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Additions and correction to “the bootstrap of the mean with arbitrary bootstrap sample size” (*)

by

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ABSTRACT. — Some inaccuracies in [2] are corrected and some additional results are presented. The bootstrap central limit theorem in the domain of attraction case is improved to include convergence of bootstrap moments. Self-normalized limit theorems for variables in the domain of attraction of a p -stable law are bootstrapped, thus freeing the bootstrap from the index p and the norming constants $\{b_n\}$. Simulations on the bootstrap of the self-normalized sums for a few values of p and n are also included.

RÉSUMÉ. — Nous corrigeons quelques inexactitudes de l'article [2] et nous présentons certains résultats complémentaires. Nous améliorons le théorème central limite « bootstrap » pour obtenir la convergence des moments « bootstrap ». Des théorèmes limites auto-normalisés pour des variables dans le domaines d'attraction d'une loi p -stable sont donnés sous forme bootstrap, ce qui libère le bootstrap de l'indice p et des constantes de normalisation (b_n). On présente aussi des simulations du bootstrap des sommes auto-normalisée pour quelques valeurs de p et n .

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1. INTRODUCTION

Remarks 2.3 and 2.4 in [2] are inaccurate, and we correct them in Section 2. We take the opportunity to broaden our previous study on the bootstrap of the mean [2] in two directions. Bickel and Freedman [3] observe that if $EX^2 < \infty$, not only does the bootstrap CLT hold a.s. in the sense that e.g. $d_{BL_1} \left(\hat{\mathcal{L}} \left(\sum_{j=1}^n (X_{nj} - \bar{X}_n) / n^{1/2} \right), N(0, \text{Var } X) \right) \rightarrow 0$ a.s. but that actually d_{BL_1} can be replaced by the Mallows distance d_2 which metrizes weak convergence plus convergence of the second moments. This can be strengthened to include convergence of exponential bootstrap moments even for different bootstrap sample sizes m_n , as long as $m_n \geq cn$ for some $c > 0$. Curiously enough, if $m_n/n \rightarrow 0$ then a.s. convergence of the p -th bootstrap moment for $p \geq 2$ implies (is equivalent to) further integrability of X , namely $\sum_{n=1}^{\infty} P\{|X| > m_n^{1/2-1/p} n^{1/p}\} < \infty$. The case $EX^2 = \infty$ is also thoroughly examined.

In another direction, we look at the bootstrap of selfnormalized (Studentized) sums, in a sense expanding on Remark 2.3 of [2]. It is well known (e.g. Logan *et al.* [6]) that if X belongs to some domain of attraction with normings b_n and centers a_n then the random vectors $\left\{ \left(b_n^{-1} \sum_{i=1}^n X_i - a_n, b_n^{-2} \sum_{i=1}^n X_i^2 \right) \right\}_{n=1}^{\infty}$ converge in law. In particular, if X is in the domain of attraction of a p -stable random variable, $1 < p \leq 2$, then $\left\{ \sum_{i=1}^n (X_i - EX) / \left(\sum_{i=1}^n X_i^2 \right)^{1/2} \right\}$ converges in law. (It is irrelevant whether one takes $\left(\sum_{i=1}^n X_i^2 \right)^{1/2}$ or $\left(\sum_{i=1}^n (X_i - \bar{X}_n)^2 \right)^{1/2}$: see e.g. [6].) We show that if $m_n/n \rightarrow 0$ then the bootstrap of this statistic,

$$\left\{ \sum_{i=1}^{m_n} (X_{ni} - \bar{X}_n) / \left(\sum_{i=1}^n X_{ni}^2 \right)^{1/2} \right\}$$

converges weakly in probability to the same limit as the original for all $1 < p \leq 2$ and all possible norming sequences $\{b_n\}$. This suggests a procedure for constructing bootstrap confidence intervals for the mean which is robust against integrability properties. Some simulations in the infinite variance case are included.

2. CORRECTIONS TO [2]

Remark 2.3 in [2] on random normings for the bootstrap CLT with normal limit refers only to the case $EX^2 = \infty$, although this is not explicitly stated there, and the norming (2.20) is only valid for $m_n < cn$ for some $c < \infty$. Under these constraints, the remark is correct. The normings described there can be modified to hold simultaneously for $EX^2 = \infty$ and $EX^2 < \infty$ as follows:

$$\hat{a}_n(\omega) = \left[(m_n/n) \sum_{i=1}^n (X_i - \bar{X}_n)^2 \right]^{1/2} \quad \text{if } m_n \geq n,$$

and $\hat{a}_n(\omega) = \text{average over all the } \binom{n}{m_n} \text{ combinations } 1 \leq j_1 < \dots < j_{m_n} \leq n$
of

$$\left[\sum_{i=1}^{m_n} \left(X_{j_i} - m_n^{-1} \sum_{i=1}^{m_n} X_{j_i} \right)^2 \right]^{1/2} \quad \text{if } m_n \leq n.$$

