W. Hazod

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Probabilities on contractible locally compact groups:
The existence of universal distributions in the sense of W. Doeblin

by

W. HAZOD
Institut für Mathematik der Universität Dortmund,
D-44221 Dortmund, Germany

Abstract. – We show that universally attractable probability distributions in the sense of W. Doeblin exist on locally compact contractible groups. As a consequence we obtain for this class of groups the famous characterisation of infinite divisibility due to Khinchine and Doeblin: A probability measure is infinitely divisible if its domain of attraction is non empty, and vice versa.

Key words: Contractible groups convolution semigroups, domains of partial attraction, Doeblin distributions.

Résumé. – Dans ce travail nous montrons l’existence de probabilités universellement attractables dans l’esprit de W. Doeblin pour des groupes localement compacts contractables. Comme conséquence on obtient aussi pour cette classe de groupes un théorème de type Khinchine-Doeblin: Les probabilités infiniment divisibles possèdent des domaines d’attraction partielle non vides et vice versa.

Classification A.M.S. : 43 A 05, 60 B 15, 60 B 10, 60 E 07 (60 F 17), 43 A 70.

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The investigation of limit theorems on locally compact groups leads in a natural way to contractible groups. Let \((X_t)_{t \geq 0}\) be an i.i.d. sequence of \(G\)-valued random variables with distribution \(v\). Let \(a \in \text{Aut}(G)\) be a contracting automorphism. Then for sequences of natural numbers \(k_n \uparrow \infty, n \in \mathbb{N}\), \(l_n \uparrow \infty, (a^{l_n}(X_n))_{i=1, \ldots, k_n; n \in \mathbb{N}}\) is an infinitesimal triangular array the row products of which are distributed according to \((a^{l_n}v)^{k_n} = a^{l_n}(v^{k_n})\). Since the group \(G\) is not supposed to be abelian we do not use centering terms to normalize the random variables. \(v\) is in the domain of (normal) partial attraction of \(\mu\) (abbreviated DNPA \((\mu; a)\) or DNPA \((\mu)\)) if \(a^{l_n}v^{k_n} \Rightarrow \mu\). (Convergence in the weak sense.)

In the "classical" situation \(G = \mathbb{R}\) or \(\mathbb{R}^d\) the possible limits \(\mu\) are infinitely divisible. W. Doeblin [Doe] proved the existence of universal distributions \(v \in \cap \{\text{DNPA}(\mu) : \mu\text{ infinitely divisible}\}\). Hence the well-known characterization of infinite divisibility by Doeblin and Khinchine [Doe] follows: \(\mu \in M^1(\mathbb{R})\) is infinitely divisible iff its domain of partial attraction is non empty. These results hold for Hilbert spaces ([Ba1], [Ba2]), more generally for Fréchet spaces [Ph] where \(a\) is an homothetical automorphism \(x \mapsto a \cdot x (a \in \mathbb{R}_+^*)\), and also for Banach spaces with contracting automorphism \(a\) [Th].

Our aim is to show analogous results for locally compact groups \(G\) with contracting automorphism \(a \in \text{Aut}(G)\).

In Section 1 we study on general locally compact groups the limit behaviour of sequences of discrete and continuous convolution semigroups and the embedability of the limit measures. For a new class of groups called strongly B-root-compact, a slight generalization of strong root-compactness ([He], [Si], [S]) we show that infinitely divisible measures are continuously embeddable up to a shift.

In Section 2 we improve our knowledge of contractible locally compact groups ([M-R], especially [Si1], [Si2]) and show that they belong to the class of groups studied in Section 1. Moreover, to any contractible group \(G\) there belongs a submonogeneous group \(S_n \subseteq \mathbb{R}\), such that to every \(x \in G\) there exists a unique homomorphism \(f : S_n \rightarrow G\) with \(f(1) = x\). Especially, we obtain that infinitely divisible laws are submonogeneously embeddable.

In Section 3 we prove the existence of universal distributions on contractible groups. The proof is similar to the "classical" proofs ([Doe], [Ba1], [Ph], [Th]) but instead of Fourier transforms we have to use the Lévy-Khinchine formula (see e.g. [He]) and the interplay between convergence of sequences of convolution semigroups and of the corresponding generating distributions. ([Ha], [Si1]; [H-S], [Kh]). Therefore the proofs only work for locally compact groups. It is an open question if on non locally compact contactible groups always Doeblin distributions exists.
DOEBLIN DISTRIBUTIONS

Notations. — Let in the following G be a locally compact group. Let $M^1(G)$ be the convolution semigroup of probability measures, endowed with the topology of weak convergence. $C_c^0(G)/C_0(G)/\mathcal{D}(G)/\mathcal{E}(G)$ are the function-spaces of bounded continuous/continuous functions vanishing at infinity/Bruchat-Schwartz test functions/resp. regular functions (i.e. $f \in \mathcal{E}(G)$ iff $f \in C_c^0(G)$ and $f \cdot g \in \mathcal{D}(G)$ for $g \in \mathcal{D}(G)$). (See e.g. [Ha], [Si1], [He]).

If $(\mu_t, t \geq 0, \mu_0 = \varepsilon_e)$ is a continuous convolution semigroup (short: c.c.s.) in $M^1(G)$ then the (infinitesimal) generating functional (or infinitesimal generating distribution) $A \in \mathcal{D}'(G)$ is defined $A := \frac{d^+}{dt} \mid_{t = 0}$. We use the notation $\text{Exp} t A := \mu_t, t \geq 0$, then, and denote $\mathcal{B}(G)$ the set of generating functionals. For $\mu \in M^1(G)$ resp. $A \in \mathcal{B}(G)$ let $R_{\mu}$ resp. $R_A$ be the convolution operators $f \mapsto \mu \ast f$ resp. $f \mapsto A \ast f$.

For a compact subgroup $K$ let $\omega_K$ be the normalized Haar measure. If $\lambda \in M^1(G), \alpha > 0$ then $A := \alpha(\lambda - \varepsilon_e) \in \mathcal{B}(G)$ is called Poisson generator, 

$$
\text{Exp} t A = \exp t \alpha(\lambda - \varepsilon_e) = e^{-\alpha t} \sum_{k \geq 0} \frac{t^k \alpha^k}{k!} \lambda^k 
$$

is called Poisson semigroup. More generally, if $\lambda = \omega_K \ast \lambda \ast \omega_K$, then $\exp_t(\alpha(\lambda - \varepsilon_e)) = \left(\omega_K + \sum_{k \geq 1} \frac{\alpha^k t^k}{k!} \lambda^k\right) e^{-\alpha t}$ is called Poisson measure with idempotent factor $\omega_K$. (For more details see e.g. [He] esp. Chap. IV, [Si1], [Ha] ch. O., ch. I.).

