \in S\) using 3., 2. and 1. as usually. Further, 1. and 2. enable to write the following four linear equations \((s \in S)\)

\[
\square_{3,m-1}(s00, \cdot) + \square_{3,m-1}(s01, \cdot) = 0, \\
[00s] \square_{3,m-1}(10s, \cdot) - [10s] \square_{3,m-1}(00s, \cdot) = 0.
\]

If the determinant \(\Delta_1(0)\) of the system of equations is nonzero then the order of dependence of the sequence decreases. Analogically as soon as

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\begin{align*}
[00] &= 0. \text{ Let } \triangle_1(0) = 0 \text{ and } [00] \neq 0. \text{ Since } [11] \neq 0 \text{ we know that } [s0t] = [s0][0t]/[0] > 0 \text{ for } s, t \in S \text{ and thus }
(110s)(10st) = (10s)(0st) = (0s)(00st) = (000s)(00st).
\end{align*}

This equality implies $0 = \Box_{3,m-1}(110, \cdot) = \Box_{3,m-1}(000, \cdot)$ and then $0 = \Box_{3,m-1}(s0t, \cdot)$ for all $s, t \in S$, having decrease of the order of dependence, too.

**Lemma 5.** - In every $(3, m, 2)$-sequence $\Delta_2(00) = 0$ or $\Delta_2(11) = 0$.

**Proof.** - By Lemma 3, Lemma 4 and symmetry we can assume $\Delta_2(10) \neq 0$ and $\Delta_2(01) = 0$. We start from the opposite $\Delta_2(00)\Delta_2(11) \neq 0$ aiming at a contradiction. Note that $[s_3^3]$ is positive for $s_3^3 \in S$. Since $\Box_{3,4}(\cdot, \cdot) = 0$ by Corollary 1, one has $\Box_{3,3}(s0t, \cdot) = 0$ and $\Box_{3,2}(00s, \cdot) = 0$ by means of $3, s, t \in S$. Owing to $4. \Box_{3,1}(00s, t1u) = 0$ and $\Box_{3,0}(00s, t11) = 0$ for any $s, t, u \in S$. The choice $st = 00$ in the latter equality gives $[01] = (0000)(0001)$. Then we substitute $st = 01$ and $st = 11$ and find $(0001) = (1111)$. On the other hand, from $\Box_{3,2}(00s, \cdot) = 0$ and $\Delta_2(00) \neq 0$ we have also $\Box_{3,1}(00s, t00) = 0$ again by $4., s, t \in S$. This provides for $st = 01$


where the left product equals $[0100]$. Thus, $[1100]$ is equal to the right product and then


Let us multiply both sides by $(0000)$ whence $(0000)[01] = [0110]$. The contradiction sounds $(0110) = (0000) = (1110)$.

**Corollary 2.** - $(3, 4, 2)$ is not an index.

**Proof.** - Let a sequence $\xi$ has the index $(3, 4, 2)$. Then index of the sequence of triples $\eta = ((\xi_i, \xi_{i+1}, \xi_{i+2}); i \geq 1)$ is $(1, 6, d)$ whence $d = 8$ by Proposition 1. From the proof of Lemma 1 we know that the transition matrix of $\eta$ has the rank 7. But, due to Lemma 3 and Lemma 5 this matrix has at least two pairs of equal rows, a contradiction.

**Lemma 6.** - Let $\Delta_2(10) \neq 0$ and $\Delta_2(01) = 0$ in a $(3, m, 2)$-sequence $\xi$. Then this sequence is 2-dependent if and only if both numbers $\Delta_2(00)$ and $\Delta_2(11)$ are equal to zero.
Proof. – Let us observe that $[s_2^2] > 0$ for $[s_1^3] \in S - \{000, 111\}$ and then the 1-Markov sequence $\eta$ constructed by grouping triples of consequent variables as above has the index $(1, m, d)$ where $6 \leq d \leq 8$. The transition matrix of $\eta$ has the two rows indexed 001 and 101 identical. If $\Delta_2(ss) = 0$ and $[sss] > 0$ then also the two rows indexed by $0ss$ and $1ss$ coincide, $s \in S$. Hence, the matrix has rank at most 5. Then $\eta$ is 4-dependent and $\xi$ must be 2-dependent.

