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Sharp large deviations estimates for simulated annealing algorithms

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ABSTRACT. — Simulated annealing algorithms are Monte-Carlo simulations of physical systems where the temperature is a decreasing function of time. The method can be used as a general purpose optimization technique to locate the minima of an arbitrary function defined on a finite but possibly very large set. It is described as a non-stationary controlled Markov chain. The aim of this paper is to build a large deviations theory in this time-inhomogeneous discrete setting. We make a careful investigation of the law of the exit point and time from sets, based on Wentzell and Freidlin's decomposition of the states space into cycles, which leads us to establish some kind of systematic "calculus" of the probability of jumps from one arbitrary subdomain into another one. We feel that giving a precise description of how trajectories escape from attractors brings a qualitative contribution to the insight one may have into the behaviour of simulated annealing. We think also that our sharp large deviation estimates are a useful tool for further investigations. We illustrate their use by addressing in this paper and in a forthcoming one four questions: convergence to the ground states, asymptotical equidistribution on the ground states, the discussion of quasi-equilibrium and the

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shape of "optimal" cooling schedules. Short cuts to "rough" but more uniform estimate will be given elsewhere.

Key words : Simulated annealing, large deviations, non-stationary Markov chains.

RÉSUMÉ. – Les algorithmes de recuit simulé permettent de simuler le comportement d'un système physique dont la température décroît au cours du temps. Ils constituent par là-même une méthode d'optimisation de portée générale capable de situer les minimums d'une fonction arbitraire définie sur un ensemble fini, fût-il de très grande taille. Leur description mathématique est celle d'une chaîne de Markov contrôlée non stationnaire. Le but de cet article est d'établir une théorie des grandes déviations pour ces systèmes discrets non homogènes dans le temps. Une étude détaillée de la loi du lieu et du temps de sortie d'un sous-ensemble d'états, fondée sur la décomposition en cycles de Wentzell et Freidlin, conduit à un calcul systématique de la probabilité des sauts d'un sous-domaine quelconque dans un autre. Cette description de la façon dont les trajectoires s'échappent des attracteurs nous semble apporter une information qualitative intéressante sur le comportement du recuit. Nous pensons aussi que ces estimées précises de grandes déviations sont susceptibles de nombreuses applications. Nous illustrons leur portée en traitant dans cet article et celui qui lui fera suite quatre thèmes : la convergence vers les états fondamentaux, l'équidistribution asymptotique sur les états fondamentaux, la question du quasi-équilibre thermique, la forme des courbes de température optimales. Un raccourci vers des estimées plus « grossières » mais plus uniformes sera présenté ultérieurement.

Contents

INTRODUCTION

Simulated annealing is a simulation algorithm from Statistical Mechanics. It was designed to simulate the canonical distribution of a physical system in contact with a heat bath when the temperature of the heat bath is progressively lowered to zero. If the cooling is not too fast, the system will go through a succession of quasi-equilibrium states towards one of its ground states. As any function defined on a finite set can be viewed as the energy levels of an "abstract" physical system, simulated annealing algorithms have been found to be a general purpose optimization technique, allowing one to locate the global minima of an arbitrary function. They have been widely used as such without further reference to their physical interpretation.

From the mathematical point of view, simulated annealing is an interesting case of non-stationary Markov chain: the transition matrices converge to a boundary point of the domain of ergodic Markov matrices. As temperature goes to zero the "instantaneous" mixing properties grow weaker and weaker and one question is to know whether the non-stationary Markov chain as a whole retains some mixing properties. The answer will depend on the cooling schedule, on how fast temperature is approaching zero. Non-stationary Markov chains are a not so recent but still widely open field of investigation (Dobrushin [7], Iosifescu and Theodorescu [17], Isaacson and Madsen [18], Seneta [20], Gidas [10]).

Let us formulate some of the questions about simulated annealing people have been interested in. One is to know for which cooling schedules quasi-equilibrium is maintained. Another one is to know whether the distribution law of the system concentrates on the ground states. In case it is so and there are more than one ground state, one is entitled to ask whether all the ground states are reached with equal probabilities. Finally when the target is to minimize the energy function, one wants to know which cooling schedules provide the fastest convergence.

The aim of this paper and of a forthcoming one is to address these questions.

Partial answers already exist in the mathematical literature. S. Geman and D. Geman pointed out in 1984 [9] that cooling schedules of the type $T_n = c/\ln n$ with c large enough ensure quasi-equilibrium (n is the discrete time variable in this equation). The question was then to know what was the smallest possible value of c.

The critical value was given by different people. Holley and Stroock showed in 1988 [13] that the law $\mathscr{L}_{X_n}$ of the system X_n undergoing simulated annealing had a bounded density $f_n = \mathscr{L}_{X_n}/\mu_{T_n}$ with respect to the equilibrium law μ_{T_n} at temperature T_n when $c > c_0$, c_o being the sharp critical value. Their argument is based on the study of the smallest non zero eigenvalue of the Dirichlet form associated with the infinitesimal generator of the process and goes through $L_p(\mu_{T_n})$ estimates of f_n derived from the Kolmogorov equations. They were not able to conclude on the L^∞ norm of f_n in the critical case $c = c_0$.

More precise results were given by Chiang and Chow and by Hwang and Sheu, both in 1988. These authors showed that $\lim_{n \to +\infty} f_n \equiv 1$ when $c > c_0$, c_0 being the critical value. Chiang and Chow uses the forward equation and an induction argument on the precision of their estimations.

Hwang and Sheu have a different approach. They use the results from Wentzell and Freidlin [8] on stationary random perturbations of dynamical systems to derive a weak ergodicity estimate in the approximately stationary case, that is for Markov chains with transitions remaining of the same order of magnitude as the transitions of annealing at one fixed temperature. Then they use their estimates on time intervals on which the variation of the temperature is small enough.

These papers, which are focused on the study of quasi-equilibrium, give also some answers for the convergence to a ground state but the sharpest results are in two papers by Hajek [11] in 1988 and Tsitsiklis [22] in 1989. They give a necessary and sufficient condition for which the probability of the system to be in a ground state tends to one when time tends to infinity. The condition is $\sum_k e^{-d/T_k} = +\infty$, where $d < c_0$ is a "critical depth".

Tsitsiklis uses the same strategy as Hwang and Sheu: he studies the almost constant temperature case and applies it on well chosen time intervals. Hajek works directly in the non-constant temperature case. He studies the exit time and point from secondary attractors of the system (he calls them "cups"). If C is one of these cups, he shows by an induction argument on the size of C that for any state i in C of minimal energy

$$P(X_s \in C,\ t \leqq s \leqq r \mid X_t = i) \geqq \exp\left(-\Gamma \sum_{s=t+1}^{r} e^{-H/T_s}\right), \tag{1}$$

where H is sharp but not Γ. He shows also the weak reverse inequality

$$\sum_{r,\, r \geqq t} P(X_s \in C,\ t \leqq s \leqq r \mid X_t = i)\, e^{-H/T_r} < +\infty, \qquad i \in C. \tag{2}$$

To my knowledge there was no necessary and sufficient answer to the problem of knowing whether the law of the system becomes equidistributed on the ground states before the one given in this paper. I did not found either any study of optimal cooling schedules (those leading to the fastest convergence).

Let us close this quick review of the literature by saying that the constant temperature case is well understood in a more general context from the works of Wentzell and Freidlin on random perturbations of dynamical systems. They have made decisive breakthroughs in the theory of large deviations of dynamical systems with small random perturbations. Large deviations theory is the right framework to study annealing algorithms because escaping from local minima is an event of small probability at low temperature.

Roughly speaking we will bring the following answers to the four aforementioned questions:

1. To study the critical case $1/T_n = 1/c_0 \ln n$, it is necessary to introduce a second constant and to consider cooling schedules of the type $1/T_n = 1/c_0 \ln n + B$. The behaviour of the system depends on whether B is larger than some critical value B_0 or not.