1. DOMAINS OF PARTIAL ATTRACTION AND EMBEDDABILITY

Let $G$ be a locally compact group.

Definition 1.1. — Let $\mu \in M^1(G)$.

a) The domain of partial attraction (in the strict sense) is defined as

$$
\text{DPA}(\mu) = \{ \nu \in M^1(G) : \text{There exist } a_n \in \text{Aut}(G), k_n \in \mathbb{N}, k_n \uparrow \infty, \text{ such that } a_n \nu \to \varepsilon_e \text{ and } a_n \nu^k \to \mu \}.
$$

The domain of normal partial attraction [w.r.t. $a \in \text{Aut}(G)$] is defined as

$$
\text{DNPA}(\mu; a) = \text{DNPA}(\mu) = \{ \nu : \text{There exists } k_n \uparrow \infty, l_n \uparrow \infty, \text{ such that } a_n^k \nu \to \varepsilon_e \text{ and } a_n^l \nu^k \to \mu \}.
$$

If $G$ is strongly root compact and aperiodic or if $a$ is contracting the infinitesimal-conditions $a_n \nu \to \varepsilon_e$, resp. $a_n^k \nu \to \varepsilon_e$ are automatically fulfilled. (cf. [No1], [No2], [H-S]).

b) Let $\nu_n := a_n \nu$ resp. $= a_n^k \nu$. If we consider $(\nu_n^{[k^a]})_{t \geq 0}$ as sequence of “discrete semigroups”, i.e. if we have a functional limit theorem in mind,
and if we suppose the existence of a c.c.s. \((\mu_t = \text{Exp } tA)\) with \(\mu_1 = \mu\), then we define

\[
\text{FDNPA}(A; a) = \text{FDNPA}(A) = \{ v : d^n v^{k_n t} \to \mu_t = \text{Exp } tA, t \geq 0 \}\.
\]

\(c\) Analogously, if we have the infinitesimal generating functionals in mind we define

\[
\text{IDNPA}(A; a) = \{ v : A_n := k_n (d^n v - e) \to A \text{ on } \mathcal{F}(G) \}.
\]

\(d\) Sometimes we have to use normalizing shifts. Then we define e.g.

\[
\text{DNPA}_S(\mu; a) = \{ v : d^n v^{l_n} e_{x_n} \to \mu \text{ for suitable sequences } l_n \uparrow \infty, k_n \uparrow \infty \text{ and } (x_n) \subseteq G \}.
\]

Remark 1.2. – In the “classical situation” (i.e. if \(G = \mathbb{R}\) or \(\mathbb{R}^d\)) the domains of attraction and the functional resp. infinitesimal versions coincide, i.e. we have

\[
\text{DNPA}(\mu_1) = \text{FDNPA}(A) = \text{IDNPA}(A).
\]

In general we can only prove a weaker result:

**Proposition 1.3.** – Let \((\mu_t = \text{Exp } tA)\) be a c.c.s. with \(\mu_1 = \mu\). Then

\(a\) \quad \text{IDNPA}(A; a) \subseteq \text{FDNPA}(A; a) \subseteq \text{DNPA}(\mu; a).

\(b\) \quad If \(G\) is Lie projective we can prove

\[
\text{IDNPA}(A; a) = \text{FDNPA}(A; a).
\]

\(c\) \quad If \(G\) is strongly root compact and aperiodic then

\[
\text{FDNPA}(A; a) = \text{DNPA}(\mu; a).
\]

(In this case we know that every \(\mu\) with DPA(\(\mu\)) \(\neq \emptyset\) is embeddable into a c.c.s., see [No1], [No2].)

\(a\) The left inclusion follows from [Ha] I. Satz 2.3, O. Section 2 Satz 4.2, see also [No1] (1.1), [No2] remark 2 b. The right inclusion is obvious.

\(b\) Convergence of discrete semigroups is equivalent to resolvent-convergence of the generating distributions ([Ha], [No1], [No2]), this is again equivalent to convergence of the c.c.s. \(\text{Exp } tA_n \to \text{Exp } tA, t \geq 0\). If \(G\) is Lie-projective this implies \(A_n \to A\) on \(\mathcal{F}(G)\). (The last assertion is proved in [Sil], p. 143, but not explicitly stated, see [Kh]. See also [H-S].)

\(c\) Let \(d^n v^{l_n} \to \mu\). According to [No1] (1.11) resp. [No2] Theorem 1 there exists a subsequence \((n') \subseteq \mathbb{N}\), and a c.c.s. \(\mu_t = \text{Exp } tA, \mu_1 = \mu\), such that \(d^n v^{l_n t} \to \mu, t \geq 0\).

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For the construction of universal distributions in paragraph 3 we need the following observation:

**Proposition 1.4**

\[
\begin{align*}
\quad & \quad (a) \quad \exists \in \text{FDNPA}(A; a) \iff \forall m \in \mathbb{Z}, \quad \exists \in \text{FDNPA}(a^m A; a) \\
\quad & \quad (a') \quad \exists \in \text{IDNPA}(A; a) \iff \forall m \in \mathbb{Z}, \quad \exists \in \text{IDNPA}(a^m A; a) \\
\quad & \quad (b) \quad \exists \in \text{FDNPA}(A; a) \iff \forall \alpha > 0, \quad \exists \in \text{FDNPA}(\alpha A; a) \\
\quad & \quad (b') \quad \exists \in \text{IDNPA}(A; a) \iff \forall \alpha > 0, \quad \exists \in \text{IDNPA}(\alpha A; a).
\end{align*}
\]

\([a), a') are obvious.\]

b) Obviously we have:

\[
d^h \nu^{[k_n \alpha]} \to \mu_t \quad \text{for any } t \geq 0 \iff d^h \nu^{[k_n \alpha]} \to \mu_{at} \quad \text{for } t \geq 0, \alpha > 0.
\]

The representation \([k_n \alpha t] = [[k_n \alpha] t] + s_n, 0 \leq s_n \leq [t] + 1\) yields

\[
d^h \nu^{[k_n \alpha t]} = d^h (\nu^{[k_n \alpha] t} \ast \nu^s) = d^h \nu^{[k_n \alpha] t} \ast d^h \nu^s \to \mu_{at}.
\]

On the other hand \(d^h \nu \to \nu_e \) implies \(d^h \nu^s \to \nu_e \), hence \(d^h \nu^{[k_n \alpha] t} \to \mu_{at}, t \geq 0.\)

b') Let \(k_n (d^h \nu - \nu_e) \to A, \) let \(\alpha > 0.\) Then

\[
\alpha k_n (d^h \nu - \nu_e) \to \alpha A,
\]

and hence because of

\[
0 \leq \alpha k_n - [\alpha k_n] \leq 1 \quad \text{and} \quad d^h \nu - \nu_e \to 0
\]

we obtain

\[
[\alpha k_n] (d^h \nu - \nu_e) \to \alpha A.]
\]

**Definition 1.6.** Let \(S \subseteq M^1(G)\) be a subset. The common domain of normal partial attraction with respect to \(a\) is defined:

\[
\text{NDPA}(S; a) = \cap_{\mu \in S} \text{NDPA}(\mu; a).
\]

And in an analogous way we define for a subset \(\mathcal{E} \subseteq \mathcal{B}(G)\) of generating functionals \(\text{FDNPA}(\mathcal{E}; a)\) resp. \(\text{IDNPA}(\mathcal{E}; a)\).