[This replaces equation (2.19).] For $m_n/n \rightarrow 0$ one may as well take $\hat{a}_n(\omega)$ to be the average of

$$\left[\sum_{i=km_n+1}^{i=(k+1)m_n} \left(X_i - m_n^{-1} \sum_{i=km_n+1}^{i=(k+1)m_n} X_i \right)^2 \right]^{1/2}, \quad k=1, \dots, [n/m_n].$$

Moreover, for $m_n \leq cn$, $c < \infty$, another possible norming is

$$\hat{a}_n^\omega(\omega') = \left[\sum_{j=1}^{m_n} (X_{n,j}^\omega(\omega') - \bar{X}_n^\omega)^2 \right]^{1/2}. \quad \text{[This replaces equation (2.20).] The}$$

proofs are as indicated in [2] using convergence of the sequence

$$\left\{ \sum_{i=1}^n (X_i - EX)^2 / b_n^2 \right\} \text{ instead of } \left\{ \sum_{i=1}^n X_i^2 / b_n^2 \right\}.$$

The computations in Remark 2.4 of [2] are correct but they do not show what we say there. In fact, in Theorem 2.2 the centering $\tilde{X}_n^\omega$ can be replaced by $\bar{X}_n^\omega$. To see this note that if $m_n > cn$, then $a_n > c' b_n$ for some constant c' and therefore

$$\begin{aligned} P \left\{ \sum_{i=1}^n I_{|X_i| \geq a_n} \neq 0 \right\} \\ = P \left\{ \sum_{i=1}^n I_{|X_i| \geq a_n} > \delta \right\} \leq \delta^{-1} n P \{ |X| > c' b_n \} \rightarrow 0 \quad \text{for } 0 < \delta < 1. \end{aligned}$$

This shows that $(m_n/n a_n) \sum_{i=1}^n X_i I_{|X_i| > a_n} \rightarrow 0$ in probability and the equivalence between the centerings $\bar{X}_n$ and $\tilde{X}_n$ follows.

We also correct some minor misprints: on page 465, line 5, $\frac{m_n}{b_{m_n}} U(b_{m_n})$ should be $\frac{m_n}{b_{m_n}^2} U(b_{m_n})$; in (2.21) the sum should be for $i \leq n'$ instead of $i \leq m_n$; on page 475, lines 9 and 13, $\bar{a}_n$ and $\bar{p}_n$ should be replaced by $\bar{a}_k$ and $\bar{p}_k$; finally in the statement of Theorem 3.4, the constant c in $m_n/m_{2n} \geq c$ should be strictly positive.

3. CONVERGENCE OF MOMENTS

The bootstrap in probability of the mean in the domains of attraction case (Theorem 2.2 and Corollary 2.6 in [2]) can be strengthened to include convergence in probability of bootstrap moments, even exponential in the normal case. Weak convergence together with convergence of the r -th absolute moment is metrizable (Mallows-Wassertein distances; see e.g. Bickel and Freedman [3]). We will call d_r any distance metrizing this convergence.

The following theorem improves on Theorem 2.1 of [3]; we only state it for real random variables but it is obvious that it extends to random vectors in $\mathbf{R}^k$, $k < \infty$.

3.1. THEOREM. — (a) If $EX^2 < \infty$ and $m_n/n \geq c > 0$ then for all $t > 0$

$$(3.1) \quad \hat{\mathbb{E}} \exp \left\{ t \sum_{i=1}^{m_n} (X_{nj} - \bar{X}_n) / m_n^{1/2} \right\} \rightarrow E e^{tg} \quad \text{a. s.}$$

where g is $N(0, \text{Var } X)$. In particular

$$(3.2) \quad d_p \left[\hat{\mathcal{L}} \left(\sum_{i=1}^{m_n} (X_{nj} - \bar{X}_n) / m_n^{1/2} \right), N(0, \text{Var } X) \right] \rightarrow 0 \quad \text{a. s. for all } p > 0.$$

(b) If X is in the domain of attraction of a normal law with norming constants $b_n \nearrow \infty$, that is $\mathcal{L} \left(\sum_{j=1}^n (X_j - EX) / b_n \right) \rightarrow_w N(0, 1)$, and if $m_n/n \geq c > 0$ and $a_n = b_n (m_n/n)^{1/2}$, then

$$(3.1)' \quad \hat{\mathbb{E}} \exp \left\{ t \sum_{i=1}^{m_n} (X_{nj} - \bar{X}_n) / a_n \right\} \rightarrow E e^{tg} \quad \text{in probability,}$$

where g is $N(0, 1)$. In particular

$$(3.2)' \quad d_p \left[\hat{\mathcal{L}} \left(\sum_{i=1}^{m_n} (X_{nj} - \bar{X}_n) / a_n \right), N(0, 1) \right] \rightarrow 0 \quad \text{in probability for all } p > 0.$$

