ANNALES DE L'I. H. P., SECTION B

MICHEL TALAGRAND

Approximating a helix in finitely many dimensions

Annales de l'I. H. P., section B, tome 28, nº 3 (1992), p. 355-363
<http://www.numdam.org/item?id=AIHPB_1992__28_3_355_0>

Ann. Inst. Henri Poincaré,

Vol. 28, n° 3, 1992, p. 355-363. *Probabilités et Statistiques*

Approximating a helix in finitely many dimensions

by

Michel TALAGRAND

Équipe d'Analyse, Tour 46, U.A. au C.N.R.S. n° 754,
Université Paris-VI, 75230 Paris Cedex 05, France

ABSTRACT. – Consider $\alpha \in]0, 1[$. We prove that there exists a constant $K(\alpha)$, depending on α only, such that for $p \geqq 1$, there exists a map F from $\mathbb{R}$ to $\mathbb{R}^p$ such that for $s, t \in \mathbb{R}$, we have

$$\left| \| F(s) - F(t) \| / | s - t |^{\alpha} - 1 \right| \leqq K(\alpha)/p^{\alpha}.$$

RÉSUMÉ. – Pour $\alpha \in]0, 1[$, il existe une constante $K(\alpha)$, dependant de α seulement, telle que pour $p \geqq 1$, il existe une application F de $\mathbb{R}$ dans $\mathbb{R}^p$ telle que, pour tous réels s, t on ait

$$\left| \| F(s) - F(t) \| / | s - t | - 1 \right| \leqq K(\alpha)/p^{\alpha}.$$

A.M.S. 1980 classification numbers primary 461320, 60617.

Annales de l'Institut Henri Poincaré - Probabilités et Statistiques - 0246-0203
Vol. 28/92/03/355/09/$ 2,90/© Gauthier-Villars

1. INTRODUCTION

A helix is a map h from $\mathbb{R}$ to a Hilbert space H such that $\|h(s)-h(t)\|=\|h(s-t)\|$ for $s, t\in\mathbb{R}$. Within isometries, a helix is determined by the function

$$\psi(t)=\|h(t)\|^2. \tag{1}$$

It is a theorem of I. J. Shoenberg that the functions $\psi(t)$ given by (1) are exactly the functions of negative type. In this note, we are interested in the case $\|h(t)\|=|t|^\alpha$, for a certain $\alpha\in]0, 1[$. The case $\alpha=1/2$ corresponds to Wilson's helix, that is realized by Brownian motion.

P. Assouad and L. A. Shepp raised the question whether the helix corresponding to $\|h(t)\|=|t|^{1/2}$ (Wilson's helix) can be approximated in the p-dimensional euclidean space. This was settled by J. P. Kahane [2] who obtained the following result. (Throughout the paper, $\|.\|$ denotes the euclidean norm.)

THEOREM 1 (J. P. Kahane). — There exists a universal constant K such that for $p\geqq 1$, there exists a map F from $\mathbb{R}$ to $\mathbb{R}^p$ such that

$$\forall s, t\in\mathbb{R},\quad 1-\frac{K}{p}\leqq\frac{\|F(s)-F(t)\|}{|s-t|^{1/2}}\leqq 1+\frac{K}{p}.$$

On the other hand, P. Assouad [1] proved that for all $\alpha\in]0, 1]$, $p\geqq p_0$, there exists a map F from $\mathbb{R}$ to $\mathbb{R}^p$ such that

$$\forall s, t\in\mathbb{R},\quad \frac{1}{K}\leqq\frac{\|F(s)-F(t)\|}{|s-t|^\alpha}\leqq K \tag{2}$$

where K depends on α only. The estimate of (2) does not improve when $p\to\infty$. The purpose of the present note is to improve upon (2).

THEOREM 2. — *Given* $\alpha\in]0, 1[$, *there exists a constant* $K(\alpha)$, *depending on* α *only, such that for* $p\geqq 1$, *there exists a map* F *from* $\mathbb{R}$ *to* $\mathbb{R}^p$ *that satisfies*

$$\forall s, t\in\mathbb{R},\quad 1-\frac{K(\alpha)}{p^\alpha}\leqq\frac{\|F(s)-F(t)\|}{|s-t|^\alpha}\leqq 1+\frac{K(\alpha)}{p^\alpha}. \tag{3}$$

In the case $\alpha=1/2$, this gives an error in $K/\sqrt{p}$, and unfortunately does *not* recover the error K/p of Kahane's Theorem 1. It is not difficult to see that this error K/p is of optimal order in Kahane's theorem; but when $\alpha\neq 1/2$, we do not have a nontrivial lower bound for the error in (3).

