ANNALES DE L'I. H. P., SECTION B

DANIEL W. STROOCK
WEIAN ZHENG

Markov chain approximations to symmetric diffusions

Annales de l'I. H. P., section B, tome 33, nº 5 (1997), p. 619-649
<http://www.numdam.org/item?id=AIHPB_1997__33_5_619_0>

Ann. Inst. Henri Poincaré,

Vol. 33, n° 5, 1997, p. 619-649 *Probabilités et Statistiques*

Markov chain approximations to symmetric diffusions

by

Daniel W. STROOCK

M.I.T., rm. 2-272, Cambridge, MA 02139-4307, USA
E-mail: dws@math.mit.edu

and

Weian ZHENG

Dept. of Math., UC Irvine, Irvine, CA 9217-0001, USA
E-mail: wzheng@math.uci.edu

ABSTRACT. – This paper contains an attempt to carry out for diffusions corresponding to divergence form operators the sort of approximation via Markov chains which is familiar in the non-divergence form context. At the heart of the our procedure are an extension to the discrete setting of the famous De Giorgi-Moser-Nash theory.

RÉSUMÉ. – Cet article contient une tentative pour établir dans le cas où le générateur donné sous forme de divergence, les approximations par des chaînes de Markov qui sont bien connues pour des diffusions dont le générateur est donné sous une forme de non-divergence. L'essentiel de notre méthode repose sur une extension de la célèbre théorie de De Giorgi, Moser, et Nash au cas discret.

The first author acknowledges partial support from NSF grant 9302709-DMS.

Annales de l'Institut Henri Poincaré - Probabilités et Statistiques - 0246-0203
Vol. 33/97/05/$ 7.00/© Gauthier-Villars

0. INTRODUCTION

Given a continuous, symmetric matrix $a : \mathbb{R}^d \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ satisfying $\lambda I \leq a(\mathbf{x}) \leq \lambda^{-1} I$ for some $\lambda \in (0,1]$ and a bounded continuous $b : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, the martingale problem for the operator

$$(0.1) \qquad \mathcal{L}^{a,b} = \sum_{i,j=1}^{d} a^{i,j}(\mathbf{x}) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i(\mathbf{x}) \frac{\partial}{\partial x_i} \quad \text{on } C_c^\infty(\mathbb{R}^d; \mathbb{R})$$

is well-posed (cf. Theorem 7.2.1 in [11]). As a consequence, one can use any one of a large variety of procedures to approximate the associated diffusion by Markov chains (cf. Section 11.2 in [11]). Indeed, aside from checking a uniform infinitesimality condition, all that one has to do is make sure that the action on $f \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$ of the approximating generators is tending, in a reasonable way, to $\mathcal{L}^{a,b} f$. If, on the other hand, one replaces the $\mathcal{L}^{a,b}$ in (0.1) by the *divergence form operator*

$$(0.2) \qquad L^a = \nabla \cdot (a \nabla) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \sum_{j=1}^{d} a^{i,j}(\mathbf{x}) \frac{\partial}{\partial x_j},$$

then, unless a is sufficiently smooth to allow one to re-write L^a as $\mathcal{L}^{a,b}$ with

$$b^i(\mathbf{x}) = \sum_{j=1}^{d} \frac{\partial a^{i,j}}{\partial x_j}(\mathbf{x}),$$

then the theory alluded to above does not apply. Nonetheless, the magnificent analytic theory of De Giorgi, Nash, Moser, and Aronson (cf. [9]) tells us that the operator L^a not only determines a diffusion, it does so via transition probability functions which are, in general, *better* than those determined by $\mathcal{L}^{a,b}$. Thus, one should hope that probability theory should be able to provide a scheme for approximating these diffusions via Markov chains, and in the present article we attempt to do just that.

In order to understand the issues involved, it may be helpful to keep in mind the essential difference between the theories for the *non-divergence form* operators $\mathcal{L}^{a,b}$ and the divergence form operators L^a. Namely, all the operators $\mathcal{L}^{a,b}$ share $C_c^\infty(\mathbb{R}^d; \mathbb{R})$ as a large subset of their domain. By contrast, when a is not smooth, finding non-trivial functions in the domain of L^a becomes a highly delicate matter. In particular, as a changes by even a little bit, the domains of the corresponding L^a's may undergo radical change. For this reason, one has to approach L^a via the quadratic form which it determines in $L^2(\mathbb{R}^d;\mathbb{R})$. Namely, if $f \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$, then one can interpret $L^a f$ as a distribution (in the sense of L. Schwartz) and one finds that

$$-\langle g, L^a f\rangle = \mathcal{E}^a(f,g) \equiv \big(\nabla g, a\nabla f\big)_0, \quad g \in C_c^\infty(\mathbb{R}^d;\mathbb{R}), \tag{0.3}$$

where we have introduced the notation $(\,\cdot\,,\,\cdot\,)_0$ to denote the inner product on the usual Lebesgue space $L^2(\mathbb{R}^d;\mathbb{R})$. For historical reasons, the quadratic form $\mathcal{E}^a$ is called a *Dirichlet form*. Clearly, for any a satisfying our hypotheses, $\mathcal{E}^a$ is well-defined on $C_c^\infty(\mathbb{R}^d;\mathbb{R})$. In fact, one finds that each $\mathcal{E}^a \restriction C_c^\infty(\mathbb{R}^d;\mathbb{R})$ admits a closure, which we continue to denote by $\mathcal{E}^a$, and that $\mathrm{Dom}(\mathcal{E}^a)$ coincides with the Sobolev space $W_2^1(\mathbb{R}^d;\mathbb{R})$ of $f \in L^2(\mathbb{R}^d;\mathbb{R})$ having one (distributional) derivative in $L^2(\mathbb{R}^d;\mathbb{R})$. Furthermore (cf. Theorem II.3.1 in [9]), there is a unique, strongly Feller continuous, Markov semigroup $\{P_t^a : t > 0\}$ with the property that, for each $f \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$, $t \rightsquigarrow P_t^a f$ is the unique continuous map $t \in [0,\infty) \longmapsto u_t \in W_2^1(\mathbb{R}^d;\mathbb{R})$ such that

$$\begin{gathered}\big(u_t, g\big)_0 = (f,g)_0 - \int_0^t \mathcal{E}^a\big(u_\tau, g\big)\, d\tau,\\ t \in [0,\infty) \text{ and } g \in C_c^\infty(\mathbb{R}^d;\mathbb{R}).\end{gathered} \tag{0.4}$$

On the basis of the characterization in (0.4), one might think that convergence of the semigroups should follow from convergence of the associated Dirichlet forms. That is, one might hope that $P_t^{a_n} f \longrightarrow P_t^a f$ for all $f \in C_c(\mathbb{R}^d;\mathbb{R})$ should follow from $\mathcal{E}^{a_n}(f,f) \longrightarrow \mathcal{E}^a(f,f)$ for all $f \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$. However, this is *not* the case, not even when $d = 1$. To wit, given a measurable $a : \mathbb{R} \longrightarrow [1,3]$, set

$$u^a(x) = \int_0^x \frac{1}{a(\xi)}\, d\xi \quad \text{and} \quad \hat{a} = \frac{1}{a \circ (u^a)^{-1}}.$$

At least when a is differentiable, it is easy to check that

$$L^a(f \circ u^a) = \big(\mathcal{L}^{\hat{a},0} f\big) \circ u^a,$$

and therefore, if, for $\lambda > 0$, R_λ^a is the resolvent operator given by $\int_0^\infty e^{-\lambda t} P_t^a\, dt$, then an elementary application of the Feynman-Kac formula leads to

$$\begin{aligned}\big[R_\lambda^a f\big](x) = 2\mathbb{E}^{\mu_{u^a(x)}}\Bigg[\int_0^\infty & a \circ (u^a)^{-1}\big(w(t)\big)\\ &\times \exp\left(-2\lambda \int_0^t a \circ (u^a)^{-1}\big(w(\tau)\big)\, d\tau\right)\\ &\times f \circ (u^a)^{-1}\big(w(t)\big)\, dt\Bigg],\end{aligned} \tag{0.5}$$

where μ_y denotes the standard Wiener measure for 1-dimensional Brownian motion starting at y. In fact, (0.5) holds whether or not a is differentiable. Now suppose that $\{a_n\}_1^\infty$ is a sequence of $[1, 3]$-valued measurable functions. Then $\mathcal{E}^{a_n}(f, f) \longrightarrow \mathcal{E}^a(f, f)$ for all $f \in C_c^\infty(\mathbb{R}; \mathbb{R})$ is equivalent to saying that $a_n \longrightarrow a$ *weakly* in the sense that

$$\int_{\mathbb{R}} a_n(\xi)\varphi(\xi)\,dx \longrightarrow \int_{\mathbb{R}} a(\xi)\varphi(\xi)\,dx \quad \text{for all } \varphi \in L^1(\mathbb{R}; \mathbb{R}).$$

On the other hand, (0.5) shows that it is weak convergence of the reciprocals $\frac{1}{a_n}$ to $\frac{1}{a}$ which determines whether the associated semigroups converge. Indeed, it is clear that $\frac{1}{a_n} \longrightarrow \frac{1}{a}$ implies first that $u^{a_n} \longrightarrow u$ and therefore that $(u^{a_n})^{-1} \longrightarrow (u^a)^{-1}$ uniformly on compacts. In particular, since $(u^{a_n})^{-1}(y) = \int_0^y a_n \circ (u^{a_n})^{-1}(\eta)\,d\eta$, this means that $a_n \circ (u^{a_n})^{-1} \longrightarrow a \circ (u^a)^{-1}$ weakly. Hence, after representing $\int_0^t a \circ (u^a)^{-1}\big(w(\tau)\big)\,d\tau$ in terms of the local time of Brownian motion, one sees from (0.5) that $R_\lambda^{a_n} f \longrightarrow R_\lambda^a f$ follows from $\frac{1}{a_n} \overset{\text{weak}}{\longrightarrow} \frac{1}{a}$. Finally, take $a_n(x) = 2 + \sin(2n\pi x)$, and observe that $a_n \longrightarrow 2$ weakly while

$$\frac{1}{a_n} \overset{\text{weak}}{\longrightarrow} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2 + \sin(2\pi\xi)}\,d\xi = \int_0^{\frac{1}{2}} \frac{1}{1 - 4^{-1}\sin(2\pi\xi)^2}\,d\xi > \frac{1}{2}.$$

As the preceding discussion indicates, even in the context of diffusions, identification of limits via Dirichlet forms is a delicate matter. Thus, it should come as no surprise that the difficulties become only worse when one is trying to approximate diffusions by Markov chains. In order to handle these problems, we devote the next section to a derivation of a number of *a priori* estimates. These estimates are extensions to the discrete setting of estimates which are familiar in the diffusion setting, and our proof of them mimics that of their diffusion analogs. In §2 we use the results in §1 to find general criteria which tell us when our Markov chains are converging to a diffusion, and in §3 we construct Markov chain approximations to the diffusion corresponding to a's satisfying (0.1), first in the case when a is continuous (cf. Theorem 3.9) and then in the general (cf. Theorem 3.14). Finally, in §4 we describe a possible application.

1. THE BASIC ESTIMATES

Our purpose in this section is to derive the *a priori* estimates (especially (1.11) and (1.32)) on which our whole program rests. The general setting in which we will be working is described as follows. We are given a function $\rho : \mathbb{Z}^d \times \mathbb{Z}^d \longrightarrow [0, \infty)$ satisfying[1]

$$\begin{aligned} &\rho(\mathbf{k},\mathbf{e}) \geq 1 \quad \text{for all } (\mathbf{k},\mathbf{e}) \in \mathbb{Z}^d \times \mathbb{Z}^d \text{ with } \|\mathbf{e}\| = 1 \\ &\qquad\qquad \text{and } \sup_{\mathbf{e}\in\mathbb{Z}^d} \sum_{\mathbf{k}\in\mathbb{Z}^d} \big|\rho(\mathbf{k},\mathbf{e})\big| < \infty. \end{aligned} \tag{1.1}$$

For $\alpha \in (0, \infty)$, we set $\alpha\mathbb{Z}^d = \{\alpha\mathbf{k} : \mathbf{k} \in \mathbb{Z}^d\}$, introduce the spaces $L^p(\alpha\mathbb{Z}^d; \mathbb{R})$ of $f : \alpha\mathbb{Z}^d \longrightarrow \mathbb{R}$ with

$$\|f\|_{\alpha,p} \equiv \Big(\alpha^d \sum_{\mathbf{k}\in\mathbb{Z}^d} \big|f(\alpha\mathbf{k})\big|^p\Big)^{\frac{1}{p}} < \infty,$$

and define the symmetric quadratic forms $(\,\cdot\,,\,\cdot\,)_\alpha$ and $\mathcal{E}^{\alpha,\rho}$ on $L^2(\alpha\mathbb{Z}^d; \mathbb{R})$ by

$$\begin{aligned} (f,g)_\alpha &= \alpha^d \sum_{\mathbf{k}\in\mathbb{Z}^d} f(\alpha\mathbf{k})g(\alpha\mathbf{k}) \quad \text{and} \\ \mathcal{E}^{\alpha,\rho}(f,g) &= \alpha^{d-2} \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{\mathbf{e}\in\mathbb{Z}^d} \rho(\mathbf{k},\mathbf{e}) \\ &\quad \times \big(f\big(\alpha(\mathbf{k}+\mathbf{e})\big) - f(\alpha\mathbf{k})\big)\big(g\big(\alpha(\mathbf{k}+\mathbf{e})\big) - g(\alpha\mathbf{k})\big). \end{aligned} \tag{1.2}$$

By the elementary theory of Markov chains, it is an easy matter to check that, for each $\alpha \in (0, \infty)$, there is a unique symmetric, Markov semigroup $\{P_t^{\alpha,\rho} : t > 0\}$ on $L^2(\alpha\mathbb{Z}^d; \mathbb{R})$ with the property that, for each $f \in L^2(\alpha\mathbb{Z}^d; \mathbb{R})$, $t \in (0, \infty) \longmapsto P_t^{\alpha,\rho} f \in L^2(\alpha\mathbb{Z}^d; \mathbb{R})$ is the unique differentiable $t \in (0, \infty) \longmapsto u_t \in L^2(\alpha\mathbb{Z}^d; \mathbb{R})$ satisfying

$$\begin{aligned} &\lim\nolimits_{t\searrow 0} u_f = f \text{ pointwise, and} \\ &\frac{d}{dt}\big(g, u_t\big)_\alpha = -\mathcal{E}^{\alpha,\rho}\big(g, u_t\big) \end{aligned} \tag{1.3}$$

for all $g \in L^2(\alpha\mathbb{Z}^d; \mathbb{R})$. Moreover, if $p^{\alpha,\rho} : (0, \infty) \times \alpha\mathbb{Z}^d \times \alpha\mathbb{Z}^d \longrightarrow [0, \infty)$ is determined by

$$\big[P_t^{\alpha,\rho} f\big](\alpha\mathbf{k}) = \alpha^d \sum_{\mathbf{l}\in\mathbb{Z}^d} f(\alpha\mathbf{l})\, p^{\alpha,\rho}(t, \alpha\mathbf{k}, \alpha\mathbf{l}), \tag{1.4}$$

then

$$\begin{aligned} &p^{\alpha,\rho}(t, \alpha\mathbf{k}, \alpha\mathbf{l}) = p^{\alpha,\rho}(t, \alpha\mathbf{l}, \alpha\mathbf{k}) \quad \text{and} \\ &\beta^d p^{\alpha\beta,\rho}\big(\beta^2 t, \alpha\beta\mathbf{k}, \alpha\beta\mathbf{l}\big) = p^{\alpha,\rho}(t, \alpha\mathbf{k}, \alpha\mathbf{l}). \end{aligned} \tag{1.5}$$

[1] for $\mathbf{v} \in \mathbb{R}^{\mathbf{d}}$, we use $\|\mathbf{v}\| \equiv \max_{1\leq i\leq d} |\mathbf{v}_i|$, as distinguished from $|\mathbf{v}|$, which reserved for the Euclidean length of $\mathbf{v}$.