Proof. — Let us recall that convexity of $f(x) = e^{tx}$ implies $E e^{t(X+Y)} \leq (E e^{2tX} + E e^{2tY})/2$ for any rv's X and Y , and that if X and Y are independent and Y is centered then $E e^{t(X+Y)} \geq E e^{tX}$. Moreover if $\{\varepsilon_i\}$ is a Rademacher sequence then $E e^{i \sum_{j=1}^n a_j \varepsilon_j} \leq e^{i^2 \sum_{j=1}^n a_j^2 / 2}$ (since $E e^{a \varepsilon} \leq e^{a^2/2}$). To

prove (b) we take a Rademacher sequence $\{\varepsilon_i\}$ independent of $\{X_{nj}\}$ and a copy $\{X'_{nj}\}$ of $\{X_{nj}\}$ independent of the rest of the variables. Then we have, for each $\omega \in \Omega$ (which we omit),

$$(3.3) \quad \hat{E} e^{t \sum_{j=1}^{m_n} (X_{nj} - \bar{X}_n)/a_n} \leq \hat{E} e^{t \sum_{j=1}^{m_n} \varepsilon_j (X_{nj} - X'_{nj})/a_n} \leq \hat{E} e^{2t \sum_{j=1}^{m_n} \varepsilon_j X_{nj}/a_n} \\ \leq \hat{E} e^{2t^2 \sum_{j=1}^{m_n} X_{nj}^2/a_n^2} = \left[n^{-1} \sum_{i=1}^n \exp(2t^2 X_i^2/a_n^2) \right]^{m_n}.$$

Since $\max_{i \leq n} X_i^2/a_n^2 \rightarrow 0$ in probability, $n^{-1} \sum_{i=1}^n \exp(2t^2 X_i^2/a_n^2) \rightarrow 1$ in probability. Therefore the logarithm of the last term in (3.3) is asymptotic to $(m_n/n) \sum_{i=1}^n (e^{2t^2 X_i^2/a_n^2} - 1)$ which in turn is asymptotic to

$$(m_n/n) \sum_{i=1}^n 2t^2 X_i^2/a_n^2 = 2t^2 b_n^{-2} \sum_{i=1}^n X_i^2 \rightarrow 2t^2$$

in probability. Hence, for all t , the sequence

$\left\{ \hat{E} \exp \left(t \sum_{i=1}^{m_n} (X_{nj} - \bar{X}_n)/a_n \right) \right\}_{n=1}^{\infty}$ is stochastically bounded. Let

$V_n = \sum_{j=1}^{m_n} (X_{nj} - \bar{X}_n)/a_n$. We have

$$P \{ |\hat{E} e^{t V_n} - E e^{tg}| > \varepsilon \} \\ \leq P \{ |\hat{E} \exp t(V_n \wedge c) - E \exp t(g \wedge c)| > \varepsilon/2 \} \\ + 2P \{ e^{-tc} \hat{E} \exp(2t V_n) > \varepsilon/2 - e^{-tc} E \exp(2tg) \}$$

for any c . The first probability tends to zero by weak convergence in probability of V_n to g , for all c , and the second tends to zero uniformly in n as $c \rightarrow \infty$ by stochastic boundedness of $\{\hat{E} \exp(2t V_n)\}$. This proves (b). For (a) we just notice that the above arguments with $a_n = m_n^{1/2}$ and $b_n = n^{1/2}$, give a.s. boundedness of the sequence $\{\hat{E} \exp(2t V_n)\}$ because

$$\sum_{i=1}^n X_i^2/b_n^2 \rightarrow EX^2 \text{ a.s. and } \max_{i \leq n} X_i^2/a_n^2 \rightarrow 0 \text{ a.s.} \quad \square$$

3.2. THEOREM. — *If for $m_n \nearrow \infty$*

$$(3.4) \quad \hat{\mathcal{P}}\left(m_n^{-1/2} \sum_{j=1}^{m_n} (X_{nj}^\omega - c_j(\omega))\right) \rightarrow_w N(0, 1) \quad a.s.$$

then

$$(3.5) \quad EX^2 < \infty \quad \text{and} \quad d_2\left(\hat{\mathcal{P}}\left(m_n^{-1/2} \sum_{i=1}^{m_n} (X_{ni} - \bar{X}_n)\right), N(0, 1)\right) \rightarrow 0 \quad a.s.$$

Proof. — We have by the converse CLT that

$$n^{-1} \sum_{i=1}^n X_i^2 I_{|X_i| \leq m_n^{1/2}} - \left(n^{-1} \sum_{i=1}^n X_i I_{|X_i| \leq m_n^{1/2}}\right)^2 \rightarrow 1 \quad a.s.$$