2. THE APPROACH

We fix $\alpha\in]0,1[$, and $p\geqq 1$. For convenience, we assume that p is a multiple of 4 (so that $p\geqq 4$). We set, for $n\geqq 0$,

$$D_n=\left\{\frac{i}{p\,2^n};\ 0\leqq i\leqq p\,2^n\right\}.$$

For $0\leqq q\leqq 2^{n+1}-2$, we set

$$I_{n,q}=\left[\frac{q}{2^{n+1}},\frac{q+2}{2^{n+1}}\right].$$

Thus $I_{n,q}\subset[0,1]=I_{0,0}$. For $n\geqq 1$, $0\leqq q\leqq 2^{n+1}-2$, we find $l(q)(=l_n(q))$ such that $I_{n,q}\subset I_{n-1,l(q)}$. When $0<q<2^{n+1}-2$, and when q is even, there are two possible choices. We make an arbitrary choice; the construction will actually not depend on that choice.

Consider a map $t\to x(t)$ from $\mathbb{R}$ to a Hilbert space H that satisfies $\|x(t)-x(s)\|=|t-s|^\alpha$. We first construct affine maps $\theta_{n,q}$ from H to $\mathbb{R}^p$ that satisfy

$$\forall s,t\in D_n\cap I_{n,q},\quad \|\theta_{n,q}(x(t))-\theta_{n,q}(x(s))\|=|t-s|^\alpha \tag{4}$$

$$\forall t\in D_{n-1}\cap I_{n,q},\quad \theta_{n,q}(x(t))=\theta_{n-1,l(q)}(x(t)). \tag{5}$$

We proceed to this easy construction, by induction over n. A basic observation is that $D_n\cap I_{n,q}$ has $p+1$ points. The affine span of these points is isometric to $\mathbb{R}^p$; thus for each n, q, one can find an affine map $\xi_{n,q}$ from H to $\mathbb{R}^p$ that satisfies

$$\forall s,t\in D_n\cap I_{n,q},\quad \|\xi_{n,q}(x(t))-\xi_{n,q}(x(s))\|=|t-s|^\alpha.$$

We take $\theta_{0,0}=\xi_{0,0}$. If all the maps $\theta_{n,q}$ have been constructed, for a certain n and for all $q\leqq 2^{n+1}-2$, we take $\theta_{n+1,q}=U\circ\xi_{n+1,q}$, where U is an isometry of $\mathbb{R}^p$ such that $U(\xi_{n+1,q}(x(t)))=\theta_{n,l(q)}(x(t))$ for $t\in D_{n-1}\cap I_{n,q}$. By isometry we mean that $\|U(x)-U(y)\|=\|x-y\|$ for $x,y\in\mathbb{R}^p$. The existence of U follows from the following elementary fact, that will be used repeatedly: if S is a map from a subset A of $\mathbb{R}^p$ to $\mathbb{R}^p$ such that $\|S(x)-S(y)\|=\|x-y\|$ for x, $y\in A$, then we can find an isometry U of $\mathbb{R}^p$ such that $U(x)=S(x)$ for $x\in A$.

For the simplicity of notation, we will write $x_{n,q,t}=\theta_{n,q}(x(t))$. The idea of the preceding construction is that the points $x_{n,q,t}$, $t\in D_n\cap I_{n,q}$ have the correct position with respect to each other. Also, a certain degree of consistency is obtained through (5). One would like to have $F(t)=x_{n,q,t}$ for $t\in D_n\cap I_{n,q}$. The problem is that it is not possible to insure that $x_{n,q,t}=x_{n,q+1,t}$ for $t\in D_n\cap I_{n,q}\cap I_{n,q+1}$. To solve that difficulty, for $t\in I_{n,q}$, we will construct an isometry $R_{n,q,t}$ of $\mathbb{R}^p$. We require the following properties.