- If $B > B_0$, then $\lim_{n \to +\infty} \|f_n\|_\infty = +\infty$.
- If $B < B_0$, then $\lim_{n \to +\infty} f_n(i)$ exists for any state i, is finite, is always $\geqq 1$

and for some states > 1.

Hence quasi-equilibrium is not maintained in the critical case $c = c_0$, but it can be almost maintained for small enough values of B.

2. We give a new proof of Hajek criterion of convergence: if F is the set of ground states, $\lim_{n \to +\infty} P(X_n \in F) = 1$ if and only if $\sum_k e^{-d/T_k} = +\infty$.

3. We prove that $\lim_{n \to +\infty} P(X_n = f) = \mathrm{Card}(F)^{-1}$ for any $f \in F$ if and only if $\sum_k e^{-c_0/T_k} = +\infty$ where $c_0 \geqq d$ is the same critical constant as in the discussion of quasi-equilibrium.

4. We say that $\hat{T}_1^N, \ldots, \hat{T}_N^N$ is optimal if

$$P(X_N^{\hat{T}^N} \in F) = \max_{T_1, \ldots, T_N} P(X_N^T \in F). \tag{3}$$

Thus our measure of convergence is the probability to be in a ground state at some finite (large) time N corresponding to the end of the simulation. There is no reason *a priori* to think that the best schedules $\hat{T}_1, \ldots, \hat{T}_N$ are independent of N, hence we have put a superscript N. We show that there is some value $\tilde{B}$ of B and some $\alpha > 0$ such that

$$1/\hat{T}_k^N \leqq \frac{1}{c_0} \ln k + \tilde{B} + \frac{1}{k^\alpha}, \tag{4}$$

as soon as k and $N - k$ are large enough.

We show that changing $1/\hat{T}^N$ into $\sup(1/\hat{T}_n^N, 1/c_0 \ln k + \tilde{B})$ produces only a small variation of $P(X_N \in F)$.

Remarks:

- We will prove elsewhere [3] that (4) does not hold for $N - k$ "small". This shows that there is no optimal cooling schedule which is independent of the simulation time N (*i. e.* such that $\hat{T}_k^N$ is constant in N for k fixed).
- Our study of $\hat{T}$ is done under some non-degeneracy asumptions on the energy landscape: we assume that there is only one ground state and that the critical depth d in Hajek criterion is reached only once.

This first paper will be mainly devoted to build the technical tool we need to prove our claims. This tool is made of a precise and systematic study of the way the system escapes from any given strict subdomain of the states space.

This program has been carried through by Wentzell and Freidlin [8] in a general context that applies to the constant temperature case. We are going to carry it for simulated annealing, that is in a non-stationary case. We have not used Wentzell and Freidlin results as Hwang and Sheu did because we needed more precise estimates: we wanted them to be uniform with respect to the values of the temperatures and we wanted to know more than the exponential order of magnitude of the probability to espace from attractors, we wanted a full equivalent. Nontheless we followed the conceptions about large deviations which Wentzell and Freidlin give along with their proofs.

There are subdomains for which the dependence of the exit time and point on the starting point within the domain fades away at low temperatures. These subdomains were called cycles by Wentzell and Freidlin. We will prove that for any cycle C and any state $i \in \mathrm{C}$

$$\mathrm{P}(\mathrm{X}_k \in \mathrm{C},\, m<k\leqq n \mid \mathrm{X}_m = i) \simeq \prod_{k=m+1}^{n} (1 - a\, e^{-\mathrm{H}/\mathrm{T}_k}) \tag{5}$$

with sharp constants a and H. We will make explicit how the "almost equal" can be considered to be uniform with respect to the temperatures sequence $\mathrm{T}_{m+1}, \ldots, \mathrm{T}_n$. Getting uniform estimates is crucial to address the problem of optimal cooling schedules. Having a sharp constant a is useful to prove convergence to the equidistributed law on the ground states. This is sharper than usual large deviation estimates where only the exponential order, that is H, is sharp.

The cycles are organized by the inclusion relation into a tree structure. Two cycles are either disjoint or one is a subset of the other.

Along with the escape from cycles, we will study the jumps from one cycle to another cycle. Now taking some general strict subdomain of the states space, we can write it as the disjoint union of cycles in the coarsest possible way. Studying the jumps from one cycle of this "maximal partition" to another, we will give the dependence of the exit time and point of the domain on the starting point.

The proofs are by induction on the size of the cycles. The study of the behaviour of the system in cycles and the study of jumps from cycle to cycle are closely related in the induction argument. To prove that the conditional law of the system knowing that it remains in some cycle becomes equidistributed on the ground states of the cycle, we study jumps between subcycles containing the ground states. Then we remove one ground state from the cycle, obtaining thus a subdomain which is no more a cycle, and we study the behaviour of the system within this domain after its last visit to the marked ground state. It has a strong probability to return to the marked ground state, and a small one to escape from the cycle. In this we follow the program traced by Wentzell and Freidlin's investigation of the escape from a domain containing one stable attractor.

What we get in the end is a systematic algebraic tool to derive estimates for the system to have lived in a tube of arbitrary shape during a given period of time as well as estimates for an arbitrary succession of jumps from one tube to another one.

This tool is therefore very general and could lead to other applications than the results we give in these two papers.

This research has been carried under the direction of R. Azencott, I am glad to thank him for having introduced me to simulated annealing and having supported me by useful suggestions and encouragements.

1. DESCRIPTION OF THE MODEL

1.1. Annealing algorithms

DEFINITION 1.1. — *An energy landscape is a couple* (E, U), *where:*

- E *is a finite set;*
- $U : E \to \mathbb{R}$ *is a non-constant real valued function.*

DEFINITION 1.2. — *Let* (E, U) *be an energy landscape. A communication kernel on* E *is an irreducible symmetric Markov kernel* q *on* E, *that is a function*

$$q: \quad E \times E \to \mathbb{R}_+ \tag{6}$$

such that

$$\left.\begin{array}{l}\sum_{j \in E} q(i, j) = 1, \qquad q(i, j) = q(j, i), \qquad \text{and} \qquad \sup_{n \in \mathbb{N}} q^n(i, j) > 0, \\ \qquad\qquad\qquad\qquad i, j \in E.\end{array}\right\} \tag{7}$$

For any subset A *of* E, *by* $q_{|A}$ *we will mean the restriction of* q *to* A, *that is the kernel*

$$q_{|A}(i, j) = \chi_A(i)\, \chi_A(j)\, q(i, j), \tag{8}$$

where χ_A *is the characteristic function of* $A : \chi_A(i) = 1$ *if* $i \in A$ *and* $\chi_A = 0$ *otherwise.*

Remark: The assumption of symmetry could be replaced by the existence of a locally invariant probability measure, that is a measure μ, such that:

$$\forall i, j \quad \mu(i)\, q(i, j) = \mu(j)\, q(j, i) \tag{9}$$

or even by a weaker assumption such as in Hajek [11]; but this would not lead us to significantly deeper results nor change the nature of the proofs.

DEFINITION 1.3. — *Let us write* x^+ *for the positive part of* x, *that is for* $\sup(x, 0)$. *Let* (E, U, q) *be an energy landscape with communications. For any* T *in* $\mathbb{R}_+$, *the transition kernel* p_T *at temperature* T *is a Markov kernel* *on* E *defined by*

$$p_T(i,j)=q(i,j)\exp(-(U_j-U_i)^+/T) \qquad \textit{for} \quad i\neq j \tag{10}$$

and

$$p_T(i,j)=1-\sum_{j\in E-\{i\}} q(i,j)\exp(-(U_j-U_i)^+/T). \tag{11}$$

By convention we will extend this definition to the case T = 0 *in the following way:*

$$\left.\begin{aligned} p_0(i,j)&=q(i,j)\,\mathbf{1}((U_j-U_i)\leqq 0), \qquad i\neq j,\\ p_0(i,i)&=1-\sum_{j\neq i} q(i,j)\,\mathbf{1}((U_j-U_i)\leqq 0). \end{aligned}\right\} \tag{12}$$

DEFINITION 1.4. — *A cooling schedule is a non-increasing sequence of positive real numbers* $(T_n)_{n\in\mathbb{N}^*}$.