The symmetry assertion in (1.5) comes from the symmetry of the operator $P_t^{\alpha,\rho}$ in $L^2(\alpha\mathbb{Z}^d;\mathbb{R})$. To see the second assertion, for $f \in L^\infty(\alpha\mathbb{Z}^d;\mathbb{R})$, define $f_\beta \in L^\infty(\alpha\beta\mathbb{Z}^d;\mathbb{R})$ so that $f_\beta(\alpha\beta\mathbf{k}) = f(\alpha\mathbf{k})$. Next, take

$$u_t(\alpha\mathbf{k}) = \left[P_{\beta^2 t}^{\alpha\beta,\rho} f_\beta\right](\alpha\beta\mathbf{k}).$$

Then $u_t \longrightarrow f$ pointwise as $t \searrow 0$ and

$$\begin{aligned}\frac{d}{dt}\left(g, u_t\right)_\alpha &= \beta^{-d}\frac{d}{dt}\left(g_\beta, P_{\beta^2 t}^{\alpha\beta,\rho} f_\beta\right)_{\alpha\beta}\\ &= -\beta^{2-d}\mathcal{E}^{\alpha\beta,\rho}\left(g_\beta, P_{\beta^2 t}^{\alpha\beta,\rho} f_\beta\right) = -\mathcal{E}^{\alpha,\rho}\left(g, u_t\right).\end{aligned}$$

Hence, $u_t = P_t^{\alpha,\rho} f$, and so we are done. Note that, as a consequence of the symmetry and Jensen's inequality, we know that

$$\left\|P_t^{\alpha,\rho} f\right\|_{\alpha,p} \le \|f\|_{\alpha,p} \quad \text{for all } t \in (0,\infty) \text{ and } p \in [1,\infty]. \tag{1.6}$$

We turn next to the derivation of pointwise estimates on $p^{\alpha,\rho}(t,x,y)$, and for this purpose we will need the following Nash inequality (cf. [1]).

1.7 Lemma. – *There is a constant $C \in [1,\infty)$, depending only on d, with the property that*

$$\|f\|_{1,2}^{2+\frac{4}{d}} \le C\mathcal{E}^{1,\rho}(f,f)\|f\|_{1,1}^{\frac{4}{d}}, \quad f \in L^1\left(\mathbb{Z}^d;\mathbb{R}\right). \tag{1.8}$$

Proof. – We begin by noticing that, since $\|f\|_{1,2} \le \|f\|_{1,1}$, we may and will assume that $0 < \mathcal{E}^{1,\rho}(f,f) \le \|f\|_{1,1}^2$. Next, set

$$\hat{f}(\boldsymbol{\xi}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} f(\mathbf{k})\exp(\sqrt{-1}\,2\pi(\boldsymbol{\xi},\mathbf{k})_{\mathbb{R}^d}), \quad \boldsymbol{\xi} \in [0,1]^d.$$

By Parseval's identity, for each $r \in (0,1]$:

$$\|f\|_{1,2}^2 = \int_{[0,1]^d}\left|\hat{f}(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} = I(r) + J(r),$$

$$\text{where } I(r) = \int_{\Gamma(r)}\left|\hat{f}(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} \text{ and } J(r) = \int_{[0,1]^d\setminus\Gamma(r)}\left|\hat{f}(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi}$$

with $\Gamma(r)$ being the set of $\boldsymbol{\xi} \in [0,1]^d$ with the property that, for each $1 \le i \le d$, either $\xi_i \le \frac{r}{2}$ or $\xi_i \ge 1 - \frac{r}{2}$. Clearly, $I(r) \le r^d\|f\|_{1,1}$. At the same time,

$$\begin{aligned}\mathcal{E}^{1,\rho}(f,f) &\ge \sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{i=1}^{d}\left(f(\mathbf{k}+\mathbf{e}^i) - f(\mathbf{k})\right)^2\\ &= \sum_{i=1}^{d}\int_{[0,1]^d}\left(1-\cos\left(2\pi\xi_i\right)\right)\left|\hat{f}(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} \ge \gamma r^2 J(r),\end{aligned}$$

where

$$\gamma \equiv 4 \sup\left\{\frac{1-\cos(2\pi\xi)}{\xi^2} : |\xi| \le \frac{1}{2}\right\} > 0.$$

Hence, we know that

$$\|f\|_{1,2}^2 \le r^d \|f\|_{1,1}^2 + \frac{\mathcal{E}^{1,\rho}(f,f)}{\gamma r^2}, \quad r \in (0,1].$$

In particular, since we are assuming that $0 < \mathcal{E}^{1,\rho}(f,f) \le \|f\|_{1,1}^2$, we can take

$$r = \frac{\mathcal{E}^{1,\rho}(f,f)}{\|f\|_{1,1}^2}$$

and thereby obtain our result. □

1.9 LEMMA. – *Assume that,*

$$K(\rho) \equiv \sup_{\mathbf{k}\in\mathbb{Z}^d} \sum_{\mathbf{e}\in\mathbb{Z}^d} \rho(\mathbf{k},\mathbf{e})\|\mathbf{e}\|^2 e^{3\|\mathbf{e}\|^2} < \infty. \tag{1.10}$$

Then there exists a $C \in [1,\infty)$, *depending only on d, such that*

$$p^{\alpha,\rho}(t,\mathbf{x},\mathbf{y}) \le C e^{K(\rho)} V_\alpha\left(t^{\frac{1}{2}}\right)^{-1} \exp\left(-\frac{\|\mathbf{y}-\mathbf{x}\|}{t^{\frac{1}{2}} \vee \alpha}\right), \tag{1.11}$$

where

$$V_\alpha(r) \equiv \alpha^d \operatorname{card}\Big(\left\{\mathbf{k} \in \mathbb{Z}^d : \|\alpha\mathbf{k}\| \le r\right\}\Big).$$

Proof. – Observe that

$$\left(\frac{\alpha \vee r}{2}\right)^d \le V_\alpha(r) \le (\alpha \vee r)^d. \tag{1.12}$$

Thus, we need only prove (1.11) with $V_\alpha(t^{\frac{1}{2}})$ replaced by $\alpha \vee t^{\frac{1}{2}}$. In addition, by the second part of (1.5), it suffices for us to handle the case when $\alpha = 1$.

Because of (1.8), we can apply Theorem 3.25 in [1] to see that there is a C, depending only on d, such that

$$p^{1,\rho}(t,\mathbf{x},\mathbf{y}) \le C t^{-\frac{d}{2}} \exp\left(-\mathbf{D}\left(\frac{3t}{2};\mathbf{x},\mathbf{y}\right)\right),$$

where

$$\mathbf{D}(t;\mathbf{x},\mathbf{y}) \ge \sup\left\{(\boldsymbol{\xi},\mathbf{y}-\mathbf{x})_{\mathbb{R}^d} - t\Gamma(\boldsymbol{\xi})^2 : \boldsymbol{\xi} \in \mathbb{R}^d\right\} \text{ with}$$

$$\Gamma(\boldsymbol{\xi})^2 \equiv \sup\{\sum_{\mathbf{e}\in\mathbb{Z}^d} \rho(\mathbf{k},\mathbf{e})(\exp\left(\left|(\boldsymbol{\xi},\mathbf{e})_{\mathbb{R}^d}\right| - 1\right)^2 : \mathbf{k} \in \mathbb{Z}^d\}.$$

Thus, when $t \geq 1$, we get (1.11) by considering

$$\boldsymbol{\xi} = t^{-\frac{1}{2}} \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}$$

and using $\left(e^{|\eta|}-1\right)^2 \leq \eta^2 e^{2|\eta|}$. When $t \in (0,1]$, we have to argue somewhat differently. Namely, by (3.19) in [1], we know that, for any $\boldsymbol{\xi} \in \mathbb{R}^d$,

$$\|\psi_{-\xi}\, P_t^{1,\rho}\left(\psi_\xi f\right)\|_{1,2} \leq e^{t\Gamma(\boldsymbol{\xi})^2} \|f\|_{1,2} \quad \text{where } \psi_\xi(x) \equiv e^{(\boldsymbol{\xi},\mathbf{x})_{\mathbb{R}^d}}.$$

Hence, since $\|g\|_{1,\infty} \leq \|g\|_{1,2} \leq \|g\|_{1,1}$,

$$\|\psi_{-\xi}\, P_t^{1,\rho}\left(\psi_\xi f\right)\|_{1,\infty} \leq e^{t\Gamma(\boldsymbol{\xi})^2} \|f\|_{1,1},$$

and so

$$p^{1,\rho}(t,\mathbf{x},\mathbf{y}) \leq \exp(-\left(\boldsymbol{\xi},\mathbf{y}-\mathbf{x}\right)_{\mathbb{R}^d} + t\Gamma(\boldsymbol{\xi})^2).$$

Thus, we can now complete the proof by taking $\boldsymbol{\xi} = \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}$. □

1.13 LEMMA. – *There is a $C \in [1,\infty)$, depending only on d, such that (cf. (1.10))*

$$\alpha^d \sum_{\mathbf{l}\in\mathbb{Z}^d} \|\alpha\mathbf{l}-\alpha\mathbf{k}\|^2 p^{\alpha,\rho}(t,\alpha\mathbf{k},\alpha\mathbf{l}) \leq Ce^{K(\rho)}t. \tag{1.14}$$

Proof. – Again, by (1.5), we need only look at $\alpha = 1$. When $t \geq 1$, (1.14) with $\alpha = 1$ is an obvious consequence of (1.11) with $\alpha = 1$. When $t \in (0,1]$, we use the description of the right continuous Markov process $X(t)$ corresponding to $\{P_t^{1,\rho} : t > 0\}$ starting at $\mathbf{k}$. Namely, because this is a pure jump process, we know that

$$\|X(t) - X(0)\| \leq \sum_{s\in(0,t]} \|X(s) - X(s-)\|.$$

Thus, if τ_n denotes the time of the nth jump and $\Delta_n = \|X(\tau_n) - X(\tau_n-)\|\mathbf{1}_{[0,t]}(\tau_n)$, then

$$\begin{aligned}\mathbb{E}\left[\|X(t)-X(0)\|^2\right] &\leq \sum_{m,n=1}^{\infty} \mathbb{E}\left[\Delta_m\Delta_n\right] \\ &= \sum_{m=1}^{\infty} \mathbb{E}\left[\Delta_m^2\right] + 2 \sum_{1\leq m<n<\infty} \mathbb{E}\left[\Delta_m\Delta_n\right].\end{aligned}$$

But, by the standard theory of jump Markov processes (cf. Chapter 3 of [5]), the number of jumps is a Poisson-like process with rate dominated by

$$\sup_{\mathbf{k}\in\mathbb{Z}^d}\sum_{\mathbf{e}\in\mathbb{Z}^d}\rho(\mathbf{k},\mathbf{e}),$$

form which it is easy to estimate the preceding by, respectively,

$$t\sup_{\mathbf{k}\in\mathbb{Z}^d}\sum_{\mathbf{e}\in\mathbb{Z}^d}\rho(\mathbf{k},\mathbf{e})\|\mathbf{e}\|^2 \quad\text{and}\quad t^2\Big(\sup_{\mathbf{k}\in\mathbb{Z}^d}\sum_{\mathbf{e}\in\mathbb{Z}^d}\rho(\mathbf{k},\mathbf{e})\|\mathbf{e}\|\Big)^2.$$

Hence, because $t \le 1$, we are done. $\square$

From (1.14) it is immediate that there is an R, depending only on d and $K(\rho)$, with the property that

$$\alpha^d \sum_{\|\alpha\mathbf{l}-\alpha\mathbf{k}\|\le Rt^{\frac{1}{2}}} p^{\alpha,\rho}(t,\alpha\mathbf{k},\alpha\mathbf{l}) \ge \frac{1}{2} \quad\text{for each } \mathbf{k}\in\mathbb{Z}^d.$$

In particular, by the Chapman-Kolmogorov equation, symmetry, and Schwarz's inequality:

$$\begin{aligned}
p^{\alpha,\rho}(2t,\alpha\mathbf{k},\alpha\mathbf{k}) &= \alpha^d\sum_{\mathbf{l}\in\mathbb{Z}^d}p^{\alpha,\rho}(t,\alpha\mathbf{k},\alpha\mathbf{l})^2\\
&\ge \alpha^d \sum_{\|\alpha\mathbf{l}-\alpha\mathbf{k}\|\le Rt^{\frac{1}{2}}} p^{\alpha,\rho}(t,\alpha\mathbf{k},\alpha\mathbf{l})^2\\
&\ge V_\alpha\big(Rt^{\frac{1}{2}}\big)^{-1}\Big(\sum_{\|\alpha\mathbf{l}-\alpha\mathbf{k}\|\le Rt^{\frac{1}{2}}} p^{\alpha,\rho}(t,\alpha\mathbf{k},\alpha\mathbf{l})\Big)^2\\
&\ge \frac{1}{4}V_\alpha\big(Rt^{\frac{1}{2}}\big)^{-1}.
\end{aligned}$$

Thus, there is an $\epsilon\in(0,1]$, depending only on d and $K(\rho)$, such that (cf. (1.12))

$$p^{\alpha,\rho}(t,\alpha\mathbf{k},\alpha\mathbf{k}) \ge \frac{\epsilon}{V_\alpha(t^{\frac{1}{2}})}. \tag{1.15}$$

Our next goal is to show that the on diagonal estimate in (1.15) extends to a neighborhood of the diagonal. Namely, we want to show that there is an $\epsilon > 0$, depending only on d and $K(\rho)$, such that

$$\begin{gathered}
p^{\alpha,\rho}(t,\alpha\mathbf{k},\alpha\mathbf{l}) \ge \frac{\epsilon}{V_\alpha(t^{\frac{1}{2}})}\\
\text{for } \alpha\in(0,1] \text{ and } (t,\mathbf{k},\mathbf{l})\in(0,\infty)\times\mathbb{Z}^d\times\mathbb{Z}^d\\
\text{with } \|\alpha\mathbf{k}-\alpha\mathbf{l}\|\le 2t^{\frac{1}{2}}.
\end{gathered}\tag{1.16}$$

Our derivation is based on the ideas of Nash, as interpreted in [4]. We begin with a number of observations.