(6) For $t \in D_n \cap I_{n,q} \cap I_{n,q+1}$, we have

$$R_{n,q,t}(x_{n,q,t}) = R_{n,q+1,t}(x_{n,q+1,t}).$$

(7) For $t \in D_{n-1} \cap I_{n,q}, y = x_{n,q,t} = x_{n-1,l(q),t}$, we have

$$R_{n,q,t}(y) = R_{n-1,l(q),t}(y).$$

(8) For $s, t \in I_{n,q}, x, y \in \mathbb{R}^p$, we have

$$\| R_{n,q,s}(x) - R_{n,q,s}(y) - (R_{n,q,t}(x) - R_{n,q,t}(y)) \| \leqq K \| x - y \| \, | t - s |.$$

(There, as in the sequel, K is a constant depending on α only, that is not necessarily the same at each occurence; on the other hand, $K_1, K_2, \ldots$ denote specific constants depending on α only).

(9) If $x = x_{n,q,u}$ for $u \in I_{n,q} \cap D_n$, then for $s, t \in I_{n,q} \cap D_n$, we have

$$\| R_{n,q,s}(x) - R_{n,q,t}(x) \| \leqq K \frac{2^{n(1-\alpha)}}{p^\alpha} | s - t |.$$

(10) For t in $[(q+1)2^{-n-1}, (q+2)2^{-n-1}]$, the isometry $R_{n,q,t}^{-1} \circ R_{n,q+1,t}$ does not depend on t.

The construction of these isometries will be done in section 3; but, before, we provide motivation by proving Theorem 2.

For $t \in D_n \cap I_{n,q}$, we set

(11) $$F(t) = R_{n,q,t}(x_{n,q,t}).$$

Givne n, there are two consecutive values of q for which $t \in I_{n,q}$; if follows from (6) that the value of $F(t)$ does not depend on which value of q we use. Also, it follows from (7) that the value of $F(t)$ does not depend on which value of n we consider. Thus, (11) actually defines $F(t)$ for $t \in D = \bigcup_{n \geqq 0} D_n$.

Consider now $u, v \in D_n$ such that $|u - v| \leqq 2^{-n-1}$. Thus $u, v \in I_{n,q}$ for some q. Let $\tau = (q+1)2^{-n-1}$. It follows from (4), since $R_{n,q,\tau}$ is an isometry, that

$$\| R_{n,q,\tau}(x_{n,q,u}) - R_{n,q,\tau}(x_{n,q,v}) \| = | u - v |^\alpha.$$

Thus, by (9), used for $s = u$, $t = \tau$, and for $s = v$, $t = \tau$, we have

(12) $$\big| \| F(u) - F(v) \| - | u - v |^\alpha \big| \leqq \| R_{n,q,u}(x_{n,q,u}) - R_{n,q,\tau}(x_{n,q,u}) \| + \| R_{n,q,v}(x_{n,q,v}) - R_{n,q,\tau}(x_{n,q,v}) \| \leqq K \frac{2^{-n\alpha}}{p^\alpha}.$$

It follows in particular that

(13) $$\| F(u) - F(v) \| \leqq K \, 2^{-n\alpha}.$$

LEMMA. — *For* s, $t \in D$, *we have* $\| F(s)-F(t) \| \leqq K |s-t|^{\alpha}$.

Proof. — Consider the largest n such that $|s-t| \leqq 2^{-n}$, so that $2^{-n} \leqq 2|s-t|$. We observe that, given $s \in [0, 1]$, we can find $u \in D_n$ such that $|s-u| \leqq 2^{-n}/p \leqq 2^{-n-2}$. We thus construct sequences (u_k), (v_k) $k \geqq n$, such that $u_k, v_k \in D_{k-2}$, $|u_k - s| \leqq 2^{-k}$, $|v_k - u| \leqq 2^{-k}$. Thus $|u_n - v_n| \leqq 2^{-n+2}$, $|u_k - u_{k+1}|$, $|v_k - v_{k+1}| \leqq 2^{-k+1}$. We can and do assume that $u_k = s$, $v_k = t$ for k large enough. Then

$$\| F(u) - F(v) \| \leqq \| F(u_n) - F(v_n) \| + \sum_{k \geqq n} (\| F(u_k) - F(u_{k+1}) \| + \| F(v_k) - F(v_{k+1}) \|).$$

By (13), this implies that

$$\| F(u) - F(v) \| \leqq K 2^{-\alpha n} \leqq K |s-t|^{\alpha}. \quad \square$$

The lemma implies in particular that F can be extended by continuity to the closure of D, *i.e.* to [0, 1], and that

$$\| F(s) - F(t) \| \leqq K |s-t|^{\alpha} \tag{14}$$

for s, $t \in [0, 1]$.

