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Markov dilations of nonconservative dynamical semigroups and a quantum boundary theory

by

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ABSTRACT. – A boundary theory for quantum Markov processes associated with nonconservative one parameter semigroups of completely positive linear contractions on a von Neumann algebra is initiated along the lines of Feller, Chung and Dynkin.

Key words: Completely positive maps, weak Markov flow, exit and entrance cocycles, exit time, boson Fock space.

RÉSUMÉ. – On développe une théorie des frontières pour des processus de Markov quantiques associés à des semi-groupes non conservatifs de contractions complètement positives sur une algèbre de von Neumann, parallèlement à la théorie classique de Feller, Chung et Dynkin.

1. INTRODUCTION

In classical probability theory it is well known that, to any one parameter semigroup of substochastic matrices or transition probability operators, one

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can associate a Markov process with an exit time which can be interpreted as a stop time at which the trajectory of the process goes out of the state space or hits a boundary. There are various possibilities for continuing the process after the exit time in such a manner that the Markov property and stationarity of transition probabilities are retained. Feller [Fe 1,2] initiated the study of this problem by a functional analytic approach based on resolvents or Laplace transforms of one parameter positive contraction semigroups whereas Chung [C1,2] and Dynkin [Dy] outlined a pathwise approach. The aim of the present paper is to investigate the same problem for quantum Markov flows when substochastic matrices or transition probability operators are replaced by completely positive linear contraction maps on a unital von Neumann algebra of operators in a Hilbert space.

In Section 2 it is shown how, by using the famous Stinespring's theorem [St, P1] and the GNS construction, one can associate a canonical weak Markov flow to any one parameter semigroup of completely positive contractions on a von Neumann or C^* algebra. To any such semigroup we introduce, in Section 3, the notion of entrance and exit cocycles and demonstrate how a Feller perturbation of the semigroup can be constructed using a pair (S, ω) where S is a cocycle and ω is a state. The resolvent of the perturbed semigroup is an exact quantum analogue of Feller's formula in [Fe 2]. This raises the basic open problem of constructing the Markov flow of the perturbed semigroup from the flow of the original semigroup. In order to study this problem we present in Section 4 a general procedure of gluing two quantum processes and their filtrations by using quantum stop times [H, PS] which are adapted spectral measures in the closed interval $[0, \infty]$. In Section 5 a quantum Markov flow for the perturbed semigroup is obtained by gluing countably many copies of a Markov flow for the unperturbed semigroup when the exit cocycle is the expectation of an exit time. We conclude the last section with several examples of nonconservative quantum Markov flows which admit exit times. Eventhough the presentation is done for the case of continuous time the reader can easily construct the discrete time analogue of many of our results.

Alternative approaches to dilations of quantum dynamical semigroups on a C^* algebra may be found in [EL], [Em], [Sa] and [Vi-S]. However, they are too weak especially because nothing concrete is mentioned about the conditional expectation of an observable at time t when the algebra of observables up to time s is fixed for some $0 < s < t$. Our approach in Theorem 2.12, 2.13 is much more direct and closer to the spirit of classical probability. This is illustrated by several examples in the course of the present exposition. A more elaborate and leisurely treatment of these ideas is included in the Ph.D. thesis of Bhat [B]. The case of dilation of a nonstationary quantum dynamical evolution is examined in the note [BP].

2. COMPLETELY POSITIVE SEMIGROUPS AND WEAK MARKOV FLOWS

Motivated by the notion of a quantum Markov process introduced by Accardi, Frigerio and Lewis [AFL] and influenced by the absence of conditional expectation in many situations in quantum probability we introduce here a weaker notion of a Markov flow and describe how such a weak Markov flow can be associated to any one parameter semigroup of completely positive linear maps of a von Neumann algebra into itself. This may be viewed as a continuous time version of Stinespring's theorem [St].

Let $\mathcal{H}$ be any complex Hilbert space with scalar product $< \cdot, \cdot >$ linear in the second variable. By a *weak filtration* F on $\mathcal{H}$ we mean a family $F = \{F(t), t \geq 0\}$ of orthogonal projection operators nondecreasing in the variable t. Denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$ and write

$$\mathcal{B}_{t]}^F = \{F(t)XF(t), X \in \mathcal{B}(\mathcal{H})\}$$

for every t. Then $\{\mathcal{B}_{t]}^F, t \geq 0\}$ is a nondecreasing family of $*$ subalgebras of $\mathcal{B}(\mathcal{H})$. The map $\mathbb{E}_{t]}^F : \mathcal{B}(\mathcal{H}) \to \mathcal{B}_{t]}^F$ defined by

$$\mathbb{E}_{t]}^F(X) = F(t)XF(t)$$

is called the *weak conditional expectation* with respect to F at time t.

PROPOSITION 2.1. – *The weak conditional expectation maps* $\{\mathbb{E}_{t]}^F, t \geq 0\}$ *satisfy the following :*

(i) $\mathbb{E}_{t]}^F$ *is a completely positive and contractive linear map;*

(ii) $\mathbb{E}_{t]}^F(I) = F(t)$;

(iii) $\mathbb{E}_{t]}^F(X) = X$ *for all* $X \in \mathcal{B}_{t]}^F$;

(iv) $\mathbb{E}_{t]}^F(XY) = X\mathbb{E}_{t]}^F(Y), \mathbb{E}_{t]}^F(YX) = \mathbb{E}_{t]}^F(Y)X$ *for all* $X \in \mathcal{B}_{t]}^F, Y \in \mathcal{B}(\mathcal{H})$;

(v) $\mathbb{E}_{s]}^F\mathbb{E}_{t]}^F = \mathbb{E}_{s\wedge t]}^F$ *where* $s \wedge t = \min(s,t)$.