DEFINITION 1.5. — *A simulated annealing algorithm is a collection* (E, U, $\mathscr{L}_0$, q, T, X), *where:*

- (E, U, q) *is an energy landscape with communications;*
- $\mathscr{L}_0$ *is a probability distribution on* E (*the initial distribution*);
- T *is a cooling schedule;*
- *and* X *is the Markov chain on* E *defined by its initial distribution* $\mathscr{L}_0$ *and its transitions*

$$P(X_n=j\,|\,X_{n-1}=i)=p_{T_n}(i,j) \qquad n>0. \tag{13}$$

Remark. — We can identify (E, U, $\mathscr{L}_0$, q, T, X) and (E, $\tilde{U}$, $\mathscr{L}_0$, q, T, X) as soon as $U-\tilde{U}$ is a constant function on E (because the law of X is then the same). Hence we can – and will – assume that

$$\min_{i\in E} U_i=0 \tag{14}$$

1.2. Decomposition into cycles

We follow here the decomposition into cycles given by Wentzell and Freidlin [8] in a more general setting.

DEFINITION 1.6. — *Let* (E, U, q) *be an energy landscape. The level set at level* λ *of* (E, U) *is the subset* E_λ *of* E *defined by*

$$E_\lambda=U^{-1}(\,]-\infty,\lambda])\tag{15}$$

DEFINITION 1.7. — *Let* (E, U, q) *be an energy landscape. Let* λ *be some real number. The communication relation at level* λ *on* (E, U) *is the equivalence relation* $\mathcal{R}_\lambda$ *defined by*

$$\mathcal{R}_\lambda = \{ (i, j) \mid \sup_{n \in \mathbb{N}} (q_{|\,E_\lambda})^n (i, j) > 0 \}. \tag{16}$$

We shall assume by convention that $(q_{|\,E_\lambda})^0 = \mathrm{Id}$, *hence that* $(i, i) \in \mathcal{R}_\lambda$, $i \in E$.

Two states i and j communicate at level λ if there is a path from i to j which does not go through any state of energy superior to λ.

DEFINITION 1.8. — *Let* (E, U, q) *be an energy landscape. Let* λ *be a real number. A cycle of level* λ *is a subset of* E *which is an equivalence class of* $\mathcal{R}_\lambda$. *The class of all cycles of level* λ *will be denoted by* $\mathscr{C}_\lambda$(E, U). *The class of all cycles will be denoted by* $\mathscr{C}$(E, U). *We have*

$$\mathscr{C}(E, U) = \bigcup_{\lambda \in \mathbb{R}} \mathscr{C}_\lambda (E, U). \tag{17}$$

The class of all sub-cycles of a cycle C *of* $\mathscr{C}$(E, U) *will be denoted by* $\mathscr{C}$(E, U, C). *Thus we have*

$$\mathscr{C}(E, U, C) = \{ G \in \mathscr{C}(E, U) : G \subset C \}. \tag{18}$$

Let us mention a few simple facts about cycles:

PROPOSITION 1.9:

- *All sets reduced to one state are cycles.*
- *If* C *and* G *are in* $\mathscr{C}$(E, U) *they are either disjoint or comparable for inclusion* (*that is one is a subset of the other*).
- $\mathscr{C}_\lambda$(E, U) *is a partition of* E.

DEFINITION 1.10. — *Let* (E, U, q) *be an energy landscape. We extend the definition of* U *to* $\mathscr{C}$(E, U) *by putting*

$$U(C) = \min_{i \in C} U_i. \tag{19}$$

The energy of a cycle is the energy of its fundamental states.

DEFINITION 1.11. — *Let* (E, U, q) *be an energy landscape. Let* A *be a subset of* E *and let* i, j *be any states in* E. *We define the minimal energy of communication between* i *and* j *through* A *to be*

$$U(i, j, A) = \inf \{ \lambda : \sup_{n \in \mathbb{N}} (q_{|\,E_\lambda \cap A})^n (i, j) > 0 \}. \tag{20}$$

DEFINITION 1.12. — *The boundary of* $A \subset E$ *is*

$$B(A) = \{ j \in E - A : \sup_{i \in A} q_{i,j} > 0 \} \tag{21}$$

DEFINITION 1.13. — *The depth function* $H:\mathscr{C}(E, U)\to\mathbb{R}$ *is defined by:*

$$H(C)=\max\Big(0, \min_{j\in B(C)} (U_j-U(C))\Big). \tag{22}$$

Remark. — The max is needed in the case when C is reduced to one point.

DEFINITION 1.14. — *The principal boundary of* $C\in\mathscr{C}(E, U)$ *is*

$$\tilde{B}(C)=\{j\in B(C): U_j\leqq U(C)+H(C)\}. \tag{23}$$

Remark. — Again the $\leqq$ is there to cover the case singletons. When C is not reduced to one point, the inequality is equivalent to the equality:

PROPOSITION 1.15. — *If* $H(C)>0$ *then*

$$\tilde{B}(C)=\{j\in B(C): U_j=U(C)+H(C)\}. \tag{24}$$

DEFINITION 1.16. — *The bottom of* $C\in\mathscr{C}(E, U)$ *is*

$$F(C)=\{i\in C: U_i=U(C)\}. \tag{25}$$

DEFINITION 1.17. — *Let* (E, U, q) *be an energy landscape with communications. The set of minimums of* E *is*

$$M(E)=\bigcup_{C\in\mathscr{C}(E, U): H(C)>0} F(C). \tag{26}$$

A minimum i is called a local minimum if $U_i>0$ [*let us recall that* $U(E)=0$ *by convention*].

DEFINITION 1.18. — *Extension of the communication kernel q to* $\mathscr{P}(E)\times\mathscr{P}(E)$ *and* $\mathscr{C}(E, U)$:

$$q(A, B)=\sum_{i\in A,\, j\in B} q(i, j), \tag{27}$$

and for any cycle C *of* $\mathscr{C}(E, U)$

$$q(C)=q(C, \tilde{B}(C)). \tag{28}$$

DEFINITION 1.19. — *Some more classes of cycles:*

Let C *be a cycle in* $\mathscr{C}(E, U)$, $\mathscr{C}''(E, U, C)$ *will denote the class of cycles* G *in* $\mathscr{C}(E, U, C)$ *such that* $U(G)>U(C)$, *and* $\mathscr{C}'(E, U, C)$ *will denote the class of cycles* G *in* $\mathscr{C}(E, U, C)$ *such that* $F(C)\not\subset G$. *We will abreviate* $\mathscr{C}'(E, U, E)$ *and* $\mathscr{C}''(E, U, E)$ *by* $\mathscr{C}'(E, U)$ *and* $\mathscr{C}''(E, U)$.

The reason for introducing $\mathscr{C}'$ and $\mathscr{C}''$ is the following: if we want to make sure that the system can reach some fundamental state of C starting from a state in C, we have to make sure that it is possible to get out of any cycle of $\mathscr{C}''(E, U, C)$. If we want to make sure that it is possible to travel from any ground state of C to any other ground state of C, we have to make sure that it is possible to get out of any cycle of $\mathscr{C}'(E, U, C)$.