Consider now $s, t \in [0, 1]$ and the largest n such that $|s-t| \leqq 2^{-n-1}$, so that $2^{-n} \leqq 4|s-t|$. Consider q such that $s, t \in I_{n, q}$. Thus we can find $u, v \in I_{n, q} \cap D_n$ such that $|s-u| \leqq 2^{-n}/p$, $|t-v| \leqq 2^{-n}/p$. By (14), we have

$$\| F(s) - F(u) \| \leqq \frac{K 2^{-n\alpha}}{p^{\alpha}}; \qquad \| F(t) - F(v) \| \leqq \frac{K 2^{-n\alpha}}{p^{\alpha}}.$$

Thus

$$\big| \| F(s) - F(t) \| - \| F(u) - F(v) \| \big| \leqq \frac{K 2^{-n\alpha}}{p^{\alpha}}.$$

From (12), we have

$$\big| \| F(u) - F(v) \| - |u-v|^{\alpha} \big| \leqq \frac{K 2^{-n\alpha}}{p^{\alpha}}.$$

Thus, since $|s-t| \geqq 2^{-n-2}$, we have

$$\left| \frac{\| F(s) - F(t) \|}{|s-t|^{\alpha}} - 1 \right| \leqq \frac{K}{p^{\alpha}} + \left| \frac{|u-v|^{\alpha}}{|s-t|^{\alpha}} - 1 \right|. \tag{15}$$

We have $\big| |u-v| - |s-t| \big| \leqq 2^{-n+1}/p$. Using that $|(1+x)^{\alpha} - 1| \leqq K|x|$ for $|x| \leqq 4$, we get that

$$\left| \frac{|u-v|^{\alpha}}{|s-t|^{\alpha}} - 1 \right| \leqq \frac{K}{p} \leqq \frac{K}{p^{\alpha}}.$$

Thus, we have constructed a map F from [0, 1] to $\mathbb{R}^p$ such that

$$\forall s, t \in [0, 1], \quad \left| \frac{\| F(s) - F(t) \|}{|s-t|^{\alpha}} - 1 \right| \leqq \frac{K}{p^{\alpha}}. \tag{16}$$

There is no loss of generality to assume $F(1/2)=0$. Consider an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and define

$$G(t)=\lim_{n\to\mathcal{U}} n^\alpha F\left(\frac{1}{2}+\frac{t}{n}\right).$$

The limit exists since, from (14) and $F\left(\frac{1}{2}\right)=0$, we have

$$n^\alpha \left\| F\left(\frac{1}{2}+\frac{t}{n}\right)\right\| \leqq K|t|^\alpha.$$

Moreover it is immediate to check that, for s, $t\in\mathbb{R}$, we have $\left|\|G(s)-G(t)\|/|s-t|^\alpha-1\right|\leqq K/p^\alpha$. This completes the proof of Theorem 2.

The reader has observed that conditions (8) and (10) have not been used. Condition (8) is used during the construction as a preliminary step for conditions (9). Condition (10) helps to keep control of the situation as the induction continues.