A probability measure \(\exists \in M^1(G)\) is called universal for \(S\) w.r.t. \(a\) if \(\exists \in \text{DNPA}(S; a)\).

Obviously we have

**Proposition 1.7.** Let \(S \subseteq M^1(G)\) and let \(S^-\) be the closure (w.r.t. the weak topology). Then if \(G\) is metrizable

\[
\text{DNPA}(S; a) = \text{DNPA}(S^-; a).
\]
Definition 1.8. – In the following we consider the following subsets of $M^1(G)$:

$\mathfrak{I} := \{\text{infinitely divisible measures}\}$,

$\mathfrak{J} := \{\text{idempotent measures } \mu = \omega_\kappa, K \text{ a compact subgroup}\}$,

$\mathcal{E} := \{\text{continuously embeddable measures}\}$

\[ = \{\mu: \text{there exists a c.c.s. } (\mu_\lambda) \text{ with } \mu_1 = \mu\}, \]

$\mathcal{E}_0 := \{\mu \in \mathcal{E}: \mu = \mu_1 \text{ where } (\mu_\lambda) \text{ is a c.c.s. with } \mu_0 = \varepsilon_{e}\}$

$\mathcal{F} := \{\text{continuously embeddable measures with trivial idempotent factor}\}$.

Definition 1.9. – $\nu$ is called Doeblin-distribution or universal distribution on $G$ w.r.t. $a$ if $\nu \in \text{DNPA}(\mathfrak{I}; a)$.

$\nu$ is universal in the wide sense if $\nu \in \text{DNPA}_S(\mathfrak{I}; a)$.

(Universal distributions in [Doe], [Bal], [Ph], [Th] are by definition universal in the wide sense.)

We define analogously

$\nu$ is called

D-universal if $\nu \in \text{DNPA}(\mathcal{E}; a)$

F-universal if $\nu \in \text{FDNPA}(\mathfrak{B}(G); a)$ ($\Rightarrow \nu \in \text{DNPA}(\mathcal{E}_0; a)$)

I-universal if $\nu \in \text{INDPA}(\mathfrak{B}(G); a)$.

It is easily shown that the following relations hold,

Proposition 1.10. – Let $G$ be metrizable.

a) $\nu$ I-universal $\Rightarrow$ $\nu$ F-universal $\Rightarrow$ $\nu$ D-universal.

b) $\nu \in \text{DNPA}(\mathcal{E}_0; a) \Rightarrow \nu \in \text{DNPA}(\mathcal{E}; a)$ ("\(\Rightarrow\) being obvious)

c) If $\mathcal{E}_0$ is dense in $\mathfrak{I}$ then

\[ \nu \in \text{DNPA}(\mathcal{E}_0; a) \Rightarrow \nu \in \text{DNPA}(\mathfrak{I}; a). \]

d) If $\mathfrak{I} \subseteq \{\mu \ast \varepsilon_x: \mu \in \mathcal{E}, x \in G\}$, i.e. if every infinitely divisible measure is embeddable up to a shift, then we have: $\nu \in \text{DNPA}(\mathcal{E}_0; a) \Rightarrow \nu$ is universal in the wide sense.

[a] See 1.3 a)

b) Follows from 1.7 since $\mathcal{E}_0$ is dense in $\mathcal{E}$.

c) Follows from 1.7.

d) is obvious.]

The assumptions 1.10c, d are fulfilled if $G$ is strongly root compact ([He], [Si2]). We need a slight generalization of these groups (see [S] for similar definitions):

Definitions 1.11. – a) Let $B \subseteq \mathbb{N}$ be a nontrivial multiplicative subsemigroup of $\mathbb{N}$. Let

\[ S_B := \left\{ \frac{k}{m}: k \in \mathbb{Z}, m \in B \right\}, \quad S_B^+ := \{ r \in S_B: r > 0 \}. \]
SB is submonogeneous, i.e. there exists a submonogeneous basis 

\((m_i) \subseteq B, r_i \in \mathbb{N} \setminus \{1\}, \text{ such that } m_i r_i = m_{i+1}, i \geq 1\)

and

\[ SB = \bigcup_{1}^{\infty} \frac{1}{m_i} \mathbb{Z} \tag{cf. e.g. [He]} \]

b) Let \(B \subseteq \mathbb{N}\). A locally compact group is B-root-compact if for any compact set \(K \subseteq G\), for any \(m \in B\) there exist compact sets \(C_m \subseteq G\), such that given \(x_1, \ldots, x_m \in G\), \(x_m = e\), with

\[ K x_i K x_j \cap K x_{i+j} \neq \emptyset \quad \text{for } 1 \leq i, j \leq i+j \leq m \]

we have \(x_i \subset C_m\), \(1 \leq i \leq m\). G is strongly B-root compact if we can choose \(C_m = C\) independently from \(m \in B\). (See e.g. [He], [Si1], [S].)

In the following we shall always suppose in addition that B is a multiplicative subsemigroup of \(\mathbb{N}\).

**PROPOSITION 1.12.** a) Let G be B-root compact. Let \(\mu \in M^1(G)\) be B-divisible, i.e. for any \(m \in B\) there exists a root \(\mu_{(m)} \in M^1(G)\) with \(\mu_{(m)}^m = \mu\).

Then \(\mu\) is \(S_B^+\)-submonogeneously embeddable, i.e. there exists a homomorphism \(S_B^+ \ni \mu_r \mapsto \mu_r, r \in M^1(G)\) with \(\mu_1 = \mu\).

b) If moreover G is strongly B-root compact then \(\{\mu_r : 0 < r \leq 1\}\) is uniformly tight. Furthermore, there exist a c.c.s. \((v_t)_{t \geq 0}\) and a homomorphism \(f : S_B^+ \rightarrow G, f(r) = x_r\), such that \(f(S_B^-)\) is compact, such that

\[ v_r \ast e_{x_r} = e_{x_r} \ast v_r, \quad r \in S_B^+, \quad t \geq 0, \]

and

\[ \mu_r = e_{x_r} \ast v_r, \quad r \in S_B^+. \]

Moreover the measures \(e_{x_r} \ast \omega_K = \omega_K \ast e_{x_r}, \text{ where } v_0 = \omega_K\), are accumulation points of \(\{\mu_r\}_{r \geq 0}, r \in S_B^+\).