On the other hand, let $\xi$ be 2-dependent. If $\Delta_2(00) \neq 0$ then by the fact 3. we have $\boxtimes_{3,1}(s0t, \cdot) = 0$ for $s, t \in S$ and then by 3. again $\boxtimes_{3,0}^*(00t, \cdot) = 0$ for $t \in S$. We can argue as in the proof of Lemma 2 to arrive at a contradiction and thus necessarily $\Delta_2(00) = 0$. If $\Delta_2(11) \neq 0$ then by 3. we get $\boxtimes_{3,1}^*(11t, \cdot) = 0$, $t \in S$, and by the fact 4.

$$\boxtimes_{3,0}^*(11t, s_5 s_7) = 0, \quad t, s_5, s_7 \in S.$$ The choice $ts_5 = 01$ leads to

$$[11s] = (1101) (1011) (011s), \quad s \in S,$$

and then (add the above equations) to $[11s] = [11] (011s)$. Hence $\Delta_2(11)$ equals zero, a contradiction.

Under the assumptions of Lemma 6, the sequence $\xi$ is 3-dependent and not 2-dependent if and only if exactly one of the numbers $\Delta_2(00)$ and $\Delta_2(11)$ is equal to zero. This follows from Corollary 1, Corollary 2, Lemma 5 and Lemma 6.

5. (3, 2, 2)-SEQUENCES

In this section we will describe parametrically all binary 2-dependent sequences that are Markov of order 3.

Proposition 2. – Let $\xi$ be a binary 3-Markov sequence such that

$$\Delta_2(10) \neq 0 = \Delta_2(00) = \Delta_2(01) = \Delta_2(11) = 0.$$ Then $\xi$ is 2-dependent if and only if $[0st1] = [0s][t1]$, $s, t \in S$.

Proof. – If $\xi$ is a (3, 2, 2)-sequence with $\Delta_2$’s as above then by 3. and 4. we derive

$$\boxtimes_{3,1}^*(10s, \cdot) = \boxtimes_{3,0}^*(10s, t10) = 0, \quad s, t \in S.$$
Since \([s_1^3]\) are positive if \(s_1^3\) is different from 000 and 111 we obtain 
\([t_{10}] = (0st)(0st1)(st10)\) and for \(st \neq 11\) immediately the desired equalities. But,

\[
\sum_{s,t \in S} [0st1] - [0s][t1] = 0
\]

by 2-dependence and we have also \([0111] = [01][11]\).

In the opposite direction, we deduce first from \([0ss1] = [0s][s1]\) that the probabilities \([000]\) and \([111]\) are positive, too. Then

\[
[s_5^1s_7s_8] = (s_2s_4s_5)(0s_4s_51)(s_4s_51s_7)(s_51s_7s_8)
\]

because \((s_20s_4s_5) = (0s_4s_5), (s_51s_7s_8) = (1s_7s_8)\) and \([0s_4s_51] = [0s_4][s_51]\). We multiply the above equation by \((s_1s_20s_4)\), sum over \(s_4\) and \(s_5\) and arrive at \(\square_{3,2}(s_1^80, 1s_7^8) = 0\).

Further, we are going to verify the equality \(\square_{3,2}(s_1^2, 1s_7^8) = 0\) for \(s_1^2, s_7^8 \in S^2\), which is equivalent to

\[
[1s_7^8] - \sum_{s_5 \in S} \frac{[s_5^1s_7s_8]}{[s_2^1][s_5^1]} \sum_{s_4 \in S} [s_2^1s_4s_5^1](1s_4s_51) = 0.
\]

This will be clear if we show that

\[
\nabla(s_2, s_5) = [s_2^1][s_5^1] - \sum_{s_4 \in S} [s_2^1s_4s_5^1](1s_4s_51) = 0
\]

for every \(s_2\) and \(s_5\) from \(S\). But,

\[
[01]^{-1}\nabla(0, 1) = [11] - [01](011) - [11](111) = 0
\]

is straightforward and

\[
\nabla(1, 0) = [11][01] - [1100](001) - (111)(1101) = 0
\]

owing to \((001) = (111)\) which is a consequence of

\[
[000][11] - [011][00] = [001]^{-1}([0001][11][00] - [0011][00][01]) = 0.
\]

The fact

\[
\sum_{s_2 \in S} \nabla(s_2, s_5) = [1][s_5^1] - \sum_{s_4 \in S} [s_4s_5^1] - [0s_4s_51] = 0
\]

implies that \(\nabla\) is identically zero, indeed.
At this moment we know \( \Box_{3,2}^*(\cdot, 1 s_8^8) = 0 \) for \( s_7^8 \in S^2 \). The multiplication by \( (s_6^8) \) and the summation over \( s_3 \in S \) will give \( \Box_{3,3}^*(\cdot, 1 s_8^8) = 0 \). Now, we sum over \( s_9 \) and compare the result with \( \Box_{3,2}^*(\cdot, 11 s_8) \). We have \( \Box_{3,2}^*(\cdot, 01 s_8) = 0 \). Repeating the same trick we arrive at \( \Box_{3,2}^*(\cdot, 001) = 0 \). But, the sum of \( \Box_{3,2}^*(\cdot, s_8^8) \) over \( s_8^8 \in S^3 \) is zero and thus \( \Box_{3,2}^*(\cdot, 000) = 0 \). Since \( \Box_{3,2} \) is identically zero the sequence \( \xi \) is 2-dependent.