3. CONSTRUCTION

The construction proceeds by induction on n. For $t\in[0,1]$, we set $R_{0,0,t}=\text{Identity}$. We now perform the induction step from $n-1$ to n. Consider q, $-1\leqq q\leqq 2^{n+2}-2$, and set

$$\tau=(q+1)2^{-n-1},\qquad \tau'=(q+2)2^{-n-1},\qquad I=[\tau,\tau'].$$

For $t\in I$, we construct isometries $T_{n,q,t}$, $S_{n,q,t}$ of $\mathbb{R}^p$, such that the following holds (where we set $l(-1)=0$)

(17) $\qquad T_{n,q,\tau}=R_{n-1,l(q),\tau};\qquad S_{n,q,\tau'}=R_{n-1,l(q+1),\tau'}.$

(18) $\qquad \forall t\in I\cap D_n,\quad T_{n,q,t}(x_{n,q,t})=S_{n,q,t}(x_{n,q+1,t})$

(19) For $t\in D_{n-1}\cap I$, we have

$$T_{n,q,t}(x_{n,q,t})=R_{n-1,l(q),t}(x_{n,q,t})$$
$$S_{n,q,t}(x_{n,q+1,t})=R_{n-1,l(q+1),t}(x_{n,q+1,t}).$$

(20) For $s,t\in I, x,y\in\mathbb{R}^p$, we have

$$\|T_{n,q,s}(x)-T_{n,q,s}(y)-(T_{n,q,t}(x)-T_{n,q,t}(y))\|\leqq K_1 2^n|s-t|\,\|x-y\|$$
$$\|S_{n,q,s}(x)-S_{n,q,s}(y)-(S_{n,q,t}(x)-S_{n,q,t}(y))\|\leqq K_1 2^n|s-t|\,\|x-y\|.$$

(21) For $u,s,t\in I\cap D_n$, $x=x_{n,q,u}$, $y=x_{n,q+1,u}$, we have

$$\|T_{n,q,s}(x)-T_{n,q,t}(x)\|\leqq K_2\frac{2^{n(1-\alpha)}}{p^\alpha}|s-t|$$
$$\|S_{n,q,s}(y)-S_{n,q,t}(y)\|\leqq K_2\frac{2^{n(1-\alpha)}}{p^\alpha}|s-t|.$$

(22) For $t\in I$, the isometry $T^{-1}_{n,q,t}\circ S_{n,q+1,t}$ does not depend on t.

Before we proceed to the construction of the isometries $T_{n,q,t}$, $S_{n,q,t}$, we show how to construct the isometries $R_{n,q,t}$ for $0 \leqq q \leqq 2^{n+1}-2$. For $t \in [q\, 2^{-n-1}, (q+1)\, 2^{-n-1}]$ we set $R_{n,q,t}=S_{n,q-1,t}$; for $t \in [(q+1)\, 2^{-n-1}, (q+2)\, 2^{-n-1}]$, we set $R_{n,q,t}=T_{n,q,t}$. Condition (17) ensures that $S_{n,q-1,\tau}=T_{n,q,\tau}$, so that $R_{n,q,\tau}$ is well defined. It is simple to see that conditions (6) to (10) follow from conditions (18) to (22) respectively.

We now construct the isometries $T_{n,q,t}$, $S_{n,q,t}$. Set $l=l(q)$, $l'=l(q+1)$. Thus, we either have $l'=l$ or $l'=l+1$. For $t \in [(l+1)\, 2^{-n}, (l+2)\, 2^{-n}]$, we have by induction hypothesis and (10) that, if $l'=l+1$,

$$R_{n-1,l,t}^{-1} \circ R_{n-1,l',t} = \text{Constant isometry} := V. \tag{23}$$

If $l'=l$, the above also holds, for $V=$ identity. We set for simplicity $A=R_{n-1,l,\tau}$; $B=R_{n-1,l',\tau}$. It is simple to see that $\tau \in [(l+1)\, 2^{-n}, (l+2)\, 2^{-n}]$; thus, by (23), we have $A^{-1} \circ B=V$.

Given $t \in I \cap D_{n-1}$, we have

$$R_{n-1,l,t}(x_{n-1,l,t}) = R_{n-1,l',t}(x_{n-1,l',t}).$$

This is obvious if $l'=l$; if $l'=l+1$, this follows from (6). Remembering that $R_{n-1,l,t}^{-1} \circ R_{n-1,l',t}=V=A^{-1} \circ B$, we get

$$\forall t \in I \cap D_{n-1}, \quad A(x_{n-1,l,t}) = B(x_{n-1,l',t}).$$

It then follows from (5) that

$$\forall t \in I \cap D_{n-1}, \quad A(x_{n,q,t}) = B(x_{n,q+1,t}). \tag{24}$$

Since A, B are isometries, it follows from (4) that

$$\forall s, t \in I \cap D_n, \quad \| A(x_{n,q,s}) - A(x_{n,q,t}) \| = \| B(x_{n,q+1,s}) - B(x_{n,q+1,t}) \|.$$