1.17 LEMMA. – (1.16) *will follow from the existence of an* ϵ, *depending only on* d *and* $K(\rho)$, *such that*

$$p^{\alpha,\rho}(1,\mathbf{0},\alpha\mathbf{l}) \geq \epsilon \quad \textit{for} \quad \alpha \in (0,2] \textit{ and } \|\alpha\mathbf{l}\| \leq 2. \tag{1.18}$$

Proof. – In the first place, notice that our hypotheses are translation invariant. Hence, there is no loss in generality when we take $\mathbf{k} = \mathbf{0}$. Further, if we know (1.18) and $\|\alpha\mathbf{l}\| \leq 2t^{\frac{1}{2}}$, then either $\mathbf{l} = \mathbf{0}$ and (1.15) applies, or $\alpha \leq 2t^{\frac{1}{2}}$. In the latter case, take $\beta = t^{-\frac{1}{2}}$, note that $\alpha\beta \leq 2$, and apply the second part of (1.5) and (1.12) to see that (1.16) follows from (1.18) with α replaced by $\alpha\beta$. □

The proof of (1.18) depends on the following Poincaré inequality.

1.19 LEMMA. – *Let* $U \in C^\infty(\mathbb{R};[0,\infty))$ *have the properties that* $\int e^{-2U(\xi)}\,d\xi = 1$ *and* $U(\xi) = |\xi|$ *for all sufficiently large* $|\xi|$. *For* $\alpha \in (0,2]$, *define*

$$g_\alpha(\mathbf{l}) = \alpha^{-d}\prod_{i=1}^{d}\int_{\alpha \mathbf{l}_i}^{\alpha(\mathbf{l}_i+1)} e^{-2U(\xi_i)}\,d\xi_i, \quad \mathbf{l} \in \mathbb{Z}^d.$$

Then there exists a $\lambda > 0$ *such that*

$$\lambda\Big\langle \big(f - \langle f\rangle_{g_\alpha}\big)^2\Big\rangle_{g_\alpha} \leq \alpha^{d-2}\sum_{\mathbf{l}\in\mathbb{Z}^d} g_\alpha(\mathbf{l})\sum_{i=1}^{d}\big(f(\mathbf{l}+\mathbf{e}^i) - f(\mathbf{l})\big)^2, \tag{1.20}$$

where

$$\langle\psi\rangle_{g_\alpha} \equiv \alpha^d \sum_{\mathbf{l}\in\mathbb{Z}^d}\psi(\alpha\ell)g_\alpha(\ell)$$

and $\mathbf{e}^i$ *is the element of* $\mathbb{Z}^d$ *whose* j*th component is* 1 *if* $i = j$ *and* 0 *otherwise.*

Proof. – Because of the product structure, we may and will restrict our attention to the case when $d = 1$.

We begin by showing that

$$\begin{gathered}\mu\int_{\mathbb{R}}\big(f(\xi) - \langle f\rangle\big)^2 e^{-2U(\xi)}\,d\xi \leq \int_{\mathbb{R}} f'(\xi)^2 e^{-2U(\xi)}\,d\xi,\\ \text{where } \langle f\rangle \equiv \int f(\xi)e^{-2U(\xi)}\,d\xi,\end{gathered} \tag{1.21}$$

for some $\mu > 0$. To this end, note that (1.21) is equivalent to

$$\int_{\mathbb{R}}\Big(\psi(\xi) - \big(\psi, e^{-U}\big)_{L^2(\mathbb{R})}\Big)^2\,d\xi \leq \int_{\mathbb{R}}\Big(\psi'(\xi)^2 + V(\xi)\psi(\xi)^2\Big)\,d\xi,$$

where $V \equiv (U')^2 - U''$. Thus, if $H \equiv -\frac{d^2}{d\xi^2} + V$, then (1.21) is equivalent to

$$\text{spec}(H) \subseteq \{0\} \cup [\mu, \infty). \tag{1.22}$$

But, because $V - 1$ has compact support, H is a compact perturbation of $-\frac{d^2}{d\xi^2} + 1$, and therefore the essential spectrum of H coincides with $[1, \infty)$, the essential spectrum of $-\frac{d^2}{d\xi^2} + 1$. On the other hand, e^{-U} is an eigenfunction of H with eigenvalue 0. Hence, 0 must be an isolated eigenvalue of H, and so (1.22) will follow once we show that it is a simple eigenvalue. However, because it lies at the bottom of spec(H), a familiar argument, based on the variational characterization of the bottom of the spectrum, shows that 0 is indeed simple.

Given (1.21), we proceed as follows. For f on $\alpha\mathbb{Z}$, extend f to $\mathbb{R}$ by linear interpolation. Thus,

$$f(\xi) = \frac{\alpha(k+1) - t}{\alpha} f(\alpha k) + \frac{t - \alpha k}{\alpha} f\big(\alpha(k+1)\big) \quad \text{for } t \in \big[\alpha k, \alpha(k+1)\big].$$

Then (remember that $d = 1$)

$$\begin{aligned}
\alpha^{-1} \sum_{\ell \in \mathbb{Z}} (f\big(\alpha(\ell+1)\big) - f(\alpha\ell))^2 g_\alpha(\ell) &= \int_{\mathbb{R}} f'(\xi)^2 e^{-2U(\xi)}\, d\xi \\
&\geq \mu \int_{\mathbb{R}} \big(f(\xi) - \langle f \rangle\big)^2 e^{-2U(\xi)}\, d\xi \\
&= \frac{\mu}{2} \iint_{\mathbb{R}^2} \big(f(\eta) - f(\xi)\big)^2 e^{-2U(\xi) - 2U(\eta)}\, d\xi d\eta \\
&\geq \frac{\mu\alpha^2}{2} \sum_{k,\ell \in \mathbb{Z}} \big(f(\alpha k) - f(\alpha\ell)\big)^2 g_\alpha(k) g_\alpha(\ell) \\
&\quad - \mu\alpha^2 \sum_{k,\ell \in \mathbb{Z}} \big|f(\alpha k) - f(\alpha\ell)\big| \big|f\big(\alpha(k+1)\big) - f(\alpha k)\big| g_\alpha(k) g_\alpha(\ell) \\
&\geq \frac{\mu\alpha^2}{4} \sum_{k,\ell \in \mathbb{Z}} \big(f(\alpha k) - f(\alpha\ell)\big)^2 g_\alpha(k) g_\alpha(\ell) \\
&\quad - 4\mu\alpha \sum_{k \in \mathbb{Z}} (f\big(\alpha(k+1)\big) - f(\alpha k))^2 g_\alpha(\mathbf{k}).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\alpha^{-1} \sum_{k \in \mathbb{Z}} (f\big(\alpha(k+1)\big) - f(\alpha k))^2 g_\alpha(k) \\
&\quad \geq \frac{\mu}{2(1 + 4\mu\alpha^2)} \alpha \sum_{k \in \mathbb{Z}} (f(\alpha k) - \langle f \rangle_{g_\alpha})^2 g_\alpha(k). \qquad \square
\end{aligned}$$

The proof of the following lemma depends on the inequality

$$\left(\frac{d}{b}-\frac{c}{a}\right)(b-a) \le -\frac{c\wedge d}{2}(\log b-\log a)^2+\frac{(d-c)^2}{2c\wedge d} \tag{1.23}$$

for all positive numbers a, , b , c and d. To verify this, first note that, by homogeneity, it suffices to treat the case when $a = 1 = c \wedge d$. In addition, by using the transformation $b \rightsquigarrow \frac{1}{b}$, one can reduce to the case when $c \le d$. In other words, (1.23) comes down to checking that

$$\left(\log x\right)^2 \le 2\left(\frac{d}{x}-1\right)(1-x)+(d-1)^2$$

for all $x \in (0,\infty)$ and $d \ge 1$. Finally, observe that, by Schwarz's inequality,

$$\left(\log x\right)^2 \le \frac{(x-1)^2}{x},$$

and so it suffices to see that

$$\frac{(x-1)^2}{x} \le 2\left(\frac{d}{x}-1\right)(1-x)+(d-1)^2.$$

When $x \in (0,1]$, this is trivial. When $x > 1$,

$$2\left(\frac{d}{x}-1\right)(1-x) = 2(d-1)\frac{1-x}{x}+2\frac{(1-x)^2}{x} \ge \frac{(x-1)^2}{x}-(d-1)^2.$$

1.24 LEMMA. – *There is an $\epsilon \in (0,1)$, depending only on d and $K(\rho)$, such that*

$$\alpha^d \sum_{\mathbf{l}\in\mathbb{Z}^d} \log\left(p^{\alpha,\rho}\left(\frac{1}{2},\alpha\mathbf{k},\alpha\mathbf{l}\right)\right)g_\alpha(\mathbf{l}) \ge \frac{1}{2}\log\epsilon \quad \textit{for all } \alpha \in (0,2] \textit{ and } \|\alpha\mathbf{k}\| \le 2. \tag{1.25}$$

Proof. – Set $u_t(\mathbf{l}) = p^{\alpha,\rho}(t,\alpha\mathbf{k},\alpha\mathbf{l})$ and

$$G(t) = \alpha^d \sum_{\mathbf{l}\in\mathbb{Z}^d} \log\big(u_t(\mathbf{l})\big)g_\alpha(\mathbf{l}),$$

and note that, by Jensen's inequality, $G(t) \le 0$. Moreover, by (1.2),

$$G'(t) = \left(\frac{\partial u_t}{\partial t},\frac{g_\alpha}{u_t}\right)_\alpha = -\mathcal{E}^{\alpha,\rho}\left(u_t,\frac{g_\alpha}{u_t}\right).$$

Hence,

$$\begin{aligned} G'(t) &= -\alpha^{d-2} \sum_{\mathbf{l}\in\mathbb{Z}^d} \sum_{\mathbf{e}\in\mathbb{Z}^d} \left(\frac{g_\alpha(\mathbf{l}+\mathbf{e})}{u_t(\mathbf{l}+\mathbf{e})} - \frac{g_\alpha(\mathbf{l})}{u_t(\mathbf{l})} \right) \Big(u_t(\mathbf{l}+\mathbf{e}) - u_t(\mathbf{l}) \Big) \rho(\mathbf{l},\mathbf{e}) \\ &\geq \alpha^{d-2} \sum_{\mathbf{l}\in\mathbb{Z}^d} \sum_{i=1}^{d} \frac{g_\alpha(\mathbf{l}+\mathbf{e}^i) \wedge g_\alpha(\mathbf{l})}{2} \Big(\log u_t(\mathbf{l}+\mathbf{e}^i) - \log u_t(\mathbf{l}) \Big)^2 \\ &\quad - \alpha^{d-2} \sum_{\mathbf{l}\in\mathbb{Z}^d} \sum_{\mathbf{e}\in\mathbb{Z}^d} \frac{\left| g_\alpha(\mathbf{l}+\mathbf{e}) - g_\alpha(\mathbf{l}) \right|^2}{2 g_\alpha(\mathbf{l}+\mathbf{e}) \wedge g_\alpha(\mathbf{l})} \rho(\mathbf{l},\mathbf{e}), \end{aligned}$$

where, in the passage from the first line, we have used (1.23).