PROPOSITION 1.20. – *We have*

$$\mathscr{C}''(E, U, C) \subset \mathscr{C}'(E, U, C) \subset \mathscr{C}(E, U, C). \tag{29}$$

DEFINITION 1.21. – *For any cycles* C_1, C_2 *such that* $C_2 \subset C_1$, *we will denote by* $\mathcal{N}(C_2, C_1)$ *and call the natural context of* C_2 *in* C_1 *the subcycle of* C_1 *which is minimal for inclusion in*

$$\{C \in \mathscr{C}(E, U, q) \mid C_2 \subset C \subset C_1 \quad \text{and} \quad C \cap F(C_1) \neq \varnothing\}. \tag{30}$$

DEFINITION 1.22. – *Let* C *be a cycle of* (E, U, *q*). *The natural partition of* C *is the partition of* C *into its maximal strict subcycles.*

LEMMA 1.23. – *The quantity* $H(G) + U(G)$ *is constant among the* G*s belonging to the natural partition of* C.

Proof. – If it were not the case, putting

$$\lambda = \max\{H(G) + U(G) \mid G \text{ in the natural partition of } C\} \tag{31}$$

the communication relation at level λ, $\mathscr{R}_\lambda$, would induce a partition of C coarser than the natural partition of C, which is a contradiction.

End of the proof of lemma 1.23.

DEFINITION 1.24. – *Let* (E, U,) *be an energy landscape with communications. For any* $C \in \mathscr{C}(E, U)$ *we define*

$$H'(C) = \max\{H(G) : G \in \mathscr{C}'(E, U, C)\}, \tag{32}$$

and

$$H''(C) = \max\{H(G) : G \in \mathscr{C}''(E, U, C)\}. \tag{33}$$

1.3. Invariant probability measures

DEFINITION 1.25. – *Let* (E, U) *be an energy landscape. The equilibrium distribution at temperature* T *is defined to be the probability*

$$\mu_T(i) = \frac{1}{Z}\exp(-U_i/T), \qquad T \in \mathbb{R}_+^*, \quad i \in E. \tag{34}$$

where

$$Z = \sum_{j \in E} \exp(-U_j/T) \tag{35}$$

is the partition function. We extend this definition to $T=0$ *by putting*

$$\mu_0(i) = \frac{1}{|F(E)|} \chi_{F(E)}(i) \tag{36}$$

where $|A|$ *is the cardinal of* A *and* χ_A *is the characteristic function of* A.

PROPOSITION 1.26. — *Let* (E, U) *be an energy landscape. We have for any* $i \in E$

$$\lim_{T \to 0} \mu_T(i) = \mu_0(i). \tag{37}$$

Proof. — Easy.

PROPOSITION 1.27. — *For any* $T \in \mathbb{R}_+$, μ_T *is the invariant probability of the transition kernel at temperature* T. *In fact we have a strong form of invariance, which is* local invariance: *for any i and j in* E

$$\mu_T(i) p_T(i, j) = \mu_T(j) p_T(j, i). \tag{38}$$

2. STUDY OF SOME LARGE DEVIATION EVENTS

We will need uniform estimates, holding for some families of cooling schedules. The simplest interesting family of cooling schedules we will encounter are the neighbourhoods of zero for the L_∞ norm. We will study how long the system remains in a given subdomain of E at low temperature. This is a way of localizing our study both in space and in time.

DEFINITION 2.1. — *Let* (E, U, q) *be an energy landscape with communications. A cooling framework is any set of cooling schedules. For any* $T_0 \in \mathbb{R}_+^*$, *we define* $\mathcal{F}(T_0)$ *to be the cooling framework:*

$$\mathcal{F}(T_0) = \{ (T_n)_{n \in \mathbb{N}^*} : T_{k+1} \leq T_k,\ k \in \mathbb{N} \} \tag{39}$$

Such cooling frameworks will be called simple cooling frameworks.

DEFINITION 2.2. — *An annealing framework is a collection* (E, U, $\mathcal{L}_0$, q, $\mathcal{F}$, $\mathcal{X}$) *where:*

• (E, U, q, $\mathcal{L}_0$) *is an energy landscape with communications and initial distribution;*

• $\mathcal{F}$ *is a cooling framework;*
$\mathcal{X}$ *is the class of Markov chains appearing in the annealing algorithms* (E, U, $\mathcal{L}_0$, q, T, X) *with* $T \in \mathcal{F}$.

An annealing framework is said to be simple if the corresponding cooling framework is simple.

DEFINITION 2.3. – *Let* (E, U, $\mathcal{L}_0$, q, $\mathcal{F}$, $\mathcal{X}$) *be an annealing framework. Let* A *be some subset of* E, *the first exit time from* A *after time* m *will be noted*

$$\tau(A, m) = \inf\{n > m : X_n \notin A\} \tag{40}$$

2.1. The most probable behaviour of X

The results from this section are very simple. They quantify the fact that in first approximation a simulated annealing algorithm strongly resembles a descent algorithm.

THEOREM 2.4. – *For any energy landscape with communications and initial distribution* (E, U, q, $\mathcal{L}_0$), *there exist a simple annealing framework* (E, U, $\mathcal{L}_0$, q, $\mathcal{F}$, $\mathcal{X}$) *and a constant* $\alpha > 0$ *such that, for any* $X \in \mathcal{X}$, *for any* $i, j \in E$ *we have*

$$P(X_{n+k} = j \mid X_n = i) \geqq (p_0)^k (i, j) \prod_{l=1}^{k} (1 - \exp(-\alpha/T_{n+l})). \tag{41}$$

Proof. – There are constants $\alpha > 0$ and $T_0 > 0$ for which for any $T \leqq T_0$, for any $i, j \in E$ we have

$$p_T(i, j) \geqq p_0(i, j)(1 - \exp(-\alpha/T)). \tag{42}$$

Consider the annealing framework (E, U, $\mathcal{L}_0$, q, $\mathcal{F}(T_0)$, $\mathcal{X}$). In this framework

$$P(X_{n+k} = j \mid X_n = i \geqq \sum_{s \in E} (P(X_{n+k-1} = s \mid X_n = i) p_0(s, j)(1 - \exp(-\alpha/T_{n+k})) \tag{43}$$

DEFINITION 2.5. – *Let* (E, U, q) *be an energy landscape with communications. For any* $i \in E$ *the set of minima of* E *under* i *is*

$$M_i = \{j \in M(E) : \sup_{n \in \mathbb{N}} p_0^n(i, j) > 0\}. \tag{44}$$

The set M_i is thus the set of local minima of the states space E which can be reached by the descent algorithm with transitions p_0 starting at i.

THEOREM 2.6. — *For any energy landscape with communications and initial distribution* $(E, U, q, \mathscr{L}_0)$, *there exist a simple annealing framework* $(E, U, \mathscr{L}_0, q, \mathscr{F}, \mathscr{X})$ *and constants* $\alpha>0$, $\lambda\in]0, 1[$, $\gamma>0$ *such that, for any* $X\in\mathscr{X}$, *for any* $i, j\in E$, *for any* $k\in\mathbb{N}$ *we have*

$$P(X_{n+k}\in M_i \mid X_n=i)\geqq(1-\gamma\lambda^k)\prod_{l=1}^{k}(1-\exp(-\alpha/T_{n+l})) \qquad (45)$$

Proof. — According to the preceding theorem, we have to prove that

$$\sum_{j\in M_i} p_0^k(i, j)\geqq 1-\gamma\lambda^k. \qquad (46)$$

Let O_i be the set of states which can be reached by the descent algorithm with transitions p_0 starting from i:

$$O_i=\{j\in E: \sup_{n\in\mathbb{N}} p_0^n(i, j)>0\}. \qquad (47)$$

The relation "to be reachable from" is clearly transitive, which can equivalently be stated in the following way:

If j belongs to O_i, then O_j is a subset of O_i.