**Theorem 1.** – Let \( \alpha, \beta \in \mathbb{R} \) satisfy the two inequalities

\[
\pm (4\alpha - 3\beta - \beta^3) \leq 1 - \beta^2.
\]

The binary 3-Markov sequence \( \zeta^{\alpha,\beta} \), which has its distribution of first four variables proportional to the function given by the following table is 2-dependent.

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Every \( (3, 2, 2) \)-sequence \( \xi \) is equal in distribution to some \( \zeta^{\alpha,\beta} \) or to the sequence obtained from some \( \zeta^{\alpha,\beta} \) by interchanging zeros and ones.

**Proof.** – The proportionality factor is 1/16. The conditions imposed on \( \alpha \) and \( \beta \) restrict the parameters to be between \(-1\) and \(1\), see Figure 1, and
guarantee that all entries of the table are nonnegative; notably the critical inequalities are $[0100] \geq 0$ and $[1101] \geq 0$.

It is easy to verify that $\sum_{t \in S} [s_t^3] - [s_t^2] = 0$, $s_t^3 \in S_3$, i.e. that every sequence $\zeta^{\alpha, \beta}$ is strictly stationary. To this end we remark that

\[
\begin{align*}
8[000] &= (1 - \beta)(1 + \beta^2 - 2\alpha) & 8[010] &= (1 - \beta)(1 - \beta^2 + 2\beta - 2\alpha) \\
8[111] &= (1 + \beta)(1 + \beta^2 + 2\alpha) & 8[101] &= (1 + \beta)(1 - \beta^2 + 2\alpha - 2\beta) \\
8[100] &= (1 + \beta)(1 + \beta^2 - 2\alpha) & 8[110] &= (1 - \beta)(1 + \beta^2 + 2\alpha)
\end{align*}
\]

and $[100] = [001]$, $[110] = [011]$. It is also not difficult to see that $\zeta^{\alpha, \beta}$, $\alpha \neq \beta$, fulfils the assumptions of Proposition 2, especially

\[
\begin{align*}
(0001) = (1001) &= \frac{1 + \beta}{2} = (0111) = (1111) \\
(0011) = (1011) &= \frac{1 + \beta^2 + 2\alpha}{2(1 + \beta)}
\end{align*}
\]

and $\Delta_2(10) \neq 0$. Hence, all sequences $\zeta^{\alpha, \beta}$ are 2-dependent. Note that $\zeta^{\alpha, \alpha}$ are i.i.d., $|\alpha| \leq 1$ (the dotted segment in Figure 1).

In the opposite direction, let $\xi$ be a (3,2,2)-sequence. By Lemmas 3 and 6 we know that up to switching between zeros and ones $\Delta_2(st) = 0$ for $st \neq 10$. If also $\Delta_2(10) = 0$ then $\xi$ is i.i.d. by Lemma 4 and thus equal in distribution to some $\zeta^{\alpha, \alpha}$. If $\Delta_2(10) \neq 0$ then $[0st1] = [0s][t1]$ for $s, t \in S$ by Proposition 2. Thus, with $st = 11$ here, the quadratic equation in $[111]$

\[
\]
has nonnegative discriminant equal to \([11]^2 (1-4[01])\). We set \(\beta^2 = 1-4[01]\)
and then we have \(2[111] = [11] (1 + \beta)\). If we take \(\alpha = 2[1] - 1\) we can
compute \(4[00] = 1 + \beta^2 - 2\alpha\) and \(4[11] = 1 + \beta^2 + 2\alpha\). Using the listed
properties of \(\xi\) and the stationarity it is easy, but a bit laborious, to compute
first the probabilities \([s_1s_2s_3]\) and then to construct the whole table. ■

**Corollary 3.** - The triple \(\langle 3, 2, 2 \rangle\) is the index of \(\zeta^{0,1,0}\).

**Remark 1.** - Let us mention that the sequence \(\zeta^{-\alpha, \alpha}, \alpha \neq 0\) small, has
the same index as \(\zeta^{0,1,0}\). In addition, every its two consequent variables are
independent. But, no of the sequences \(\zeta^{\alpha, \beta}, \alpha \neq \beta\), is 2-independent.