Because U' is bounded and $\alpha \in (0, 2]$, we can find $\gamma \in (0, 1]$ such that

$$\min_{1\leq i\leq d} g_\alpha\big(\mathbf{l}+\mathbf{e}^i\big) \geq \gamma g_\alpha(\mathbf{l})$$

and

$$\alpha^{d-2} \sum_{\mathbf{l}\in\mathbb{Z}^d} \sum_{\mathbf{e}\in\mathbb{Z}^d} \frac{\left| g_\alpha(\mathbf{l}+\mathbf{e}) - g_\alpha(\mathbf{l}) \right|^2}{2 g_\alpha(\mathbf{l}+\mathbf{e}) \wedge g_\alpha(\mathbf{l})} \rho(\ell,\mathbf{e}) \leq \frac{K(\rho)}{\gamma}.$$

Hence, there is an $M \in [1, \infty)$, depending only on d and $K(\rho)$, such that (cf. (1.20))

$$\begin{aligned} G'(t) &\geq \frac{1}{M} \alpha^{d-2} \sum_{\mathbf{l}\in\mathbb{Z}^d} \sum_{i=1}^{d} (\log u_t\big(\mathbf{l}+\mathbf{e}^i\big) - \log u_t(\mathbf{l}))^2 g_\alpha(\mathbf{l}) - M \\ &\geq \frac{\lambda}{M} \alpha^d \sum_{\mathbf{l}\in\mathbb{Z}^d} (\log u_t(\mathbf{l}) - G(t))^2 g_\alpha(\mathbf{l}) - M. \end{aligned}$$

Next, for $\sigma > 0$, set

$$A_t(\sigma) = \big\{ \mathbf{l} \in \mathbb{Z}^d : u_t(\mathbf{l}) \geq e^{-\sigma} \big\}.$$

Then, for each $\sigma > 0$,

$$\begin{aligned} &\alpha^d \sum_{\mathbf{l}\in\mathbb{Z}^d} (\log u_t(\mathbf{l}) - G(t))^2 g_\alpha(\mathbf{l}) \\ &\quad \geq \alpha^d \sum_{\mathbf{l}\in\mathbb{Z}^d} (-\log^- u_t(\mathbf{l}) - G(t))^2 g_\alpha(\mathbf{l}) \\ &\quad \geq \alpha^d \sum_{\mathbf{l}\in A_t(\sigma)} (\sigma + G(t))^2 g_\alpha(\mathbf{l}) \geq G(t)^2 \frac{\alpha^d}{2} \sum_{\mathbf{l}\in A_t(\sigma)} g_\alpha(\mathbf{l}) - \sigma^2. \end{aligned}$$

Thus, we now know that

$$G'(t) \geq \frac{\lambda |A_t(\sigma)|_{g_\alpha}}{2M} G(t)^2 - (M+\sigma^2), \quad \text{where } |A_t(\sigma)|_{g_\alpha} \equiv \alpha^d \sum_{\mathbf{l} \in A_t(\sigma)} g_\alpha(\mathbf{l}).$$

Finally, we use (1.14) to find an $r \in [1, \infty)$ such that

$$\alpha^d \sum_{\|\alpha \mathbf{l}\| \leq r} p^{\alpha,\rho}(t, \alpha\mathbf{k}, \alpha\mathbf{l}) \geq \frac{3}{4} \quad \text{for all } \alpha \in (0, 2],\ t \in (0, 1],\ \text{and } \|\alpha\mathbf{k}\| \leq 2.$$

In particular, if μ is the smallest value e^{-2U} takes on $[-r, r]$, then

$$\frac{3}{4} \leq \alpha^d \sum_{\|\alpha \mathbf{l}\| \leq r} u_t(\mathbf{l}) \leq e^{-\sigma} r^d + \frac{\|u_t\|_{\alpha,\infty}}{\mu^d} \left|A_t(\sigma)\right|_{g_\alpha}.$$

Thus, by taking $\sigma = \log 4r^d$ and using (1.11), we conclude first that $\left|A_t(\sigma)\right|_{g_\alpha} \geq \frac{\mu^d e^{-K(\rho)}}{2^{d+1}C}$ and then that there exists a $\delta \in (0, 1)$, depending only on d and $K(\rho)$, such that

$$G'(t) \geq \delta G(t)^2 - \delta^{-1} \quad \text{for all } \alpha \in (0, 2], \quad t \in \left[\frac{1}{4}, \frac{1}{2}\right], \text{ and } \|\alpha\mathbf{k}\| \leq 2. \tag{1.26}$$

To complete the proof starting from (1.26), suppose that $G(\frac{1}{2}) \leq -\frac{5}{2\delta}$. By (1.26),

$$G\left(\frac{1}{2}\right) - G(t) \geq -(2\delta)^{-1}, \quad t \in \left[\frac{1}{4}, \frac{1}{2}\right],$$

and so, again by (1.26),

$$G'(t) \geq \frac{\delta}{2} G(t)^2, \quad t \in \left[\frac{1}{4}, \frac{1}{2}\right].$$

But this means that

$$G\left(\frac{1}{2}\right)^{-1} \leq G\left(\frac{1}{2}\right)^{-1} - G\left(\frac{1}{4}\right)^{-1} \leq -\frac{\delta}{8};$$

and therefore $G(\frac{1}{2}) \geq -8\delta^{-1}$. In other words, we can take $\epsilon^{\frac{1}{2}} = \frac{1}{2}\exp(-8\delta^{-1})$. □

The passage from (1.25) to (1.18), and therefore (1.16), is easy. Namely, by the Chapman-Kolmogorov equation and symmetry,

$$p^{\alpha,\rho}(1,\mathbf{0},\alpha\mathbf{l}) \geq \alpha^d \sum_{\mathbf{j}} p^{\alpha,\rho}\left(\frac{1}{2},\mathbf{0},\alpha\mathbf{j}\right)\mathbf{p}^{\alpha,\rho}\left(\frac{\mathbf{1}}{\mathbf{2}},\alpha\mathbf{l},\alpha\mathbf{j}\right)\mathbf{g}_\alpha(\mathbf{j}).$$

Hence, by Jensen's inequality, if $\|\alpha\mathbf{l}\| \leq 2$, then (1.25) gives (1.18).

In the following statement, $\{P^{\alpha,\rho}_{\alpha\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^d\}$ denotes the Markov family of measures on $\Omega_\alpha \equiv D([0,\infty);\alpha\mathbb{Z}^d)$ for which transition probability function has $p^{\alpha,\rho}(t,\alpha\mathbf{k},\alpha\mathbf{l})$ as its density. Also, for each $r > 0$ and $\mathbf{k} \in \mathbb{Z}^d$, $\zeta^\alpha_{\mathbf{k},r} : \Omega_\alpha \longrightarrow [0,\infty]$ is given by

$$\zeta^\alpha_{\mathbf{k},r}(\omega) = \inf\{t \geq 0 : \|\omega(t) - \alpha\mathbf{k}\| \geq r\}.$$

Finally, for $\Gamma \subseteq \mathbb{R}^d$, $|\Gamma|_\alpha$ denotes α^d times the number of $\mathbf{k} \in \mathbb{Z}^d$ with $\alpha\mathbf{k} \in \Gamma$.

1.27 LEMMA. – *There is a $\theta \in (0,\frac{1}{2})$, depending only on d and $K(\rho)$, such that*

$$P^{\alpha,\rho}_{\alpha\mathbf{l}}(\omega(t) \in \Gamma \;\&\; \zeta^\alpha_{\mathbf{k},r}(\omega) > t) \geq \frac{\theta|\Gamma|_\alpha}{V_\alpha(t^{\frac{1}{2}})} \tag{1.28}$$

for all

$$\begin{gathered}\alpha \in (0,1],\ r \in (0,\infty),\ t \in \left(0,(2\theta r)^2\right],\\ (\mathbf{k},\mathbf{l}) \in \mathbb{Z}^d \times \mathbb{Z}^d \textit{ with } \|\alpha\mathbf{k} - \alpha\mathbf{l}\| \leq t^{\frac{1}{2}},\\ \textit{and } \Gamma \subseteq \alpha\mathbf{k} + \left[-t^{\frac{1}{2}}, t^{\frac{1}{2}}\right]^d \cap \alpha\mathbb{Z}^d.\end{gathered}$$

Proof. – First suppose that $t^{\frac{1}{2}} < \alpha$. Then $\mathbf{l} = \mathbf{k}$ and either Γ is empty, and there is nothing to do, or $|\Gamma|_\alpha = V_\alpha(t^{\frac{1}{2}})$. In the latter case,

$$\begin{aligned}P^{\alpha,\rho}_{\alpha\mathbf{l}}(\omega(t) \in \Gamma \;\&\; \zeta^\alpha_{\mathbf{k},r}(\omega) > t) &\geq P^{\alpha,\rho}_{\alpha\mathbf{k}}(\omega(s) = \mathbf{k} \text{ for } s \in [0,t])\\ &\geq \exp\left(-K(\rho)\frac{t}{\alpha^2}\right) \geq \exp(-K(\rho)).\end{aligned}$$

Thus, we need only worry about $t^{\frac{1}{2}} \geq \alpha$. But then, for $\|\alpha\mathbf{k} - \alpha\mathbf{j}\| \leq \mathbf{t}^{\frac{1}{2}}$:

$$\begin{aligned}&\alpha^{-d}P^{\alpha,\rho}_{\alpha\mathbf{l}}(\omega(t) = \alpha\mathbf{j} \;\&\; \zeta^\alpha_{\mathbf{k},\mathbf{r}}(\omega) > \mathbf{t})\\ &= p^{\alpha,\rho}(t,\alpha\mathbf{l},\alpha\mathbf{j}) - \mathbb{E}^{\mathbf{P}^{\alpha,\rho}_{\alpha\mathbf{l}}}[\mathbf{p}^{\alpha,\rho}(\mathbf{t} - \zeta^\alpha_{\mathbf{k},\mathbf{r}}(\omega), \omega(\zeta^\alpha_{\mathbf{k},\mathbf{r}}), \alpha\mathbf{j}),\ \zeta^\alpha_{\mathbf{k},\mathbf{r}}(\omega) < \mathbf{t}]\\ &\geq \frac{\epsilon}{V_\alpha(t^{\frac{1}{2}})} - Ce^{K(\rho)} \sup_{0<s\leq t} s^{-\frac{d}{2}} \exp\left(-\frac{(1-2\theta)r}{s^{\frac{1}{2}}}\right)\\ &\geq \frac{\epsilon}{V_\alpha(t^{\frac{1}{2}})} - Ce^{K(\rho)}t^{-\frac{d}{2}} \sup_{\sigma\geq 1} \sigma^d \exp\left(-\frac{(1-2\theta)\sigma}{2\theta}\right).\end{aligned}$$

Hence (cf. (1.12)) we can choose $\theta \in (0, \frac{1}{2})$ to achieve

$$P^{\alpha,\rho}_{\alpha \mathbf{l}}(\omega(t) = \alpha\mathbf{j} \;\&\; \zeta^{\alpha}_{\mathbf{k},\mathbf{r}}(\omega) > \mathbf{t}) \geq \frac{\theta\alpha^{\mathbf{d}}}{\mathbf{V}_{\alpha}(\mathbf{t}^{\frac{1}{2}})};$$

and (1.28) follows immediately after summing over $\alpha\mathbf{j} \in \mathbf{\Gamma}$. □

Our main interest in (1.28) is that it allows us to derive a Nash continuity estimate (cf. Theorem 1.31 below). However, in order to do so, we will have to replace (1.10) by the much more stringent condition:

$$\text{(1.29)} \qquad \begin{aligned} &\|\mathbf{e}\| > R(\rho) \Rightarrow \rho(\mathbf{k}, \mathbf{e}) = 0 \\ \text{and } &\sum_{\mathbf{e} \in \mathbb{Z}^d} \rho(\mathbf{k}, \mathbf{e}) \leq R(\rho) \quad \text{for all } \mathbf{k} \in \mathbb{Z}^d, \end{aligned}$$

where $R(\rho) \in [1, \infty)$.

In the following lemma,

$$Q^{\alpha}\big((T, \mathbf{k}); r\big) \equiv \big[T - r^2, T\big] \times \big\{\alpha\ell : \|\alpha\ell - \alpha\mathbf{k}\| \leq r\big\}$$
$$\text{for } r > 0 \text{ and } (T, \alpha\mathbf{k}) \in \big[r^2, \infty\big) \times \alpha\mathbb{Z}^d.$$

Also, given a function u on a set S, define

$$\operatorname{Osc}(u; S) = \sup\big\{|u(y) - u(x)| : x, y \in S\big\}.$$

1.30 LEMMA. – *Set (cf. Lemma 1.27 and* (1.29)*)* $\eta = \big(1 + R(\rho)\big)^{-1}\theta$ *and determine* $\sigma \in (0, \infty)$ *by* $\eta^{\sigma} = \big(1 - 2^{-d-3}\theta\big)$. *Then, for all* $\alpha \in (0, 1]$ *and bounded functions* f *on* $\alpha\mathbb{Z}^d$,

$$\operatorname{Osc}\big(u^{\alpha}; Q^{\alpha}\big((T, \mathbf{k}); \eta r\big)\big) \leq \eta^{\sigma}\operatorname{Osc}\big(u^{\alpha}; Q^{\alpha}\big((T, \mathbf{k}); r\big)\big),$$
$$r > 0 \text{ and } (T, \mathbf{k}) \in [r^2, \infty) \times \alpha\mathbb{Z}^d,$$

when $u^{\alpha}(t, \cdot) = P^{\alpha,\rho}_{t} f$.

Proof. – Without loss in generality, we will assume that $\mathbf{k} = \mathbf{0}$. Moreover, because there is nothing to do when $\eta r < \alpha$, we will assume that $\eta r \geq \alpha$.

For $\beta \in \big[0, T^{\frac{1}{2}}\big]$, set

$$M(\beta) = \max\big\{u^{\alpha}(s, \alpha\mathbf{j}) : (s, \alpha\mathbf{j}) \in Q^{\alpha}\big((T, \mathbf{0}); \beta\big)\big\}$$
$$\text{and } m(\beta) = \min\big\{u^{\alpha}(s, \alpha\mathbf{j}) : (s, \alpha\mathbf{j}) \in Q^{\alpha}\big((T, \mathbf{0}); \beta\big)\big\}.$$

Then $\operatorname{Osc}\Big(u^{\alpha}; Q^{\alpha}\big((T, \mathbf{0}); \beta\big)\Big) = M(\beta) - m(\beta)$. Next, set

$$r' = \frac{\eta r}{\theta} = \frac{r}{1 + R(\rho)}$$

and

$$\Gamma = \left\{ \alpha\mathbf{j} : \|\alpha\mathbf{j}\| \le \eta r \text{ and } u^\alpha\big(T - (2\eta r)^2, \alpha\mathbf{j}\big) \ge \frac{M(r) + m(r)}{2} \right\}.$$

Obviously, either $|\Gamma|_\alpha \ge \frac{1}{2} V_\alpha(\eta r)$ or $|\Gamma|_\alpha \le \frac{1}{2} V_\alpha(\eta r)$. If $|\Gamma|_\alpha \ge \frac{1}{2} V_\alpha(\eta r)$, take $w = u^\alpha - m(r)$. Given $T - (\eta r)^2 \le s \le T$, set $t = (2\eta r)^2 - T + s \in \big[(\eta r)^2, (2\eta r)^2\big]$, and note that, because $w \ge 0$ on $Q^\alpha\big((T, \mathbf{0}); r\big)$ and

$$\zeta^\alpha_{1,r'}(\omega) < \infty \Rightarrow \big\|\omega(\zeta^\alpha_{0,r'})\big\| \le r' + \alpha R(\rho) \le \frac{r + \theta r R(\rho)}{1 + R(\rho)} \le r$$

$$P^{\alpha,\rho}_{\alpha\ell} - \text{almost surely when } \|\alpha\mathbf{l}\| \le \eta r,$$