Proof. – Immediate. ■

DEFINITION 2.2. – Let $\mathcal{A}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}_0$ and let $\{T_t, t \geq 0\}$ be a one parameter semigroup of contractive and completely positive linear maps of $\mathcal{A}$ into itself with T_0 being identity. A triple $(\mathcal{H}, F, j_t)$ is called a *weak Markov flow* with *expectation semigroup* $\{T_t\}$ if $\mathcal{H}$ is a Hilbert space containing $\mathcal{H}_0$ as a subspace, F is a weak filtration on $\mathcal{H}$ with $F(0)$ having range $\mathcal{H}_0$ and $\{j_t, t \geq 0\}$ is a family of * homomorphisms from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ satisfying the following

(i) $\mathbb{E}^F_{0]} j_0(X) = XF(0)$ and $j_t(X)F(t) = F(t)j_t(X)F(t)$ for all $t \geq 0, X \in \mathcal{A}$;

(ii) $\mathbb{E}^F_{s]} j_t(X) = j_s(T_{t-s}(X))F(s)$ for all $0 \leq s \leq t < \infty, X \in \mathcal{A}$.

The flow is said to be *subordinate* if $j_t(I) \leq F(t)$ for all t. If $j_t(I) = F(t)$ for all t it is said to be *conservative.*

Condition (i) describes faithfulness of j_0 and adaptedness of the flow to the filtration F whereas condition (ii) describes the Markov property of the flow. In the case of a subordinate flow the factor $F(s)$ on the right hand side of (ii) may be dropped. It may be noted that if $(\mathcal{H}, F, j_t)$ is a weak Markov flow then $(\mathcal{H}, F, j_t(\cdot)F(t))$ is a subordinate weak Markov flow.

For any $X \in \mathcal{A}$ denote by L_X and R_X respectively the linear maps from $\mathcal{A}$ into itself defined by $L_X Y = XY$ and $R_X Y = YX$ for all $Y \in \mathcal{A}$. L_X and R_Y commute with each other for any X, Y. For any finite sequence $\underline{t} = (t_1, ..., t_n)$ in $\mathbb{R}_+$ and $\underline{X} = (X_1, ..., X_n)$ in $\mathcal{A}$ (of *length* n) write $j(\underline{t}, \underline{X}) = j(t_1, t_2, ..., t_n, X_1, ..., X_n) = j_{t_1}(X_1)j_{t_2}(X_2)...j_{t_n}(X_n)$. In particular, $j(t, X) = j_t(X)$. For $\underline{s} = (s_1, ..., s_m), \underline{X} = (X_1, ..., X_m), \underline{t} = (t_1, ..., t_n), \underline{Y} = (Y_1, ..., Y_n)$ we have $j(\underline{s}, \underline{X})j(\underline{t}, \underline{Y}) = j((\underline{s}, \underline{t}), (\underline{X}, \underline{Y}))$ where $(\underline{s}, \underline{t}) = (s_1, ..., s_m, t_1, ..., t_n), (\underline{X}, \underline{Y}) = (X_1, ..., X_m, Y_1, ..., Y_n)$. Since each j_t is a homomorphism we have $j(\underline{s}, t, \underline{X}, Y)j(t, Z) = j(\underline{s}, t, \underline{X}, YZ)$, and $j(s, X)j(s, \underline{t}, Y, \underline{Z}) = j(s, \underline{t}, XY, \underline{Z})$. With these conventions we shall establish a few elementary propositions concerning the operators $j(\underline{t}, \underline{X})$ and their expectation values.

PROPOSITION 2.3. – *Let $(\mathcal{H}, F, j_t)$ be a weak Markov flow with expectation semigroup $\{T_t\}$ on a von Neumann algebra of operators on a Hilbert space $\mathcal{H}_0$. Then the following holds :*

(i) $j_t(X)F(t) = F(t)j_t(X) = F(t)j_t(X)F(t)$ *for all* $t \geq 0, X \in \mathcal{A}$;

(ii) *If* $0 \leq s \leq t_1 \leq \cdots \leq t_n, X_1, X_2, ..., X_n \in \mathcal{A}$ *then*

$$\mathbb{E}^F_{s]} j(\underline{t}, \underline{X}) = j(s, T_{t_1-s}L_{X_1}T_{t_2-t_1} \cdots L_{X_{n-1}}T_{t_n-t_{n-1}}(X_n))F(s);$$

(iii) *If* $t_1 \geq t_2 \geq \cdots \geq t_n \geq s \geq 0$ *then*

$$\mathbb{E}^F_{s]} j(\underline{t}, \underline{X}) = j(s, T_{t_n-s}R_{X_n}T_{t_{n-1}-t_n} \cdots R_{X_2}T_{t_1-t_2}(X_1))F(s).$$

Proof. – From property (i) in Definition 2.2 we have

$$\begin{aligned} F(t)j(t,X) &= \{j(t,X^*)F(t)\}^* \\ &= \{F(t)j(t,X^*)F(t)\}^* \\ &= F(t)j(t,X)F(t) \\ &= j(t,X)F(t). \end{aligned}$$