[The proofs are similar to [Si2] Section 2, Section 6, [He] 3.1, 3.2-3.5, [S] Section 1. Note that if \(B \neq \mathbb{N}\) the group \(\{x_r K\}_{r \in S_B^+}\) is in general not connected.]

Next we prove, that on strongly B-root compact groups, measures with non-void domains of partial attraction are shifts of embeddable measures. Indeed, we prove more generally:

**THEOREM 1.13.** Let \(B \subseteq \mathbb{N}\) be a nontrivial multiplicative semigroup. Let G be strongly B-root-compact. Let \(v \in M^1(G), v_n \rightarrow e_v\), and assume \(v_n \rightarrow \mu \in M^1(G), k_n \uparrow \infty\).

a) Then there exists a c.c.s. \((\lambda_t)_{t \geq 0}\), a homomorphism \(f : S_B \rightarrow G, f(r) = x_r\), such that \(e_{x_r} \ast \lambda_t = \lambda_t \ast e_{x_r}, r \in S_B^+, t \geq 0\), such that \(f(S_B^-)\) is compact, and such that \(\mu_r = e_{x_r} \ast \lambda_t, r \in S_B^+\) is a submonogeneous convolution semigroup with \(\mu_1 = \mu\).
b) Moreover, for any $r \in S^+_B$ and any $t > 0$ there exist sequences $\alpha_n, \beta_n, \gamma_n, \delta_n$ in $\mathbb{N}$ (depending on $r$ resp. $t$) such that

$$\nu_{\alpha_n} \rightarrow \mu_r, \quad \nu_{\beta_n} \rightarrow \lambda_t, \quad n \rightarrow \infty.$$  

**Proof.** We start with the following simple observation:

Let $\xi_n, \eta_n, \rho \in M^1(G)$, such that $\eta_n \rightarrow \varepsilon$ and $\xi_n \ast \eta_n \rightarrow \rho$. Then $\xi_n \rightarrow \rho$.

Fix $m \in \mathbb{N}$ and put $\xi_n := \nu_{\frac{k_n}{m} \cdot m}$, $\eta_n := \nu_{\frac{k_n}{m} \cdot m}$, $\rho := \mu$.

Since $\nu_{\alpha_n} \rightarrow \mu$, $\nu_{\beta_n} \rightarrow \varepsilon$ and $0 \leq k_n - \frac{k_n}{m} \cdot m \leq m$ (hence $\eta_n \rightarrow \varepsilon$) the observation above yields $\nu_{\frac{k_n}{m} \cdot m} \rightarrow \mu$.

Let $(m_i), (r_i) \subseteq \mathbb{N}$ be a submonogeneous basis for $S_B$, i.e.

$$m_i r_i = m_{i+1}, \quad i \in \mathbb{N}, \quad \bigcup_{i=1}^{\infty} \frac{1}{m_i} \mathbb{Z} = S_B.$$  

We obtain $\nu_{\frac{k_n}{m_1} \cdot m_1} \rightarrow \mu$, and applying the considerations above again,

$$\nu_{\left(\frac{k_n}{m_1} \cdot m_1\right) r_1 \cdot m_1} \rightarrow \nu_{\left(\frac{k_n}{m_1} \cdot m_1\right) r_1 \cdot m_2} \rightarrow \mu,$$  

etc.

So, for $i \in \mathbb{N}$ we obtain a sequence $k_n^{(i)} \uparrow \infty$, such that

$$\nu_{\frac{k_n^{(i)}}{m_i}} \rightarrow \mu \quad \text{and} \quad m_j \cdot k_n^{(i)}, \quad j \leq i, \quad n \in \mathbb{N}.$$  

Put $\mathcal{N} := \bigcup_{i=1}^{\infty} \{ \nu_{\frac{k_n^{(i)}}{m_i}} \} \cup \{ \mu \}$, and define the root sets

$$R_i := \left\{ \nu_{\frac{k_n^{(i)}}{m_i} \cdot m} : 0 \leq s \leq m_i, \quad n \in \mathbb{N} \right\}$$  

and

$$R_B := \left( \bigcup_{i=1}^{\infty} R_i \right)^{-} \subseteq \{ \rho^s : \exists m_i \in B \text{ such that } 0 \leq s \leq m_i \text{ and } \rho^m \in \mathcal{N} \}.$$  

An essential step in the proof of the embedding theorem for strongly root compact groups ([Si2] Section 6, Satz 1, [He] Thm. 3.1.13, $B = \mathbb{N}$) and [S]) yields the compactness of the root set $R_B$.

Now we continue as in the case $B = \mathbb{N}$ and construct a submonogeneous convolution semigroup $(\mu_r)_{r \in S^+_B}$ in $R_B$ with $\mu_1 = \mu$.

The compactness of $R_i$, $i \in \mathbb{N}$, yields the existence of $m_i$-th roots $\rho_i \in R_i^-$, $\rho_i^{m_i} = \mu$. Hence $\mu$ is $B$-divisible and we can apply proposition 1.12 a) to obtain $(\mu_r)_{r \in S^+_B}$.

Apply now 1.12 b) to obtain a c.c.s. $(\lambda_r)$ and $(x_r)_{r \in S^+_B}$ such that $\mu_r = e_{x_r} \ast \lambda_r$, $r \in S^+_B$.

Moreover the construction of $(\mu_r)$ yields the existence of a subsequence $(n') \subseteq \mathbb{N}$ such that

$$\nu_{\frac{k_n}{m_i} \cdot m_i} : n \rightarrow \mu_{i/m_i}, \quad \mu \subseteq \mathbb{N}, \quad n \in \mathbb{N}.$$  

On the other hand the measures $e_{x_r} \ast \omega_k$, $s \in S_B$, $\omega_k = \lambda_0$, are accumulation points of $\{ \mu_r \}_{r \rightarrow 0}$, therefore for $r \in S^+_B$, $\lambda_r = e_{x_r} \ast \omega_k \ast \mu_r$ is an accumulation.
2. THE STRUCTURE OF CONTRACTIBLE GROUPS

G is called contractible if there exists \( a \in \text{Aut}(G) \), such that \( a^n x \to e \), \( x \in G \). In this case [Si3] we have a representation \( G = G_0 \otimes D \), where \( G_0 \) is an \( a \)-invariant contractible Lie group (hence especially nilpotent and simply connected) and \( D \) is an \( a \)-invariant contractible totally disconnected group. There exist compact neighbourhoods \( (U_n)_{n \in \mathbb{Z}} \subseteq G \) of \( e \), such that

\[
U_n \downarrow, \quad a U_n = U_{n+1}, \quad n \in \mathbb{Z}, \quad \bigcup_n U_n = G, \quad \bigcap_n U_n = \{ e \}.
\]

If \( G = D \) we can choose \( U_n \) as compact open subgroups, such that \( U_{n+1} \triangleleft U_n, \quad n \in \mathbb{Z} \) and \( U_n/U_{n+1} \cong F \), a fixed finite group. Let \( \Delta := \text{card}(F) \). \( (U_n) \) is called a filtration then.