Thus, there exists an isometry U of $\mathbb{R}^p$ such that

$$\forall t \in I \cap D_n, \quad U \circ A(x_{n,q,t}) = B(x_{n,q+1,t}). \tag{25}$$

Since $\operatorname{card} I \cap D_n = p/2+1 < p$, we can assume that $\det U=1$ (by composing if necessary U by a reflection through a hyperplane containing the points $A(x_{n,q,t})$, $t \in I \cap D_n$.) It is then clear that we can find a semi-group $U(t)$ of isometries of $\mathbb{R}^p$, with $U(1)=U$, such that

$$\left\{\begin{array}{l} \forall a, b \in \mathbb{R}, \quad \forall x, y \in \mathbb{R}^q, \\ \| U(a)(x) - U(a)(y) - U(b)(x) + U(b)(y) \| \leqq K_3 |b-a| \, \|x-y\| \end{array}\right. \tag{26}$$

(actually one can take $K_3 = 2\pi$).

For $t \in I$, we set

$$T_{n,q,t} = R_{n-1,l,t} \circ A^{-1} \circ U(\varphi(t)) \circ A$$
$$S_{n,q,t} = R_{n-1,l',t} \circ B^{-1} \circ U(\varphi(t)-1) \circ B$$

where $\varphi(t) = 2^{n+1}(t-\tau)$. Thus $\varphi(\tau)=0$, $\varphi(\tau')=1$. Thus (17) holds.

It remains to prove (18) to (22).

Proof of (18). — It follows from (25) that, for $t \leqq D_n \cap I$, we have

$$A(x_{n,q,t}) = U^{-1} \circ B(x_{n,q+1,t})$$

so that

$$U(\varphi(t)) \circ A(x_{n,q,t}) = U(\varphi(t) - 1) \circ B(x_{n,q+1,t}). \tag{27}$$

Since $R_{n-1,l,t}^{-1} \circ R_{n-1,l',t} = A^{-1} \circ B$, we have

$$R_{n-1,l',t} \circ B^{-1} = R_{n-1,l,t} \circ A^{-1},$$

and, combined with (27) and the definition of $T_{n,q,t}$, $S_{n,q,t}$, this implies (18).

Proof of (19). — We consider only the case of $T_{n,q,t}$, and leave the other case to the reader. By (24), (25), we have

$$t \in I \cap D_{n-1} \quad \Rightarrow \quad U \circ A(x_{n,q,t}) = A(x_{n,q,t})$$

so have

$$U(s) \circ A(x_{n,q,t}) = A(x_{n,q,t})$$

for all $s \in \mathbb{R}$. Thus

$$A^{-1} \circ U(\varphi(t)) \circ A(x_{n,q,t}) = x_{n,q,t},$$

which implies the result.

Proof of (20). — We prove this inequality for the constant $K_1 = 4K_3$, where K_3 occurs in (26) and we again consider only the case of $T_{n,q,t}$. We have

$$\| T_{n,q,s}(x) - T_{n,q,s}(y) - (T_{n,q,t}(x) - T_{n,q,t}(y)) \| \leqq (I) + (II)$$

where

$$\begin{aligned}(I) = \| & R_{n-1,l,t} \circ A^{-1} \circ U(\varphi(t)) \circ A(x) - R_{n-1,l,t} \circ A^{-1} \circ U(\varphi(t)) \circ A(y) \\ & - R_{n-1,l,t} \circ A^{-1} \circ U(\varphi(s)) \circ A(x) + R_{n-1,l,t} \circ A^{-1} \circ U(\varphi(s)) \circ A(y)) \| \\ (II) = \| & R_{n-1,l,t}(x') - R_{n-1,l,t}(y') - R_{n-1,l,s}(x') + R_{n-1,l,s}(y') \|\end{aligned}$$

for $x' = A^{-1} \circ U(\varphi(s)) \circ A(x)$, $y' = A^{-1} \circ U(\varphi(s)) \circ A(y)$. Since A and $U(\varphi(s))$ are isometries, we have $\| x' - y' \| = \| x - y \|$. We observe that (8) holds with the same value $K = K_1$ of the constant K than (20); thus, by induction hypothesis, we have

$$(II) \leqq K_1 2^{n-1} |s - t| \, \| x - y \|.$$

Since $R_{n-1,l,t} \circ A^{-1}$ is an isometry, we have

$$(I) = \| U(\varphi(t)) \circ A(x) - U(\varphi(t)) \circ A(y) - U(\varphi(s)) \circ A(x) + U(\varphi(s)) \circ A(y) \|.$$