This proves (i). To prove (ii) we use property (i) of this proposition and the increasing nature of $F(t)$ repeatedly. Thus

$$\begin{aligned} &\mathbb{E}^F_{s]}j(\underline{t},\underline{X}) \\ &= F(s)F(t_1)j(t_1,X_1)...j(t_n,X_n)F(t_{n-1})F(s) \\ &= F(s)j(t_1,X_1)F(t_1)F(t_2)j(t_2,X_2)...j(t_n,X_n)F(t_{n-1})F(s) \\ &= F(s)j(t_1,X_1)j(t_2,X_2)F(t_2)j(t_3,X_3)...j(t_n,X_n)F(t_{n-1})F(s) \\ &= F(s)j(t_1,...,t_{n-1},X_1,...,X_{n-1})F(t_{n-1})j(t_n,X_n)F(t_{n-1})F(s) \\ &= F(s)j(t_1,...,t_{n-1},X_1,...,X_{n-1})j(t_{n-1},T_{t_n-t_{n-1}}(X_n))F(s) \\ &= \mathbb{E}^F_{s]}j(t_1,...,t_{n-1},X_1,...,X_{n-2},X_{n-1}T_{t_n-t_{n-1}}(X_n)). \end{aligned}$$

Now (ii) follows by induction on n. A similar argument yields (iii). ■

PROPOSITION 2.4. – *Let $(\mathcal{H},F,j_t)$ be a subordinate weak Markov flow with expectation semigroup $\{T_t\}$ on a unital von Neumann algebra $\mathcal{A}$ of operators on a Hilbert space $\mathcal{H}_0$. Then the following holds:*

(i) *If* $0 \leq t_n \leq t_1 < t_2 < ... < t_{n-1}, X_1, ..., X_n \in \mathcal{A}$ *then* $j(\underline{t},\underline{X}) = j(t_1,t_n,Y,X_n)$ *where*

$$Y = L_{X_1}T_{t_2-t_1}L_{X_2}T_{t_3-t_2}\cdots L_{X_{n-2}}T_{t_{n-1}-t_{n-2}}(X_{n-1});$$

(ii) *If* $0 \leq t_1 < t_2 < \cdots < t_{i-1} \leq t_n \leq t_i < t_{i+1} < \cdots < t_{n-1}$ *then* $j(\underline{t},\underline{X}) = j(t_1,...,t_{i-1},t_n,X_1,...,X_{i-1},Y)$ *where*

$$Y = R_{X_n}T_{t_i-t_n}L_{X_i}T_{t_{i+1}-t_i}\cdots L_{X_{n-2}}T_{t_{n-1}-t_{n-2}}(X_{n-1}).$$

Proof. – First we prove (i). Since $F(t)$ is increasing and j_t is a homomorphism Definition 2.2 together with the hypothesis that $j(t,I) \leq F(t)$ implies

$$\begin{aligned} &j(\underline{t},\underline{X}) \\ &= j(t_1,...,t_{n-2},X_1,...,X_{n-2})j(t_{n-2},I)j(t_{n-1},X_{n-1})j(t_n,I)j(t_n,X_n) \\ &= j(t_1,...,t_{n-2},X_1,...,X_{n-2})F(t_{n-2})j(t_{n-1},X_{n-1})F(t_{n-2})j(t_n,X_n) \\ &= j(t_1,...,t_{n-3},t_{n-2},t_n,X_1,...,X_{n-3},X_{n-2}T_{t_{n-1}-t_{n-2}}(X_{n-1}),X_n). \end{aligned}$$

Now (i) follows by induction on n. To prove (ii) we apply (i) to the sequence $t_n \leq t_i < t_{i+1} < \ldots < t_{n-1}$ and obtain

$$j(t_i, t_{i+1}, \ldots, t_n, X_i, X_{i+1}, \ldots, X_n) = j(t_i, t_n, Y', X_n)$$

where

$$Y' = L_{X_i} T_{t_{i+1}-t_i} \cdots L_{X_{n-2}} T_{t_{n-1}-t_{n-2}} (X_{n-1}).$$

Now observe that

$$\begin{aligned} & j(t_{i-1}, t_i, t_n, X_{i-1}, Y', X_n) \\ & \quad = j(t_{i-1}, X_{i-1}) F(t_n) j(t_i, Y') F(t_n) j(t_n, X_n) \\ & \quad = j(t_{i-1}, t_n, X_{i-1}, T_{t_i - t_n}(Y') X_n) \end{aligned}$$

which implies (ii). ■

The next theorem is of particular importance in reducing the computation of moments.

THEOREM 2.5. – *Let* $(\mathcal{H}, F, j_t)$ *satisfy the conditions of Proposition* 2.4. *Then for any sequence* $t_1, t_2, \ldots, t_n$ *in* $\mathbb{R}_+$ *and* $X_1, \ldots, X_n$ *in* $\mathcal{A}$ *there exists a sequence* $s_1, s_2, \ldots, s_m$ *in* $\mathbb{R}_+$ *and* $Y_1, \ldots, Y_m$ *in* $\mathcal{A}$ *such that* $m \leq n, s_1 = t_1, s_m = t_n$, *either* $s_1 < s_2 < \ldots < s_m$ *or* $s_1 > s_2 > \ldots > s_m$ *or* $s_1 > s_2 > \ldots > s_k < s_{k+1} < \ldots < s_m$ *for some* k *and* $j(\underline{t}, \underline{X}) = j(\underline{s}, \underline{Y})$.