**Definition 2.1.** — With the notations above put

\[
B := \{ m \in \mathbb{N} : (m, \Delta) = 1 \}.
\]

\( B \) is an infinite multiplicative semi-group \( \not\subseteq \mathbb{N} \). Hence \( S_B^+ = \left\{ \frac{k}{m} : k \in \mathbb{Z}, \quad m \in B \right\} \) and \( S_B^+ \) are well defined by the totally disconnected part \( D \) of the contractible group \( G \). If \( G = G_0 \), i.e. \( D = \{ e \} \), we put \( B := \mathbb{N}, \quad S_B := \mathbb{Q} \).

**Proposition 2.2.** — Let \( G = G_0 \otimes D \) be contractible. Let \( F, \Delta, B, S_B \) and \( S_B^+ \) as above. Then \( D \) and hence \( G \) are strongly B-root-compact.

\( [G_0 \) is strongly root compact ([Si2], [He]), hence strongly B-root-compact. It is sufficient to prove the strong B-root-compactness of \( D \).

Let \( K \subseteq D \) be a compact subset. Let \( (U_n)_{n \in \mathbb{Z}} \) be a filtration, \( U_n/U_{n+1} \cong F \). Fix \( n_0 \in \mathbb{Z} \), such that \( K \subseteq U_{n_0} \).

Let \( m \in B, \quad x_1, \ldots, x_m \in D, \quad x_m = e, \quad m_0 \in \mathbb{N} \), such that \( m_0 < n_0 \), and

\[
\bigcup U_{n_0} x_i \subseteq U_{m_0}, \quad U_{n_0} x_i U_{n_0} x_j \cap U_{n_0} x_i + j \neq \emptyset, \quad 1 \leq i, j \leq i + j \leq m.
\]

From \( x_m = e \), and \( U_{m_0+1} \triangleleft U_{m_0} \) we obtain

\[
U_{m_0+1} x_i U_{m_0+1} x_j = U_{m_0+1} x_i + j.
\]
therefore the elements \( \{ \bar{x}_i : = U_{m_0+1} x_i : 1 \leq i \leq m \} \) form a cyclic subgroup of order \( o_m | m \) in \( U_{m_0}/U_{m_0+1} \). But \( U_{m_0}/U_{m_0+1} = F \) and \((o(F), m) = 1\), hence \( \bar{x}_i = \bar{e}, \ 1 \leq i \leq m \), i.e. \( x_i \in U_{m_0+1} \). Repeating these arguments we obtain finally \( x_i \in C : = U_{n_0}, \ 1 \leq i \leq m. \]

**Theorem 2.3.** Let \( G \) be as above. Then the elements of \( G \) are uniquely m-divisible for any \( m \in B \) and for \( x \in D \) the (unique) m-th root of \( x \) is contained in the (monothetic, compact, abelian) group \( \langle x \rangle \) generated by \( x \).

Hence for any \( x \in G \) there exists a uniquely defined homomorphism \( f: S_B \to G \) such that \( f(1) = x \), and \( f(S_B) \subseteq \langle x \rangle \).

[Let \( G = G_0 \otimes D \). For \( x \in G_0 \) the assertion is obvious since \( G_0 \) is nilpotent and simply connected. Consider therefore the case \( x \in D \). Let \( m \in B \). Assume \( y \in D, x \in D \), such that \( y^m = x \). Hence \( x \) and \( y \) are contained in the compact abelian group \( \langle y \rangle : = A \). We show next that given any compact abelian group \( A \subseteq D, x \in A, m \in B \) there exists a unique \( y \in A \), such that \( y^m = x \), and moreover \( y \in \langle x \rangle \).

[We have \( A \subseteq U_{n_0} \) for some \( n_0 \). Put \( A_n : = A \cap U_n, \ n \in \mathbb{Z} \) (hence \( A_{n_0} = A \)). Then \( A_n \downarrow \{ e \} \) and \( \text{card}(A_n/A_n+1) | \Delta \).

Since \((m, \Delta) = 1\) the group \( A/A_n(n \geq n_0) \) is uniquely divisible by \( m \). Hence for fixed \( n \geq n_0 \) for any \( x \in A \) there exists a unique coset \( y A_n \in A/A_n \), such that \( (y A_n)^m = y^m A_n = x A_n \), and \( y A_n \in \langle x A_n \rangle \).

The compactness of \( A = A_{n_0} \) and the filtration property \( A_n \downarrow \{ e \} \) yield the existence of \( y \in A \), such that \( y^m = x \). Moreover, \( y \) is unique and \( y \in \langle x \rangle \).

Now choose a submonogeneous basis \((m_i)\) for \( S_B = \bigcup_{1}^{\infty} \frac{1}{m_i} \mathbb{Z} \) and apply the \( m_i \)-divisibility successively.

The assertion is proved.]

**Remarks 2.4.** a) For \( p \)-adic groups \( Q_p \) the strong B-root compactness is proved in [S]. \( p \)-adic groups \( Q_p \) (and e.g. \( p \)-adic Heisenberg groups) are examples of contractible groups which are root compact but not strongly root-compact ([Si3], ex. 3.5).

b) There exist contractible totally disconnected groups which are strongly B-root compact (for suitable \( B \subseteq \mathbb{N} \) but not root-compact: Let \( F \) be a fixed finite group with order \( o(F) = \Delta \in \mathbb{N} \setminus \{1\} \). Let \( \Gamma : = \otimes F, U_n : = \otimes F \). The groups \( (U_n)_{n=1}^{\infty} \) endowed with the product topology are a (normal) filtration of the contractible group \( \Gamma^* : = \bigcup_{n} U_n \) [Si3].

It is easily seen that for any \( 1 < \delta | \Delta^k(k \in \mathbb{N}) \) and any \( n \in \mathbb{Z} \) the unit element \( e = e U_n \) has infinitely many \( \delta \)-roots in \( \Gamma^*/U_n \). Hence \( \Gamma^* \) is not root compact.
c) The results of 1.12, 1.13, 1.14 hold especially for contractible groups G= \( G_0 \otimes D \). Moreover, in the representation \( \mu_r = \epsilon_{x_r} \star \lambda_r \star \omega_k \) obtained in 1.12 the shifts \( x_r \) are uniquely determined by \( x_1 \) (mod K).

d) For p-adic matrix groups infinitely divisible measures are continuously embeddable up to a shift. This is proved in [S].

e) Contractible groups are in general not Lie-projective ([Si3], 3.5 b). Indeed, the totally disconnected part D is Lie projective if it has a normal filtration.