Then if $EX^2 = \infty$, by inequality (2.7) in [2] this reduces to $n^{-1} \sum_{i=1}^n X_i^2 I_{|X_i| \leq m_n^{1/2}} \rightarrow 1$ a.s. which implies, by the law of large numbers, $\sup_{c>0} EX^2 I_{|X| \leq c} \leq 1$ i.e. $EX^2 \leq 1$, contradiction. Thus, $EX^2 < \infty$. Then

$$\hat{\mathcal{P}}\left(\sum_{i=1}^{m_n} (X_{ni} - \bar{X}_n)/m_n^{1/2}\right) \rightarrow_w N(0, 1) \quad a.s.$$

and, since

$$\hat{E}\left(\sum_{i=1}^{m_n} (X_{ni} - \bar{X}_n)/m_n^{1/2}\right)^2 = n^{-1} \sum_{i=1}^n X_i^2 - (\bar{X}_n)^2 \rightarrow 1 \quad a.s.,$$

the result follows. $\square$

3.3. THEOREM. — *For any $p \geq 2$ and $m_n \nearrow \infty$, consider*

- (i) $EX^2 < \infty$;
- (ii) $\sum_{i=1}^{\infty} P\{|X| > m^{1/2-1/p} n^{1/p}\} < \infty$;
- (iii) $d_p\left(\hat{\mathcal{P}}\left(m_n^{-1/2} \sum_{i=1}^{m_n} (X_{ni} - \bar{X}_n)\right), N(0, 1)\right) \rightarrow 0$ a.s.

Then (i) and (ii) together are equivalent to (iii).

Proof. — Suppose (iii) holds. Then $EX^2 < \infty$ by Theorem 3.2. From randomization by a Rademacher sequence independent of $\{X_{nj}\}$, convexity

of $y = |x|^p$, $p \geq 1$, and Kinchin's inequality (e. g. [1], p. 176) we obtain

$$\begin{aligned} 2 \hat{E} \left| m_n^{-1/2} \sum_{j=1}^{m_n} (X_{nj} - \bar{X}_n) \right|^p &\geq \hat{E} \left| m_n^{-1/2} \sum_{j=1}^{m_n} \varepsilon_i (X_{nj} - \bar{X}_n) \right|^p \\ &\geq c_p \hat{E} \left| m_n^{-1} \sum_{j=1}^{m_n} (X_{nj} - \bar{X}_n)^2 \right|^{p/2} \end{aligned}$$

for some $c_p > 0$. Therefore, by (iii), there is $c < \infty$ such that

$$\limsup_{n \rightarrow \infty} \hat{E} \left| m_n^{-1} \sum_{i=1}^{m_n} X_{nj}^2 \right|^{p/2} \leq c \text{ a. s. (since } \bar{X}_n \rightarrow 0 \text{ a. s.).}$$

Since

$$\hat{E} \left| m_n^{-1} \sum_{j=1}^{m_n} X_{nj}^2 \right|^{p/2} \geq \hat{E} m_n^{-p/2} \sum_{j=1}^{m_n} |X_{nj}|^p = m_n^{1-p/2} n^{-1} \sum_{i=1}^n |X_i|^p$$

we have $\limsup_{n \rightarrow \infty} n^{-1} m_n^{1-p/2} \sum_{i=1}^n |X_i|^p \leq c$ a. s. Then, by Feller's theorem in

e. g. Stout [6], p. 132, we have either $E|X|^p < \infty$ or $\sum_{n=1}^{\infty} P\{|X| > n^{1/p} m_n^{1/2-1/p}\} < \infty$, hence $\sum_{n=1}^{\infty} P\{|X| > n^{1/p} m_n^{1/2-1/p}\} < \infty$.

Suppose now that (i) and (ii) hold. Then by uniform integrability (e. g. [1], Exercise 13, p. 69) the proof of (iii) reduces to showing:

$$(a) \lim_{t \rightarrow \infty} \sup_n (m_n^{1-p/2}/n) \sum_{i=1}^n |X_i|^p I_{|X_i| \geq t m_n^{1/2}} = 0 \text{ a. s. and}$$

$$(b) (m_n/n) \sum_{i=1}^n X_i I_{|X_i| \geq m_n^{1/2}} \rightarrow 0 \text{ a. s.}$$

Now condition (ii) implies $(m_n^{1-p/2}/n) \sum_{i=1}^n |X_i|^p \rightarrow 0$ a. s. again by Feller's theorem (the case $E|X|^p < \infty$ is obvious). So condition (a) holds. As for (b) we note

$$\left| (m_n^{1/2}/n) \sum_{i=1}^n X_i I_{|X_i| \geq m_n^{1/2}} \right| \leq n^{-1} \sum_{i=1}^n X_i^2 I_{|X_i| \geq m_n^{1/2}} \rightarrow 0 \text{ a. s.}$$

by the law of large numbers. $\square$

3.4. Remark. — If $m_n/n \rightarrow 0$ then the proof of the Theorem 3.1 shows that the condition $\sum_{i=1}^{\infty} P\{|X| > m_n^{1/2}\} < \infty$ [i. e. $p = \infty$ in condition (ii) of

Theorem 3.3] implies

$$\hat{\mathbb{E}} \exp \left\{ t \sum_{i=1}^{m_n} (X_{n_j} - \bar{X}_n) / m_n^{1/2} \right\} \rightarrow \mathbb{E} e^{tg} \quad \text{a.s.}$$

for all $t \in \mathbb{R}$ but we do not know if the converse holds.