Proof. – Without loss of generality we may assume that $t_1 \neq t_2 \neq \cdots \neq t_n$. If $\{t_i\}$ itself is either monotonic increasing or decreasing there is nothing to prove. If $t_1 < \cdots < t_i > t_{i+1}$ then either $t_{i+1} \leq t_1 < \ldots < t_i$ or $t_1 < \ldots < t_{k-1} \leq t_{i+1} \leq t_k < \ldots < t_i$ for some k. By Proposition 2.4 we may then express $j(t_1, \ldots, t_{i+1}, X_1, \ldots, X_{i+1})$ as $j(t_1, t_{i+1}, Y, X_{i+1})$ or $j(t_1, \ldots, t_{k-1}, t_{i+1}, X_1, \ldots, X_{k-1}, Y)$. In any case the length of the $\underline{t}$-sequence gets reduced in $j(\underline{t}, \underline{X})$. If $t_1 > \ldots > t_k < t_{k+1} < \ldots < t_{k+\ell} > t_{k+\ell+1}$ we may once again express $j(t_k, t_{k+1}, \ldots, t_{k+\ell+1}, X_k, X_{k+1}, \ldots, X_{k+\ell+1})$ in terms of a sequence of length not exceeding $\ell + 1$. Rest follows by induction on the length. ■

COROLLARY 2.6. – *Let* $(\mathcal{H}, F, j_t)$ *satisfy the conditions of Proposition* 2.4. *Then for any sequence* $t_1, t_2, \ldots, t_n$ *in* $\mathbb{R}_+$ *and* $X_1, X_2, \ldots, X_n$ *in* $\mathcal{A}$ *there exist* $t_1 = s_1 > s_2 > \ldots > s_m > 0, m \leq n$ *and* $Y_1, Y_2, \ldots, Y_m$ *in* $\mathcal{A}$ *such that* $j(\underline{t}, \underline{X}) F(0) = j(\underline{s}, \underline{Y}) F(0)$.

Proof. – In view of Theorem 2.5 we may assume without loss of generality that $t_1 > t_2 > \ldots > t_m < t_{m+1} < \ldots < t_n$. Now by Proposition 2.4 and the fact that $F(0) = j_0(I)$ we have

$$\begin{aligned} &j(t_m, t_{m+1}, \ldots, t_n, X_m, X_{m+1}, \ldots, X_n)F(0) \\ &\quad = j(t_m, t_{m+1}, \ldots, t_n, 0, X_m, X_{m+1}, \ldots, X_n, I) \\ &\quad = j(t_m, 0, Y, I) = j(t_m, Y)F(0) \end{aligned}$$

for some Y in $\mathcal{A}$. This completes the proof. ■

PROPOSITION 2.7. – *Let* $(\mathcal{H}, F, j_t)$ *be as in Proposition* 2.4. *Suppose that it is also conservative. If* $s_1 > s_2 > \ldots > s_m \geq 0, t_1 > t_2 > \ldots > t_n \geq 0$ *and* $\{s_1, s_2, \ldots, s_m\} \subset \{t_1, t_2, \ldots, t_n\}$ *then for any* $X_1, X_2, \ldots, X_m$ *in* $\mathcal{A}$

$$j(\underline{s}, \underline{X})F(0) = j(\underline{t}, \underline{Y})F(0)$$

where

$$Y_i = \begin{cases} X_j & if\ t_i = s_j\ for\ some\ j, \\ I & otherwise. \end{cases}$$

Proof. – Let $t_{i_1} = s_1, \ldots, t_{i_m} = s_m$. Then

$$\begin{aligned} j(s_r, X_r) &= j(s_r, I)j(t_{i_r}, X_r) \\ &= F(s_r)j(t_{i_r}, X_r) \\ &= F(t_{i_{r-1}+1})F(t_{i_{r-1}+2}) \cdots F(t_{i_{r-1}})j(t_{i_r}, X_r) \\ &= j(t_{i_{r-1}+1}, t_{i_{r-1}+2}, \ldots, t_{i_r-1}, t_{i_r}, I, I, \ldots, I, X_r) \end{aligned}$$

from which the required result follows. ■

PROPOSITION 2.8. – *Let* $(\mathcal{H}, F, j_t)$ *be as in Proposition* 2.4. *Suppose* $t \geq s_1 > \ldots > s_k \geq 0, X_1, X_2, \ldots, X_k, Y, Z_1, Z_2, \ldots, Z_k \in \mathcal{A}$. *Then*

$$\begin{aligned} &F(0)j(s_k, s_{k-1}, \ldots, s_1, t, s_1, \ldots, s_k, X_k, X_{k-1}, \ldots, X_1, Y, Z_1, \ldots, Z_k)F(0) \\ &\quad = F(0)\{T_{s_k} L_{X_k} R_{Z_k} T_{s_{k-1}-s_k} \cdots L_{X_1} R_{Z_1} T_{t-s_1}(Y)\}F(0). \end{aligned}$$

Proof. – We have

$$\begin{aligned} &j(s_1, t, s_1, X_1, Y, Z_1) \\ &\quad = j(s_1, X_1)F(s_1)j(t, Y)F(s_1)j(s_1, Z_1) \\ &\quad = j(s_1, L_{X_1} R_{Z_1} T_{t-s_1}(Y)). \end{aligned}$$

Now the required result follows by repeating the same argument successively replacing the role of t by that of $s_1, s_2, \ldots, s_k$. ■

PROPOSITION 2.9. – *Let* $(\mathcal{H}, F, j_t)$ *be as in Proposition* 2.4. *Suppose* $s_1 > s_2 > \ldots > s_k \geq t \geq 0, X_1, X_2, \ldots, X_k, Y, Z_1, Z_2, \ldots, Z_k \in \mathcal{A}$. *Then there exist elements* X_k', Z_k' *depending only on* $s_1, \ldots, s_k, X_1, \ldots, X_k$ *and* $Z_1, \ldots, Z_k$ *such that*

$$\begin{aligned} &F(0)j(s_k, \ldots, s_1, t, s_1, \ldots, s_k, X_k, \ldots, X_1, Y, Z_1, \ldots, Z_k)F(0) \\ &\quad = F(0)T_t\{T_{s_k - t}(X_k')YT_{s_k - t}(Z_k')\}F(0). \end{aligned}$$

