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Probabilistic interpretation of a system of semi-linear parabolic partial differential equations

by

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ABSTRACT. – We obtain a probabilistic interpretation for systems of second order nonlinear parabolic partial differential equations by using a backward stochastic differential equation associated to a diffusion-transmutation process.

Key words: System of parabolic partial differential equations, backward stochatic differential equation, viscosity solution.

RÉSUMÉ. – Nous obtenons une formule probabiliste pour la solution de viscosité d'un système d'équations aux dérivées partielles paraboliques semilinéaires, à l'aide d'une équation différentielle stochastique rétrograde couplée à un processus de diffusion-transmutation. La nouveauté de ce travail tient à ce que l'opérateur linéaire du second ordre est différent d'une ligne à l'autre du système d'équations aux dérivées partielles.

1. INTRODUCTION

The well-known Feynman-Kac formula (*see* Kac [8]) expresses the solution of a large class of linear second order partial differrential equations of elliptic and parabolic type as the expectation of a functional of a diffusion process. Until recently, there existed three versions of the Feynman-Kac formula for nonlinear PDEs. The first one identifies the value function of an optimal stochastic control problem for a diffusion process with the solution of a Hamilton-Jacobi-Bellman equation, *see* e.g. Fleming-Soner [5]. The second relates a "nonlinear diffusion process", where the evolution of each trajectory depends not only on the current position but also on the probability law of that position (*i.e.* on the other trajectories), to the solution of a nonlinear PDE. This topics was initiated by McKean [9] and plays an essential role in the probabilistic approach to the Boltzman equation, *see* Sznitman [19]. The third one relates the law of a branching-diffusion process – or of a superprocess – with the solution of a semilinear equation, *see* e.g. Dynkin [4] for the case of superprocesses.

A new probabilistic approach to systems of semilinear PDEs has been invented recently, based on the notion of "backward stochastic differential equation" (which consists in fact rather in an inverse problem for a forward stochastic differential equation), *see* Peng [16], Pardoux-Peng [14], Pardoux [12]. So far, those systems of semilinear PDEs had the same linear second order operator appearing on each line. The aim of this paper is to treat the case of systems where the equation is completely different on each line. This is done by coupling the diffusion process with a so-called "transmutation process" which jumps from one state to another, thus modifying the dynamics of the diffusion. This idea seems to be originally due to Milstein [10] in the case of systems of linear PDEs.

Note that the results of this paper have already been exploited by one of the authors, *see* Pradeilles [17], in order to study the propagation of fronts in systems of reaction-diffusion equations.

The paper is organized as follows. In section 2, we introduce a class of backward stochastic differential equations with respect to both a Brownian motion and a finite sequence of Poisson processes. Some properties of the solution are discussed. Section 3 is devoted to the proof of a formula relating the components Z and Y of the solution of the BSDE. In section 4, we obtain a stochastic interpretation for the viscosity solution of a system of nonlinear parabolic partial differential equations by using the result of section 2 and a comparison theorem. Finally, we prove the uniqueness of the viscosity solution of our system of nonlinear parabolic partial differential equations in section 5.

2. BACKWARD STOCHASTIC DIFFERENTIAL EQUATION

Let $k \geq 2$ be an integer and $\mathbf{K} = \{1, 2, ..., k\}$. Let $b_i \in C(\mathbb{R}^+ \times \mathbb{R}^d; \mathbb{R}^d)$, $\sigma_i \in C(\mathbb{R}^+ \times \mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d)$, $i \in \mathbf{K}$. We define $b \in C(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbf{K}; \mathbb{R}^d)$ and $\sigma \in C(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbf{K}; \mathbb{R}^d \times \mathbb{R}^d)$ by :

$$b(t, x, i) = b_i(t, x), \quad \sigma(t, x, i) = \sigma_i(t, x).$$

We fix a terminal time $T > 0$. For each $t \in [0, T]$, we define the following differential operators L_t^i, L_t by:

$$(L_t^i \varphi)(x) = \frac{1}{2} \sum_{j,l=1}^{d} (\sigma_i \sigma_i^*)^{jl}(t, x) \frac{\partial^2 \varphi}{\partial x_l \partial x_j}(x) + \sum_{j=1}^{d} b_i^j(t, x) \frac{\partial \varphi}{\partial x_j}(x)$$

$$(L_t \varphi)(x, i) = \frac{1}{2} \sum_{j,l=1}^{d} (\sigma \sigma^*)^{jl}(t, x, i) \frac{\partial^2 \varphi}{\partial x_l \partial x_j}(x) + \sum_{j=1}^{d} b^j(t, x, i) \frac{\partial \varphi}{\partial x_j}(x)$$

where $i \in \mathbf{K}$, $\varphi \in C^2(\mathbb{R}^d)$.

For each $i \in \mathbf{K}$, we are given $f_i \in C(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^d)$, $g_i \in C(\mathbb{R}^d)$. Throughout the paper, we assume that there exist $K, p > 0$ such that for all $i \in \mathbf{K}; (t, x) \in [0, T] \times \mathbb{R}^d; u, u' \in \mathbb{R}^k; z, z' \in \mathbb{R}^d$,

$(\mathcal{H}1)$ $|b_i(t, x) - b_i(t, x')| \leq K(|x - x'|)$, $|\sigma_i(t, x) - \sigma_i(t, x')| \leq K(|x - x'|)$

$(\mathcal{H}2)$ $|f_i(t, x, u, z) - f_i(t, x, u', z')| \leq K(|u - u'| + |z - z'|)$

$(\mathcal{H}3)$ $|f_i(t, x, 0, 0)| \leq K(1 + |x|^p)$, $|g_i(x)| \leq K(1 + |x|^p)$

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a right continous and complete stochastic basis. We are given:

– a d-dimensional standard Wiener process $\{W_t\}_{t \geq 0}$ which is a $\mathcal{F}_t$–martingale;

– a Poisson random measure N, independent of $\{W_t\}_{t \geq 0}$, on $\mathbb{R}^+ \times \mathbf{L}$, where $\mathbf{L} = \mathbf{K} - \{k\}$ is the set of marks equipped with the field $\mathcal{L}$ of all subsets of $\mathbf{L}$, such that $M([0, t] \times A) = N([0, t] \times A) - t\lambda Card(A)\}_{t \geq 0}$ is a $\mathcal{F}_t$-martingale for all $A \subset \mathbf{L}$ and some fixed $\lambda > 0$.

Let $\mathcal{P}$ denote the σ-algebra of $\mathcal{F}_t$-predictable subsets of $\Omega \times [0, T]$. Now let us define some spaces of processes. We denote by $\mathcal{M}^2$ the set of one dimensional $\mathcal{F}_t$-adapted processes $\{Y_t : 0 \leq t \leq T\}$ such that

$$E \int_0^T Y_t^2 dt < +\infty.$$

Let $\mathcal{S}^2$ denote the set of $\mathcal{F}_t$-adapted càdlàg one-dimensional processes $\{Y_t, 0 \leq t \leq T\}$ such that

$$||Y||_{\mathcal{S}^2} \triangleq || \sup_{t\in[0,T]} |Y_t| \; ||_{L^2(\Omega)} < +\infty.$$

Let $[L^2(\mathcal{P})]^d$ be the set of $\mathcal{F}_t$-progressively mesurable d-dimensional processes $\{Z_t : 0 \leq t \leq T\}$ such that

$$||Z||_{[L^2(\mathcal{P})]^d} \triangleq \sqrt{E\int_0^T |Z_t|^2 dt} < +\infty.$$

By $[L^2(\mathcal{P}\otimes\mathcal{L})]^{k-1}$ we denote the set of mappings $H : \Omega \times [0,T] \times L \rightarrow I\!R^{k-1}$ which are $\mathcal{P}\otimes\mathcal{L}$ mesurable and such that

$$||H||_{[L^2(\mathcal{P})]^{k-1}} \triangleq \sqrt{E\sum_{l\in\mathbf{L}}\int_0^T H_t^2(l)dt} < +\infty.$$

Finally, we define $\mathcal{B}^2 = \mathcal{S}^2 \times [L^2(\mathcal{P})]^d \times [L^2(\mathcal{P}\otimes\mathcal{L})]^{k-1}$.