We conclude with the case $m_n/n \rightarrow 0$ and $EX^2 = \infty$.

3.5. THEOREM. — *If X is in the domain of attraction of a p -stable law $0 < p \leq 2$, that is*

$$\mathcal{L} \left(\sum_{i=1}^n (X_i - EX I_{|X| \leq \tau b_n}) / b_n \right) \rightarrow_d \mathcal{L}(\theta)$$

where we can take $\tau = \infty$ for $1 < p \leq 2$ and $\tau = 0$ for $0 < p < 1$, and if $m_n/n \rightarrow 0$, then

$$d_r \left[\hat{\mathcal{L}} \left(\sum_{j=1}^{m_n} \left(X_{n_j} - n^{-1} \sum_{i=1}^n X_i I_{|X_i| \leq \tau b_{m_n}} \right) / b_{m_n} \right), \mathcal{L}(\theta) \right] \rightarrow 0 \quad \text{in probability,}$$

for all $r \in (0, p)$.

Proof. — Given the bootstrap limit theorems 2.2 and 2.6 in [2], it suffices to show convergence in probability of the corresponding bootstrap moments. We only consider the case $1 < p \leq 2$ (the case $0 < p \leq 1$ is somewhat simpler). Let $1 < r < p \leq 2$. Let $\{\varepsilon_i\}$ be a Rademacher sequence independent of $\{X_{ni}\}$. Then, using symmetrization and Khinchin's inequality we have

$$\begin{aligned} \hat{\mathbb{E}} \left| \sum_{j=1}^{m_n} (X_{n_j} - \bar{X}_n) / b_{m_n} \right|^r &\leq c_r \left(\hat{\mathbb{E}} \left[\sum_{j=1}^{m_n} (X_{n_j} I_{|X_{n_j}| \leq b_{m_n}} - \hat{\mathbb{E}} X_{n_j} I_{|X_{n_j}| \leq b_{m_n}}) / b_{m_n} \right]^2 \right)^{r/2} \\ &\quad + c_r \hat{\mathbb{E}} \left| \sum_{j=1}^{m_n} \varepsilon_j X_{n_j} I_{|X_{n_j}| > b_{m_n}} / b_{m_n} \right|^r \\ &\leq c_r \left[(m_n / n b_{m_n}^2) \sum_{i=1}^n X_i^2 I_{|X_i| \leq b_{m_n}} \right]^{r/2} + c'_r \hat{\mathbb{E}} \left(\sum_{j=1}^{m_n} X_{n_j}^2 I_{|X_{n_j}| > b_{m_n}} / b_{m_n}^2 \right)^{r/2} \\ &\leq c_r \left[(m_n / n b_{m_n}^2) \sum_{i=1}^n X_i^2 I_{|X_i| \leq b_{m_n}} \right]^{r/2} + c'_r (m_n / b_{m_n}^r) \hat{\mathbb{E}} |X_{ni}|^r I_{|X_{ni}| \leq b_{m_n}} \end{aligned}$$

Each of these summands is bounded in probability because $(m_n / b_{m_n}^2) EX^2 I_{|X| \leq b_{m_n}}$ converges to a constant and

$$\mathbb{E} (m_n / b_{m_n}^r) \hat{\mathbb{E}} |X_{n_j}|^r I_{|X_{n_j}| > b_{m_n}} = (m_n / b_{m_n}^r) \mathbb{E} |X|^r I_{|X| > b_{m_n}}$$

also converges to a constant by regular variation. Stochastics boundedness

of the sequences $\left\{ \hat{E} \left| \sum_{j=1}^n (X_{nj} - \bar{X}_n) / b_{m_n} \right|^r \right\}_{n=1}^{\infty}$, $r < p$, together with weak convergence in probability give the result. $\square$

Theorem 3.5 is sharp. There are sequences m_n so that the conclusion of the theorem does not hold for $r=p$ (for $r=p < 2$ the conclusion does not even make sense since $E|\theta|^p = \infty$).