Since $\| A(x) - A(y) \| = \| x - y \|$, $|\varphi(t)| \leqq 2^{n+1} |s - t|$, by (26) we have

$$(I) \leqq K_3 2^{n+1} |s - t| \, \| x - y \|.$$

Thus

$$(I) + (II) \leqq (K_1 2^{n-1} + K_3 2^{n+1}) |s - t| \, \| x - y \| \leqq K_1 2^n |s - t| \, \| x - y \|$$

since $K_1 = 4K_3$.

Proof of (21). – We prove (21) for $K_2 = 2^\alpha K_1 (1 - 2^{-(1-\alpha)})^{-1}$, and again we consider only the case of T. We proceed by induction, observing that (9) holds with the same constant K_2. We find v in $I \cap D_{n-1}$ with $|u - v| \leqq 2^{-n+1}/p$. We set $z = x_{n, q, v}$. We have

$$\| T_{n, q, s}(x) - T_{n, q, t}(x) \| \leqq (I) + (II)$$

where

$$(I) = \| T_{n, q, s}(z) - T_{n, q, t}(z) \|$$
$$(II) = \| T_{n, q, t}(x) - T_{n, q, t}(z) - (T_{n, q, s}(x) - T_{n, q, s}(z)) \|.$$

We recall that by (24), (25)

$$U \circ A(z) = B(x_{n, q+1, v}) = A(z),$$

so that, by definition of $T_{n, q, t}$

$$(I) = \| R_{n-1, l(q), s}(z) - R_{n-1, l(q), t}(z) \|$$

and, by induction hypothesis,

$$(I) \leqq K_2 \frac{2^{(n-1)(1-\alpha)}}{p^\alpha} . |s - t|.$$

If we recall that (20) holds for the constant K_1 we have

$$(II) \leqq K_1 2^n |s - t| \, \| x - z \|.$$

Since, by (4),

$$\| x - z \| = |u - v|^\alpha \leqq 2^{(-n+1)\alpha}/p^\alpha,$$

we have $(II) \leqq K_1 2^\alpha 2^{n(1-\alpha)} |s - t| p^{-\alpha}$. Thus

$$(I) + (II) \leqq \frac{2^{n(1-\alpha)}}{p^\alpha} [2^\alpha K_1 + K_2 2^{-(1-\alpha)}] |s - t| = \frac{2^{n(1-\alpha)}}{p^\alpha} K_2 |s - t|.$$

Proof of (22). – We have, for $t \in J$,

$$T_{n, q, t}^{-1} \circ S_{n, q, t} = A^{-1} \circ U(-\varphi(t)) \circ A \circ R_{n-1, l, t}^{-1} \circ R_{n-1, l', t} \circ B^{-1} \circ U(\varphi(t) - 1) \circ B.$$

Since, by (23), we have $R_{n-1, l, t}^{-1} \circ R_{n-1, l', t} = A^{-1} \circ B$, we have

$$T_{n, q, t}^{-1} \circ S_{n, q, t} = A^{-1} \circ U(-\varphi(t)) \circ U(\varphi(t) - 1) \circ B = A^{-1} \circ U^{-1} \circ B,$$

and this does not depend on t.

The proof is complete.

REFERENCES

[1] P. Assouad, Plongements Lipschitziens dans $\mathbb{R}^n$, *Bull. Soc. Math. Fr.*, Vol. 111, 1983, pp. 429-448.

[2] J. P. Kahane, Hélices et quasi-hélices, *Adv. Math.*, Vol. 7 B, 1981, pp. 417-422.

(*Manuscript received February* 13, 1991.)