Let $0 \leq t \leq s < T$, $n \in \mathbf{K}$ and $l \in \mathbf{L}$. We denote $N_s = N((0,s]\times\mathbf{L})$, $N_s(l) = N((0,s]\times\{l\})$ and $M_s(l) = N_s(l) - \lambda s$. We define a Markov process $N_s^{t,n}$ by

$$N_s^{t,n} = n + \sum_{l=1}^{k-1} lN((t,s]\times\{l\}) \;\; mod[k].$$

Let $\{X_s^{t,x,n}, s\in[t,T]\}$ the unique strong solution of the following SDE:

$$\text{(1)} \qquad \begin{cases} dX_s^{t,x,n} = b(s, X_s^{t,x,n}, N_s^{t,n})ds + \sigma(s, X_s^{t,x,n}, N_s^{t,n})dW_s \\ X_t = x \in I\!R^d, \quad s\in[t,T] \end{cases}$$

We notice that $(X_s^{t,x,n}, N_s^{t,n})_{t\leq s\leq T}$ is a Markov process.
We define $\tilde{f}_i \in C(I\!R^+ \times I\!R^d \times I\!R \times I\!R^{k-1} \times I\!R^d)$ by

$$\tilde{f}_i(t,x,y,h,z) = f_i(t,x,u,z)$$

where

$$\begin{cases} u_j = y + h_{k-i+j} & j < i \\ u_i = y \\ u_j = y + h_{j-i} & j > i \end{cases}$$

It is easy to see that

$$
(2) \qquad \begin{cases} f_1(t,x,u,z) = \tilde{f}_1(t,x,u_1,h^1,z) \\ f_2(t,x,u,z) = \tilde{f}_2(t,x,u_2,h^2,z) \\ \cdots\cdots \\ f_k(t,x,u,z) = \tilde{f}_k(t,x,u_k,h^k,z) \end{cases}
$$

for all $(t,x) \in [0,T] \times I\!R^d$, where

$$
h^i_j = \begin{cases} u_{i+j} - u_i & 1 \le j \le k-i \\ u_{i+j-k} - u_i & k-i+1 \le j \le k-1 \end{cases}
$$

Now we claim the

PROPOSITION 2.1. – *Under the conditions* $(\mathcal{H}1)$, $(\mathcal{H}2)$ *and* $(\mathcal{H}3)$, *for each* $(t,x,n) \in [0,T] \times I\!R^d \times \mathbf{K}$, *the following BSDE:*

$$
(3) \qquad \begin{cases} dY^{t,x,n}_s = -\tilde{f}_{N^{t,n}_s}(s, X^{t,x,n}_s, Y^{t,x,n}_s, H^{t,x,n}_s, Z^{t,x,n}_s)ds \\ \qquad\qquad + \lambda \sum_{l=1}^{k-1} H^{t,x,n}_s(l)ds + Z^{t,x,n}_s dW_s + \sum_{l=1}^{k-1} H^{t,x,n}_s(l)dM_s(l) \\ \qquad s \in [t,T] \\ Y^{t,x,n}_T = g_{N^{t,n}_T}(X^{t,x,n}_T) \end{cases}
$$

where

$$
H^{t,x,n}_s = (H^{t,x,n}_s(1), H^{t,x,n}_s(2), \ldots, H^{t,x,n}_s(k-1)),
$$

admits a unique strong solution $(Y^{t,x,n}, Z^{t,x,n}, H^{t,x,n}) \in \mathcal{B}^2$ *and* $u_n(t,x) \triangleq Y^{t,x,n}_t$ *defines a deterministic function on* $[0,T] \times I\!R^d$.

Proof. – For all $n \in \mathbf{K}$, it is easy to see that $\tilde{f}_n$ is a Lipschitz function since f_n is a Lipschitz function.

So we only need to note that the BSDE (3) is a special case of the class of BSDEs considered in [1]. ■

Now we prove some technical lemmas which will be useful in what follows.

LEMMA 2.1. – u_n *grows at most polynomialy in* x.

Proof. – Using Itô's formula applied to $|Y^{t,x,n}_s|^2$, we obtain

$$
\begin{aligned} E|Y^{t,x,n}_s|^2 &= E|g(X^{t,x,n}_T)|^2 \\ &\quad + 2E\int_s^T Y^{t,x,n}_r \tilde{f}_{N^{t,n}_r}(r, X^{t,x,n}_r, Y^{t,x,n}_r, H^{t,x,n}_r, Z^{t,x,n}_r)dr \\ &\quad - E\int_s^T |Z^{t,x,n}_r|^2 dr - \lambda E\int_s^T \sum_{l=1}^{k-1} |H^{t,x,n}_r(l)|^2 dr \end{aligned}
$$

Now we notice that $E|X_s^{t,x,n}|^p \leq K'(1+|x|^p)$ for all $p \geq 1$, g has polynomial growth and $\tilde{f}(t,x,y,h,z)$ has polynomial growth with respect to x and is a Lipchitz function with respect to y, h, z. So we deduce, for some $c, p > 0$,

$$
\begin{aligned}
&E|Y_s^{t,x,n}|^2 + E\int_s^T |Z_r^{t,x,n}|^2 dr + \lambda E\int_s^T \sum_{l=1}^{k-1} |H_r^{t,x,n}(l)|^2 dr \\
&\leq c(1+|x|^{2p}) + cE\int_s^T |Y_r^{t,x,n}| \\
&\quad \times \left(1+|X_r^{t,x,n}|^p+|Y_r^{t,x,n}|+|Z_r^{t,x,n}|+\sum_{l=1}^{k-1}|H_r^{t,x,n}(l)|\right) dr
\end{aligned}
$$

and $\sup_{s\in[t,T]} E|X_s^{t,x,n}|^{2p} \leq c(1+|x|^{2p})$. Thus we get

$$
\begin{aligned}
&E|Y_s^{t,x,n}|^2 + \frac{1}{2}E\int_s^T |Z_r^{t,x,n}|^2 dr + \frac{\lambda}{2}E\int_s^T \sum_{l=1}^{k-1} |H_r^{t,x,n}(l)|^2 dr \\
&\leq c(1+|x|^{2p}) + \left(\frac{c}{2} + \frac{c^2(1+|x|^{2p})}{2}\right)(T-t) \\
&\quad + \left(2c + \frac{c^2}{2} + \frac{(k-2)c^2}{2\lambda}\right)\int_t^T E|Y_r^{t,x,n}|^2 dr
\end{aligned}
$$

which yields, from Gronwall's lemma, $E|Y_s^{t,x,n}|^2 \leq K'(1+|x|^{2p}|)$ for all $s \in [t,T]$. And we deduce finally $|u(t,x)| = |Y_t^{t,x,n}| \leq K'(1+|x|^p)$. ∎

A immediate consequence of this lemma and proposition 2.1 is the

LEMMA 2.2. – *Define* $u := (u_1, ..., u_k)$. *For all* $(t,x,n,l) \in [0,T] \times \mathbb{R}^d \times \mathbf{K} \times \mathbf{L}$,

$$
Y_s^{t,x,n} = u_{N_s^{t,n}}(s, X_s^{t,x,n})
$$

$$
H_s^{t,x,n}(l) = u_{N_{s-}^{t,n}+l}(s, X_s^{t,x,n}) - u_{N_{s-}^{t,n}}(s, X_s^{t,x,n})
$$

$$
\tilde{f}_{N_s^{t,n}}(r, X_s^{t,x,n}, Y_s^{t,x,n}, H_s^{t,x,n}, Z_s^{t,x,n}) = f_{N_s^{t,n}}(s, X_s^{t,x,n}, u(s, X_s^{t,x,n}), Z_s^{t,x,n})
$$

if $s \in [t,T]$.

Proof. – The identification of Y comes from the uniqueness of the solution of our BSDE (3):

$$
Y_s^{t,x,n} = Y_s^{s,X_s^{t,x,n},N_s^{t,n}} = u_{N_s^{t,n}}(s, X_s^{t,x,n}).
$$

Now we verify the identification of H. Let

$$\widetilde{H}_s^{t,x,n}(l) = u_{N_{s-}^{t,n}+l}(s, X_s^{t,x,n}) - u_{N_{s-}^{t,n}}(s, X_s^{t,x,n})$$

We first use the equation (3) and then the representation of Y in terms of u and get

$$\sum_{l=1}^{k-1} H_s^{t,x,n}(l)\Delta N_s(l) = \Delta Y_s^{t,x,n} = u_{N_s^{t,n}}(s, X_s^{t,x,n}) - u_{N_{s-}^{t,n}}(s, X_s^{t,x,n})$$
$$= \sum_{l=1}^{k-1} \widetilde{H}_s^{t,x,n}(l)\Delta N_s(l).$$

Hence, $E\left[\int_t^T (H_s(l) - \tilde{H}(l))^2 dN_s(l)\right] = 0$ which implies that

$$E\left[\int_t^T (H_s(l) - \tilde{H}(l))^2 ds\right] = 0.$$

Then, we have the second equality. Using the two previous equalities and the definition of $\tilde{f}$, we get the third one. ■

The last lemma establishes the continuity of the function $u_n(t,x)$ in (t,x).