4. RANDOM NORMINGS FOR THE BOOTSTRAP OF THE MEAN IN GENERAL

If X is in the domain of attraction of the normal law, random normings in the bootstrap CLT have been discussed by several authors for $m_n = n$ (Bickel and Freedman [3] and others) and in [2] and in Section 1 above for any $\{m_n\}$. The normal case is easy to handle because $\left\{ \sum_{i=1}^n X_i^2 / b_n^2 \right\}$ converges in probability to a constant (a.s. if $EX^2 < \infty$). If X is in the domain of attraction of a p -stable law, $1 < p \leq 2$ (the only values of p we will consider here), then

$$(4.1) \quad \left\{ \sum_{i=1}^n (X_i - EX) / \left(\sum_{i=1}^n X_i^2 \right)^{1/2} \right\}_{n=1}^{\infty}$$

still converges in law even though $\sum_{i=1}^n X_i^2 / b_n^2$ does not converge in probability for $p \neq 2$. This limit theorem can be bootstrapped:

4.1. THEOREM. — *Let X be in the domain of attraction of a p -stable law, $1 < p \leq 2$, and let $m_n/n \rightarrow 0$. Then*

$$(4.2) \quad w\text{-}\lim_{n \rightarrow \infty} \mathcal{L} \left[\sum_{j=1}^{m_n} (X_{nj} - \bar{X}_n) / \left(\sum_{j=1}^{m_n} X_{nj}^2 \right)^{1/2} \right] \\ = w\text{-}\lim_{n \rightarrow \infty} \mathcal{L} \left[\sum_{j=1}^n (X_j - EX) / \left(\sum_{i=1}^n X_i^2 \right)^{1/2} \right]$$

in probability.

Proof. — The case $p=2$ has already been discussed above. So, let $1 < p < 2$. It is well known that the sequence (4.1) has a limit in law (Logan *et al.* [6], Csörgö and Horvath [4]), actually the sequence

$$(4.3) \quad \left\{ \sum_{i=1}^n ((X_i - EX) / b_n, X_i^2 / b_n^2) \right\}$$

converges in law to an infinitely divisible law in $\mathbb{R}^2$ without normal part. (4.2) will follow if we show that the sequence

$$(4.4) \quad \left\{ \sum_{i=1}^n ((X_{ni} - \bar{X}_n)/b_{m_n}, X_{ni}^2/b_{m_n}^2) \right\}$$

converges weakly to the same limit as (4.3) in probability. The triangular array $\{(X_{nj}/b_{m_n}, X_{nj}^2/b_{m_n}^2), j \leq m_n, n \in \mathbb{N}\}$ is infinitesimal ω -a. s. ([2]). Hence, by the classical limit theory (e. g. [1]), proving that the limits of (4.3) and (4.4) coincide reduces to proving:

(i) $m_n \hat{P} \{(X_{n1}/b_{m_n}, X_{n1}^2/b_{m_n}^2) \in A\}$ converges in probability to

$$v(A) = \lim_{n \rightarrow \infty} n P \{(X/b_n, X^2/b_n^2) \in A\}$$

for all Borel sets A such that $0 \in (A^c)^0$ and $v(\delta A) = 0$;

(ii) for each $\delta > 0$ $m_n \hat{E} |(X_{n1}/b_{m_n}, X_{n1}^2/b_{m_n}^2)|^2 I_{|(X_{n1}/b_{m_n}, X_{n1}^2/b_{m_n}^2)| \leq \delta}$ converges in probability to some h_δ , with $h_\delta \rightarrow 0$ as $\delta \rightarrow 0$, where $|\cdot|$ denotes any norm in $\mathbb{R}^2$; we will take $|(x, y)| = |x| \vee |y|$.

(iii)

$$(m_n/b_{m_n}) \hat{E} X_{n1} I_{|X_{n1}| > b_{m_n}} \rightarrow \lim_{n \rightarrow \infty} (n/b_n) EX I_{|X| > b_n}$$

in probability and

$$(m_n/b_{m_n}^2) \hat{E} X_{n1}^2 I_{|X_{n1}| \leq b_{m_n}} \rightarrow \lim_{n \rightarrow \infty} (n/b_n^2) EX^2 I_{|X| \leq b_n}$$

in probability.

[(i) ensures that the Lévy measures are the same, (ii) that the normal part of the limit is degenerate and (iii) that centering X_{ni} and not centering X_{ni}^2 in (4.4) have the same effect in the limit as centering X_i and not centering X_i^2 in (4.3)]. Note that an easy proof of weak convergence of (4.3) could be constructed along similar lines, that is, by checking that the triangular array $\{(X_i/b_n, X_i^2/b_n^2), i \leq n\}_{n=1}^\infty$ satisfies the classical conditions for the CLT.

Proof of (i). — We have

$$m_n \hat{P} \{(X_{n1}/b_{m_n}, X_{n1}^2/b_{m_n}^2) \in A\} = (m_n/n) \sum_{i=1}^n I_{(X_i/b_{m_n}, X_i^2/b_{m_n}^2) \in A}.$$

The expected value tends to $v(A)$ and the variance is dominated by

$$(m_n^2/n^2) n P \{(X/b_{m_n}, X^2/b_{m_n}^2) \in A\} \leq (v(A) + \varepsilon) (m_n/n)$$

for some $\varepsilon > 0$ and large n , which tends to zero.