2. From the topological point of view the class of \(\langle 3, 2, 2 \rangle\)-sequences with
the weak topology is homeomorphic to two closed circles (disks) pasted
together along its diameters; the common diameter corresponds to the i.i.d.
sequences and switching between circles to switching between 0 and 1.

3. Reversing time in a \(\langle 3, 2, 2 \rangle\)-sequence indexed by integer numbers one
obtains again a \(\langle 3, 2, 2 \rangle\)-sequence. In our parametrization this corresponds
to the transition \(0 \leftrightarrow 1\) and \((\alpha, \beta) \leftrightarrow (-\alpha, -\beta)\) simultaneously.

### 6. \(\langle 3, 3, 2 \rangle\)-SEQUENCES

In this section we will describe parametrically all binary 3-dependent
sequences that are Markov of order 3. Since all of them which are
2-dependent were investigated in the previous section we concentrate
on the non-2-dependent ones. This will close the description of all
\(\langle 3, m, 2 \rangle\)-sequences.

**Proposition 3.** - Let \(\xi\) be a binary 3-Markov sequence satisfying
\(\Delta_3(1s) \neq 0 = \Delta_2(0s)\) for \(s \in S\). Then \(\xi\) is 3-dependent if and only if
\([111] = (001)^3, [0s01] = [0s][01], s \in S, and (1st) = (00t), s, t \in S\).

**Proof.** - Obviously, \([s_1^3] > 0\) for \(s_1^3 \neq 000\). If \(\xi\) is 3-dependent then by 3.
\(\square_{3,2}(1st, \cdot) = 0\) and by 4. \(\square_{3,1}(1st, u1v) = 0\) and \(\square_{3,0}(1st, u11) = 0\),
\(s, t, u, v \in S\). For \(st = 00\) the last equation gives \([000] > 0\) and
\([011] = (0t0)(0t01)(011)\), i.e. \([0t][01] = [0t01]\); as a useful consequence we
have \([01] = (000)(001)\). The choice \(su = 01\) provides \([01] = (11t0)(1t01)\)
what reads as \((000) = (1100)\) if \(t = 0\) and as \((000) = (1110)\) if \(t = 1\).
And finally, \(stv = 111\) leads to \([111] = (1111)^3 = (001)^3\).

For the reverse implication we will first demonstrate \(\square_{3,0}(1st, u11) = 0\).
For \(u = 0\) this is equivalent to \([01] = (1st0)(st01)\) which certainly holds
if \(s = 0\); if \(s = 1\) we have \([01] = (000)(001)\). For \(u = 1\) we want to see
that \[111\] = (1st1)(st11)(t111) or, rewritten, \([t11] = (1st1)(st11)(111t)\). If \(t = 0\) this means \([011] = (001)(011)(000)\). If \(t = 1\) this amounts \([s11] = (1s11)(111s)(1111)\) which is fulfilled for \(s = 1\) as \((1111) = (001)\) and which holds also for \(s = 0\) having \([011] = (011)(000)(001)\).

The next step is to show \(\square_{3,1}^*(1st, u1v) = 0\). For \(u = 1\) this can be obtained from \(\square_{3,0}^*(1st, u'11) = 0\) by multiplication with \((u'11v)\) and summation over \(u'\). For \(u = 0\) we want to verify

\[ [01] = \sum_{v \in S}(1stv)(stv0)(tv01) \]

which is clear for \(t = 0\): the product equals \((1s0v)[01]\). If \(t = 1\) then we rewrite it into

\[ (000) = \sum_{v \in S}(1s1v)(s1v0) \]

being satisfied for \(s = 1\). For \(s = 0\) we have

\[
\begin{align*}
(000)[01] &= [0100] + [0110] = [01] - [0101] - [0111] \\
&= [01] - [01]^2 - (1110)[111],
\end{align*}
\]

which can be casted into \((001)[01] = [01]^2 + (00)(001)^3\) and \((001) = [01] + (001)^2\).