LEMMA 2.3. – *$u_n(t,x)$ is continuous if f and g are continuous.*

Proof. – We fix $(t,x,n,x') \in [0,T] \times I\!R^d \times \mathbf{K} \times I\!R^d$ and $t' \in [t,(t+1)\wedge T]$. For all $s \in [t',T]$, we denote $\breve{X}_s = X_s^{t,x,n} - X_s^{t',x',n}$, $\breve{Y}_s = Y_s^{t,x,n} - Y_s^{t',x',n}$, $\breve{H}_s = H_s^{t,x,n} - H_s^{t',x',n}$, $\breve{Z}_s = Z_s^{t,x,n} - Z_s^{t',x',n}$, and

$$\breve{b}_s = b(X_s^{t,x,n}, N_s^{t,n}) - b(X_s^{t',x',n}, N_s^{t',n})$$
$$\breve{\sigma}_s = \sigma(X_s^{t,x,n}, N_s^{t,n}) - \sigma(X_s^{t',x',n'}, N_s^{t',n})$$
$$\breve{g}_T = g_{N_T^{t,n}}(X_T^{t,x,n}) - g_{N_T^{t',n}}(X_T^{t',x',n})$$

$$\breve{f}_s = \tilde{f}_{N_s^{t,n}}(s, X_s^{t,x,n}, Y_s^{t,x,n}, H_s^{t,x,n}, Z_s^{t,x,n}) - \lambda \sum_{l=1}^{k-1} H_s^{t,x,n}(l)$$
$$- \tilde{f}_{N_s^{t',n}}(s, X_s^{t',x',n}, Y_s^{t',x',n}, H_s^{t',x',n}, Z_s^{t',x',n}) + \lambda \sum_{l=1}^{k-1} H_s^{t',x',n}(l)$$

We get, from (1),

$$\breve{X}_s = X_{t'}^{t,x,n} - x' + \int_{t'}^s \breve{b}_r dr + \int_{t'}^s \breve{\sigma}_r dW_r$$

and

$$\begin{aligned}E|\breve{X}_s|^2 &= E|X_{t'}^{t,x,n} - x'|^2 + 2E\int_{t'}^{s} \breve{b}_r\breve{X}_r dr + E\int_{t'}^{s} |\breve{\sigma}_r|^2 dr \\ &\leq E|X_{t'}^{t,x,n} - x'|^2 + K'\left(E\int_{t'}^{s} |\breve{X}_r|^2 dr\right) + K'(\sqrt{P(N_{t'}^{t,n} \neq n)}) \\ &\leq K'(|x-x'|^2 + \sqrt{|t-t'|}) + K'\left(E\int_{t'}^{s} |\breve{X}_r|^2 dr\right)\end{aligned}$$

which yields, from Gronwall's lemma,

$$\sup_{s\in[t',T]} E|\breve{X}_s|^2 \leq K'(|x-x'|^2 + \sqrt{|t-t'|}).$$

where $K' > 0$ can change from line to line. Using again Itô's formula yields

$$E|\breve{Y}_s|^2 = E|\breve{g}_T|^2 + 2E\int_s^T \breve{f}_r\breve{Y}_r dr - E\int_s^T |\breve{Z}_r|^2 dr - \lambda E\int_s^T \sum_{l=1}^{k-1} |\breve{H}_r(l)|^2 dr$$

But we have

$$\begin{aligned}E|\breve{g}_T|^2 &\leq E|g_{N_T^{t',n}}(X_T^{t,x,n}) - g_{N_T^{t',n}}(X_T^{t',x',n})|^2 + E(|\breve{g}_T|^2 \mathbf{1}_{\{N_{t'}^{t,n}\neq n\}}) \\ &\leq K'(\sqrt{|t-t'|}) + E(\eta_R(|\breve{X}_T|)\mathbf{1}_{\{|X_T^{t,x,n}|<R\}}\mathbf{1}_{\{|X_T^{t',x',n}|<R\}}) \\ &\quad + K'(E((1+|X_T^{t,x,n}|^{2p} \\ &\qquad + |X_T^{t',x',n}|^{2p})(\mathbf{1}_{\{|X_T^{t,x,n}|\geq R\}} + \mathbf{1}_{\{|X_T^{t',x',n}|\geq R\}}))) \\ &\leq \epsilon_R(|t-t'| + |x-x'|) + K'\left(\frac{1}{\sqrt{R}} + \sqrt{|t-t'|}\right)\end{aligned}$$

where $\eta_R(.)$ and $\epsilon_R(.)$ are positive functions tending to 0 at 0^+. In the same way,

$$\begin{aligned}E\int_s^T \breve{f}_r\breve{Y}_r dr &\leq E\int_s^T \eta'_R(|\breve{X}_r|)dr + K'\left(\frac{1}{\sqrt{R}} + \sqrt{|t-t'|}\right) \\ &\quad + \frac{1}{2}E\int_s^T |\breve{Z}_r|^2 dr \\ &\quad + \frac{\lambda}{2}E\int_s^T \sum_{l=1}^{k-1} |\breve{H}_r(l)|^2 dr + K'\left(E\int_s^T |\breve{Y}_r|^2 dr\right) \\ &\leq \epsilon_R(|t-t'| + |x-x'|) + K'\left(\frac{1}{\sqrt{R}} + \sqrt{|t-t'|}\right) \\ &\quad + \frac{1}{2}E\int_s^T |\breve{Z}_r|^2 dr \\ &\quad + \frac{\lambda}{2}E\int_s^T \sum_{l=1}^{k-1} |\breve{H}_r(l)|^2 dr + K'\left(E\int_s^T |\breve{Y}_r|^2 dr\right).\end{aligned}$$

So we can get

$$
\begin{aligned}
&E|\breve{Y}_s|^2 + \frac{1}{2}E\int_s^T |\breve{Z}_r|^2 dr + \frac{\lambda}{2}E\int_s^T \sum_{l=1}^{k-1} |\breve{H}_r(l)|^2 dr \\
&\le K'(\epsilon_R(|t-t'| + |x-x'|)) \\
&\quad + K'\Big(\frac{1}{\sqrt{R}} + \sqrt{|t-t'|}\Big) + K'\Big(E\int_s^T |\breve{Y}_r|^2 dr\Big)
\end{aligned}
$$

which yields,

$$
\sup_{s\in[t',T]} E|\breve{Y}_s|^2 \le K'(\epsilon_R(|t-t'| + |x-x'|)) + K'\Big(\frac{1}{\sqrt{R}} + \sqrt{|t-t'|}\Big)
$$

for all $R \ge 1$.

Finaly, it is easy to see that $E(Y_s^{t,x,n})$ is continuous in $s \in [t, t']$. So the function $u_n(t,x) = Y_t^{t,x,n}$ is continuous. ■

3. THE LINK BETWEEN Z AND Y

In Pardoux-Peng [14], it is shown that Z and Y are connected in the following sense under appropriate asumptions:

$$
Z_t^{t,x} = \partial Y_t^{t,x} \sigma(x),
$$

where ∂Y is, in some sense, the gradient of $Y^{t,x}$ with respect to x. In this section, we extend this result to our case. It is extensively used in Pradeilles [17] and it is very usefull to get sharp estimates on $|Z_t^{t,x}|$.

C_b^1 denotes the set of functions of class C^1 which are bounded, together with their partial derivatives of first order. $D : L^2(\Omega) \to L^2(\Omega\times[0,T], \mathbb{R}^d)$ denotes the Malliavin derivation operator with respect to the Brownian motion and

$$
\mathrm{D}^{1,2} = \Big\{\xi \in L^2(\Omega) \;:\; E(\xi^2) + E\int_0^T |D_r\xi|^2 dr < \infty\Big\}.
$$

We make the following asumptions: for all $i \in \mathbf{K}$ and all $t \in [0,T]$

– $b_i \in C_b^1(\mathbb{R}^d, \mathbb{R}^d)$

– $\sigma_i \in C_b^1(\mathbb{R}^d, \mathbb{R}^{d\times d})$

– $g_i \in C^1(\mathbb{R}^d, \mathbb{R})$ with partial derivatives which grow at most like a polynomial function of the variable x at infinity

– $f_i(t,.,.,.) \in \mathcal{C}_b^1(I\!R^d \times I\!R^k \times I\!R^d, I\!R)$ with bounded partial derivatives with respect to (y, z) and polynomial growth of the partial derivatives with respect to x.