The next thing to remark is that M_j is never empty, because either j is a minimum or there is in O_j some state s with energy $U_s<U_j$. For each j in O_i consider

$$r_j=\inf\{k\in\mathbb{N}: p_0^k(j, M_j)>0\}, \qquad (48)$$

where

$$p_0^k(j, M_j)=\sum_{s\in M_j} p_0^k(j, s).$$

Consider

$$r=\max_{j\in O_i} r_j \qquad (49)$$

and

$$(50) \qquad A=\min_{j\in O_i} p_0^r(j, M_j).$$

If s is in M_j, we have

$$p_0^n(s, M_j)=1, \qquad n\in\mathbb{N}. \qquad (51)$$

Thus

$$p_0^r(j, M_j) \geq p_{0'}^r(j, M_j) > 0. \tag{52}$$

Thus $A > 0$. Then

$$p_0^{n+r}(i, M_i) \geq p_0^n(i, M_i) + p_0^n(i, O_i - M_i) \times A. \tag{53}$$

But $p_0^n(i, O_i - M_i) = 1 - p_0^n(i, M_i)$, hence

$$p_0^{n+r}(i, M_i) \geq p_0^n(i, M_i)(1 - A) + A. \tag{54}$$

Equation (46) is an easy consequence of this last inequality.

2.2. Generalized transition kernels

DEFINITION 2.7. – *Let* $\mathscr{A} = (E, U, \mathscr{L}_0, q, T, X)$ *be an annealing algorithm. The family* M *of generalized transition kernels (GTKs) of* $\mathscr{A}$ *is a family* M *of* 2×2 *tensors indexed by* $\mathscr{P}(E) \times \mathscr{P}(E)$ *defined by*

$$M(A, B) : \mathbb{N}^2 \times E^2 \to \mathbb{R}_+$$
$$M(A, B)_{m,\, i}^{n,\, j} = P(\tau(A, m) \geq n, X_{n-1} \notin B, X_n = j \mid X_m = i),$$
$$m < n, \qquad i \in E, \qquad j \in B,$$

and

$$M(A, B)_{m,\, i}^{n,\, j} = 0, \quad \textit{otherwise}. \tag{55}$$

The product of GTKs will be the usual inner tensor product:

$$M(A, B)\, M(C, D)_{m,\, i}^{n,\, j} = \sum_{k \in \mathbb{N},\, s \in E} M(A, B)_{m,\, i}^{k,\, s}\, M(C, D)_{k,\, s}^{n,\, j}. \tag{56}$$

Remark. – $M(A, B)_{m,\, i}^{n,\, j}$ is the probability of jumping from A into B at time n and point j of B, coming from i at time m and having stayed in A in the mean time.

It will be helpful to introduce some "characteristic kernels":

DEFINITION 2.8. – *For any subset* A *of* E *we define the characteristic kernel of* A *to be*

$$I(A)_{i,\, m}^{j,\, n} = \begin{cases} 1 & \textit{if} \quad m = n, \quad i = j \quad \textit{and} \quad i \in A \\ 0 & \textit{otherwise} \end{cases} \tag{57}$$

Remark. – The characteristic kernel of E, I(E) is a unit for the generalized transition kernels of $\mathscr{A}$:

$$M(A, B)\, I(E) = I(E)\, M(A, B) = M(A, B). \tag{58}$$

The following decomposition formulas will allow us to write the jumps from A into B in terms of smaller subdomains.

PROPOSITION 2.9. — *Let* A *and* C *be subsets of* E *and let* B *be a subset of* A. *We have:*

$$M(A, C) = M(B, C) + M(B, A-B)\, M(A, C). \tag{59}$$

Proof. — We have

$$\begin{aligned} M(A, C)_{m, i}^{n, j} &= \sum_{k=m+1}^{n-1} P(X_n = j,\ \tau(B, m) = k,\ \tau(A, k) \geqq n,\ X_{n-1} \notin C \mid X_m = i) \\ &\quad + P(X_n = j,\ \tau(B, m) \geqq n,\ X_{n-1} \notin C \mid X_m = i), \end{aligned} \tag{60}$$

from which it is easy to deduce equation (59).

PROPOSITION 2.10. — *Let* A *and* C *be subsets of* E. *Let* $\mathscr{A}$ *be a partition of* A. *We have*

$$M(A, C) = I(E-A)\, M(\varnothing, C) + \sum_{G \in \mathscr{A}} M(A, G)\, M(G, C) + I(G)\, M(G, C). \tag{61}$$

Proof. — Let us introduce the time of last jump into one of the components of $\mathscr{A}$: let us call $\sim$ the equivalence relation induced by $\mathscr{A}$ on E, $i \sim j$ if and only if $i = j$ or there is $G \in \mathscr{A}$ such that $(i, j) \in G^2$. We put

$$\sigma(n) = \sup\{ m < n : X_{m-1} \nsim X_m,\ X_m \in A \}. \tag{62}$$

We have

$$\left.\begin{aligned} M(A, C)_{i, m}^{j, n} &= \sum_{k=m+1}^{n-1} P(\sigma(n) = k,\ \tau(A, m) \geqq n,\ X_{n-1} \notin C,\ X_n = j \mid X_m = i) \\ &\quad + P(\sigma(n) \leqq m,\ \tau(A, m) \geqq n,\ X_{n-1} \notin C,\ X_n = j \mid X_m = i), \\ &\qquad j \in C, \end{aligned}\right\} \tag{63}$$

but for $k = m+1, \ldots, n-1$

$$\begin{aligned} P(\sigma(n) = k,\ \tau(A, m) \geqq n,\ X_{n-1} \notin C,\ X_n = j \mid X_m = i) &= \sum_{G \in \mathscr{A}} \sum_{s \in G} P(\tau(A, m) \geqq k,\ X_{k-1} \notin G,\ X_k = s \mid X_m = i) \\ &\quad \times P(\tau(G, k) \geqq n,\ X_{n-1} \notin C,\ X_n = j \mid X_k = s). \end{aligned} \tag{64}$$

Moreover, if $i \in G$,

$$P(\sigma(n) \leqq m,\ \tau(A, m) \geqq n, X_{n-1} \notin C,\ X_n = j \mid X_m = i) = M(G, C)_{i, m}^{j, n}, \tag{65}$$

and if $i \notin A$

$$P(\sigma(n) \leqq m, \tau(A, m) \geqq n, X_{n-1} \notin C, X_n = j \mid X_m = i) = M(\varnothing, C)^{j, n}_{i, m}. \quad (66)$$

End of the proof of proposition 2.10.

We can write the jump from A to C as a series of jumps between smaller subdomains, as expressed in the following proposition.

PROPOSITION 2.11. – *Decomposition of* M:

Let A *and* C *be subsets of* E, *and let* $(B_s)_{s \in S}$ *be a partition of* A. *We have*

$$M(A, C) = \sum_{k=0}^{+\infty} \sum_{(i_1, \dots, i_k) \in S^k} (I(E-A)M(\varnothing, B_{i_1}) + I(B_{i_1})) \times M(B_{i_2}, B_{i_2}) \dots M(B_{i_{k-1}}, B_{i_k}) M(B_{i_k}, C), \quad (67)$$

with the convention that in the first sum the term corresponding to $k=0$ *is*

$$I(E-A)M(\varnothing, C).$$

Remark. – In equation (67) the summation in k is finite for any $M(A, C)^{i, n}_{i, m}$ (namely it can be restricted to $k \leqq n-m$).

Proof. – Consider the equivalence relation $\sim$ associated with the partition $(B_s)_{s \in S}$ of A. Define the sequence of stopping times ρ^k_m by

$$\rho^0_m = m,$$

and

$$\rho^k_m = \inf\{n > \rho^{k-1}_m \mid X_{n-1} \not\sim X_n\}, \qquad k > 0. \quad (68)$$

We have for any $i \in E$ and any $j \in C$:

$$M(A, C)^{n, j}_{m, i} = \sum_{k=0}^{+\infty} P(\tau(A, m) \geqq n, \rho^k_m < n \leqq \rho^{k+1}_m, X_{n-1} \notin C, X_n = j \mid X_m = i). \quad (69)$$

Conditioning by ρ^l_m, $1 \leqq k$ in each term in the right member gives the desired result.

End of the proof.