Proof. – Since $t \leq s_2 < s_1$ we have

$$\begin{aligned} &j(t, s_1, s_2, Y, Z_1, Z_2) \\ &\quad = j(t, Y)F(s_2)j(s_1, Z_1)F(s_2)j(s_2, Z_2) \\ &\quad = j(t, Y)j(s_2, T_{s_1 - s_2}(Z_1)Z_2). \end{aligned}$$

Repeating this argument we get

$$j(t, s_1, \ldots, s_k, Y, Z_1, \ldots, Z_k) = j(t, s_k, Y, Z_k')$$

where Z_k' depends only on $s_1, \ldots, s_k, Z_1, \ldots, Z_k$. Since $s_k < s_{k-1} < \ldots < s_1 \geq t$ and $t \leq s_k$ we have from (i) in Proposition 2.4

$$j(s_k, s_{k-1}, \ldots, s_1, t, X_k, X_{k-1}, \ldots, X_1, Y) = j(s_k, X_k')j(t, Y)$$

where X_k' depends only on $X_1, \ldots, X_k, s_1 \ldots, s_k$. Combining the two we obtain

$$\begin{aligned} &j(s_k, s_{k-1}, \ldots, s_1, t, s_1, \ldots, s_k, X_k, X_{k-1}, \ldots, X_1, Y, Z_1, \ldots, Z_k) \\ &\quad = j(s_k, t, s_k, X_k', Y, Z_k'). \end{aligned}$$

Since $0 \leq s_k \geq t$ we have

$$\begin{aligned} &F(0)j(s_k, t, s_k, X_k', Y, Z_k')F(0) \\ &\quad = F(0)F(t)j(s_k, X_k')F(t)j(t, Y)F(t)j(s_k, Z_k')F(t)F(0) \\ &\quad = F(0)T_t(T_{s_k - t}(X_k')YT_{s_k - t}(Z_k'))F(0). \quad \blacksquare \end{aligned}$$

PROPOSITION 2.10. – *Let* $(\mathcal{H}, F, j_t)$ *be as in Proposition* 2.4. *Suppose that* $s_1 > s_2 > \cdots > s_{i-1} \geq t \geq s_i > \cdots > s_k$, X_1, $X_2, \ldots, X_k$, Y, $Z_1, \ldots, Z_k \in \mathcal{A}$. *Then*

$$\begin{aligned} &F(0)j(s_k, s_{k-1}, \ldots, s_1, t, s_1, s_2, \ldots, s_k, X_k, X_{k-1}, \ldots, X_1, Y, Z_1, \ldots, Z_k)F(0) \\ &\quad = F(0)\{T_{s_k}L_{X_k}R_{Z_k}T_{s_{k-1} - s_k} \\ &\quad \cdots L_{X_i}R_{Z_i}T_{t - s_i}(T_{s_{i-1} - t}(X_{i-1}')YT_{s_{i-1} - t}(Z_{i-1}'))\}F(0) \end{aligned}$$

where X_{i-1}', Z_{i-1}' *depend only on* $s_1, \ldots, s_{i-1}, X_i, \ldots, X_{i-1}, Z_1, \ldots, Z_{i-1}$.

Proof. – By (i) in Proposition 2.4 we have

$$j(s_{i-1}, s_{i-2}, \ldots, s_1, t, X_{i-1}, X_{i-2}, \ldots, X_1, Y) = j(s_{i-1}, t, X'_{i-1}, Y) \quad (2.1)$$

where X'_{i-1} depends only on $s_1, \ldots, s_{i-1}, X_1, \ldots, X_{i-1}$. Since $t \leq s_2 < s_1$ we have

$$\begin{aligned} j(t, s_1, s_2, Y, Z_1, Z_2) &= j(t, Y)F(s_2)j(s_1, Z_1)F(s_2)j(s_2, Z_2) \\ &= j(t, s_2, Y, T_{s_1 - s_2}(Z_1)Z_2). \end{aligned}$$

Repeating this argument up to the pair s_{i-2}, s_{i-1} we get

$$j(t, s_1, s_2, \ldots, s_{i-1}, Y, Z_1, \ldots, Z_{i-1}) = j(t, s_{i-1}, Y, Z'_{i-1}). \quad (2.2)$$

Since $s_i \leq t \leq s_{i-1}$ we have

$$\begin{aligned} &j(s_i, s_{i-1}, t, s_{i-1}, s_i, X_i, X'_{i-1}, Y, Z'_{i-1}, Z_i) \\ &\quad = j(s_i, t, s_i, X_i, T_{s_{i-1}-t}(X'_{i-1})YT_{s_{i-1}-t}(X'_{i-1}), Z_i) \quad (2.3) \end{aligned}$$

Combining (2.1)-(2.3) and using Proposition 2.8 for the sequence $s_k, s_{k-1}, \ldots, s_i, t, s_i, s_{i+1}, \ldots, s_k$ we obtain the required result. ■

PROPOSITION 2.11. – *Suppose $(\mathcal{H}, F, j_t)$ is a conservative weak Markov flow with a strongly continuous expectation semigroup $\{T_t\}$ on a unital von Neumann algebra $\mathcal{A}$ of operators on a Hilbert space $\mathcal{H}_0$. Then for any $u, u' \in \mathcal{H}_0$, finite sequences $\underline{s} = (s_1, \ldots, s_k), \underline{s}' = (s'_1, \ldots, s'_{k'})$ in $\mathbb{R}_+$ and $X_1, \ldots, X_k, Y, X'_1, \ldots, X'_{k'} \in \mathcal{A}$ the function*

$$\phi(t) = \langle j(\underline{s}, \underline{X})u, j_t(Y)j(\underline{s}', \underline{X}')u' \rangle$$

is continuous in $t \in \mathbb{R}_+$.