Hence, since the key result of E. Siebert on convergence of semigroups (see Prop. 1.3) is only proved for Lie projective groups, it was necessary in Section 1 to distinguish between the domains FDNPA and IDNPA.

f) Let G be contractible, let \( \Delta, B \) as above. Let \( \mu \) be infinitely divisible and let \( r \mapsto \mu_r, \ r \mapsto x_r \) be submonogeneous homomorphisms, such that \( \mu_r = \epsilon_{x_r} \star \lambda_r, \ r \in S_B^+ \), where \( \mu_1 = \mu \) and \( (\lambda_r) \) is a c.c.s. Then for \( (m_n) \subseteq B, m_n \uparrow \infty, k_n \in \mathbb{N}, \) such that \( \epsilon_n = \frac{k_n \Delta^n}{m_n} \) we obtain \( \mu_r^{(n)} = \mu_r \star_{n} \epsilon_{[x_r \cdot k_n/m_n]} \Delta^n \to \lambda_r, r \in S_B^+ \).

[The construction of the homomorphism \( f: r \mapsto x_r \) yields: For any sequence \( (s_n) \subseteq S_B \) we have \( (x_{s_n})^\Delta = x_{s_n e} \to e \). Hence the continuity of \( (\lambda_r) \) yields

\[ \mu_r^{(n)} = \lambda_r \star_{n} \epsilon_{[x_r \cdot k_n/m_n]} \Delta^n \to \lambda_r. \]

\]

g) Let G be contractible, B, S_B as above. Let \( (\lambda_t)_{t \geq 0} \) be a continuous convolution semigroup in \( M^1(G) \). Let \( x \in D \) such that \( \lambda_t \epsilon_x = \epsilon_x \star \lambda_t, t > 0 \). Let \( f: r \to x_r \) be the submonogeneous homomorphism with \( x_1 = x \). Then \( (\epsilon_r \star \lambda_r)_{r \in S_B^+} \) is a submonogeneous (in general non-continuous) convolution semigroup.

\[ \epsilon_{x_r} \star \lambda_r = \lambda_r \star \epsilon_{x_r} \text{ since } x_r \in \langle x_1 \rangle^{-}. \]

3. THE EXISTENCE OF UNIVERSAL DISTRIBUTIONS ON CONTRACTIBLE GROUPS

We show that on contractible locally compact groups with contracting automorphism \( a \in \text{Aut}(G) \) there exist I-universal distributions \( v \) (see Definition 1.9). For connected \( G = G_0 \), especially for vector-spaces \( \mathbb{R}^d \), \( v \) is a Doeblin distribution then (proposition 1.10 c). For general contractible groups remark 2.5 c and corollary 1.14 yield that \( v \) is universal in the wide sense.
3.1. In order to prove the existence of I-universal distributions on $G$ w.r.t. $a \in \text{Aut}(G)$ we reduce the problem along the following steps:

3.1. $a$ The set of Poisson generators $\mathcal{P} := \{\alpha(\lambda - \varepsilon_\alpha) : \lambda \in M^1(G), \alpha > 0\}$ is dense in $\mathcal{B}(G)$ with respect to the weak topology $\sigma(\mathcal{B}', \mathcal{B})$. Therefore, if $v \in \text{IDNPA} (\mathcal{P}; a)$ then $v$ is I-universal.

[Let $P_m \in \mathcal{P}$, $P_m \to A \in \mathcal{B}(G)$ on $\mathcal{B}'(G)$. For any $m \in \mathbb{N}$ there exist $k^{(m)}_n$, $l^{(m)}_n \uparrow \infty$ such that $k^{(m)}_n(a^{(m)}_n v - \varepsilon_\alpha) \to P_m$. Since $G$ is metrizable we can find suitable subsequences $k^{(m)}_n, l^{(m)}_n$ such that $a^{(m)}_n(v^{k^{(m)}_n} - \varepsilon_\alpha) \to A.$]

3.1. $b$ Let $(U_n)$ be a basis of neighbourhoods of $e$, such that $U_n \cap \{e\} = U_n = G$. Then $\Gamma := \{a^m \lambda : m \in \mathbb{Z}, \lambda \in M^1(G), \text{supp}(\lambda) \subseteq U_0\}$ is dense in $M^1(G)$. Let $\mathcal{P}_1 := \{\alpha(\lambda - \varepsilon_0) : \alpha > 0, \lambda \in M^1(G), \text{supp}(\lambda) \subseteq U_0\}$. We apply proposition 1.4. $a$, $a'$, and similar to 3.1. $a$ we have:

If $v \in \text{IDNPA} (\mathcal{P}_1; a)$ then $v$ is I-universal.

3.1. $c$ According to proposition 1.4. $b$, $b'$ it is sufficient to prove the existence of $v \in \text{IDNPA} (\mathcal{P}_2; a)$, where $\mathcal{P}_2 := \{\lambda - \varepsilon_0 : \lambda \in M^1(G), \text{supp}(\lambda) \subseteq U_0\} \subseteq \mathcal{P}_1$.

3.1. $d$ Let $\{\lambda_m : m \in \mathbb{N}\}$ be a fixed dense countable subset of $\{\lambda \in M^1(G) : \text{supp}(\lambda) \subseteq U_0\}$. Hence $\{\lambda_m - \varepsilon_0\}$ is dense in $\mathcal{P}_2$. Let further $\alpha(n), \beta(n), \gamma(n)$ be sequences of positive numbers such that

(i) $\sum_{n=1}^{\infty} \alpha(n) = 1$,

(ii) $\alpha(n)^{-1} \in \mathbb{N}, \quad \alpha(n) \downarrow 0$

(iii) $\alpha(N)^{-1} \sum_{n=N+1}^{\infty} \alpha(n) \to 0$

(iv) $\beta(n) \in \mathbb{N}, \quad \beta(n) \uparrow \infty$

(v) $\gamma(n) := \beta(n) - \beta(n-1) \uparrow \infty$

(vi) For any $\rho \in (0, 1)$, $\rho^{(n)}/\alpha(n) \to 0$.

[The sequences used in [Doe], [Bal], [Ph], [Th] $\alpha(n) \sim c.2^{-n^2}$, $\beta(n) \sim n^3$, $\gamma(n) \sim 3n^2 - 3n + 1$ fulfill (i) – (vi).]