We have the following classical result:

LEMMA 3.1. – *For all* $(t, x, n) \in [0, T] \times I\!R^d \times \mathbf{K}$ *and all* $s \in [t, T]$, $X_s^{t,x,n} \in (\mathrm{D}^{1,2})^d$ *and a version of* $\{D_\theta X_s^{t,x,n} \ : \ \theta, s \in [t, T]\}$ *is given by:*

i) $D_\theta X_s^{t,x,n} = 0$, *if* $\theta > s$

ii) $\{D_\theta X_s^{t,x,n} \ : \ s \in [\theta, T]\}$ *is the unique solution of*

$$D_\theta X_s^{t,x,n} = \sigma(X_\theta^{t,x,n}, N_\theta^{t,n}) + \int_\theta^s \nabla b(X_r^{t,x,n}, N_r^{t,n}) D_t X_r^{t,x,n} dr + \int_\theta^s \sum_{j=1}^d \nabla\sigma_i(X_r^{t,x,n}, N_r^{t,n}) D_t X_r^{t,x,n} dW_r^j$$

where σ_i *denotes the i-th column of the matrix* σ.

Proof. – It is a particular case of a result in [2]. ■

Let I_d be the unity matrix of $I\!R^{d\times d}$ and $\partial X^{t,x,n}$ be the solution of

$$\partial X_s^{t,x,n} = I_d + \int_t^s \nabla b(X_r^{t,x,n}, N_r^{t,n}) \partial X_r^{t,x,n} dr + \int_t^s \sum_{j=1}^d \nabla\sigma_j(X_r^{t,x,n}, N_r^{t,n}) \partial X_r^{t,x,n} dW_r^j. \tag{4}$$

LEMMA 3.2. – *For all* $(t, x, n) \in [0, T] \times I\!R^d \times \mathbf{K}$ *and all* $t \le \theta \le s \le T$,

$$D_\theta X_s^{t,x,n} = \partial X_s^{t,x,n} (\partial X_\theta^{t,x,n})^{-1} \sigma(X_\theta^{t,x,n}, N_\theta^{t,n})$$

Proof. – It is an immediate consequence of the uniqueness of the solution of the S.D.E. satisfied by $D_\theta X_s^{t,x,n}$. ■

We are going to follow the technique developped in Pardoux-Peng [14] in order to obtain the link between Y and Z. We introduce some notations:

$$f'_x(r) = \frac{\partial f}{\partial x}(r, X_r^{t,x,n}, N_r^{t,n}, Y_r^{t,x,n}, H_r^{t,x,n}, Z_r^{t,x,n})$$
$$f'_y(r) = \frac{\partial f}{\partial y}(r, X_r^{t,x,n}, N_r^{t,n}, Y_r^{t,x,n}, H_r^{t,x,n}, Z_r^{t,x,n})$$
$$f'_h(r) = \frac{\partial f}{\partial h}(r, X_r^{t,x,n}, N_r^{t,n}, Y_r^{t,x,n}, H_r^{t,x,n}, Z_r^{t,x,n})$$
$$f'_z(r) = \frac{\partial f}{\partial z}(r, X_r^{t,x,n}, N_r^{t,n}, Y_r^{t,x,n}, H_r^{t,x,n}, Z_r^{t,x,n}).$$

PROPOSITION 3.1. – *$(Y^{t,x,n}, H^{t,x,n}, Z^{t,x,n})$ is in $L^2(t,T;(\mathrm{D}^{1,2})^{k+d})$ and a version of*

$$\{D_\theta Y_s^{t,x,n}, D_\theta H_s^{t,x,n}, D_\theta Z_s^{t,x,n} \; : \; t \leq \theta \leq T, t \leq s \leq T\}$$

is given by:

i) $D_\theta Y_s^{t,x,n} = 0$, $D_\theta H_s^{t,x,n}$, $D_\theta Z_s^{t,x,n} = 0$; $t \leq s < \theta \leq T$

ii) for all $\theta \in [t,T]$, $\{D_\theta Y_s^{t,x,n}, D_\theta H_s^{t,x,n}, D_\theta Z_s^{t,x,n} \; : \; \theta \leq s \leq T\}$ is the unique solution of

$$\begin{aligned}(5) \quad D_\theta Y_s^{t,x} &= g'(X_T^{t,x,n}, N_T^{t,n}) D_\theta X_T^{t,x,n} \\ &- \int_s^T D_\theta Z_r^{t,x,n} dW_r - \int_s^T \sum_{l=1}^{k-1} D_\theta H_r^{t,x,n}(l) dM_r(l) \\ &+ \int_s^T [f'_x(r) D_\theta X_r^{t,x,n} + f'_y(r) D_\theta Y_r^{t,x,n} \\ &\qquad + f'_h(r) D_\theta H_r^{t,x,n} + f'_z(r) D_\theta Z_r^{t,x,n}] dr.\end{aligned}$$

Moreover, $\{D_s Y_s^{t,x,n} \; : \; t \leq s \leq T\}$ is a version of $\{Z_s^{t,x,n} \; : \; t \leq s \leq T\}$.

Proof. – Before proving the first part of the proposition, we need two lemmas. The second one is an extension of a result given in [14].

LEMMA 3.3. – *If $H \in L^2(t,T;\mathrm{D}^{1,2})$ is $\mathcal{F}_r$-predictable, then*

$$\int_t^T H_r dM_r \in \mathrm{D}^{1,2}$$

and

$$D_s\left(\int_t^T H_r dM_r\right) = \int_s^T D_s H_r dM_r.$$

Proof. – Let $(T_j)_{1 \leq j}$ be the sequence of times when N jumps after t.

$$\int_t^T H_r dM_r = \sum_{j=1}^{N_T} H_{T_j} - \int_t^T H_r dr.$$

The second term is in $\mathrm{D}^{1,2}$ and

$$D_s \int_t^T H_r dr = \int_s^T D_s H_r dr.$$

We only need to prove that the first one is in $\mathrm{D}^{1,2}$ too. By asumption on H,

$$\int_s^T D_s H_r dM_r = \sum_{j=1}^{N_T} D_s H_{T_j} \mathbf{1}_{s\leq T_j} - \int_s^T D_s H_r dr$$

is defined and is in $L^2(\Omega)$. We just have to condition by N to get

$$D_s \sum_{j=1}^{N_T} H_{T_j} \mathbf{1}_{s\leq T_j} = \sum_{j=1}^{N_T} D_s H_{T_j} \mathbf{1}_{s\leq T_j} \text{ a.s.}$$

We can then conclude because $D_s H_{T_j} = 0$ if $T_j < s$. ■

LEMMA 3.4. – *Let $H \in L^2(N)$ and $Z \in L^2(W)$. If*

$$\xi = \int_t^T Z_s dW_s + \int_t^T \sum_{l=1}^{k-1} H_s(l) dM_s(l) \in \mathrm{D}^{1,2}$$

then

$$(H, Z) \in L^2(t, T; (\mathrm{D}^{1,2})^{k-1+d}) \tag{6}$$

and

$$D_s\xi = Z_s + \int_s^T D_s Z_r dW_r + \int_s^T \sum_{l=1}^{k-1} D_s H_r(l) dM_r(l). \tag{7}$$

Proof. – According to Nualart-Pardoux [11], we know that, if $Z \in L^2(t, T; (\mathrm{D}^{1,2})^d)$ then

$$\int_0^t Z_r dW_r \in \mathrm{D}^{1,2} \text{ and } D_s\left(\int_t^T Z_r dW_r\right) = Z_s + \int_s^T D_s Z_r dW_r.$$

Using the previous lemma, we can see that property (6) implies $\xi \in \mathrm{D}^{1,2}$ and the equality (7). Moreover, if (6) is true, then

$$\begin{aligned}||\xi||_{1,2}^2 = E\int_t^T &\left[2|Z_s|^2 ds + |H_s|^2\right] ds \\ &+ E\int_t^T\int_t^T \left[||D_s Z_r||^2 + ||D_s H_r||^2\right] ds dr.\end{aligned}$$