Proof. – Since $F(0)u = u, F(0)u' = u'$ we can apply Corollary 2.6 and assume without loss of generality that $s_1 > s_2 > \cdots > s_k$ and $s'_1 > s'_2 > \cdots > s'_{k'}$. Since the flow is conservative we can apply Proposition

2.7 and assume without loss of generality that the sequences $\underline{s}$ and $\underline{s}'$ are same and strictly decreasing. Then $\phi(t)$ assumes the form

$$\phi(t) = \langle u, F(0)j(s_k, s_{k-1}, ..., s_1, t, s_1, ..., s_k, X_k^*, ..., \\ X_1^*, Y, X_1', ..., X_k')F(0)u'\rangle.$$

Now the strong continuity and contractivity properties of $\{T_t\}$ together with Proposition 2.8, 2.9 and 2.10 respectively imply the continuity of $\phi(t)$ in the intervals $[s_1, \infty), [0, s_k]$ and $[s_i, s_{i-1}], i = k, k-1, ..., 2$. ■

THEOREM 2.12. – *Let $\mathcal{A}$ be a unital von Neumann algebra of operators in a Hilbert space $\mathcal{H}_0$ and let $\{T_t\}$ be a semigroup of completely positive linear maps of $\mathcal{A}$ into itself such that T_0 is identity and $T_t(I) = I$ for all t. Then there exists a conservative weak Markov flow $(\mathcal{H}, F, j_t)$ on $\mathcal{A}$ satisfying the following :*

(i) *$\mathcal{H}_0 \subset \mathcal{H}$ and $\mathcal{H}_0$ is the range of $F(0)$;*

(ii) *The set $\{j(\underline{t}, \underline{X})u, u \in \mathcal{H}_0, \underline{t} = (t_1, t_2, ..., t_n), t_i \geq 0, \underline{X} = (X_1, X_2, ..., X_n), X_i \in \mathcal{A}, 1 \leq i \leq n, n = 1, 2, ...\}$ is total in $\mathcal{H}$;*

(iii) *The expectation semigroup of $(\mathcal{H}, F, j_t)$ is $\{T_t\}$;*

(iv) *If $(\mathcal{H}', F', j_t')$ is another subordinate weak Markov flow with expectation semigroup $\{T_t\}$ such that the range of $F'(0)$ is $\mathcal{H}_0$ and* (ii) *holds with $j, \mathcal{H}$ replaced by $j', \mathcal{H}'$ then there exists a unitary isomorphism $U : \mathcal{H} \to \mathcal{H}'$ satisfying*

$$UF(t)U^{-1} = F'(t), Uj_t(X)U^{-1} = j_t'(X)$$

for all $t \geq 0, X \in \mathcal{A}$;

(v) *If $\{T_t\}$ is strongly continuous on the Banach space $\mathcal{A}$ then the maps $t \to F(t)$ and $t \to j_t(X)$ are strongly continuous for each $X \in \mathcal{A}$.*

Proof. – From [P2] it is known that there exists a family $\{h_t, t \geq 0\}$ of Hilbert spaces with $h_0 = \mathcal{H}_0$, $*$ unital homomorphisms $J_t : \mathcal{A} \to \mathcal{B}(h_t)$ and isometries $V(s,t) : h_s \to h_t$ for $0 \leq s \leq t < \infty$ such that (a) $J_0(X) = X$; (b) $V(s,s) = I$; (c) $V(s,t)^* J_t(X) V(s,t) = J_s(T_{t-s}(X))$; (d) $V(t,u)V(s,t) = V(s,u)$ for all $0 \leq s \leq t \leq u < \infty$. Let $\mathcal{M} = \bigcup_{s \geq 0} h_s$ be the disjoint union of all the h_s considered as abstract sets. Define the map $K : \mathcal{M} \times \mathcal{M} \to \mathbb{C}$ by

$$K(u,v) = \langle V(s, s \vee t)u, V(t, s \vee t)v\rangle_{h_{s \vee t}}$$

whenever $u \in h_s, v \in h_t, s \vee t$ denoting the maximum of s and t. We claim that K is a positive definite kernel on $\mathcal{M}$. Indeed, consider arbitrary scalars

$c_i \in \mathbb{C}, u_i \in h_{s_i}, 1 \leq i \leq n$ and put $s = s_1 \vee s_2 \vee \cdots \vee s_n$. Then

$$\begin{aligned}
&\sum_{i,j} \bar{c}_i c_j K(u_i, u_j) \\
&\quad = \sum_{i,j} \bar{c}_i c_j \langle V(s_i, s_i \vee s_j) u_i, V(s_j, s_i \vee s_j) u_j \rangle \\
&\quad = \sum_{i,j} \bar{c}_i c_j \langle V(s_i \vee s_j, s) V(s_i, s_i \vee s_j) u_i, V(s_i \vee s_j, s) V(s_j, s_i \vee s_j) u_j \rangle \\
&\quad = \sum_{i,j} \bar{c}_i c_j \langle V(s_i, s) u_i, V(s_j, s) u_j \rangle \\
&\quad = ||\sum_i c_i V(s_i, s) u_i||^2_{h_s} \geq 0,
\end{aligned}$$

which proves the claim. Hence by the GNS theorem there exists a Hilbert space $\mathcal{K}$ and a map $\lambda : \mathcal{M} \to \mathcal{K}$ such that $\{\lambda(u), u \in \mathcal{M}\}$ is total in $\mathcal{K}$ and

$$K(u, v) = \langle \lambda(u), \lambda(v) \rangle \text{ for all } u, v \in \mathcal{M}.$$