Proof of (ii). — Only $\delta < 1$ needs to be considered. Then the sequence in (ii) is just $m_n b_{m_n}^{-2} \hat{E} |X_{n1}|^2 I_{|X_{n1}| \leq \delta b_{m_n}}$ and it is already proved in [2], pp. 469-470, that this sequence converges in probability for every $\delta > 0$ to

the limit h_δ of its expected values $\{(m_n/b_{m_n}^2) U(\delta b_{m_n})\}$. Then $h_\delta \rightarrow 0$ as $\delta \rightarrow 0$ because X is in the domain of attraction of a p -stable law, $p < 2$.

Proof of (iii). — The second limit is already proved in [2] [see the proof of (ii) above]. The proof of the first limit is omitted in [2] although it is used in Corollary 2.6 there. We give it here. Since

$$E(m_n/b_{m_n}) \hat{E}X_{n1} I_{|X_{n1}| > b_{m_n}} = (m_n/b_{m_n}) EX I_{|X| > b_{m_n}}$$

we only need to prove

$$E|(m_n/b_{m_n}) \hat{E}X_{n1} I_{|X_{n1}| > b_{m_n}} - (m_n/b_{m_n}) EX I_{|X| > b_{m_n}}|^r \rightarrow 0$$

for some $r > 0$. We take $1 < r < p$ and use symmetrization by a Rademacher sequence together with Khinchin's inequality to obtain (for suitable constants c and c')

$$\begin{aligned} E|(m_n/b_{m_n}) \hat{E}X_{n1} I_{|X_{n1}| > b_{m_n}} - (m_n/b_{m_n}) EX I_{|X| > b_{m_n}}|^r \\ \leq c E \left| (m_n/nb_{m_n}) \sum_{i=1}^n \varepsilon_i X_i I_{|X_i| > b_{m_n}} \right|^r \\ \leq E \left| c' (m_n^2/n^2 b_{m_n}^2) \sum_{i=1}^n X_i^2 I_{|X_i| > b_{m_n}} \right|^{r/2} \\ \leq c' (nm_n^r/n^r b_{m_n}^r) E|X|^r I_{|X| > b_{m_n}} = c' (m_n/n)^{r-1} (m_n/b_{m_n}^r) E|X|^r I_{|X| > b_{m_n}}. \end{aligned}$$

Since $r-1 > 0$, $m_n/n \rightarrow 0$ and $\{(m_n/b_{m_n}^r) E|X|^r I_{|X| > b_{m_n}}\}$ converges by regular variation, (iii) follows. $\square$

Theorem 4.1 may be useful if it is only known that X is in some domain of attraction. In that case one could take $\hat{t}_\alpha$ such that

$$\hat{P} \left\{ \left| \sum_{i=1}^{m_n} (X_{ni} - \bar{X}_n) \right| / \left(\sum_{i=1}^{m_n} X_{ni}^2 \right)^{1/2} > \hat{t}_\alpha \right\} \cong \alpha \text{ to obtain that}$$

$$P \left\{ |\bar{X}_n - EX| / \left(\sum_{i=1}^n X_i^2 \right)^{1/2} > \hat{t}_\alpha \right\} \cong \alpha,$$

and $\hat{t}_\alpha$ is asymptotically correct in probability. (See Logan *et al.* [6] for properties of the limiting distributions of these sequences: the limits have densities which are Gaussian like at $\pm \infty$.) Of course m_n must be taken so that $m_n/n \rightarrow 0$. It is an open question what $\{m_n\}$ gives best results; some results in [2] seem to suggest that $m_n = n/(\log \log n)^{1+\delta}$ for some

$\delta > 0$ should not be a bad choice. We should also remark that $\sum_{i=1}^n X_i^2$ and

$$\sum_{i=1}^{m_n} X_{ni}^2 \text{ can be replaced by } \sum_{i=1}^n (X_i - EX)^2 \text{ and } \sum_{i=1}^{m_n} (X_{ni} - \bar{X}_n)^2.$$

SIMULATIONS

The following simulations were performed. For each value of $p=1.1$, 1.5 and 1.9 and $n=50$ and 100 , $1,000$ samples of size n from the symmetric distribution of F_p were drawn. Here F_p is the symmetric distribution $2F_p(-t)=t^{-1/p}$, $t>1$. These samples were used to compute, for each (n, p) , the $\alpha=.90$, $.95$ and $.99$ sample quantiles of the statistic $S=\sum_{i=1}^n X_i / \left(\sum_{i=1}^n (X_i - \bar{X}_n)^2 \right)^{1/2}$. These are t_α in the Tables below [one t -value for each choice of (n, p, α)]. They should be regarded as very good approximations of the true quantiles of S . From each of these samples, say $\mathbf{X}(n, p; i)=(X_1(n, p; i), \dots, X_n(n, p; i))$, $i=1, \dots, 1,000$, $1,000$ bootstrap samples of size m_n were drawn, where $m_{50}=35$ and $m_{100}=65$ (*i.e.* m_n is slightly smaller than $n/\log \log n$), giving, for each n and p , $1,000$ values of

$$\hat{S}(n, p) = \sum_{i=1}^{m_n} (X_{ni} - \bar{X}_n) / \left(\sum_{i=1}^{m_n} (X_{ni} - \bar{X}_{nn})^2 \right)^{1/2}.$$