(1.28) implies that

$$\begin{aligned} w(s, \alpha\ell) &= \mathbb{E}^{P^{\alpha,\rho}_{\alpha\ell}}\Big[w\Big(s - \zeta^\alpha_{0,r'}(\omega) \wedge t, \omega\big(\zeta^\alpha_{0,r'} \wedge t\big)\Big)\Big] \\ &\ge \mathbb{E}^{P^{\alpha,\rho}_{\alpha\ell}}\Big[w\Big(T - (2\eta r)^2, \omega(t)\Big),\ \zeta^\alpha_{0,r'}(\omega) > t\Big] \\ &\ge \frac{M(r) - m(r)}{2} \frac{\theta |\Gamma|_\alpha}{V_\alpha(t^{\frac{1}{2}})} \\ &\ge \frac{\big(M(r) - m(r)\big)\theta}{4} \frac{V_\alpha(\eta r)}{V_\alpha(t^{\frac{1}{2}})} \ge \frac{\big(M(r) - m(r)\big)\theta}{2^{d+3}} \end{aligned}$$

for $\|\alpha\ell\| \le \eta r$. Thus, $m(\eta r) - m(r) \ge 2^{-d-3}\theta\big(M(r) - m(r)\big)$, which means that

$$\begin{aligned} M(\eta r) - m(\eta r) &\le M(r) - m(\eta r) \\ &\le \big(1 - 2^{-d-3}\theta\big)\big(M(r) - m(r)\big) = \eta^\sigma\big(M(r) - m(r)\big). \end{aligned}$$

When $|\Gamma|_\alpha \le \frac{1}{2} V_\alpha(\eta r)$, one takes $w = M(r) - u^\alpha$ and proceeds as above to get first that $M(r) - M(\eta r) \le \big(1 - 2^{-d-3}\eta\big)\big(M(r) - m(r)\big)$ and then the same estimate as we just arrived at. □

1.31 THEOREM. – *Assume that* (1.29) *holds for some* $R(\rho) \in [1, \infty)$, *and define* σ *as in Lemma 1.30. Then there is a* $B \in (0, \infty)$, *depending only on* d *and* $R(\rho)$, *such that, for all* $\alpha \in (0, 1]$ *and* $f \in L^\infty(\alpha\mathbb{Z}^d; \mathbb{R})$,

$$(1.32) \qquad \begin{aligned} &\big|\big[P^{\alpha,\rho}_t f\big](\alpha\mathbf{l}) - \big[P^{\alpha,\rho}_s f\big](\alpha\mathbf{k})\big| \\ &\quad \le B\|f\|_{\alpha,\infty}\left(\frac{|t - s|^{\frac{1}{2}} \vee \|\alpha\mathbf{k} - \alpha\mathbf{l}\|}{(t \wedge s)^{\frac{1}{2}}}\right)^\sigma. \end{aligned}$$

In particular, there is a C, depending only on d and $R(\rho)$, such that

$$(1.33)\quad \begin{aligned} &\left|p^{\alpha,\rho}(t',\alpha\mathbf{k}',\alpha\mathbf{l}') - p^{\alpha,\rho}(t,\alpha\mathbf{k},\alpha\mathbf{l})\right| \\ &\quad \le C\left(|t'-t|^{\frac{1}{2}} + \|\alpha\mathbf{k}-\alpha\mathbf{k}'\| + \|\alpha\mathbf{l}-\alpha\mathbf{l}'\|\right)^{\sigma}(t'\wedge t)^{-\frac{d+\sigma}{2}}. \end{aligned}$$

Proof. – Clearly (1.33) follows from (1.32) together with (1.11) and

$$p^{\alpha,\rho}(t,\alpha\mathbf{k},\alpha\mathbf{l}) = p^{\alpha,\rho}(t,\alpha\mathbf{l},\alpha\mathbf{k}) = \left[P^{\alpha,\rho}_{\frac{t}{2}} p^{\alpha,\rho}\left(\frac{t}{2},\cdot,\alpha\mathbf{l}\right)\right](\alpha\mathbf{k}).$$

To prove (1.32), set $T = t\wedge s$ and $r = \sqrt{(1-\eta^2)T}$. If

$$|t-s|^{\frac{1}{2}} \vee \|\alpha\ell - \alpha\mathbf{k}\| \ge r,$$

then (1.32) is easy. Otherwise, determine $n \ge 0$ so that

$$\eta^{n+1} \le \frac{|t-s|^{\frac{1}{2}} \vee \|\alpha\mathbf{l}-\alpha\mathbf{k}\|}{r} \le \eta^n.$$

Then, by repeated application of (1.30), one finds that

$$\begin{aligned} &\left|\left[P^{\alpha,\rho}_t f\right](\alpha\ell) - \left[P^{\alpha,\rho}_s f\right](\alpha\mathbf{k})\right| \\ &\quad \le \operatorname{Osc}\left(u^\alpha; Q^\alpha\left((T,\mathbf{0});\eta^n r\right)\right) \\ &\quad \le \eta^{n\sigma}\operatorname{Osc}\left(u^\alpha; Q^\alpha\left((T,\mathbf{0}); r\right)\right) \\ &\quad \le 2\eta^{n\sigma}\|f\|_{\alpha,\infty} \le 2\eta^{-\sigma}\|f\|_{\alpha,\infty}\left(\frac{|t-s|^{\frac{1}{2}}\vee\|\alpha k - \alpha\ell\|}{r}\right)^{\sigma}. \quad \square \end{aligned}$$

2. PRELIMINARY APPLICATIONS

Throughout this section we will be dealing with the following situation. For each $\alpha \in (0,1]$, we are given $\rho_\alpha : \mathbb{Z}^d \times \mathbb{Z}^d \longrightarrow [0,\infty)$. Our basic assumption is that the ρ_α's satisfy (1.1). In addition, we will assume that there is a $R \in [1,\infty)$ with the properties that

$$(2.1)\quad \begin{gathered} \|\mathbf{e}\| > R \Rightarrow \rho_\alpha(\mathbf{k},\mathbf{e}) = 0 \\ \text{and} \quad \sum_{\mathbf{e}\in\mathbb{Z}^d} \rho_\alpha(\mathbf{k},\mathbf{e}) \le R \quad \text{for all } \mathbf{k}\in\mathbb{Z}^d. \end{gathered}$$

2.2 LEMMA. – *Set (cf. 1.4)) $p^\alpha \equiv p^{\alpha,\rho_\alpha}$, and, for $\mathbf{x}\in\mathbb{R}^d$, use $[\mathbf{x}]_\alpha$ to denote the element of $\alpha\mathbb{Z}^d$ obtained by replacing each coordinate x_i with α times* *the integer part of* $\alpha^{-1}x_i$. *Then, for any sequence* $\{\alpha_n\}_1^\infty \subseteq (0,1]$ *which decreases to* 0 *there is a decreasing subsequence* $\{\alpha_{n'}\}$ *and a continuous* $q : (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ *such that*

$$(2.3)\quad \lim_{n'\to\infty} \sup_{t\in(0,\infty)} \sup_{\mathbf{x},\mathbf{y}\in\mathbb{R}^d} t^{\frac{d}{2}} e^{\beta|\mathbf{y}-\mathbf{x}|} |p^{\alpha_{n'}}(t, [\mathbf{x}]_{\alpha_{n'}}, [\mathbf{y}]_{\alpha_{n'}}) - q(t,\mathbf{x},\mathbf{y})| = 0,$$

where $\beta \in (0,\infty)$ *depends only on* d *and the number* R *in* (2.1). *In particular,*

(2.4)

$$\begin{aligned} q(s,\mathbf{x},\mathbf{y}) &= q(s,\mathbf{y},\mathbf{x}), \quad (s,\mathbf{x},\mathbf{y}) \in (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d, \\ \int_{\mathbb{R}^d} q(s,\mathbf{x},\boldsymbol{\xi})\, d\boldsymbol{\xi} &= 1, \quad (s,\mathbf{x}) \in (0,\infty)\times\mathbb{R}^d, \\ \lim_{s\searrow 0} \sup_{\mathbf{x}\in\mathbb{R}^d} \int_{|\boldsymbol{\xi}-\mathbf{x}|\geq r} q(s,\mathbf{x},\boldsymbol{\xi})\, d\boldsymbol{\xi} &= 0, \quad \mathbf{x}\in\mathbb{R}^d \textit{ and } r \in (0,\infty), \\ q(s+t,\mathbf{x},\mathbf{y}) &= \int_{\mathbb{R}^d} q(s,\mathbf{x},\boldsymbol{\xi}) q(t,\boldsymbol{\xi},\mathbf{y})\, d\boldsymbol{\xi}, \quad s,t\in(0,\infty) \textit{ and } \mathbf{x},\mathbf{y}\in\mathbb{R}^d. \end{aligned}$$

Proof. – Because of (1.11) and (1.32), there is essentially nothing to do except comment that the existence of a locally uniform limit is guaranteed by (1.32) and the Arzela-Ascoli compactness criterion and that the other statements in (2.3) and (2.4) follows from this combined with (1.11). □

From (2.4), it is clear that if Q_t is determined by

(2.5)

$$Q_t f(\mathbf{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{\xi}) q(t,\mathbf{x},\boldsymbol{\xi})\, d\boldsymbol{\xi}, \quad (t,\mathbf{x}) \in (0,\infty)\times\mathbb{R}^d \text{ and } f \in C_c(\mathbb{R}^d;\mathbb{R}),$$

then $\{Q_t : t > 0\}$ determines both a Markov semigroup on $C_b(\mathbb{R}^d;\mathbb{R})$ and a strongly continuous semigroup of self-adjoint contractions on $L^2(\mathbb{R}^d;\mathbb{R})$. Our goal now is to give criteria which will enable us to identify the semigroup $\{Q_t : t > 0\}$. Notice that if $\{Q_t : t > 0\}$ is uniquely determined, in the sense that it is independent of the subsequence selected, then there will be no need to pass to a subsequence in the statement of (2.3).

In order to carry out our program, we first impose the regularity condition that (cf. (2.1)), for each $r \in [1,\infty)$,

$$(2.6)\quad \lim_{\alpha\searrow 0} \sum_{\mathbf{e}\in\mathbb{Z}^d} \sup_{\|\alpha\mathbf{k}\|\leq r} \sup_{\|\mathbf{l}-\mathbf{k}\|\leq R} \left|\rho_\alpha(\mathbf{l},\mathbf{e}) - \rho_\alpha(\mathbf{k},\mathbf{e})\right| = 0.$$

Second, we define $a_\alpha : \alpha\mathbb{Z}^d \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ by

$$(2.7)\quad a_\alpha(\alpha\mathbf{k}) = \sum_{\mathbf{e}\in\mathbb{Z}^d} \rho_\alpha(\mathbf{k},\mathbf{e})\mathbf{e}\otimes\mathbf{e}, \quad \mathbf{k}\in\mathbb{Z}^d.$$

Finally, we suppose that there exists a Borel measurable $a : \mathbb{R}^d \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ with the property that, for each $r > 0$:

$$\lim_{\alpha \searrow 0} \int_{\|\mathbf{x}\| \le r} \left| a_\alpha([\mathbf{x}]_\alpha) - a(\mathbf{x}) \right| dx = 0. \tag{2.8}$$

As a consequence of (1.1), (2.1), and (2.8), it is an easy matter to check that, without loss in generality, we may assume that $2I \le a(\mathbf{x}) \le RI$ for all $\mathbf{x} \in \mathbb{R}^d$. In particular, the general theory (cf. either [3] or [8]) of Dirichlet forms applies and says that there is a unique strongly continuous, Markov semigroup $\{\bar{P}^a_t : t > 0\}$ self-adjoint contractions on $L^2(\mathbb{R}^d; \mathbb{R})$ with the properties that, for each $f \in L^2(\mathbb{R}^d; \mathbb{R})$, $t \in (0, \infty) \rightsquigarrow \bar{P}^a_t f$ is the unique continuous function $t \in (0, \infty) \longmapsto u_t \in W^1_2(\mathbb{R}^d; \mathbb{R})$ (the Sobolev space of square integrable functions with one square integrable derivative) such that

$$\frac{d}{dt}(\varphi, u_t)_0 = - \int_{\mathbb{R}^d} (\nabla \varphi, a \nabla u_t)_{\mathbb{R}^d} \, dx, \quad \varphi \in C^\infty_c(\mathbb{R}^d; \mathbb{R}), \\ \text{and} \quad u_t \longrightarrow f \text{ in } L^2(\mathbb{R}^d; \mathbb{R}). \tag{2.9}$$

In fact (cf. [9]) there is a unique $p^a \in C((0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d; (0, \infty))$ with the properties that

$$\begin{aligned} p^a(s, \mathbf{x}, \mathbf{y}) = p^a(s, \mathbf{y}, \mathbf{x}), &\quad (s, \mathbf{x}, \mathbf{y}) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\ \int_{\mathbb{R}^d} p^a(s, \mathbf{x}, \boldsymbol{\xi}) \, d\boldsymbol{\xi} = 1, &\quad (s, \mathbf{x}) \in (0, \infty) \times \mathbb{R}^d, \\ \lim_{s \searrow 0} \sup_{\mathbf{x} \in \mathbb{R}^d} \int_{|\boldsymbol{\xi} - \mathbf{x}| \ge r} p^a(s, \mathbf{x}, \boldsymbol{\xi}) \, d\boldsymbol{\xi} = 0, &\quad \mathbf{x} \in \mathbb{R}^d \text{ and } r \in (0, \infty), \\ p^a(s+t, \mathbf{x}, \mathbf{y}) = \int_{\mathbb{R}^d} p^a(s, \mathbf{x}, \boldsymbol{\xi}) \, p^a(t, \boldsymbol{\xi}, \mathbf{y}) \, d\boldsymbol{\xi}, &\quad s, t \in (0, \infty), \ \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, \end{aligned} \tag{2.10}$$

and, for each $f \in C_c(\mathbb{R}^d; \mathbb{R})$,

$$\bar{P}^a_t f = P^a_t f \quad \text{a.e.} \quad \text{where } P^a_t f(\mathbf{x}) \equiv \int_{\mathbb{R}^d} f(\mathbf{y}) p^a(t, \mathbf{x}, \mathbf{y}) \, dy. \tag{2.11}$$

What we want to show is that, after the addition of (2.6) and (2.8), we can show that the q in (2.3) must be p^a, and for this purpose we will need a few preparations. Namely, given $\psi_\alpha : \alpha \mathbb{Z}^d \longmapsto \mathbb{R}$, let $\bar{\psi}_\alpha$ denote the function on $\mathbb{R}^d$ obtained by baricentric extension. That is, for each $\mathbf{k} \in \mathbb{Z}^d$, set

$$Q_\alpha(\mathbf{k}) = \prod_{j=1}^d [\alpha k_j, \alpha(k_j + 1)), \tag{2.12}$$

and define $\bar{\psi}_\alpha$ on $Q_\alpha(\mathbf{k})$ to be the multilinear extension of ψ_α restricted to the extreme points of $\overline{Q_\alpha(\mathbf{k})}$. The crux of our argument is contained in the following lemma.