So, we just have to prove that the set

$$\mathcal{H} = \left\{ \int_t^T Z_s dW_s + \int_t^T \sum_{l=1}^{k-1} H_s(l) dM_s(l) : \right.$$
$$\left. (H,Z) \in L^2(t,T;(\mathrm{D}^{1,2})^{k-1+d}) \right\}$$

is dense in $\mathrm{D}^{1,2} \bigcap L^2(\Omega, \mathcal{F}_T, P) \bigcap \{E\xi = 0\}$ with respect to the $\|.\|_{1,2}$ norm. In order to do it, we respectively note $S(W)$ and $S(N)$ the sets of random variables ξ^W and ξ^N defined by:

$$\xi^W = \phi^W(W(z_1), \cdots, W(z_n))$$

$$\xi^N = \phi^N(N(h_1), \cdots, N(h_p))$$

where $\phi^W \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ and $\phi^N \in C_b^\infty(\mathbb{R}^p, \mathbb{R})$, $z_1, \cdots, z_n \in L^2(t, T; \mathbb{R}^d)$ and $h_1, \cdots, h_p \in L^2(t, T; \mathbb{R}^{k-1})$, $W(z_j)$ is the Wiener integral of z_j between t and T and $N(h_j)$ is the Stieljes integral of h_j with respect to M between t and T. By definition of $\mathrm{D}^{1,2}$,

$$Vect\{\xi \ : \xi \in L^2(\Omega, \mathcal{F}_T, P),\ \xi = \xi^W.\xi^N,\ E\xi = 0\}$$

is dense in $\mathrm{D}^{1,2} \bigcap L^2(\Omega, \mathcal{F}_T, P) \bigcap \{E\xi = 0\}$ with respect to the $\|.\|_{1,2}$ norm. Moreover, for ξ^W we have an Ocone's formula:

$$\xi^W = E\xi^W + \int_t^T E(D_s\xi^W / \mathcal{F}_s^W) dW_s$$

and for ξ^N, we have (*cf* Jacod [7])

$$\xi^N = E\xi^N + \int_t^T \sum_{l=1}^{k-1} \psi_s(l) dM_s(l)$$

where ψ is $\mathcal{F}_s^N$-predictable and independant of W. Applying Itô's formula with the following notations

$$\xi_s^W = E\xi^W + \int_t^s E(D_r\xi^W / \mathcal{F}_r^W) dW_r$$

$$\xi_s^N = E\xi^N + \int_t^s \sum_{l=1}^{k-1} \psi_r(l) dM_r(l),$$

yields:

$$\xi = \int_t^T E(D_s\xi^W.\xi_s^N/\mathcal{F}_s)dW_s + \int_t^T \sum_{l=1}^{k-1} E(\xi_s^W.\psi_s(l)/\mathcal{F}_s)dM_s(l),$$

which implies that such ξ belong to $\mathcal{H}$. ■

Let us go back to the proof of the proposition 3.1. We omit the index (t, x, n). Let

$$(U, Z, V) \in \mathcal{B}^2 \bigcap (L^2(t, T; \mathrm{D}^{1,2}))^{k+d}.$$

Then

$$g(X_T, N_T) + \int_t^T f(X_r, N_r, U_r, Z_r, V_r)dr \in \mathrm{D}^{1,2}$$

so that $(Z, H) \in L^2(t, T; (\mathrm{D}^{1,2})^{k-1+d})$, according to the lemma 3.4. We can then deduce that for all $s \in [t, T]$, $Y_s \in \mathrm{D}^{1,2}$ and that if

$$\begin{aligned} F(\theta, r) = {} & f'_x(r, X_r, N_r, U_r, Z_r, V_r)D_\theta X_r + f'_y(r, X_r, N_r, U_r, Z_r, V_r)D_\theta U_r \\ & + f'_z(r, X_r, N_r, U_r, Z_r, V_r)D_\theta V_r + f'_h(r, X_r, N_r, U_r, Z_r, V_r)D_\theta Z_r, \end{aligned}$$

for all $s \in [t, T]$, then for all $\theta \in [t, s]$,

$$\begin{aligned} D_\theta Y_s = {} & g'(X_T, N_T)D_\theta X_T + \int_s^T F(\theta, r)dr \\ & - \int_s^T D_\theta Z_r dW_r - \int_s^T \sum_{l=1}^{k-1} D_\theta H_r(l)dM_r(l). \end{aligned}$$

So $Y \in L^2(t, T; \mathrm{D}^{1,2})$. This result and the fact that (Y, H, Z) is the fixed point of an appropriate problem in $L^2(t, T; (\mathrm{D}^{1,2})^{k+d})$ with respect to a $||\cdot||_{1,2,\beta}$ norm defined by

$$||\xi||_{1,2,\beta}^2 = E[\xi^2 + \int_t^T e^{\beta s}||D_s\xi||^2 ds]$$

where $\beta \geq 0$ is well choosen, yield to the first part of our proposition.

Now, let us show the second part. For all $\theta \in]t, s]$, we have

$$\begin{aligned} D_\theta Y_s = {} & Z_\theta + \int_\theta^s D_\theta Z_r dW_r + \int_\theta^s \sum_{l=1}^{k-1} D_\theta H_r(l)dM_r(l) \\ & - \int_\theta^s [f'_x(r)D_\theta X_r + f'_y(r)D_\theta Y_r + f'_h(r)D_\theta H_r + f'_z(r)D_\theta Z_r]dr, \end{aligned}$$

hence

$$Z_\theta = g'(X_T, N_T)D_\theta X_T - \int_\theta^T D_\theta Z_r dW_r - \int_\theta^T \sum_{l=1}^{k-1} D_\theta H_r(l) dM_r(l)$$
$$+ \int_\theta^T [f'_x(r)D_\theta X_r + f'_y(r)D_\theta Y_r + f'_h(r)D_\theta H_r + f'_z(r)D_\theta Z_r]dr.$$

With the version we have choosen, this means

$$Z_\theta = D_\theta Y_\theta, \text{ p.p.} \quad \blacksquare$$

Let $(\partial Y^{t,x,n}, \partial H^{t,x,n}, \partial Z^{t,x,n})$ be the solution of

$$\begin{aligned}(8) \qquad \partial Y_s^{t,x,n} &= g'(X_T^{t,x,n}, N_T^{t,n})\partial X_T^{t,x,n} \\ &- \int_s^T \partial Z_r^{t,x,n} dW_r - \int_s^T \sum_{l=1}^{k} \partial H_r^{t,x,n}(l) dM_r(l) \\ &+ \int_s^T [f'_x(r)\partial X_r^{t,x,n} + f'_y(r)\partial Y_r^{t,x,n} \\ &\qquad + f'_h(r)\partial H_r^{t,x,n} + f'_z(r)\partial Z_r^{t,x,n}]dr.\end{aligned}$$

THEOREM 3.1. – *The process* $\{Z_s^{t,x,n} \ : \ t \leq s \leq T\}$ *has a càd-làg version and for this version, for all* $s \in [t, T]$,

$$(9) \qquad Z_s^{t,x,n} = \partial Y_s^{t,x,n}(\partial X_s^{t,x,n})^{-1}\sigma(X_s^{t,x,n}, N_s^{t,n}).$$

A particular case is

$$Z_t^{t,x,n} = \partial Y_t^{t,x,n}\sigma(x, n).$$

Proof. – The uniqueness of the solution of equation (5) and the lemma 3.2 imply that for all $s \in [t, T]$ and all $\theta \in [t, s]$

$$D_\theta Y_s = \partial Y_s(\partial X_\theta)^{-1}\sigma(X_\theta, N_\theta),$$

which allows to say that $\{D_s Y_s \ : t \leq s \leq T\}$ has a càd-làg version. We just have to remind that $\{D_s Y_s \ : t \leq s \leq T\}$ is a version of $\{Z_s \ : \ t \leq s \leq T\}$. $\blacksquare$

4. A VISCOSITY SOLUTION FOR A PARABOLIC SYSTEM

Let

$$\mathbf{u}(t,x) = (u_1(t,x), u_2(t,x), \dots, u_k(t,x)) \quad 0 \le t \le T, x \in I\!R^d. \tag{10}$$

We are interested in the following system of backward parabolic PDEs:

$$\begin{cases} \dfrac{\partial u_i}{\partial t}(t,x) + L^i u_i(t,x) + f_i(t,x,\mathbf{u}(t,x),(\nabla u_i \sigma_i)(t,x)) = 0, \\ \qquad (t,x) \in [0,T] \times I\!R^d, \quad 1 \le i \le k \\ u_i(T,x) = g_i(x), \; x \in I\!R^d, \quad 1 \le i \le k, \end{cases} \tag{11}$$