If $u, v \in h_t$ then

$$\langle \lambda(u), \lambda(v) \rangle = K(u, v) = \langle u, v \rangle_{h_t}.$$

Thus λ is an isometry from h_t onto a subspace $\mathcal{K}_t$ of $\mathcal{K}$. If $s < t$ and $u \in h_s$ then $V(s,t)u \in h_t$ and

$$\begin{aligned}
&||\lambda(u) - \lambda(V(s,t)u)||^2 \\
&\quad = 2||u||^2 - 2 \operatorname{Re} K(u, V(s,t)u) \\
&\quad = 2||u||^2 - 2\operatorname{Re} \langle V(s,t)u, V(s,t)u \rangle_{h_t} = 0.
\end{aligned}$$

Thus $\mathcal{K}_s \subseteq \mathcal{K}_t$ whenever $s \leq t$. Denote by $E(t)$ the projection onto $\mathcal{K}_t$ and define j_t by

$$j_t(X) = \lambda J_t(X) \lambda^{-1} E(t) \qquad \text{for} \quad X \in \mathcal{A}, \quad t \geq 0$$

where λ^{-1} is the inverse of the map $\lambda : h_t \to \mathcal{K}_t$. Since the range of $j_t(X)$ is contained in $\mathcal{K}_t$ and J_t is a * homomorphism from $\mathcal{A}$ into $\mathcal{B}(h_t)$ it follows that $j_t(X) = E(t) \lambda J_t(X) \lambda^{-1} E(t), X \in \mathcal{A}$ is a * homomorphism from $\mathcal{A}$ into $\mathcal{B}(\mathcal{K})$ and $j_t(I) = E(t)$.

Now consider $u, v \in h_s, s < t$. Then

$$\begin{aligned}
&\langle \lambda(u), j_t(X) \lambda(v) \rangle \\
&\quad = \langle \lambda(V(s,t)u), j_t(X) \lambda(V(s,t)v) \rangle \\
&\quad = \langle V(s,t)u, J_t(X) V(s,t)v \rangle \\
&\quad = \langle u, V(s,t)^* J_t(X) V(s,t)v \rangle \\
&\quad = \langle u, J_s(T_{t-s}(X))v \rangle \\
&\quad = \langle \lambda(u), j_s(T_{t-s}(X)) \lambda(v) \rangle.
\end{aligned}$$

Thus

$$E(s)j_t(X)E(s) = j_s(T_{t-s}(X)) \qquad \text{if} \quad X \in \mathcal{A}, \quad s \leq t.$$

Denote by $\mathcal{H} \subset \mathcal{K}$ the closed subspace spanned by the set M of all vectors of the form $j(\underline{t}, \underline{X})u, u \in \mathcal{H}_0, \underline{t} = (t_1, ..., t_n), \underline{X} = (X_1, ..., X_n), t_i \geq 0$, $X_i \in \mathcal{A}$, $n = 1, 2, ...$ Denote by $\mathcal{H}_t \subset \mathcal{H}$ the closed subspace spanned by the set $M_t \subset M$ of all vectors of the same form $j(\underline{t}, \underline{X})u$ with $t_i \leq t$ for every i. We now claim that $\mathcal{H}_t = \mathcal{H} \cap \mathcal{K}_t$. Indeed, let $\xi = j(\underline{s}, \underline{X})u$ where $\underline{s} = (s_1, ..., s_n), \underline{X} = (X_1, ..., X_n), t \geq s_i \geq 0, X_i \in \mathcal{A}, u \in \mathcal{H}_0$. Then ξ is in the range of $j(s_1, X_1)$ which is contained in $\mathcal{K}_{s_1} \subset \mathcal{K}_t$. Thus $M_t \subset \mathcal{H} \cap \mathcal{K}_t$ and therefore $\mathcal{H}_t \subset \mathcal{H} \cap \mathcal{K}_t$. Now consider an element of the form $\eta = E(t)j(\underline{s}, \underline{X})u$ where $u \in \mathcal{H}_0, \underline{s} = (s_1, ..., s_n)$ and $s_i \geq 0$ Then $\eta = j(t, s_1, ..., s_n, I, X_1, ..., X_n)u$. Since $(\mathcal{K}, E, j_t)$ is a conservative weak Markov flow it follows from Corollary 2.6 that we can express $\eta = j(t, s'_1, ..., s'_m, Y_0, Y_1, ..., Y_m)u$ where $t > s'_1 > \cdots > s'_m \geq 0$ and hence $\eta \in \mathcal{H}_t$. Thus $E(t)M \subset \mathcal{H}_t$ and therefore $\mathcal{H} \cap \mathcal{K}_t \subset \mathcal{H}_t$ proving the claim. Denote by $F(t)$ the projection on $\mathcal{H}_t$ in the Hilbert space $\mathcal{H}$ and $j_t(X)$ the restriction of $j_t(X)$ to $\mathcal{H}$. Then $(\mathcal{H}, F, j_t)$ is a conservative weak Markov flow satisfying properties (i) - (iii) of the theorem.