2.13 LEMMA. – *For each $\alpha \in (0,1]$, let $\psi_\alpha : \alpha\mathbb{Z}^d \longrightarrow \mathbb{R}$ be given, and assume that*

$$(2.14) \qquad \sup_{\alpha\in(0,1]} \alpha^{d-2} \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{\|\mathbf{e}\|=1} \left(\psi_\alpha\big(\alpha(\mathbf{k}+\mathbf{e})\big) - \psi_\alpha\big(\alpha\mathbf{k}\big)\right)^2 < \infty.$$

If $\bar{\psi}_\alpha$ converges to ψ in $L^2(\mathbb{R}^d;\mathbb{R})$, then $\psi \in W_2^1(\mathbb{R}^d;\mathbb{R})$ and

$$(2.15) \qquad \int_{\mathbb{R}^d} (\nabla\varphi, a\nabla\psi)_{\mathbb{R}^d}\, d\mathbf{x} = \lim_{\alpha\searrow 0} \mathcal{E}^\alpha(\varphi,\psi_\alpha), \quad \varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}),$$

where we have taken (cf. (1.2)) $\mathcal{E}^\alpha = \mathcal{E}^{\alpha,\rho_\alpha}$.

Proof. – We begin by observing that, for $\mathbf{x} \in Q_\alpha(\mathbf{k})$,

$$\left\|\nabla\bar{\psi}_\alpha(\mathbf{x})\right\| \leq \alpha^{-1}\max\left\{\left|\psi_\alpha(\alpha\mathbf{l}') - \psi_\alpha(\alpha\mathbf{l})\right| : \alpha\mathbf{l},\alpha\mathbf{l}' \in Q_\alpha(\mathbf{k}) \text{ with } \|\mathbf{l}'-\mathbf{l}\| = 1\right\}.$$

In particular, this means that

$$\left\|\left|\nabla\bar{\psi}_\alpha\right|\right\|_{L^2(\mathbb{R}^d;\mathbb{R})} \leq 2^d\alpha^{d-2} \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{\|\mathbf{e}\|=1} \Big(\psi_\alpha\big(\alpha(\mathbf{k}+\mathbf{e})\big) - \psi_\alpha\big(\alpha\mathbf{k}\big)\Big)^2,$$

and so, by (2.14), the L^2-norm of $\left|\nabla\bar{\psi}_\alpha\right|$ is bounded uniformly in $\alpha \in (0,1]$. Hence, from the elementary functional analysis of the space $W_2^1(\mathbb{R}^d;\mathbb{R})$, we know that $\psi \in W_2^1(\mathbb{R}^d;\mathbb{R})$ and that $\nabla\bar{\psi}_\alpha$ tends weakly in $L^2(\mathbb{R}^d;\mathbb{R}^d)$ to $\nabla\psi$; and, after combining this observation with (2.8), we see that it suffices for us to check that

$$\lim_{\alpha\searrow 0}\left|\mathcal{E}^\alpha(\varphi,\psi_\alpha) - \int_{\mathbb{R}^d}\Big(\nabla\varphi(\mathbf{x}), a_\alpha\big([\mathbf{x}]_\alpha\big)\nabla\bar{\psi}_\alpha(\mathbf{x})\Big)_{\mathbb{R}^d} d\mathbf{x}\right| = 0.$$

Moreover, because $\varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$, it is clear from (2.14) that the preceding can be replaced by

$$(2.16) \qquad \lim_{\alpha\searrow 0}\left|\alpha^{d-1}\sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{\mathbf{e}\in\mathbb{Z}^d} c_\alpha(\mathbf{k},\mathbf{e})\delta_{\alpha\mathbf{e}}\psi_\alpha - \int_{\mathbb{R}^d}\Big(\nabla\varphi(\mathbf{x}), a_\alpha\big([\mathbf{x}]_\alpha\big)\nabla\bar{\psi}_\alpha(\mathbf{x})\Big)_{\mathbb{R}^d} d\mathbf{x}\right| = 0,$$

where

$$c_\alpha(\mathbf{k},\mathbf{e}) \equiv \rho_\alpha(\mathbf{k},\mathbf{e})\big(\mathbf{e},\nabla\varphi(\alpha\mathbf{k})\big)_{\mathbb{R}^d},$$

and we have introduced the notation $\delta_{\mathbf{v}}\psi(\mathbf{x}) \equiv \psi(\mathbf{x}+\mathbf{v}) - \psi(\mathbf{x})$.

To prove (2.16), we define a rectilinear, nearest-neighbor *path* from $\mathbf{k}$ to $\mathbf{k}+\mathbf{e}$. Namely, for $(\mathbf{k},\mathbf{e}) \in \mathbb{Z}^d \times \mathbb{Z}^d$ and $1 \le j \le d$, let

$$P_j(\mathbf{k},\mathbf{e}) = \begin{cases} \emptyset & \text{if } e_j = 0, \\ \{\ell : \ell_i = k_i + e_i \text{ for } i < j,\ k_j \le \ell_j < k_j + e_j, \\ \qquad \text{and } \ell_i = k_i \text{ for } i > j\} & \text{if } e_j \ge 1 \\ \{\ell : \ell_i = k_i + e_i \text{ for } i < j,\ k_j - e_j \le \ell_j < k_j, \\ \qquad \text{and } \ell_i = k_i \text{ for } i > j\} & \text{if } e_j \le -1. \end{cases}$$

Then,

$$\delta_{\alpha\mathbf{e}}\psi_\alpha(\alpha\mathbf{k}) = \sum_{j=1}^d \operatorname{sgn}(e_j) \sum_{\ell\in P_j(\mathbf{k},\mathbf{e})} \delta_{\alpha,j}\psi_\alpha(\alpha\ell),$$

where

$$\delta_{\alpha,j}\psi(\mathbf{x}) \equiv \psi\big((x_1,\dots,x_{j-1},x_j+\alpha,x_{j+1},\dots,x_d)\big) - \psi(\mathbf{x});$$

and so

$$\begin{aligned} &\alpha^{d-1}\sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{\mathbf{e}\in\mathbb{Z}^d} c_\alpha(\mathbf{k},\mathbf{e})\delta_{\alpha\mathbf{e}}\psi_\alpha(\alpha\mathbf{k}) \\ &= \alpha^{d-1}\sum_{\mathbf{l}\in\mathbb{Z}^d}\sum_{\mathbf{e}\in\mathbb{Z}^d} c_\alpha(\mathbf{l},\mathbf{e})\sum_{j=1}^d e_j\delta_{\alpha,j}\psi_\alpha(\alpha\mathbf{l}) \\ &\quad + \alpha^{d-1}\sum_{\mathbf{l}\in\mathbb{Z}^d}\sum_{\mathbf{e}\in\mathbb{Z}^d}\sum_{j=1}^d \operatorname{sgn}(e_j)\delta_{\alpha,j}\psi_\alpha(\mathbf{l}) \sum_{\{\mathbf{k}:\mathbf{l}\in P_j(\mathbf{k},\mathbf{e})\}} \big(c_\alpha(\mathbf{k},\mathbf{e}) - c_\alpha(\mathbf{l},\mathbf{e})\big). \end{aligned}$$

Thus, because of (2.6), (2.14), and the fact that $\varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$, we see that (2.16) reduces to

$$\lim_{\alpha\searrow 0}\left|\alpha^{d-1}\sum_{\mathbf{l}\in\mathbb{Z}^d}\sum_{\mathbf{e}\in\mathbb{Z}^d} c_\alpha(\mathbf{l},\mathbf{e})\sum_{j=1}^d e_j\delta_{\alpha,j}\psi_\alpha(\alpha\mathbf{l}) - \int_{\mathbb{R}^d}\Big(\nabla\varphi(\mathbf{x}), a_\alpha\big([\mathbf{x}]_\alpha\big)\nabla\bar{\psi}_\alpha(\mathbf{x})\Big)_{\mathbb{R}^d} d\mathbf{x}\right| = 0. \tag{2.17}$$

Finally, to prove (2.17), observe that

$$\int_{Q_\alpha(\mathbf{k})} \frac{\partial\bar{\psi}_\alpha}{\partial x_j}(\mathbf{x})\,d\mathbf{x} = \left(\frac{\alpha}{2}\right)^{d-1} \sum_{\mathbf{l}\in Q(\mathbf{k}|j)} \delta_{\alpha,j}\psi_\alpha(\alpha\mathbf{l}),$$

where

$$Q(\mathbf{k}|j) \equiv \{\mathbf{k} : \mathbf{l}_j = k_j \text{ and } \mathbf{l}_i \in \{k_i, k_i + 1\} \text{ for } i \neq j\}.$$

Hence, after elementary manipulation, we see that

$$\begin{aligned}
&\alpha^{d-1} \sum_{\mathbf{l}\in\mathbb{Z}^d} \sum_{\mathbf{e}\in\mathbb{Z}^d} c_\alpha(\mathbf{l},\mathbf{e}) \sum_{j=1}^{d} e_j \delta_{\alpha,j} \psi_\alpha(\alpha\mathbf{l}) \\
&= \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{\mathbf{e}\in\mathbb{Z}^d} c_\alpha(\mathbf{k},\mathbf{e}) \int_{Q_\alpha(\mathbf{k})} (\mathbf{e}, \nabla\bar{\psi}_\alpha(\mathbf{x}))_{\mathbb{R}^d}\, d\mathbf{x} \\
&+ \left(\frac{\alpha}{2}\right)^{d-1} \sum_{\mathbf{l}\in\mathbb{Z}^d} \sum_{\mathbf{e}\in\mathbb{Z}^d} \sum_{j=1}^{d} e_j \delta_{\alpha,j} \psi_\alpha(\alpha\mathbf{l}) \sum_{\{\mathbf{k}:\mathbf{l}\in Q(\mathbf{k}|j)\}} \big(c_\alpha(\mathbf{l},\mathbf{e}) - c_\alpha(\mathbf{k},\mathbf{e})\big).
\end{aligned}$$

Thus, because of (2.6), all that remains is to note that

$$\begin{aligned}
\lim_{\alpha\searrow 0} \Bigg| &\sum_{\mathbf{l}\in\mathbb{Z}^d} \sum_{\mathbf{e}\in\mathbb{Z}^d} c_\alpha(\mathbf{l},\mathbf{e}) \int_{Q_\alpha(\mathbf{l})} (\mathbf{e}, \nabla\bar{\psi}_\alpha(\mathbf{x}))_{\mathbb{R}^d}\, d\mathbf{x} \\
&- \int_{\mathbb{R}^d} \Big(\nabla\varphi(\mathbf{x}), a_\alpha\big([\mathbf{x}]_\alpha\big)\nabla\bar{\psi}_\alpha(\mathbf{x})\Big)_{\mathbb{R}^d} d\mathbf{x}\Bigg| = 0. \quad \square
\end{aligned}$$

By combining Lemma 2.2 and 2.13, we get the our main result.

2.18 THEOREM. – *Let* $a : \mathbb{R}^d \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ *be a Borel measurable, symmetric matrix valued function which satisfies*

$$2I \leq a(\mathbf{x}) \leq RI, \quad \mathbf{x} \in \mathbb{R}^d, \tag{2.19}$$

for some $R \in [1, \infty)$, *and determine* $p^a : (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow (0,\infty)$ *accordingly, as in* (2.11). *Also, for each* $\alpha \in (0,1]$, *let* $\rho_\alpha : \alpha\mathbb{Z}^d \times \mathbb{Z}^d \longrightarrow [0,\infty)$ *be a function which satisfies the conditions in* (1.1) *and* (2.1). *If, in addition,* (2.6) *and* (2.8) *hold, then*

$$\lim_{\alpha\searrow 0} \sup_{t\in(0,\infty)} \sup_{\mathbf{x},\mathbf{y}\in\mathbb{R}^d} t^{\frac{d}{2}} e^{\beta|\mathbf{y}-\mathbf{x}|} \Big| p^\alpha\big(t, [\mathbf{x}]_\alpha, [\mathbf{y}]_\alpha\big) - p^a(t,\mathbf{x},\mathbf{y})\Big| = 0, \tag{2.20}$$

where $\beta \in (0,\infty)$ *depends only on d and R (cf.* (2.1) *and* (2.6)*).*

Proof. – In view of Lemma 2.2 and the discussion which follows its proof, we need only check that (cf. (2.3)) $q = p^a$. Moreover, starting from the characterization of $\{\bar{P}^a_t : t > 0\}$ given in (2.9) and using the continuity of p^a, one can apply standard soft arguments to see that it suffices for us to prove that, for each $f \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$ and $\lambda \in (0,\infty)$,

$$u_\lambda \in W_2^1(\mathbb{R}^d;\mathbb{R})$$

and

$$\lambda\big(\varphi, u_\lambda\big)_0 + \int_{\mathbb{R}^d} \big(\nabla\varphi, a\nabla u_\lambda\big)_{\mathbb{R}^d}\, d\mathbf{x} = \big(\varphi, f\big)_0, \quad \varphi \in C_c(\mathbb{R}^d;\mathbb{R}),$$

when

$$u_\lambda \equiv \int_0^\infty e^{-\lambda t} Q_t f\, dt.$$

To this end, define $u_{\lambda,\alpha} : \alpha\mathbb{Z}^d \longrightarrow \mathbb{R}$ by

$$u_{\lambda,\alpha} = \int_0^\infty e^{-\lambda t} P_t^{\alpha,\rho_\alpha} f\, dt.$$

From the spectral theorem, it is clear that

$$\mathcal{E}^\alpha\big(u_{\lambda,\alpha}, u_{\lambda,\alpha}\big) \le \frac{1}{e\lambda}\|f\|_{2,\alpha}^2.$$

Hence, because of (1.1), we know that (2.14) holds with $\psi_\alpha = u_{\lambda,\alpha}$. Moreover, from (2.3), it is an easy step to $\bar{u}_{\lambda,\alpha} \longrightarrow u_\lambda$ in $L^2(\mathbb{R}^d;\mathbb{R})$. Thus, by Lemma 2.13, $u_\lambda \in W_2^1(\mathbb{R}^d;\mathbb{R})$ and

$$\begin{aligned}
&\lambda\big(\varphi, u_\lambda\big)_0 + \int_{\mathbb{R}^d} \big(\nabla\varphi, a\nabla u_\lambda\big)_{\mathbb{R}^d}\, d\mathbf{x} \\
&= \lim_{n'\to\infty} \Big[\lambda\big(\varphi, u_{\lambda,\alpha_{n'}}\big)_{\alpha_{n'}} + \mathcal{E}^{\alpha_{n'}}\big(\varphi, u_{\lambda,\alpha_{n'}}\big)\Big] \\
&= \lim_{n'\to\infty} \big(\varphi, f\big)_{\alpha_{n'}} = \big(\varphi, f\big)_0. \quad \square
\end{aligned}$$

3. CONSTRUCTION OF MARKOV CHAIN APPROXIMATIONS

Theorem 2.18 provides us with a criterion on which to base the construction of Markov chain approximation schemes. The only ingredient which is still missing is a procedure for going from a given coefficient matrix a to a family $\{\rho_\alpha : \alpha \in (0,1]\}$ which satisfies the hypotheses of that theorem. Thus, let a Borel measurable $a : \mathbb{R}^d \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ symmetric matrix-valued function satisfying

(3.1) $$3I \le a(\mathbf{x}) \le (2+M)I, \quad \mathbf{x} \in \mathbb{R}^d,$$

be given. We want to construct an associated family $\{\rho_\alpha : \alpha \in [0,1]\}$ which satisfies (1.1), (2.1), (2.6), (2.8), and therefore, by Theorem 2.18, (2.20).