For well behaved domains A (*i.e.* for cycles) the probability to stay in A between times m and n is roughly speaking $\prod_{k=m+1}^{n} (1 - a e^{-H(A)/T_k})$ where H(A) is the depth of A and the probability of jumping out of A into C is roughly

$$b \prod_{k=m+1}^{n-1} (1 - a e^{-H(A)/T_k}) e^{-V(A, C)/T_n}. \tag{70}$$

The following definitions will make more precise what "roughly" could mean in the previous sentences.

DEFINITION 2.12. — *A KI (Kernel on the Integers)* G *is a function*

$$G: \quad \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}_+$$

A KI G *will be said to be increasing if* $G_m^n = 0$ *for* $m > n$, *it will be said to be decreasing if* $G_m^n = 0$ *for* $m < n$, *it will be said to be finite if*

$$\sum_{n \in \mathbb{Z}} G_m^n < +\infty, \qquad m \in \mathbb{Z}. \tag{71}$$

A RKI (Right Kernel on the Integers) is an increasing KI Q *which has the Markov property:*

$$\sum_{n \in \mathbb{Z}} Q_m^n = 1, \qquad m \in \mathbb{Z}. \tag{72}$$

A LKI is a KI Q *such that* $\tilde{Q}$ *defined by* $\tilde{Q}_m^n = Q_n^m$ *is decreasing and has the Markov property.*

Let $(E, U, \mathscr{L}_0, q, T, X)$ *be an annealing algorithm. For any positive constants* H, *a*, *b*, *a RKI of class* $\mathscr{D}^r(H, a, b)$ *is a RKI* Q *satisfying*

$$\sum_{k=n}^{+\infty} Q_m^k \leq (1+b) \prod_{k=m+1}^{n-1} (1 - a \exp(-H/T_k)) \tag{73}$$

where the cooling schedule T *is extended to* $\mathbb{Z}$ *by putting* $T_k = T_1$ *for* $k < 1$.

In the same way a LKI of class $\mathscr{D}^l(H, a, b)$ *is a LKI* Q *satisfying*

$$\sum_{k=-\infty}^{m} Q_k^n \leq (1+b) \prod_{k=m+1}^{n-1} (1 - a \exp(-H/T_k)), \qquad m < n. \tag{74}$$

DEFINITION 2.13 *We will use the following notations for KIs and GTKs:*

$$Q_m^{n\rightarrow} = \sum_{k=n}^{+\infty} Q_m^k \tag{75}$$

and

$$Q_{\leftarrow m}^n = \sum_{k=-\infty}^{m} Q_k^n. \tag{76}$$

Using these decomposition formulas, we are going estimate M (A, B) in terms of RKIs and LKIs.

Let us study first the simplest case as an example to be generalized.

We will study the jumps from A into B by induction on the size of A. Hence the first thing to do is to describe the simple case when A is reduced to one point.

Example: Study of $M(\{s\}, E-\{s\})_{i,m}^{j,n}$

We will study the following cases:

- The pair $\{i, s\}$ forms a cycle C, and $\{i\}=F(C)$.
- The starting state i is equal to s.

The first corresponds to the simplest case when A is a cycle with one of its ground states removed. This situation is interesting in the study of the last excursion of the system from one of the ground states of a cycle before it jumps out of it.

The second case is the initialization of the induction argument on the size of A and is mentioned for the sake of completeness.

PROPOSITION 2.14. – *Let* $(E, U, \mathscr{L}_0, q)$ *be an energy landscape with initial distribution and communications. Let* $C=\{i, s\}$ *be a cycle with two elements. Assume that* $U_i \leqq U_s$. *Let* j *be a state of* $E-C$. *There are positive constants* a *and* b, *there is a simple annealing framework* $(E, U, \mathscr{L}_0, q, \mathscr{F}(T_0), \mathscr{X})$ *in which there is a RKI* Q_1 *and a LKI* Q_2 *of class* $\mathscr{D}(H_{\{s\}}, a, b)$ *such that*

$$\left.\begin{aligned} & q(C, j)(1-e^{-\alpha/T_1}) \exp(-(U_j-U_i)/T_n)(Q_l)_m^n \\ & \qquad \leqq M(C-\{i\}, E-C)_{i,m}^{j,n} \\ & \leqq q(C, j)(1+e^{-\alpha/T_1}) \exp(-(U_j-U_i)/T_m)(Q_l)_m^n, \\ & \qquad l=1, 2. \end{aligned}\right\} \tag{77}$$

Proof. – Let us put

$$Q_m^n = \prod_{k=m+1}^{n-1} p_{T_k}(s, s)(1-p_{T_n}(s, s)), \qquad m<n, \tag{78}$$

$$\tilde{Q}_m^n = \delta_{m+1}^n \tag{79}$$

where $\delta_k^l = 1$ if $k=l$ and $\delta_k^l=0$ otherwise. In the same way, let us put:

$$R_m^n = (1-p_{T_m}(s, s)) \prod_{k=m+1}^{n-1} p_{T_k}(s, s). \tag{80}$$

It is easy to see that Q is a RKI and R a LKI of class $\mathscr{D}(0, a, b)$. We have

$$\left.\begin{aligned} &M(\{s\}, E-C)_{i,m}^{j,n} = Q_{m+1}^n (1-p_{T_n}(s,s))^{-1} q(i,s) \\ &\times \exp(-(U_s - U_i)/T_{m+1})\, q(s,j) \exp(-(U_j-U_s)/T_n), \\ &\qquad n > m+1, \end{aligned}\right\} \tag{81}$$

and

$$M(\{s\}, E-C)_{i,m}^{j,m+1} = q(i,j) \exp(-(U_j - U_i)/T_{m+1})$$

There exist positive constants T_0 and α such that, in the annealing framework associated with the cooling framework $\mathscr{F}(T_0)$ we have:

$$q(i,s)/(1-p_{T_n}(s,s)) = (1+\varepsilon_n),$$

with

$$|\varepsilon_n| \leqq \exp(-\alpha/T_1).$$

The choice of T_0 and α are linked, it is possible, for example, to take

$$\alpha = \frac{1}{2} \min\{U_l - U_s : l \notin \tilde{B}(\{s\})\}.$$

Let us put

$$\lambda = q(i,j)/(q(i,j)+q(s,j)), \tag{82}$$

and

$$(Q_1)_m^n = (1-\lambda) Q_{m+1}^n + \lambda \tilde{Q}_m^n. \tag{83}$$

it is easy to deduce the part of proposition 2.14 concerning RKIs from the preceding equations.

The proof for LKIs is in the same trend, putting

$$Q_{2m}^n = (1-\lambda) R_m^{n-1} + \lambda \tilde{Q}_m^n. \tag{84}$$

End of the proof.

PROPOSITION 2.15. — *Let* $(E, U, q, \mathscr{L}_0)$ *be an energy landscape with communications and initial distribution. Let* $C = \{i\}$ *be a singleton in* E *and let* j *be in* $E-C$. *There are positive constants* α, T_0, *a and a simple annealing framework* $(E, U, q, \mathscr{L}_0, \mathscr{F}(T_0), \mathscr{X})$ *in which there exist a RKI* Q_1 *and a LKI* Q_2 *of class* $\mathscr{D}(H(C), a, 0)$ *such that:*

$$\begin{aligned} &q(s,j)/q(C)\,(Q_1)_m^n \exp(-(U_j - U_i - H(C))^+/T_n)(1-e^{\alpha/T_1}) \\ &\qquad \leqq M(\{i\}, j)_{s,m}^{j,n} \\ &\quad \leqq q(s,j)/q(C)\,(Q_1)_m^n \exp(-(U_j - U_i - H(C))^+/T_{m+1}) \\ &\qquad\qquad \times (1+e^{-\alpha/T_1}) \end{aligned} \tag{85}$$

and

$$q(s,j)/q(\mathrm{C})(\mathrm{Q}_2)^n_m \exp(-(\mathrm{U}_j-\mathrm{U}_i)^+/\mathrm{T}_n+\mathrm{H}(\mathrm{C})/\mathrm{T}_{m+1})(1-e^{\alpha/\mathrm{T}_1})$$
$$\leqq \mathrm{M}(\{i\},j)^{j,n}_{s,m}$$
$$\leqq q(s,j)/q(\mathrm{C})(\mathrm{Q}_1)^n_m \exp(-(\mathrm{U}_j-\mathrm{U}_i-\mathrm{H}(\mathrm{C}))^+/\mathrm{T}_{m+1})$$
$$\times(1+e^{-\alpha/\mathrm{T}_1}) \quad (86)$$

The proof is not difficult and is left to the reader.