To prove (iv) we observe that the proofs of Theorem 2.5, Corollary 2.6 and Proposition 2.8 imply that

$$\langle j(\underline{s}, \underline{X})u, j(\underline{t}, \underline{Y})v\rangle = \langle j'(\underline{s}, \underline{X})u, j'(\underline{t}, \underline{Y})v\rangle$$

for all $u, v \in \mathcal{H}_0, \underline{s} = (s_1, ..., s_m), \underline{t} = (t_1, ..., t_n), \underline{X} = (X_1, ..., X_m), \underline{Y} = (Y_1, ..., Y_n)$. This shows that the correspondence $j(\underline{s}, \underline{X})u \to j'(\underline{s}, \underline{X})u$ is isometric and hence extends uniquely to a unitary isomorphism from $\mathcal{H}$ onto $\mathcal{H}'$ satisfying (iv). Observe that cyclicity (property (ii)) forces j'_t to be conservative.

Property (v) of the theorem is immediate from Proposition 2.11 and the fact that j_t is a homomorphism for every $t \geq 0$. ■

Now we extend Theorem 2.12 to non-conservative contractive semigroups.

THEOREM 2.13. – *Let $\mathcal{A}$ be a unital von Neumann algebra of operators in a Hilbert space $\mathcal{H}_0$ and let $\{T_t\}$ be a semigroup of completely positive linear maps of $\mathcal{A}$ into itself such that T_0 is identity and $T_t(I) \leq I$ for all t. Then there exists a subordinate weak Markov flow $(\mathcal{H}, F, j_t)$ on $\mathcal{A}$ satisfying* (i) - (v) *of Theorem* 2.12.

Proof. – Consider the extended von Neumann algebra $\widehat{\mathcal{A}} = \mathcal{A} \bigoplus \mathbb{C}$ acting on the Hilbert space $\widehat{\mathcal{H}}_0 = \mathcal{H}_0 \bigoplus \mathbb{C}$. For convenience we denote the element

$X \bigoplus c$ of $\widehat{\mathcal{A}}$, for $X \in \mathcal{A}$ and $c \in \mathbb{C}$, by the column vector $\binom{X}{c}$. Define the maps $\widehat{T}_t : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$ by

$$\widehat{T}_t\begin{pmatrix} X \\ c \end{pmatrix} = \begin{pmatrix} T_t(X) + c(I - T_t(I)) \\ c \end{pmatrix}, \qquad X \in \mathcal{A}, \quad c \in \mathbb{C}. \qquad (2.4)$$

Then $\{\widehat{T}_t\}$ is a conservative one parameter semigroup of completely positive linear maps. If $\{T_t\}$ is strongly continuous so is $\{\widehat{T}_t\}$. Thus Theorem 2.12 becomes applicable for $\{\widehat{T}_t\}$ and we have a conservative weak Markov flow $(\widehat{\mathcal{H}}, \widehat{F}, \hat{j}_t)$ on $\mathcal{A}$ with expectation semigroup $\{\widehat{T}_t\}$. Define the operators $F(t)$ and $j_t(X)$ on $\widehat{\mathcal{H}}$ by,

$$F(t) = \hat{j}_t\begin{pmatrix} I \\ 1 \end{pmatrix} - \hat{j}_0\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$j_t(X) = \hat{j}_t\begin{pmatrix} X \\ 0 \end{pmatrix} \qquad \text{for} \quad t \geq 0 \text{ and } X \in \mathcal{A}.$$

Before obtaining the required Markov flow we prove the following statements. For $0 \leq s \leq t, X \in \mathcal{A}$ and $c \in \mathbb{C}$

(a) $\left\{\hat{j}_t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ is a family of projections nondecreasing in t;

(b) $j_t(X)\hat{j}_0\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hat{j}_0\begin{pmatrix} 0 \\ 1 \end{pmatrix}j_t(X) = 0$;

(c) $\{F(t)\}$ is a family of projections nondecreasing in t;

(d) Range of $F(0)$ is $\mathcal{H}_0$ and range of $F(t)$ increases to the orthogonal complement of range of $\hat{j}_0\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as t increases to ∞;

(e) $F(t)\hat{j}_s\begin{pmatrix} X \\ c \end{pmatrix} = j_s(X - cI)F(s) + cF(s)$.

Property (a) follows from the identity

$$\begin{aligned} \hat{j}_s\begin{pmatrix} 0 \\ 1 \end{pmatrix}\hat{j}_t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\hat{j}_s\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \hat{j}_s\begin{pmatrix} 0 \\ 1 \end{pmatrix}\widehat{F}(s)\hat{j}_t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\widehat{F}(s)\hat{j}_s\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \hat{j}_s\begin{pmatrix} 0 \\ 1 \end{pmatrix}\hat{j}_s(\widehat{T}_{t-s}\begin{pmatrix} 0 \\ 1 \end{pmatrix})\hat{j}_s\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \hat{j}_s\begin{pmatrix} 0 \\ 1 \end{pmatrix}\hat{j}_s\begin{pmatrix} I - T_{t-s}(1) \\ 1 \end{pmatrix}\hat{j}_s\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \hat{j}_s\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Now make use of (a) to obtain

$$j_t(X)\hat{j}_0\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hat{j}_t\begin{pmatrix} X \\ 0 \end{pmatrix}\hat{j}_t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\hat{j}_0\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 = \hat{j}_0\begin{pmatrix} 0 \\ 1 \end{pmatrix}j_t(X)$$