An important role in our construction will be played by the following simple application of elementary linear algebra. Here, and elsewhere, elements of $\mathbb{R}^d$ will be thought of as column vectors. In particular, if $\mathbf{e}^1, \dots, \mathbf{e}^d$ are given elements of $\mathbb{R}^d$, then $[\mathbf{e}^1, \dots, \mathbf{e}^d]$ will be the $d \times d$-matrix whose jth column is $\mathbf{e}^j$.

3.2 LEMMA. – *Given a symmetric* $S \in \mathbb{R}^d \otimes \mathbb{R}^d$, *there exist linearly independent* $\mathbf{e}^1, \dots, \mathbf{e}^d \in \mathbb{Z}^d$ *such that* $\max_{1 \le j \le d} \|\mathbf{e}^j\| \le 32d^2(\|S\|_{\text{op}} \vee 1)$ *(*$\|\cdot\|_{\text{op}}$ *stands for the operator norm), and*

$$\max_{1 \le i \le d} \sum_{j \ne i} \left| \left(ESE^\top\right)_{i,j} \right| \le \frac{1}{2} \min_{\mathbf{v} \in \mathbf{S}^{d-1}} \left| E\mathbf{v} \right|^2 \tag{3.3}$$

when $E^{-1} = [\mathbf{e}^1, \dots, \mathbf{e}^d]$.

Proof. – Set $R = 32d^2(\|S\|_{\text{op}} \vee 1)$. Choose an orthogonal matrix O so that $\Lambda \equiv O^\top S O$ is diagonal, determine $\mathbf{e}^j \in \mathbb{Z}^d$ so that, for $1 \le i \le d$, e_i^j is the integer part $[RO_{i,j}]$ of $RO_{i,j}$, and set $B = R^{-1}[\mathbf{e}^1, \dots, \mathbf{e}^d]$. We must show that B is invertible and

$$\max_{1 \le i \le d} \sum_{j \ne i} \left| \left(B^{-1}S(B^{-1})^\top\right)_{i,j} \right| \le \frac{1}{2} \min_{\mathbf{v} \in \mathbf{S}^{d-1}} \left| B^{-1}\mathbf{v} \right|^2. \tag{3.4}$$

For this purpose, set $\Delta = O - B$. Then $\|\Delta\|_{\text{op}} \le \frac{d}{R} \le \frac{1}{2}$ and $B = (I - \Delta O^\top)O$. Thus, B is invertible and

$$B^{-1} = O^\top \left(I - \Delta O^\top\right)^{-1} = O^\top + H \quad \text{where } \|H\|_{\text{op}} \le \frac{2d}{R}.$$

In particular,

$$\left\|B^{-1}S(B^{-1})^\top - \Lambda\right\|_{\text{op}} \le \frac{8d\|S\|_{\text{op}}}{R} \quad \text{and} \quad \left\|B^{-1}(B^{-1})^\top - I\right\|_{\text{op}} \le \frac{8d}{R}.$$

Hence, on the one hand,

$$B^{-1}(B^{-1})^\top \ge \left(1 - \frac{8d}{R}\right) I \ge \frac{3}{4} I,$$

while, on the other hand,

$$\max_{1 \le i \le d} \sum_{j \ne i} \left| \left(B^{-1}S(B^{-1})^\top\right)_{i,j} \right| \le \frac{8d(d-1)^{\frac{1}{2}}\|S\|_{\text{op}}}{R} \le \frac{1}{4}. \quad \square$$

At this point it is useful to make a distinction between the cases when a is assumed to be continuous and when no continuity is assumed. The next statement is a more or less immediate consequence of Lemma 3.2 plus elementary point-set topology. Indeed, when a is continuous, Lemma 3.2 makes it clear that each $x \in \mathbb{R}^d$ admits a neighborhood U and a choice of basis $\mathbf{e}^1, \dots, \mathbf{e}^d \in \mathbb{Z}^d$ such that $\max_{1\le j\le d} \|\mathbf{e}^j\| \le 32d^2M$ and

$$E = \left[\mathbf{e}^1, \dots, \mathbf{e}^d\right]^{-1} \Rightarrow \sum_{j\neq i}\left|\left(E\big(a(y)-2I\big)E^\top\right)_{i,j}\right| \le \left(E\big(a(y)-2I\big)E^\top\right)_{i,i}$$

for all $1 \le i \le d$ and $y \in U$. Hence, the following statement comes from the preceding and the fact that $\mathbb{R}^d$ is σ-compact.

3.5 LEMMA. – *If a is continuous, then there is a countable, locally finite, open cover $\{U_n\}_0^\infty$ of $\mathbb{R}^d$ such that, for each $n \in \mathbb{N}$, $a \restriction U_n$ is uniformly continuous and there exists a basis $\mathbf{e}^{1,n}, \dots, \mathbf{e}^{d,n}$ such that $\max_{1\le j\le d} \|\mathbf{e}^{j,n}\| \le 32d^2M$ and, for each $1 \le i \le d$ and $\mathbf{x} \in U_n$,*

$$\sum_{j\neq i}\left|\left(E_n\big(a(\mathbf{x}) - 2I\big)E_n^\top\right)_{i,j}\right| \le \left(E_n\big(a(\mathbf{x}) - 2I\big)E_n\top\right)_{i,i} \tag{3.6}$$

when $E_n = \left[\mathbf{e}^{1,n}, \dots, \mathbf{e}^{d,n}\right]^{-1}$.

Next, choose $\{\eta_n : n \in \mathbb{N}\} \subseteq C^\infty\big(\mathbb{R}^d; [0,1]\big)$ to be a partition of unity which is subordinate to $\{U_n\}$, set

$$\Xi_n(\mathbf{x}) = E_n^{-1}\big(a(\mathbf{x}) - 2I\big)E_n^\top, \tag{3.7}$$

and, for each n, define $\sigma_n : \mathbb{R}^d \times \mathbb{Z}^d \longrightarrow [0,\infty)$ by

$$\sigma_n(\mathbf{x},\mathbf{e}) = \begin{cases} \frac{1}{2}\eta_n(\mathbf{x})\big(\Xi_n(\mathbf{x})_{i,j}\big)^+ \\ \qquad \text{when } \mathbf{x} \in U_n,\ i \neq j, \text{ and } \mathbf{e} = \big(\mathbf{e}^{i,n} + \mathbf{e}^{j,n}\big) \\ \frac{1}{2}\eta_n(\mathbf{x})\big(\Xi_n(\mathbf{x})_{i,j}\big)^- \\ \qquad \text{when } \mathbf{x} \in U_n,\ i \neq j, \text{ and } \mathbf{e} = \big(\mathbf{e}^{i,n} - \mathbf{e}^{j,n}\big) \\ \eta_n(\mathbf{x})\Big(\Xi_n(\mathbf{x})_{i,i} - \sum_{j\neq i}\big|\Xi_n(\mathbf{x})_{i,j}\big|\Big) \\ \qquad \text{when } \mathbf{x} \in U_n \text{ and } \mathbf{e} = \mathbf{e}^{i,n} \\ 0 \\ \qquad \text{when } \mathbf{x} \notin U_n. \end{cases}$$

Finally, set

$$\rho_\alpha(\mathbf{k},\mathbf{e}) = \mathbf{1}_{\{1\}}\big(\|\mathbf{e}\|\big) + \sum_n \sigma_n(\mathbf{k},\mathbf{e}). \tag{3.8}$$

3.9 THEOREM. – *If* $a:\mathbb{R}^d\longrightarrow\mathbb{R}^d\times\mathbb{R}^d$ *is a continuous, symmetric matrix valued function which satisfies* (3.1) *and if* $\{\rho_\alpha:\alpha\in(0,1]\}$ *is determined by the prescription in* (3.8), *then Theorem 2.18 applies and so* (2.20) *holds.*

Proof. – There is nearly nothing to do. Indeed, since each of the $\sigma_n(\,\cdot\,,\mathbf{e})$'s is continuous and vanishes identically when $\|\mathbf{e}\|>64d^2M$, it suffices to observe that, by construction, $\sum_{\mathbf{e}}\sigma_n(\mathbf{x},\mathbf{e})=a(\mathbf{x})-2I$ for all $\mathbf{x}\in\mathbb{R}^d$. □

The situation when a is not continuous is hardly different. The only substantive change is that one must begin by doing a little smoothing. For example, choose some $\psi\in C_c^\infty\big(\mathbb{R}^d;[0,\infty)\big)$ with total integral 1, and set

$$(3.10)\quad a_\alpha(\mathbf{x})=\alpha^{-\frac{d}{2}}\int_{\mathbb{R}^d}\psi\big(\alpha^{-\frac12}(\mathbf{y}-\mathbf{x})\big)\,a(\mathbf{y})\,d\mathbf{y},\quad \alpha\in(0,1]\text{ and }\mathbf{x}\in\mathbb{R}^d.$$

Then

$$(3.11)\qquad \|a_\alpha(\mathbf{y})-a_\alpha(\mathbf{x})\|_{\mathrm{op}}\le C\alpha^{-\frac12}\|\mathbf{y}-\mathbf{x}\|,\quad \mathbf{x},\,\mathbf{y}\in\mathbb{R}^d.$$

Hence, by Lemma 3.2, one can find an $\alpha_0\in(0,1]$ such that, for each $\alpha\in(0,\alpha_0]$ and $\mathbf{l}\in\mathbb{Z}^d$, there is a basis $\{\mathbf{e}^{1,(\alpha,\mathbf{l})},\dots,\mathbf{e}^{d,(\alpha,\mathbf{l})}\}$ in $\mathbb{R}^d$ with (cf. (2.12))

$$\max_{1\le j\le d}\|\mathbf{e}^{j,(\alpha,\mathbf{l})}\|\le 32d^2M\quad\text{and, for each }1\le i\le d,$$
$$(3.12)\quad \sum_{j\ne i}\Big|\big(E_{\alpha,\mathbf{l}}\big(a_\alpha(\mathbf{x})-2I\big)E_{\alpha,\mathbf{l}}^\top\big)_{i,j}\Big|\le\Big(E_{\alpha,\mathbf{l}}\big(a(\mathbf{x})-2I\big)E_{\alpha,\mathbf{l}}^\top\Big)_{i,i},$$
$$\mathbf{x}\in Q_{\alpha^{\frac34}}(\mathbf{l}),$$

when $E_{\alpha,\mathbf{l}}=\big[\mathbf{e}^{1,(\alpha,\mathbf{l})},\dots,\mathbf{e}^{d,(\alpha,\mathbf{l})}\big]^{-1}$. Set

$$\Xi_{\alpha,\mathbf{l}}(\mathbf{x})=E_{\alpha,\mathbf{l}}\big(a_\alpha(\mathbf{x})-2I\big)E_{\alpha,\mathbf{l}}^\top,\quad \mathbf{x}\in Q_\alpha(\mathbf{l}).$$

Next, choose an $\eta\in C_c^\infty\big((-1,1);[0,1]\big)$ so that $\eta\equiv1$ on $\big[-\frac34,\frac34\big]$, take

$$\tilde\eta(t)=\frac{\eta(t)}{\eta(t-1)+\eta(t)+\eta(t-1)}\quad\text{and}\quad \eta_{\alpha,\mathbf{l}}(\mathbf{x})=\prod_{j=1}^d\tilde\eta\big(\alpha^{-\frac34}\mathbf{x}_j-\mathbf{l}_j\big),$$

and define $\sigma_{\alpha,\mathbf{l}}:\mathbb{R}^d\times\mathbb{Z}^d\longrightarrow[0,\infty)$ so that, when $\mathbf{x}\in Q_{\alpha^{\frac34}}(\mathbf{l})$,

$$\sigma_{\alpha,\mathbf{l}}(\mathbf{x},\mathbf{e})=\begin{cases}\dfrac{\eta_{\alpha,\mathbf{l}}(\mathbf{x})}{2}\big(\Xi_{\alpha,\mathbf{l}}(\mathbf{x})_{i,j}\big)^+ & \text{if } i\ne j\ \&\ \mathbf{e}=\big(\mathbf{e}^{i,(\alpha,\mathbf{l})}+\mathbf{e}^{j,(\alpha,\mathbf{l})}\big)\\[2mm] \dfrac{\eta_{\alpha,\mathbf{l}}(\mathbf{x})}{2}\big(\Xi_{\alpha,\mathbf{l}}(\mathbf{x})_{i,j}\big)^- & \text{if } i\ne j\ \&\ \mathbf{e}=\big(\mathbf{e}^{i,(\alpha,\mathbf{l})}-\mathbf{e}^{j,(\alpha,\mathbf{l})}\big)\\[2mm] \eta_{\alpha,\mathbf{l}}(\mathbf{x})\Big(\Xi_{\alpha,\mathbf{l}}(\mathbf{x})_{i,i}-\sum_{j\ne i}\big|\Xi_{\alpha,\mathbf{l}}(\mathbf{x})_{i,j}\big|\Big) & \text{if } \mathbf{e}=\mathbf{e}^{i,(\alpha,\mathbf{l})}\end{cases}$$

and $\sigma_{\alpha,\mathbf{l}}(\mathbf{x},\,\cdot\,) = 0$ when $\mathbf{x} \notin Q_{\alpha^{\frac{3}{4}}}(\mathbf{l})$. Finally, set

$$\rho_\alpha(\mathbf{k},\mathbf{e}) = \mathbf{1}_{\{1\}}\big(\|\mathbf{e}\|\big) + \sum_{\mathbf{l}\in\mathbb{Z}^d} \sigma_{\alpha\wedge\alpha_0,\mathbf{l}}(\alpha\mathbf{k},\mathbf{e}). \tag{3.13}$$