Let $a \in \text{Aut}(G)$ be a fixed contracting automorphism. Define

$v := \sum_{n=1}^{\infty} \alpha(n) a^{-\beta(n)}(\lambda_n) \in M^1(G).$

**Theorem 3.2.** The measure $v$ defined above is I-universal w.r.t. $a$.

**Proof.** According to the reduction steps 3.1. $a$-3.1. $c$ it is sufficient to show that for any $\lambda \in M^1(G)$ with $\text{supp}(\lambda) \subseteq U_0$ there exist $L(k) \uparrow \infty$,
Let \( \lambda \) be a subsequence such that \( \lambda_{n_k} \to \lambda \) and \( n_k \uparrow \infty \). (This is of course possible even if \( \lambda \in \{ \lambda_m : m \in \mathbb{N} \} \).) We define then \( L(k) := \beta(n_k) \), \( M(k) := \alpha(n_k)^{-1} \).

The approximating Poisson generators are represented in the form

\[
M(k)(a^{L(k)}v - \varepsilon_e) = M(k) \left( \sum_{n=1}^{\infty} \alpha(n) \left( a^{L(k)} - \beta(n) \lambda_n - \varepsilon_e \right) \right)
= P_k + (\lambda_{n_k} - \varepsilon_e) + Q_k,
\]

where

\[
P_k := M(k) \sum_{n=1}^{n_k-1} \alpha(n) \left( a^{L(k)} - \beta(n) \lambda_n - \varepsilon_e \right)
\]

and

\[
Q_k := M(k) \sum_{n_{k+1}}^{\infty} \alpha(n) \left( a^{L(k)} - \beta(n) \lambda_n - \varepsilon_e \right).
\]

We have to show \( P_k \to 0 \) and \( Q_k \to 0 \) on \( \mathcal{E}(G) \).

3. \( \| Q_k \| \to 0 \).

\[
\| Q_k \| = 2 \cdot Q_k(G \setminus \{e\}) \leq 2 Q_k(G) = \frac{2}{\alpha(n_k)} \sum_{n_k+1}^{\infty} \alpha(n) \to 0 \quad \text{[by 3.1. d. (iii)].}
\]

4. Let \( V \) be a neighbourhood of the unit. Then we obtain for the Lévy measures \( P_k(\mathcal{E}(V)) \to 0 \).

\[
\left[ \text{Let } \eta_k := P_k|_{\mathcal{E}(V)} \right]
\]

Since

\[
\text{supp}(\lambda_n) \subseteq U_0, \quad \text{supp}(\eta_k) \subseteq \bigcup_{n=1}^{n_k-1} U_{\beta(n_k) - \beta(n)} \subseteq U_{\beta(n_k) - \beta(n_k - 1)} = U_\tau(n_k).
\]

Since \( U_n \downarrow \{e\} \) and \( \gamma(n_k) \uparrow \infty \) we have \( \text{supp}(\eta_k) \subseteq V \) for sufficiently large \( k \), hence \( \eta_k = 0 \).

Remark 5. – If \( G = D \) is totally disconnected, 3. and 4. already imply \( P_k + Q_k \to 0 \), hence \( M(k)(a^{L(k)}v - \varepsilon_e) \to \lambda - \varepsilon_e \).

6. Let \( G = G_0 \) be a contractible Lie group. Let \( \zeta_1, \ldots, \zeta_k \) be local coordinates and let \( \varphi \) be a Hunt function (cf. [He] 4.5).
Let \( \mathcal{G} \) be the Lie algebra. For functions \( f \) on \( G \) let \( \hat{f} = f \circ \exp \) on \( \mathcal{G} \), analogously define \( \hat{a}, \hat{\mu}, \hat{A} \) for automorphisms, measures resp. generating distributions. W.l.o.g. we may assume that

\[
\hat{a} \text{ is a contracting automorphism on the vector space, hence for some } r \geq 0 \| \hat{a}^r \| = : \delta < 1.
\]

6.1.

\[
|\langle P_k, \xi_i \rangle| = |\langle \hat{P}_k, \xi_i \rangle| \leq \sum_{n=1}^{n_k} \frac{\alpha(n)}{\alpha(n_k)} \int_{\hat{U}_0} |\hat{\xi}_i(\hat{a}^{\beta(n_k) - \beta(n)} x)| d\hat{\lambda}_n(x).
\]

Here \( \hat{U}_0 = \exp^{-1}(U_0) \). Let \( R = \max_{x \in \hat{U}_0} \| x \|, \quad K = \max_{0 \leq j \leq r-1} \| \hat{a}^j \| \). Then

\[
|\langle P_k, \xi_i \rangle| \leq \frac{1}{\alpha(n_k)} \sum_{n=1}^{n_k} \alpha(n) \sup_{x \in \hat{U}_0} \| \hat{\xi}_i(\hat{a}^{\beta(n_k) - \beta(n)} x) \|
\]

\[
\leq \frac{R}{\alpha(n_k)} \sum_{n=1}^{n_k} \alpha(n) \| \hat{a}^{\beta(n_k) - \beta(n)} \|
\]

\[
\leq \frac{R \cdot K}{\alpha(n_k)} \sum_{n=1}^{n_k} \alpha(n) \delta^{(\beta(n_k) - \beta(n))/r}
\]

\[
\leq \frac{R \cdot K}{\alpha(n_k)} \sum_{n=1}^{n_k} \alpha(n) \delta^{\gamma(n_k)/r}.
\]

Using \( \gamma(n_k)/r = \gamma(n_k)/r - \varepsilon, \varepsilon \in [0, 1] \), we obtain by 3.1.d (vi)

\[
|\langle P_k, \xi_i \rangle| \leq \frac{R \cdot K}{\alpha(n_k)} \sum_{n=1}^{n_k} \alpha(n) (\delta^{1/r})^{\gamma(n_k)} \cdot \delta^{-1} \leq \frac{(\delta^{1/r})^{\gamma(n_k)}}{\alpha(n)} \cdot (R \cdot K \cdot \delta^{-1}) \rightarrow 0.
\]

6.2. Analogously we obtain

\[
0 \leq \langle P_k, \varphi \rangle = \langle \hat{P}_k, \varphi \rangle
\]

\[
= \frac{1}{\alpha(n_k)} \sum_{n=1}^{n_k} \alpha(n) \int_{\hat{U}_0} \hat{\varphi}(\hat{a}^{\beta(n_k) - \beta(n)} x) d\hat{\lambda}_n(x)
\]

\[
\leq \frac{1}{\alpha(n_k)} \cdot R \cdot K^2 \cdot \left[(\delta^{1/r})^{\gamma(n_k)}\right]^2
\]

\[
= \text{const. } (\delta^{2/r})^{\gamma(n_k)} / \alpha(n_k) \rightarrow 0
\]

[again by 2.1 (vi)].

Remark 6.3. If \( G = G_0 \) is a Lie group we obtain by steps 4., 6.1. and 6.2. \( P_k \rightarrow 0 \) (see e.g. [Si 1] 5.4, 8.1 remark 4.7).