DEFINITION 4.1. – *Let* $\mathbf{u} = (u_1, u_2, \dots, u_k)$ *belong to* $C([0,T] \times I\!R^d; I\!R^k)$. $\mathbf{u}$ *is said to be a viscosity sub-solution (resp. super-solution) of the system* (11) *if*

$$u_i(T,x) \le g_i(x), \quad \forall i \in \mathbf{K}, x \in I\!R^d$$

$$\big(\textit{resp.} \quad u_i(T,x) \ge g_i(x), \quad \forall i \in \mathbf{K}, x \in I\!R^d\big)$$

and for all $i \in \mathbf{K}$, $(t,x) \in (0,T) \times I\!R^d$, $\psi \in C^{1,2}((0,T) \times I\!R^d)$ *such that* (t,x) *is a local minimum (resp. maximum) point of* $\psi - u_i$, *we have*

$$\frac{\partial \psi}{\partial t}(t,x) + (L_t^i \psi)(x) + f_i(t,x,u_1(t,x),\dots,u_k(t,x),\nabla\psi(t,x)\sigma_i(t,x)) \ge 0.$$

$$\left(\textit{resp.} \quad \frac{\partial \psi}{\partial t}(t,x) + (L_t^i \psi)(x) \right.$$
$$\left. + f_i(t,x,u_1(t,x),\dots,u_k(t,x),\nabla\psi(t,x)\sigma_i(t,x)) \le 0.\right)$$

$\mathbf{u}$ *is said to be a viscosity solution of the system* (11) *if* $\mathbf{u}$ *is both a viscosity sub-solution and a viscosity super-solution of the system* (11).

THEOREM 4.1. – *Under the conditions* $(\mathcal{H}1)$, $(\mathcal{H}2)$, $(\mathcal{H}3)$,*the function* $\mathbf{u}$ *defined by* (10) *is a viscosity solution of the system of backward parabobic PDEs* (11).

Proof. – We show that $\mathbf{u}$ is a viscosity sub-solution of (11). The property of being a viscosity super-solution can be proved analogously.

Let $i \in \mathbf{K}$ and $(t,x) \in [0,T] \times I\!R^d$. We suppose that $\varphi \in C^{1,2}([0,T] \times I\!R^d)$ satisfies $\varphi \ge u_i$ on $[0,T] \times I\!R^d$ and $\varphi(t,x) = u_i(t,x)$. We need to prove that

$$\frac{\partial \varphi}{\partial t}(t,x) + (L_t^i \varphi)(x) + f_i(t,x,u_1(t,x),\dots,u_k(t,x),\nabla\varphi(t,x)\sigma_i(t,x)) \ge 0$$

Let us argue by contradiction. We assume that

$$\frac{\partial \varphi}{\partial t}(t,x) + (L_t^i \varphi)(x) + f_i(t,x,u_1(t,x),...,u_k(t,x),\nabla\varphi(t,x)\sigma_i(t,x)) < 0$$

By continuity, there exists $\alpha > 0$ such that for all $s \in [t, t+\alpha]$ and all $y \in I\!R^d$, $|y-x| \le \alpha$,

$$\begin{cases} u_i(s,y) \le \varphi \\ \psi(s,y) \triangleq \dfrac{\partial\varphi}{\partial t}(s,y) + (L_s^i\varphi)(y) \\ \qquad\qquad + f_i(s,y,u_1(s,y),...,u_k(s,y),\nabla\varphi(s,y)\sigma_i(s,y)) < 0. \end{cases}$$

Define τ by

$$\tau = \inf\{s \ge t;\ |X_s^{t,x,i} - x| \ge \alpha,\ N_s^{t,i} \ne i\} \wedge (t+\alpha).$$

It follows from the last statement of lemma 2.2 that $(\overline{Y}_s, \overline{Z}_s) = (Y_{s\wedge\tau}^{t,x,i}, \mathbf{1}_{[t,\tau]} Z_{s\wedge\tau}^{t,x,i})$, $s \in [t, t+\alpha]$, is a solution of

$$\begin{aligned} \overline{Y}_s = {} & u_i((t+\alpha)\wedge\tau, X_{(t+\alpha)\wedge\tau}^{t,x,i}) \\ & + \int_s^{t+\alpha} \mathbf{1}_{[t,\tau]}(r) f_i(r, X_r^{t,x,i}, u(r, X_r^{t,x,i}), \overline{Z}_r^{t,x,i})dr - \int_s^{t+\alpha} \overline{Z}_r dW_r. \end{aligned}$$

In the same way, we define

$$(\hat{Y}_s, \hat{Z}_s) \triangleq (\varphi(s\wedge\tau, X_{s\wedge\tau}^{t,x,i}), \mathbf{1}_{[t,\tau]}(s)(\nabla\varphi\sigma)(s, X_s^{t,x,i})),$$

for all $s \in [t, t+\alpha]$. Notice that $\overline{Y}_t = \hat{Y}_t = u_i(t,x)$. Applying Itô's formula to $\varphi(s, X_s^{t,x,i})$, we obtain

$$\begin{aligned} \hat{Y}_s = {} & \hat{Y}_{t+\alpha} - \int_s^{t+\alpha} \mathbf{1}_{[t,\tau](r)} \left[\frac{\partial\varphi}{\partial t}(s,y) + (L_s^i\varphi)(y)\right] \\ & \times (r, X_r^{t,x,i})dr - \int_s^{t+\alpha} \hat{Z}_r dW_r \\ = {} & \hat{Y}_{t+\alpha} + \int_s^{t+\alpha} \mathbf{1}_{[t,\tau]}(r) f_i(r, X_r^{t,x,i}, u(r, X_r^{t,x,i}), \hat{Z}_r^{t,x,i})dr \\ & - \int_s^{t+\alpha} \hat{Z}_r dW_r - \int_s^{t+\alpha} \mathbf{1}_{[t,\tau]}(r)\psi(r, X_r^{t,x,i})dr \end{aligned}$$

But $\psi(r, X_r^{t,x,i}) > 0$ on $[t,\tau]$ and $\tau > t$ a.s. Then according to the strong comparison theorem 1.6 in Pardoux [12], $\overline{Y}_t < \hat{Y}_t$ which yields a contradiction. ■

5. UNIQUENESS OF THE VISCOSITY SOLUTION

We consider now the system (11) of partial differential equations of parabolic type which we rewrite as

$$(12)\quad \begin{cases} -\dfrac{\partial u_i}{\partial t}(t,x) - L^i u_i(t,x) - f_i(t,x,\mathbf{u}(t,x),(\nabla u_i \sigma_i)(t,x)) = 0, \\ \qquad (t,x) \in [0,T] \times I\!R^d, \quad 1 \le i \le k \\ u_i(T,x) = g_i(x), \ x \in I\!R^d, \quad 1 \le i \le k, \end{cases}$$

where for each i the second-order differential operator L^i takes the form

$$L^i\varphi(x) = \frac{1}{2}Tr\left(a_i(x)\frac{\partial^2\varphi}{\partial x^2}(x)\right) + < b_i(x), \nabla\varphi(x) >, \quad \varphi \in C^2(I\!R^d),$$
$$a_i^{ij}(x) = (\sigma_i(x)\sigma_i(x)^*)_{ij},$$

The functions σ_i, b_i, f_i and g_i are supposed to satisfy the assumptions made above. Moreover, we shall need the following result to be true: for all $R > 0$, there exists a positive function $\eta_R(.)$ tending to 0 at 0^+ such that

$$(\mathcal{H}4)\quad |f_i(t,x,u,z) - f_i(t,x',u,z)| \le \eta_R(|x - x'|(1 + |z|))$$

when $|x|, |x'|, |u| \le R$, $i \in \mathbf{K}$, $t \in [0,T]$, $z \in I\!R^d$. Then we have:

THEOREM 5.1. – *Under the conditions* $(\mathcal{H}1)$, $(\mathcal{H}2)$, $(\mathcal{H}3)$ *and* $(\mathcal{H}4)$, *there exists at most one viscosity solution* $\mathbf{u}$ *of* (12) *such that*

$$(13)\qquad \lim_{|x|\to+\infty} |\mathbf{u}(t,x)|e^{-A[\log|x|]^2} = 0,$$

uniformly for $t \in [0,T]$, *for some* $A > 0$.