Knowing that \(\square_{3,1}^*(1st, u1v) = 0\) we can obtain \(\square_{3,2}^*(1st, 1vu) = 0\) and \(\square_{3,2}^*(s1t, u1v) = 0\). These two equalities imply \(\sum_{t \in S}\square_{3,2}^*(11s, 00t) = 0\) where both summands must equal zero due to \((001)\square_{3,2}^*(\cdot, 000) = (000)\square_{3,2}^*(\cdot, 001)\) (cf. the fact 2.). Hence, \(\square_{3,2}^*(11s, \cdot) = 0\) and then \(\square_{3,3}^*(s11, \cdot) = 0\) and \(\square_{3,3}^*(s11, \cdot) = 0\) arguing as usually. But owing to the equality \((110t)(10tu) = (s00)(00tu)\) we have \(\square_{3,2}^*(110, \cdot) = \square_{3,2}^*(s00, \cdot)\) whence \(\square_{3,3}^*(s00, \cdot) = 0\) and \(\square_{3,4}^*(s00, \cdot) = 0\). We can write now

\[ 0 = \square_{3,4}^*(000, \cdot) + \square_{3,4}^*(100, \cdot) = \square_{3,3}^*(000, \cdot) + \square_{3,3}^*(001, \cdot) \]

and deduce \(\square_{3,3}^*(s01, \cdot) = 0\). Thence \(\square_{3,3}^*(\cdot, \cdot) = 0\) and \(\xi\) is 3-dependent.

**Theorem 2.** - Let \(\alpha, \beta \in \mathcal{R}\) satisfy the two inequalities

\[ -(1 - \beta^2)^2 \leq 8\alpha - 8\beta \leq (1 - \beta)^3 (1 + \beta). \]

The binary 3-Markov sequence \(\theta_{\alpha, \beta}\), which has its distribution of first four variables proportional to the function given by the following table is 3-dependent.
Every $(3,3,2)$-sequence $\xi$ which is not 2-dependent equals in distribution to some $\theta^{\alpha,\beta}$, $\alpha \neq \beta$, up to the switching of zeros and ones or up to the time reversal.

**Proof.** – The proportionality factor is $1/16$. The conditions imposed on $\alpha$ and $\beta$ restrict the parameters to be strictly between $-1$ and $1$, see Figure 2, and guarantee that all entries of the table are nonnegative (the critical inequalities are $[0100] \geq 0$ and $[0110] \geq 0$). By continuity, $\theta^{1,1}$ will be a constant sequence. The dashed curves in Figure 2 remind the restrictions from Theorem 1 which can be here interpreted as $[010] \geq 0$ and $[011] \geq 0$.

It is easy to verify that the sequences $\theta^{\alpha,\beta}$ are strictly stationary. For this purpose we write down

\[
\begin{align*}
8[000] &= (1 - \beta)(1 + \beta^2 - 2\alpha) & 8[111] &= (1 + \beta)^3 \\
8[100] &= (1 + \beta)(1 + \beta^2 - 2\alpha) & 8[010] &= (1 + 3\beta - \beta^2 + \beta^3 - 4\alpha)/(1 - \beta) \\
8[101] &= (1 + \beta)(1 - \beta^2 + 2\alpha - 2\beta) & 8[110] &= (1 - 3\beta - \beta^2 - \beta^3 + 4\alpha)/(1 - \beta)
\end{align*}
\]

From

\[
\begin{align*}
(0001) &= (1001) = \frac{1 + \beta}{2} = (1101) = (1111) \\
(0011) &= (1011) = \frac{1 - 3\beta - \beta^2 - \beta^3 + 4\alpha}{2(1 - \beta^2)}
\end{align*}
\]
we see that the sequences $\theta^{\alpha,\beta}$ satisfy the assumptions of Proposition 3 and are thus 3-dependent.

Let $\xi$ be a $(3,3,2)$-sequence which is not 2-dependent. Switching zeros and ones and reversing time, if necessary, we know that $\Delta_2(0s) = 0$ and $\Delta_2(1s) \neq 0$, $s \in S$; see the lemmas of Section 4. By Proposition 3, $[000][001] = [0001][00] = [00]^2[01]$, which can be casted into a quadratic equation in $[000]$ similarly as in the proof of Theorem 1. The equation has nonnegative discriminant equal to $[11]^2 (1 - 4[01])$. We set $\beta^2 = 1 - 4[01]$, $\alpha = 2[1] - 1$ and then we can easily compute $[000]$, $[001]$ and $[101]$. Using $[111] = (011)^2$ we obtain $[111]$, $[110]$ and $[010]$. From $(11st) = (00t)$ and $(s0tu) = (0tu)$ the whole table can be computed with a little effort.

**COROLLARY 4.** - The triple $(3, 3, 2)$ is the index of $\theta^{0,0,1}$.

**Remark.** – From the topological point of view the union of the classes of $(3, m, 2)$-sequences over $m \geq 0$ finite, with the week topology, is homeomorphic to six closed disks pasted together along its diameters. The common diameter corresponds to the i.i.d. sequences, two disks to the $(3, 2, 2)$-sequences and the remaining four disks without their common diameter to the $(3, 3, 2)$-sequences that are not 2-dependent.

**ACKNOWLEDGEMENT**

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REFERENCES


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