These values were used to compute the $.90$, $.95$ and $.99$ sample quantiles of $\hat{S}(n, p)$, $\hat{t}_{.95}(\mathbf{X}(n, p))$, $\hat{t}_{.90}(\mathbf{X}(n, p))$ and $\hat{t}_{.99}(\mathbf{X}(n, p))$. So, for each choice of (n, p, α) , we obtained $1,000$ independent replications of $\hat{t}_\alpha(\mathbf{X}(n, p))$ [one for each original sample $\mathbf{X}(n, p; i)$] and with these the distribution of $\hat{t}_\alpha(\hat{S}(n, p))$ was estimated. The Tables below show the median $m\hat{t}_\alpha$; the $.25$ and the $.75$ quantiles, $Q_1\hat{t}_\alpha$ and $Q_3\hat{t}_\alpha$ respectively; the mean $av\hat{t}_\alpha$ and the 10% trimmed mean $tav\hat{t}_\alpha$ of the distribution of $\hat{t}_\alpha$ for each n and p .

Note that the median of $\hat{t}_\alpha$ approximates t_α quite well and that the approximation of t_α by $\hat{t}_\alpha$ is acceptable at least 50% of times (actually more because the empirical distribution of $\hat{t}_\alpha$ is quite concentrated). Note however that the mean of $\hat{t}_\alpha$ is far off t_α , *particularly for* $p=1.1$: $\hat{t}_\alpha$ does take infrequent very large values which have a considerable effect on the mean (the trimmed mean is also quite close to $\hat{t}_\alpha$). The distribution of $\hat{t}_\alpha$ deserves thus further study. The results become better for larger p , and for each p fixed $m\hat{t}_\alpha$ is closer to t_α when $n=100$, as was to be expected. However the interquantile range $Q_3\hat{t}_\alpha - Q_1\hat{t}_\alpha$ is essentially the same for $n=50$ and for $n=100$; this suggests that the convergence of $\hat{t}_\alpha$ to t_α in probability takes place at a slow rate. These data do not show $\hat{t}_\alpha \rightarrow t_\alpha$ in pr. since m_n/n is too large. Analogous simulations were made for

$$S = \sum_{i=1}^n X_i / \left(\sum_{i=1}^n X_i^2 \right)^{1/2}, \text{ with similar results which we omit.}$$

TABLES.

$p=1.9$

$n=100, m=65$							$n=50, m=35$						
	t	$m\hat{t}$	$Q_1\hat{t}$	$Q_3\hat{t}$	$tav\hat{t}$	$ave\hat{t}$		t	$m\hat{t}$	$Q_1\hat{t}$	$Q_3\hat{t}$	$tav\hat{t}$	$ave\hat{t}$
$\alpha=.90$	1.32	1.35	0.91	1.77	1.34	1.42		1.32	1.34	0.97	1.79	1.38	1.39
.95	1.69	1.71	1.27	2.14	1.71	1.81		1.57	1.71	1.33	2.19	1.77	1.79
.99	2.40	2.33	1.90	2.83	2.37	2.48		2.21	2.37	1.90	2.96	2.46	2.51

$p=1.5$

$n=100, m=65$							$n=50, m=35$						
	t	$m\hat{t}$	$Q_1\hat{t}$	$Q_3\hat{t}$	$tav\hat{t}$	$ave\hat{t}$		t	$m\hat{t}$	$Q_1\hat{t}$	$Q_3\hat{t}$	$tav\hat{t}$	$ave\hat{t}$
$\alpha=.90$	1.32	1.37	0.93	1.85	1.41	1.90		1.29	1.37	0.99	1.89	1.47	1.56
.95	1.70	1.70	1.28	2.25	1.78	2.34		1.59	1.74	1.34	2.33	1.86	2.00
.99	2.33	2.31	1.83	2.98	2.44	3.07		2.11	2.38	1.83	3.07	2.55	2.78

$p=1.1$

$n=100, m=65$							$n=50, m=35$						
	t	$m\hat{t}$	$Q_1\hat{t}$	$Q_3\hat{t}$	$tav\hat{t}$	$ave\hat{t}$		t	$m\hat{t}$	$Q_1\hat{t}$	$Q_3\hat{t}$	$tav\hat{t}$	$ave\hat{t}$
$\alpha=.90$	1.32	1.39	0.97	2.17	1.65	10.78		1.27	1.40	1.03	2.21	1.81	2.47
.95	1.58	1.71	1.29	2.66	2.07	12.17		1.53	1.74	1.35	2.71	2.26	3.29
.99	2.15	2.25	1.72	3.52	2.82	14.29		2.00	2.33	1.78	3.61	3.04	5.00

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