3.14 THEOREM. – *Given a Borel measurable, symmetric $a : \mathbb{R}^d \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$, define $\{\rho_\alpha : \alpha \in (0,1]\}$ as in* (3.13). *Then Theorem 2.18 applies and so* (2.20) *holds.*

4. CONVERGENCE OF PROCESSES AND AN APPLICATION TO THE DIRICHLET PROBLEM

As a more or less immediate dividend of Theorems 3.9 or Theorem 3.14, we get an accompanying statement about the associated stochastic processes. Namely, for each $\alpha \in (0,1]$, let Ω_α denote the Skorohod space $D\big([0,\infty);\alpha\mathbb{Z}^d\big)$ of right continuous, $\alpha\mathbb{Z}^d$-valued paths on $[0,\infty)$ with left limits, and note that each Ω_α is a closed subspace of $D\big([0,\infty);\mathbb{R}^d\big)$. Next, let $\big\{P^\alpha_{\alpha\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^d\big\}$ be the Markov family of probability measures on Ω_α with transition density p^α. Also, let $\big\{P^a_{\mathbf{x}} : \mathbf{x} \in \mathbb{R}^d\big\}$ be the Markov family of probability measures on $C\big([0,\infty);\mathbb{R}^d\big)$ for which p^a is the transition density. Because of (1.14) and standard compactness criteria (e.g., Chenčov's in Theorem 8.8 on page 139 of [2]) for the Skorohod topology, it is clear that, as a subset of probability measures on Ω_0,

$$\big\{P^\alpha_{\alpha\mathbf{k}} : \alpha \in (0,1],\ \mathbf{k} \in \mathbb{Z}^d,\ \text{and } \|\alpha\mathbf{k}\| \le r\big\} \text{ is tight for each } r \in (0,\infty).$$

At the same time, by (2.14), if $\alpha_n \searrow 0$ and $\alpha_n\mathbf{k}_n \longrightarrow \mathbf{x} \in \mathbb{R}^d$, then the only limit $\big\{P^{\alpha_n}_{\alpha_n\mathbf{k}_n}\big\}_1^\infty$ can have is $P^a_{\mathbf{x}}$. Hence, we have now proved the following.

4.1 COROLLARY. – *Under the conditions described in either Theorem 3.9 or Theorem 3.14, for any bounded $F : \Omega_0 \longrightarrow \mathbb{R}$ which is continuous and any $r \in (0,\infty)$,*

$$\lim_{\alpha\searrow 0}\sup_{|\mathbf{x}|\le r}\Big|\mathbb{E}^{P^\alpha_{[\mathbf{x}]_\alpha}}[F] - \mathbb{E}^{P^a_{\mathbf{x}}}[F]\Big| = 0. \tag{4.2}$$

In fact, (4.2) *continues to hold for bounded F's on Ω_0 which, for each $\mathbf{x} \in \mathbb{R}^d$, are $P^a_{\mathbf{x}}$-almost surely continuous with respect to the topology of uniform convergence on finite intervals.*

Proof. – The only point not already covered is the final one. However, because $\{P_{\mathbf{x}} : |\mathbf{x}| \le r\}$ is compact as probability measures on

$C\big([0,\infty);\mathbb{R}^d\big)$, it follows from standard facts about the Skorohod topology that the preceding weak convergence result self-improves to cover functions which are continuous with respect to the topology of uniform convergence on finite intervals. □

We conclude with an application of the preceding to the *Dirichlet problem for* L^a in a bounded, connected region $\mathfrak{G} \subseteq \mathbb{R}^d$. That is, given an $f \in C(\partial\mathfrak{G};\mathbb{R})$ we seek a function $u_f^a \in C\big(\overline{\mathfrak{G}};\mathbb{R}\big)$ with the properties that:

$$\tag{4.3} \begin{aligned} &u_f^a \restriction \partial\mathfrak{G} = f, \quad u_f^a \in W^1_{2,\mathrm{loc}}(\mathfrak{G};\mathbb{R}), \quad \text{and} \\ &\int_{\mathfrak{G}} \big(\nabla u_f^a, a\nabla\varphi\big)_{\mathbb{R}^d} = 0, \quad \varphi \in C_c^\infty(\mathfrak{G};\mathbb{R}). \end{aligned}$$

Set

$$\zeta_{\mathfrak{G}}(\omega) = \inf\{t \in [0,\infty] : \overline{\omega \restriction [0,t]} \not\subseteq \mathfrak{G}\}$$

and

$$\zeta_{\overline{\mathfrak{G}}}(\omega) = \inf\{t \in [0,\infty] : \omega \restriction [0,t] \not\subseteq \overline{\mathfrak{G}}\}.$$

Because $\mathfrak{G}$ is bounded, (1.16) can be applied to check that there are $T \in (0,\infty)$ and $\epsilon \in (0,1)$ with the property that

$$P^\alpha_{\alpha\mathbf{k}}\big(\zeta_{\overline{\mathfrak{G}}} \le T\big) \ge \epsilon \quad \text{for all } \alpha \in (0,1] \text{ and } \alpha\mathbf{k} \in \overline{\mathfrak{G}}.$$

Hence, by a familiar argument, there exists another $\epsilon > 0$ such that

$$\sup_{\alpha\in(0,1]} \sup_{\alpha\mathbf{k}\in\overline{\mathfrak{G}}} \mathbb{E}^{P^\alpha_{\alpha\mathbf{k}}}\Big[e^{\epsilon\zeta_{\overline{\mathfrak{G}}}}\Big] < \infty.$$

Similarly, from the lower bound in (I.0.10) of [9], one knows that

$$\sup_{\mathbf{x}\in\overline{\mathfrak{G}}} \mathbb{E}^{P^a_{\mathbf{x}}}\Big[e^{\epsilon\zeta_{\overline{\mathfrak{G}}}}\Big] < \infty$$

for a suitable choice of $\epsilon > 0$.

Obviously $\zeta_{\mathfrak{G}} \le \zeta_{\overline{\mathfrak{G}}}$ always. In addition, $\zeta_{\mathfrak{G}}$ and $\zeta_{\overline{\mathfrak{G}}}$ are, respectively, lower and upper semicontinuous with respect to the topology of uniform convergence on finite intervals. Finally, the lower bound from (I.1.10) in [9] leads (cf. part (iii) of Exercise 3.2.38 in [10]) to

$$\tag{4.4} P^a_{\mathbf{x}}\big(\zeta_{\mathfrak{G}} = \zeta_{\overline{\mathfrak{G}}}\big) = 1, \quad \mathbf{x} \in \mathbb{R}^d,$$

whenever $\partial\mathfrak{G}$ is *Lebesgue regular in* $\overline{\mathfrak{G}\mathfrak{C}}$ in the sense that

$$\tag{4.5} \varlimsup_{r\searrow 0} \frac{|\overline{\mathfrak{G}\mathfrak{C}} \cap B_{\mathbb{R}^d}(\mathbf{x},r)|}{r^d} > 0, \quad \mathbf{x} \in \partial\mathfrak{G}.$$

4.6 THEOREM. – *Assume that* (4.4) *holds. Given* $f \in C(\partial\mathfrak{G};\mathbb{R})$, *choose* $\bar{f} \in C_b(\mathbb{R}^d;\mathbb{R})$ *so that* $\bar{f} \upharpoonright \partial\mathfrak{G} = f$. *Then (cf.* (4.1))

$$u_f^a(\mathbf{x}) = \lim_{\alpha\searrow 0} \mathbb{E}^{P^a_{[\mathbf{x}]_\alpha}}\left[\bar{f}\big(\omega(\zeta_{\mathfrak{G}})\big),\ \zeta_{\mathfrak{G}}(\omega) < \infty\right],\quad \mathbf{x} \in \mathfrak{G}. \tag{4.7}$$

In fact, the convergence in (4.4) *takes place uniformly on compact subsets of* $\mathfrak{G}$.

Proof. – In view of Corollary 4.1 and the preceding discussion, the derivation of (4.4) comes down to checking that

$$u_f^a(\mathbf{x}) = \mathbb{E}^{P^a_{\mathbf{x}}}\left[f\big(\omega(\zeta_{\mathfrak{G}})\big),\ \zeta_{\mathfrak{G}}(\omega) < \infty\right],\quad \mathbf{x} \in \mathfrak{G}. \tag{4.8}$$

Indeed, (4.4) guarantees that $\bar{f}\big(\omega(\zeta_{\mathfrak{G}})\big)$ is $P^a_{\mathbf{x}}$-almost surely continuous in a topology with respect to which we know that $P^\alpha_{[\mathbf{x}]_\alpha}$ is converging weakly to $P^a_{\mathbf{x}}$.

To prove (4.8), one needs to use three facts. First, (4.8) is essentially trivial when a is smooth. Second, the convergence result in Theorem II.3.1 together with the tightness which comes from the upper bound in (I.1.10) of [9] and the continuity discussion just given, show that

$$\mathbb{E}^{P^{a_n}_{\mathbf{x}}}\left[f\big(\omega(\zeta_{\mathfrak{G}})\big),\ \zeta_{\mathfrak{G}} < \infty\right] \longrightarrow \mathbb{E}^{P^a_{\mathbf{x}}}\left[f\big(\omega(\zeta_{\mathfrak{G}})\big),\ \zeta_{\mathfrak{G}} < \infty\right]$$

uniformly on compacts in $\mathfrak{G}$

if $\{a_n\}_1^\infty$ is a sequence of coefficients matrices satisfying (3.1) and tending to a in measure. Third, one must know that, under the same conditions, $u_f^{a_n} \longrightarrow u_f^a$ uniformly on compacts, a conclusion which can be easily drawn from the facts in [9] and is explicitly contained in results of [7] or §5 of Chapter II in [6]. □

Remark. – For actual computation, it may be useful to observe that

$$\mathbb{E}^{P^\alpha_{\alpha\mathbf{k}}}\left[\bar{f}\big(\omega(\zeta_{\mathfrak{G}})\big),\ \zeta_{\mathfrak{G}} < \infty\right] = \mathbb{E}^{Q^\alpha_{\alpha\mathbf{k}}}\left[\bar{f}\big(\omega(\zeta_{\mathfrak{G}})\big),\ \zeta_{\mathfrak{G}} < \infty\right], \tag{4.9}$$

where $\{Q^\alpha_{\alpha\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^d\}$ is the discrete-time parameter Markov family on $\alpha\mathbb{Z}^d$ with transition probability

$$\Pi^\alpha(\alpha\mathbf{k}, \alpha\boldsymbol{\ell}) = R^\alpha(\mathbf{k})^{-1}\Big(\rho_\alpha(\mathbf{k}, \boldsymbol{\ell} - \mathbf{k}) + \rho_\alpha(\boldsymbol{\ell}, \mathbf{k} - \boldsymbol{\ell})\Big),$$

when

$$R^\alpha(\mathbf{k}) = \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d}\Big(\rho_\alpha(\mathbf{k}, \boldsymbol{\ell} - \mathbf{k}) + \rho_\alpha(\boldsymbol{\ell}, \mathbf{k} - \boldsymbol{\ell})\Big).$$

Indeed, one can construct $Q^\alpha_{\alpha\mathbf{k}}$ from $P^\alpha_{\alpha\mathbf{k}}$ by simply recording the size of the jumps and taking the time between successive jumps to be 1. Hence, the distribution of unparameterized trajectories is the same under $P^\alpha_{\alpha\mathbf{k}}$ and $Q^\alpha_{\alpha\mathbf{k}}$. In particular, the distribution of $\omega(\zeta_{\mathfrak{G}})\mathbf{1}_{(0,\infty)}(\zeta_{\mathfrak{G}})$ is the same under $Q^\alpha_{\mathbf{x}}$ as it is under $P^\alpha_{\mathbf{x}}$.

REFERENCES

[1] E. Carlen, S. Kusuoka and D. Stroock, Upper bounds for symmetric Markov transition functions, *Ann. Inst. H. Poincaré*, Vol. **35**, Sup. au #2, 1987, pp. 245-287.
[2] S. Ethier and T. Kurtz, *Markov Processes*, J. Wiley & Sons, 1986.
[3] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, De Gruyter Series in Math., W. De Gruyter, 1994.
[4] E. Fabes and D. Stroock, The De Giorgi-Moser Harnack principle via the old ideas of Nash, *Arch. Ratl. Mech. & Anal.*, Vol. **96**, #4, 1987, pp. 327-338.
[5] I. I. Gihman and A. Skorohod, *The Theory of Stochastic Processes II*, Springer-Verlag, Grund. #218, 1975.
[6] D. Kinderlehrer and G. Stampacchia, An Introduction to Variational Inequalities and their Applications, Academic Press, *Pure & Appl. Math.*, #88, 1980.
[7] W. Littman, G. Stampacchia and H. Weinberger, Regular Points for elliptic equations with discontinuous coefficients, *Ann. Scuola Norm. Sup Pisa*, #17, 1963, pp. 43-77.
[8] Z.-M. Ma and M. Röckner, *Dirichlet Forms*, Springer-Verlag, Universitext Series, 1992.
[9] D. Stroock, *Diffusion semigroups corresponding to uniformly elliptic divergence form operators*, Séminaire de Probabililité XXII (J. Azema & M. Yor, ed.), Springer-Verlag LNMS #1321, 1988, pp. 316-348.
[10] D. Stroock, *Probability Theory, An Analytic View*, Cambridge Univ. Press, 1993.
[11] D. Stroock and S. R. S. Varadhan, *Multidimensional Diffusion Theory*, Springer-Verlag, Grundlehren 233, 1979.

(Manuscript received May 14, 1996;
Revised December 12, 1996.)