End of the example.

2.3. Jumping out of cycles

The induction argument is a generalization of the preceding example. The crucial step is to study the jumps from a cycle C. It is decomposed into two parts:

• Study the jumps between sub-components of C to prove that the law of the system knowing that it remains in C becomes equidistributed on the bottom of C (the fundamental states of C).

• Study the jumps from $\mathrm{C}-\{f\}$ into $\mathrm{E}-\mathrm{C}$ where f is some marked ground state of C, that is consider the behaviour of the system after its last visit to f. The kernel $\mathrm{M}(\mathrm{C}-\{f\}, \mathrm{E}-\mathrm{C})$ is not that of a simple jump, because $\mathrm{C}-\{f\}$ is not a cycle. This is the reason why the induction argument is complex and involves the case when A is not a cycle.

We need some definitions to introduce the induction hypothesis.

DEFINITION 2.16. – *Let* $(\mathrm{E}, \mathrm{U}, \mathscr{L}_0, q, \mathscr{F}, \mathscr{X})$ *be some annealing framework. Let* $(\mathrm{G}_\mathrm{T})_{\mathrm{T}\in\mathscr{F}}$ *be some family of KIs indexed by* $\mathscr{F}$ i.e. *depending on the choice of the annealing algorithm within the given annealing framework. Let* a, b, H *and* D *be positive constants. The family* G *is said to be of class* $\mathscr{E}^r(\mathrm{H}, a, \mathrm{D}, b)$ [*resp. of class* $\mathscr{E}^l(\mathrm{H}, a, \mathrm{D}, b)$] *if there exist positive constants* α *and* c *and RKIs* $(\mathrm{Q}^1_\mathrm{T})_{\mathrm{T}\in\mathscr{F}}$ *and* $(\mathrm{Q}^2_\mathrm{T})_{\mathrm{T}\in\mathscr{F}}$ [*resp. LKIs* $(\mathrm{Q}^3_\mathrm{T})_{\mathrm{T}\in\mathscr{F}}$ *and* $(\mathrm{Q}^4_\mathrm{T})_{\mathrm{T}\in\mathscr{F}}$] *of class* $\mathscr{D}(\mathrm{D}, c, b)$ *such that*

$$\left.\begin{array}{l}(1-e^{-\alpha/\mathrm{T}_1})\,a(\mathrm{Q}^l_\mathrm{T})^n_m \exp(-\mathrm{H}/\mathrm{T}_n)\\ \leqq(\mathrm{G}_\mathrm{T})^n_m\leqq(1+e^{-\alpha/\mathrm{T}_1})\,a(\mathrm{Q}^r_\mathrm{T})^n_m \exp(-\mathrm{H}/\mathrm{T}_{m+1}),\\ \text{for } l=1,3,\ r=2,4.\end{array}\right\} \quad (87)$$

The family G *is said to be of class* $\mathscr{E}(\mathrm{H}, a, \mathrm{D}, b)$ *if it is both of class* $\mathscr{E}^r(\mathrm{H}, a, \mathrm{D}, b)$ *and of class* $\mathscr{E}^l(\mathrm{H}, a, \mathrm{D}, b)$.

In the same way, G *is said to be of class* $\mathscr{E}^r_-(\mathrm{H}, \mathrm{D})$ [*resp. of class* $\mathscr{E}^l_-(\mathrm{H}, \mathrm{D})$] *if there exist positive constants* K, c, d, *and a RKI* [*resp. a LKI*]

$(Q_T)_{T \in \mathcal{F}}$ *of class* $\mathcal{D}(D, c, d)$ *such that*

$$G_m^n \leqq K(Q_T)_m^n \exp(-H/T_{m+1}). \tag{88}$$

It is said to be of class $\mathcal{E}_-(H, D)$ *if it is both of class* $\mathcal{E}^r_-(H, D)$ *and of class* $\mathcal{E}^l_-(H, D)$.

We have just introduced definitions allowing one to express that some GTK $M(A, B)_i^j$ is roughly of the form $b \prod_{k=m+1}^{n-1} (1 - a e^{-H/T_k}) e^{-V/T_n}$. We will need also a sharper notion of comparison when H=V. We will call it being α-adjacent to $\sum a e^{-H/T_k}$.

We say that $M(A, B)_i^j$ is α-adjacent to $\sum a e^{H/T_k}$ if

$$\sum_{k=m+1}^{n} M(A, B)_{i,m}^{j,k} = (1 + \varepsilon_n) M(A, B)_{i,m}^{j,m\to} a \sum_{k=m+1}^{n} e^{-H/T_k} \tag{89}$$

with $|\varepsilon_n| \leqq e^{-\alpha/T_{m+1}}$ for "good n's" and if more generally

$$\sum_{k=n_1+1}^{n_2} M(A, B)_{i,m}^{j,k} = (1 + \varepsilon_{n_1,n_2}) M(A, B)_{i,m}^{j,n_1\to} a \sum_{k=n_1+1}^{n_2} e^{-H/T_k} \tag{90}$$

with $|\varepsilon_{n_1,n_2}| \leqq e^{-\alpha/T_{m+1}}$ [*i.e.* we can roughly speaking apply the Markov property to M(A, B)]. The definition of α-adjacent has been devised such that general composition rules could be easily formulated (*cf.* the last composition lemma).

DEFINITION 2.17. – *Let* $(E, U, \mathcal{L}_0, q, \mathcal{F}, \mathcal{X})$ *be some annealing framework. Let* H *be some positive real number. The sequence* $\mathcal{J}(H, n)$, *characterizing the rate at which* X *is escaping from a cycle of depth* H, *is defined to be*

$$\mathcal{J}(H, n) = \exp(-H/T_n). \tag{91}$$

Associated with $\mathcal{J}$ *we define*

$$N(H, T_0, \alpha, m) = \inf\left\{ n > m : \sum_{k=m+1}^{n} \mathcal{J}(H, n) \geqq \exp(\alpha/T_0) \right\}. \tag{92}$$

where T_0 *is a positive constant, but where* α *may be negative.*

Let $\mathcal{F}$ *be a cooling framework, let* α *be a positive constant, let* G *be some finite increasing KI (Kernel on the Integers) defined in this framework. Let* $\mathcal{S}$ *be some sequence depending on the cooling schedule, such as* $\mathcal{J}(H)$.

The KI G *is said to be* α*-adjacent to* $\mathscr{S}$ *if,* $G_m^m = 0$ *and for any* $m \leqq n_1 \leqq n_2$ *such that*

$$\sum_{k=n_1+1}^{n_2} \mathscr{S}_k > \exp(-\alpha/T_{m+1}) \tag{93}$$

we have:

$$\left| \frac{\sum_{k=n_1+1}^{n_2} G_m^{k\to} \mathscr{S}_k}{\sum_{k=n_1+1}^{n_2} G_m^k} - 1 \right| \leqq \exp(-\alpha/T_{m+1}). \tag{94}$$

When we will say only that G *is adjacent to* $\mathscr{S}$ *we will mean that there is a positive constant* α *such that* G *is* α*-adjacent to* $\mathscr{S}$.

Remark. — If G is α-adjacent to $\mathscr{S}$ and if $\beta \leqq \alpha$, then G is β-adjacent to $\mathscr{S}$.