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7. Let $G = G_0 \otimes D$ be a contractible group. Let $\mathcal{D} := \mathcal{D}(G_0) \otimes \mathcal{D}(D)$. $\mathcal{D}$ is a core for the generators of convolution semigroups (e.g. [Ha] 0. Section 4, Satz of F. Hirsch). Moreover the distributions $P_k$ are concentrated on the fixed compact neighbourhood $U_0$. Hence the assertion $\langle P_k, f \rangle \to 0, f \in \mathcal{D}_d(G)$, is equivalent to $\langle P_k, f \rangle \to 0, f \in \mathcal{D}$. Therefore we have to show $\langle P_k, f \otimes g \rangle \to 0$ for $f \in \mathcal{D}_d(G), g \in \mathcal{D}_d(D)$.

For any $g \in \mathcal{D}_d(D)$ there exists a compact open subgroup $V_g \subseteq D$, such that $g(\kappa x) = g(x)$ for $x \in D$, $\kappa \in V_g$.

Let $\lambda_n^1$ be the projection of $\lambda_n$ onto $G_0$ and let $U_0^1$ be the projection of $U_0$ to $G_0$. Hence we have

$$\langle P_k, f \otimes g \rangle = \frac{1}{\alpha(n_k)} \sum_{n=1}^{n_k-1} \alpha(n) \int_{G_0 \otimes D} \left[ f(x) g(\kappa) - f(e) g(e) \right] d\lambda_0^{\beta(n_k) - \beta(n)}(\lambda_n)(x, \kappa)$$

$$= \frac{1}{\alpha(n_k)} \sum_{n=1}^{n_k-1} \alpha(n) \left\{ \int_{D} \left[ (f (a^{\beta(n_k) - \beta(n)} x) - f(e)) \right] \cdot \lambda_n(x, \kappa) \right\}$$

If $k$ is sufficiently large, such that $a^{\beta(n_k) - \beta(n)} \kappa \in V_g$ for $(x, \kappa) \in U_0$, then the second integral is zero and therefore

$$\langle P_k, f \otimes g \rangle = \frac{1}{\alpha(n_k)} \sum_{n=1}^{n_k-1} \alpha(n)$$

$$\times \int_{U_0} (f (a^{\beta(n_k) - \beta(n)} x) - f(e)) \cdot g(e)$$

$$= \frac{g(e)}{\alpha(n_k)} \sum_{n=1}^{n_k-1} \int_{U_0} (f (a^{\beta(n_k) - \beta(n)} x) - f(e)) \frac{d\lambda_0^{\beta(n_k) - \beta(n)}(x)}{k \to \infty}$$

(as proved in step 6.2).

Theorem 3.2 is proved. □

Now we are able to prove a characterization of infinite divisibility (for $G = \mathbb{R}$ due to Doeblin and Khinchine, see [Doe]):

**Theorem 3.3.** - [Doe], [Ba1], [Ba2], [Ph], [Th])

Let $G$ be locally compact with contracting automorphism $a$. Let $\mu \in M^1(G)$.

Consider the following assertions:

(i) $\mu$ is infinitely divisible

(ii) $\mu$ is $B$-divisible (i.e. for $m \in B$ there exists a root $\mu_{(m)} \in M^1(G)$ with $\mu_{(m)}^m = \mu$).

[Here, if $G = G_0$ is a Lie group, $B$ is any infinite subset of $\mathbb{N}$. If $G = G_0 \otimes D$ is contractible and $D \neq \{e\}$ then $B$ is defined as in Definition 2.1.]

(ii) $\mu$ is continuously embeddable

(ii') There exists a shift $\varepsilon_x$ such that $\mu \ast \varepsilon_x = \varepsilon_x \ast \mu$ is embeddable into a c.c.s. $\lambda_t$ with $\varepsilon_x \ast \lambda_t = \lambda_t \ast \varepsilon_x$, $t \geq 0$.

(iii) DNPA $(\mu; a) \neq \emptyset$

(iv) DPA $(\mu) \neq \emptyset$.

a) We have

$$(ii) \Rightarrow (i) \Rightarrow (i') \Leftrightarrow (ii')$$

and

$$(ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i')$$

b) If $G = G_0$ is a Lie group, then all the assertions (i) to (iv) are equivalent.

[a] (ii)$\Rightarrow (i) \Rightarrow (i'), (iii) \Rightarrow (iv)$ obvious.

(iv)$\Rightarrow (i)$ by Theorem 1.13. a. (ii')$\Rightarrow (i')$ since by Theorem 2.3 every shift $\varepsilon_x$ is submonogeneously embeddable. (i')$\Rightarrow (ii')$ by Proposition 1.12. (ii)$\Rightarrow (iii)$ by Theorem 3.2.

b) (i)$\Leftrightarrow (ii)$ see [He] 3.5.8, 3.5.9 since $G = G_0$ is nilpotent. (i)$\Rightarrow (i')$ obvious, (i')$\Rightarrow (ii)$ by Proposition 1.12 b: $\mu = \varepsilon_x \ast \lambda = \lambda \ast \varepsilon_x$, with continuously embeddable $\lambda$. But $\varepsilon_x$ is continuously embeddable, hence (ii).

(ii)$\Rightarrow (iii)$ by Theorem 3.1, (iii)$\Rightarrow (iv)$ obvious. (iv)$\Rightarrow (ii)$ by [No1] I 1.11, [No2] 4, Thm. 1.]

Remark 3.4. Let $G = G_0$ be a contractible Lie group with Lie algebra $\mathcal{G}$. Let $\mu$, $\nu \in M^1(G)$ and let $a \in \text{Aut}(G)$ be contracting. Let $\mu$, $\nu$, $a$ be the corresponding objects on the tangent space $\mathcal{G}$ (see e.g. [H-S]). Then we have:

$$a^n \nu \circ [k,n] t \to \mu_t = \text{Exp} \ t A, \ \mu_1 = \mu, \ t \geq 0$$

iff

$$a^n \circ [k,n] t \to \gamma_t = \text{Exp} \ t \dot{A}, \ t \geq 0$$

(See [H-S]). Therefore we obtain:

$\nu$ is a Doeblin distribution w.r.t. $a$ on the group $G$ iff $\dot{\nu}$ is a Doeblin distribution w.r.t. $\dot{a}$ on the vectorspace $\mathcal{G}$.

Remark added in proof:

Recently it could be shown that Siebert's characterization of convergence of convolution semigroups and hence Proposition 1.3 b is valid for arbitrary locally compact groups. Hence the distinction between different domains of attraction is superfluous for contractible groups.

("A generalization of E. Siebert's theorem on convergence of c.c.s. and accompanying laws."

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