In particular, the function $\mathbf{u}(t,x) = (Y_t^{t,x,1}, Y_t^{t,x,2}, ..., Y_t^{t,x,k})$ *is the unique viscosity solution of* (12) *in the class of solutions which satisfy* (13) *for some* $A > 0$. ■

Remark 5.2. – Notice that, by lemma 2.1, $\mathbf{u}(t,x) = (Y_t^{t,x,1}, Y_t^{t,x,2}, ..., Y_t^{t,x,k})$ has at most polynomial growth at infinity and therefore it satisfies (13).

Proof of theorem 5.1. – The result is a particular case of the uniqueness result in Barles, Buckdahn and Pardoux [1]. We give again the proof in the present particular case for the convenience of the reader.

Let $\mathbf{u}$ and $\mathbf{v}$ be two viscosity solutions of (12). The proof consists in two steps. We first show that $\mathbf{u} - \mathbf{v}$ and $\mathbf{v} - \mathbf{u}$ are viscosity subsolutions of a partial differential system; then we build a suitable sequence of smooth supersolutions of this system to show that $|\mathbf{u}-\mathbf{v}| = 0$ in $[0,T] \times I\!R^d$. Here and below, we denote by $|\cdot|$ the sup norm in $I\!R^k$.

LEMMA 5.3. – *Let* $\mathbf{u}$ *be a subsolution and* $\mathbf{v}$ *a supersolution of* (12). *Then the function* $\mathbf{w} \triangleq \mathbf{u} - \mathbf{v}$ *is a viscosity subsolution of the system*

$$-\frac{\partial w_i}{\partial t} - L^i w_i - K[|\mathbf{w}| + |\nabla w_i \sigma_i|] = 0 \text{ in } [0,T] \times I\!R^d \tag{14}$$

for $1 \le i \le k$, *where* K *is the Lipschitz constant of* f *in* (h, g).

Proof. – Let $\varphi \in C^2([0,T] \times I\!R^d)$ and let $(t_0, x_0) \in (0,T) \times I\!R^d$ be a strict global maximum point of $w_i - \varphi$ for some $i \in \mathbf{K}$.

We introduce the function

$$\psi_{\varepsilon,\alpha}(t,x,s,y) = u_i(t,x) - v_i(s,y) - \frac{|x-y|^2}{\varepsilon^2} - \frac{(t-s)^2}{\alpha^2} - \varphi(t,x)$$

where ε, α are positive parameters which are devoted to tend to zero.

Since (t_0, x_0) is a strict global maximum point of $u_i - v_i - \varphi$, by a classical argument in the theory of viscosity solutions, there exists a sequence $(\bar{t}, \bar{x}, \bar{s}, \bar{y})$ such that

(i) $(\bar{t}, \bar{x}, \bar{s}, \bar{y})$ is a global maximum point of $\psi_{\varepsilon,\alpha}$ in $([0,T] \times \overline{B}_R)^2$ where B_R is a ball with a large radius R.

(ii) $(\bar{t}, \bar{x})$, $(\bar{s}, \bar{y}) \to (t_0, x_0)$ as $(\varepsilon, \alpha) \to 0$.

(iii) $\dfrac{|\bar{x}-\bar{y}|^2}{\varepsilon^2}$, $\dfrac{(\bar{t}-\bar{s})^2}{\alpha^2}$ are bounded and tend to zero when $(\varepsilon, \alpha) \to 0$.

We have dropped above the dependence of $\bar{t}, \bar{x}, \bar{s}$ and $\bar{y}$ in ε and α for the sake of simplicity of notations.

It follows from theorem 8.3 in [3] that there exists $X, Y \in \mathcal{S}^d$ such that

$$\left(\bar{a} + \frac{\partial \varphi}{\partial t}(\bar{t}, \bar{x}), \bar{p} + D\varphi(\bar{t}, \bar{x}), X\right) \in \overline{D}^{2,+} u_i(\bar{t}, \bar{x})$$

$$(\bar{a}, \bar{p}, Y) \in \overline{D}^{2,-} v_i(\bar{s}, \bar{y})$$

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{4}{\varepsilon^2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} D^2\varphi(\bar{t}, \bar{x}) & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$\bar{a} = \frac{2(\bar{t}-\bar{s})}{\alpha^2} \quad \text{and} \quad \bar{p} = \frac{2(\bar{x}-\bar{y})}{\varepsilon^2}.$$

Modifying if necessary $\psi_{\varepsilon,\alpha}$ by adding terms of the form $\chi(x)$ and $\chi(y)$ with supports in $B^c_{R/2}$, we may assume that $(\bar{t}, \bar{x}, \bar{s}, \bar{y})$ is a global maximum point of $\psi_{\varepsilon,\alpha}$ in $([0,T] \times \mathbb{R}^d)^2$. Since $\mathbf{u}$ and $\mathbf{v}$ are respectively sub and supersolution of (12), we have

$$-\overline{a} - \frac{\partial \varphi}{\partial t}(\overline{t}, \overline{x}) - \frac{1}{2} Tr(a_i(\overline{x})X) - < b_i(\overline{x}), \overline{p} + D\varphi(\overline{t}, \overline{x}) >$$
$$-f_i(\overline{t}, \overline{x}, \mathbf{u}(\overline{t}, \overline{x}), (\overline{p} + D\varphi(\overline{t}, \overline{x}))\sigma_i(\overline{x})) \leq 0,$$

and

$$-\overline{a} - \frac{1}{2} Tr(a_i(\overline{y})Y) - < b_i(\overline{y}), \overline{p} > - f_i(\overline{s}, \overline{y}, \mathbf{v}(\overline{s}, \overline{y}), \overline{p}\sigma_i(\overline{y})) \geq 0.$$

Of course, we are going to substract these inequalities and we need to estimate differences between terms of the same type.

It is worth noticing that the χ terms we have to add to $\psi_{\varepsilon,\alpha}$ to have a global maximum point do not appear in the two inequalities above because they have a support which is included in $\overline{B}^c_{R/2}$, and since R is large, for α and ε small enough, $|\overline{x}| < R/2$ and $|\overline{y}| < R/2$.

First, if $(e_1 \ldots e_d)$ is an orthonormal basis of $\mathbb{R}^d$,

$$Tr(a_i(\overline{x})X) - Tr(a_i(\overline{y})Y) = Tr(\sigma_i^*(\overline{x})X\sigma_i(\overline{x})) - Tr(\sigma_i^*(\overline{y})Y\sigma_i(\overline{y}))$$
$$= \sum_{n=1}^{d} [< X\sigma_i(\overline{x})e_n, \sigma_i(\overline{x})e_n > - < Y\sigma_i(\overline{y})e_n, \sigma_i(\overline{y})e_n >]$$

To estimate this sum, we use the matrix inequality above together with the Lipschitz continuity of σ. We get

$$Tr(a_i(\overline{x})X) - Tr(a_i(\overline{y})Y) \leq C\frac{|\overline{x} - \overline{y}|^2}{\varepsilon^2} + Tr(a_i(\overline{x})D^2\varphi(\overline{t}, \overline{x}))$$

for some constant C. Then

$$| < b_i(\overline{x}), \overline{p} > - < b_i(\overline{y}), \overline{p} > | \leq C_1|\overline{x} - \overline{y}||\overline{p}| \leq C\frac{|\overline{x} - \overline{y}|^2}{\varepsilon^2}$$

because of the Lipschitz continuity of b_i.

Finally, we consider the difference between the nonlinear terms

$$|f_i(\overline{t}, \overline{x}, \mathbf{u}(\overline{t}, \overline{x}), (\overline{p} + D\varphi(\overline{t}, \overline{x}))\sigma_i(\overline{x})) - f_i(\overline{s}, \overline{y}, \mathbf{v}(\overline{s}, \overline{y}), \overline{p}\sigma_i(\overline{y}))|$$
$$\leq \rho_{\varepsilon,\delta}(|\overline{t} - \overline{s}|) + \eta(|\overline{x} - \overline{y}|(1 + |\overline{p}\sigma_i(\overline{y})|)) + K|\mathbf{u}(\overline{t}, \overline{x}) - \mathbf{v}(\overline{s}, \overline{y})|$$
$$+ K|\overline{p}(\sigma_i(\overline{x}) - \sigma_i(\overline{y})) + D\varphi(\overline{t}, \overline{x})\sigma_i(\overline{x})|.$$

The first term in the right–hand side comes from the continuity of f_i in t : $\rho_{\varepsilon,\delta}(s) \to 0$ when $s \to 0^+$ for fixed ε and δ. The second term comes from $(\mathcal{H}4)$: we have denoted by η the modulus η_R which appears in this assumption for R large enough. The two last terms come from the Lipschitz continuity of f_i w.r.t. the two last variables.

We notice that

$$|\overline{p}(\sigma_i(\overline{x}) - \sigma_i(\overline{y}))| \leq C\frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2}$$

because of the Lipschitz continuity of σ_i and that

$$|\overline{x}-\overline{y}| \cdot |\overline{p}\sigma_i(\overline{y})| \leq C\frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2}.$$

Now we substract the viscosity inequalities for $\mathbf{u}$ and $\mathbf{v}$: thanks to the above estimates, we can write the obtained inequality in the following way

$$\begin{aligned} &-\frac{\partial\varphi}{\partial t}(\overline{t},\overline{x}) - L^i\varphi(\overline{t},\overline{x}) - K|u(\overline{t},\overline{x}) - v(\overline{s},\overline{y})| - K|D\varphi(\overline{t},\overline{x})\sigma_i(\overline{x})| \\ &\leq \rho_{\varepsilon,\delta}(|\overline{t}-\overline{s}|) + \omega(\varepsilon,\alpha) \end{aligned}$$

where we have gathered in the $\omega(\varepsilon,\alpha)$ term, all the term of the form $\frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2}$ and $|\overline{x}-\overline{y}|$; $\omega(\varepsilon,\alpha) \to 0$ when (ε,α) tends to 0. To conclude we first let α go to zero : since $\frac{|\overline{t}-\overline{s}|^2}{\alpha^2}$ is bounded, $|\overline{t}-\overline{s}| \to 0$ and we get rid of the first term of the right–hand side above. Then we let δ go to zero keeping ε fixed and finally we let $\varepsilon \to 0$. Since $(\overline{t},\overline{x})$, $(\overline{s},\overline{y}) \to (t_0,x_0)$, we obtain :

$$-\frac{\partial\varphi}{\partial t}(t_0,x_0) - L^i\varphi(t_0,x_0) - K|\mathbf{w}(t_0,x_0)| - K|D\varphi(t_0,x_0)\sigma(x_0)| \leq 0$$

and therefore $\mathbf{w}$ is a subsolution of the desired equation by lemma 5.3. ■

Now we are going to build suitable smooth supersolutions for the equation (14).

LEMMA 5.4. – *For any $A > 0$, there exists $C_1 > 0$ such that the function*

$$\chi(t,x) = \exp\left[(C_1(T-t) + A)\psi(x)\right]$$

where

$$\psi(x) = [\log((|x|^2+1)^{1/2}) + 1]^2,$$

satisfies

$$-\frac{\partial\chi}{\partial t} - L^i\chi - K\chi - K|D\chi\sigma_i| > 0 \textit{ in } [t_1,T] \times I\!R^d$$

for $1 \leq i \leq k$ where $t_1 = T - A/C_1$.

Proof. – We first give estimates on the first and second derivatives of ψ: easy computations yield

$$|D\psi(x)| \leq \frac{2[\psi(x)]^{1/2}}{(|x|^2+1)^{1/2}} \leq 4 \text{ in } I\!R^d,$$

and

$$|D^2\psi(x)| \leq \frac{C(1+[\psi(x)]^{1/2})}{|x|^2+1} \text{ in } I\!R^d.$$

These estimates imply that, if $t \in [t_1, T]$,

$$\begin{aligned} |D\chi(t,x)| &\leq (C_1(T-t)+A)\chi(t,x)|D\psi(x)| \\ &\leq C\chi(t,x)\frac{[\psi(x)]^{1/2}}{(|x|^2+1)^{1/2}}, \end{aligned}$$

and, in the same way

$$|D^2\chi(t,x)| \leq C\chi(t,x)\frac{\psi(x)}{|x|^2+1}.$$

Because of our choice of t_1, the above estimates do not depend on C_1. Since σ_i and b_i grow at most linearly at infinity, we have

$$\begin{aligned} &-\frac{\partial\chi}{\partial t}(t,x) - L^i\chi(t,x) - K\chi(t,x) - K|D\chi(t,x)\sigma(x)| \\ &\geq \chi(t,x)\Big[C_1\psi(x) - C\psi(x) - C(\psi(x))^{1/2} - K - CK(\psi(x))^{1/2}\Big], \end{aligned}$$

for some constant C. Since $\psi(x) \geq 1$ in $I\!R^d$, it is clear enough that for C_1 large enough the quantity in the brackets is positive and the proof is complete. ■

To conclude the proof, we are going to show that $\mathbf{w} = \mathbf{u} - \mathbf{v}$ satisfies

$$|\mathbf{w}(t,x)| \leq \alpha\chi(t,x) \text{ in } [0,T] \times I\!R^d$$

for any $\alpha > 0$. Then we will let α tend to zero.

To prove this inequality, we first remark that because of (13)

$$\lim_{|x|\to+\infty} |\mathbf{w}(t,x)|e^{-A[\log((1+|x|^2)^{1/2})+1]^2} = 0$$

uniformly for $t \in [0,T]$, for some $A > 0$. This implies, in particular, that $|w_i| - \alpha\chi$ is bounded from above in $[t_1, T] \times I\!R^d$ for any $1 \leq i \leq k$ and that

$$M = \max_{1\leq i\leq k} \max_{[t_1,T]\times I\!R^d} (|w_i| - \alpha\chi)(t,x)e^{K(T-t)}$$

is achieved at some point (t_0, x_0) and for some i_0.

We first remark that, since $|\cdot|$ is the sup norm in $\mathbb{R}^k$, we have

$$M = \max_{[t_1,T]\times\mathbb{R}^d}(|\mathbf{w}| - \alpha\chi)(t,x)e^{K(T-t)}$$

and $|w_{i_0}(t_0,x_0)|=|\mathbf{w}(t_0,x_0)|$. We may assume w.l.o.g. that $|w_{i_0}(t_0,x_0)|>0$, otherwise we are done.

Then two cases: either $w_{i_0}(t_0,x_0) > 0$ or $w_{i_0}(t_0,x_0) < 0$. We treat the first case, the second one is treated in a similar way since the roles of $\mathbf{u}$ and $\mathbf{v}$ are symmetric.

From the maximum point property, we deduce that

$$w_{i_0}(t,x) - \alpha\chi(t,x) \le (w_{i_0} - \alpha\chi)(t_0,x_0)e^{K(t-t_0)}$$

and this inequality can be interpreted as the property for the function $w_{i_0} - \phi$ to have a global maximum point at (t_0,x_0) where

$$\phi(t,x) = \alpha\chi(t,x) + (w_{i_0} - \alpha\chi)(t_0,x_0)e^{K(t-t_0)}$$

Since $\mathbf{w}$ is a viscosity subsolution of (14), if $t_0 \in [t_1,T[$, we have

$$-\frac{\partial\phi}{\partial t}(t_0,x_0) - L^i\phi(t_0,x_0) - K|\mathbf{w}(t_0,x_0)| - K|D\phi(t_0,x_0)\sigma_i(x_0)| \le 0.$$

But the left-hand side of this inequality is nothing but

$$\alpha\left[-\frac{\partial\chi}{\partial t}(t_0,x_0) - L^i\chi(t_0,x_0) - K\chi(t_0,x_0) - K|D\chi(t_0,x_0)\sigma_i(x_0)|\right]$$

since $w_{i_0}(t_0,x_0) = |\mathbf{w}(t_0,x_0)|$; so, by lemma 5.4, we have a contradiction. Therefore $t_0 = T$ and since $|\mathbf{w}(T,x)| = 0$, we have

$$|\mathbf{w}(t,x)| - \alpha\chi(t,x) \le 0 \text{ in } [t_1,T]\times\mathbb{R}^d.$$

Letting α tend to zero, we obtain

$$|\mathbf{w}(t,x)| = 0 \text{ in } [t_1,T]\times\mathbb{R}^d.$$

Applying successively the same argument on the intervals $[t_2,t_1]$ where $t_2 = (t_1 - A/C_1)^+$ and then, if $t_2 > 0$ on $[t_3,t_2]$ where $t_3 = (t_2 - A/C_1)^+$ ... etc. We finally obtain that

$$|\mathbf{w}(t,x)| = 0 \text{ in } [0,T]\times\mathbb{R}^d,$$

and the proof is complete. ∎

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