We will have to express the fact that the law of the system knowing that it stays in C concentrates on F (C).

DEFINITION 2.18. — *Concentration subsets of a cycle:*

Let T_0 *be a positive temperature. Let* $(E, U, \mathscr{L}_0, q, \mathscr{F}, \mathscr{X})$ *be some annealing framework such that* $\mathscr{F} \subset \mathscr{F}(T_0)$. *Let* C *be a cycle of* E. *Let* A *be a subset of* C. *Let* S, β *be positive constants, let* $\alpha \in \mathbb{R}$ *be a non necessarily positive constant. We will call* A *a concentration set of* C *of class* $\mathscr{O}(S, T_0, \alpha, \beta)$ *if for any* $T_{1/2} \in \mathbb{R}_+$ *such that* $T_0 \geqq T_{1/2} \geqq T_1$ *we have*

$$\sup_{n \geqq N(S, T_{1/2}, \alpha, m)} \left| P(X_n \in A \mid X_m = i, \tau(C, m) > n) - \frac{|F(C) \cap A|}{|F(C)|} \right| \leqq \exp(-\beta/T_{1/2}). \tag{95}$$

Remark:

- If $S' \geqq S$, $T'_0 \leqq T_0$, $\alpha' \geqq \alpha$, $\beta' \leqq \beta$, then

$$\mathscr{O}(S, T_0, \alpha, \beta) \subset \mathscr{O}(S', T'_0, \alpha', \beta') \tag{96}$$

- If each point of F (C) is a concentration set, then the conditional law converges towards the equidistributed law on F (C).

DEFINITION 2.19. — *We will call* $\mathscr{G}(T_0, H)$ *the cooling framework:*

$$\mathscr{G}(T_0, H) = \Big\{ (T_n)_{n \in \mathbb{N}^*} : T_{k+1} \leqq T_k,\ k \in \mathbb{N},\ \sum_{n \in \mathbb{N}^*} \exp(-H/T_n) = +\infty \Big\}. \tag{97}$$

We will now state the induction hypothesis, introducing five more definitions.

DEFINITION 2.20. — *Let* (E, U, $\mathscr{L}_0$, q) *be an energy landscape with communications and initial distribution. Let* C *be a cycle of* E. *We will say that* C *is of class* $\mathscr{P}_1$ *if there exist positive constants* T_0, d, *such that in the annealing framework* (E, U, $\mathscr{L}_0$, q, $\mathscr{G}(T_0, H'(C))$, $\mathscr{X}$), *for any* f *in* F(C) *and* j *in* B(C), $M(C-\{f\}, E-C)^j_f$ *is of class* $\mathscr{E}((U_j-U(C))^+, q(C,j), H'(C), d)$, *and more generally for any* f *in* F(C), i *in* C *and* j *in* B(C), $M(C-\{f\}, E-C)^j_i$ is of class $\mathscr{E}_-((U_j-U(i,f,C))^+, H'(C))$.

Comment. — In this definition, we compare $M(C-\{f\}, E-C)^{j,\,n}_{f,\,m}$ with

$$\prod_{k=m+1}^{n-1}(1-a\,e^{-H'(C)/T_k})\,a\,e^{-H'(C)/T_n}\,q(C,j)\,e^{-(U_j-U(C))^+/T_n}$$

(case of RKIs) and with

$$q(C,j)\,e^{-(U_j-U(C))^+/T_{m+1}}\,a\,e^{-H'(C)/T_{m+1}}\prod_{k=m+2}^{n}(1-a\,e^{-H'(C)/T_k})$$

(case of LKIs). In the same way we compare $M(C-\{f\}, E-C)^{j,\,n}_{i,\,m}$ with

$$\prod_{k=m+1}^{n-1}(1-a\,e^{-H'(C)/T_k})\,a\,e^{-H'(C)/T_n}\,b\,e^{-(U_j-U(i,f,C))^+/T_n}$$

and

$$b\,e^{-(U_j-U(i,f,C))^+/T_{m+1}}\,a\,e^{-H'(C)/T_{m+1}}\prod_{k=m+2}^{n}(1-a\,e^{-H'(C)/T_k}).$$

Let us recall that U(i, f, C) is the minimum over paths from i to f in C of the maximum energy level reached on the path. If G is the maximal cycle in $\mathscr{C}$(E, U, C) such that $i \in G$ and $f \notin G$, then $U(i,f,C)=U(G)+H(G)$.

DEFINITION 2.21. — *Let* (E, U, $\mathscr{L}_0$, q) *be an energy landscape with communications and initial distribution. Let* C *be a cycle of* E. *We will say that* C *is of class* $\mathscr{P}_2$ *if there exists a positive constant* T_0 *such that in the annealing framework* (E, U, $\mathscr{L}_0$, q, $\mathscr{G}(T_0, H(C))$, $\mathscr{X}$), *for any* $i \in C$, $j \in B(C)$, $M(C, E-C)^{j,\,n}_{i,\,m}$ is of class

$$\mathscr{E}^r((U_j-U(C)-H(C))^+, q(C,j)/q(C), H(C), e^{-\alpha/T_1}),$$

and $\exp(-H(C)/T_{m+1})\,M(C, E-C)^{j,\,n}_{i,\,m}$ *is of class*

$$\mathscr{E}^l((U_j-U(C))^+, q(C,j)/q(C), H(C), e^{-\alpha/T_1}).$$

Remarks:

- If C is of class $\mathcal{P}_2$, then $\exp(-H(C)/T_{m+1}) M(C, E-C)_{i, m}^{j, n}$ is of class

$$\mathcal{E}((U_j - U(C))^+, q(C, j)/q(C), H(C), e^{-\alpha/T_1}).$$

- We compare $M(C, E-C)_{i, m}^{j, n}$ with

$$\prod_{k=m+1}^{n-1} (1 - a e^{-H(C)/T_k}) a e^{-H(C)/T_n} \frac{q(C, j)}{q(C)} e^{-(U_j - U(C) - H(C))^+/T_n}.$$

and

$$\frac{q(C, j)}{q(C)} e^{(U_j - U(C) - H(C))^+/T_{m+1}} a e^{-H(C)/T_{m+1}} \prod_{k=m+2}^{n} (1 - a e^{-H(C)/T_k})$$

DEFINITION 2.22. – *Let* (E, U, $\mathcal{L}_0$, q) *be an energy landscape with communications and initial distribution. Let* C *be a cycle of* E. *We will say that* C *is of class* $\mathcal{P}_3$ *if for any* $\lambda > 0$ *there exist positive constants* T_0, α, *such that in the annealing framework* (E, U, $\mathcal{L}_0$, q, $\mathcal{G}(T_0, H'(C))$, $\mathcal{X}$), *for any* $f \in F(C)$, $\{f\}$ *is a concentration set of class* $\mathcal{O}(H'(C), T_0, \lambda, \alpha)$.

DEFINITION 2.23. – *Let* (E, U, $\mathcal{L}_0$, q) *be an energy landscape with communications and initial distribution. Let* C *be a cycle of* E. *We will say that* C *is of class* $\mathcal{P}_4$ *if for any* $\lambda > 0$ *there exist positive constants* T_0, α, *such that in the annealing framework* (E, U, $\mathcal{L}_0$, q, $\mathcal{G}(T_0, H''(C))$, $\mathcal{X}$), F (C) *is a concentration set of class* $\mathcal{O}(H''(C), T_0, \lambda, \alpha)$.

DEFINITION 2.24. – *Let* (E, U, $\mathcal{L}_0$, q) *be an energy landscape with communications and initial distribution. Let* C *be a cycle of* E. *We will say that* C *is of class* $\mathcal{P}_5$ *if there exists a positive* T_0 *such that in the annealing framework* (E, U, $\mathcal{L}_0$, q, $\mathcal{G}(T_0, H(C))$, $\mathcal{X}$), *for any* $i \in C$ *and* $j \in \tilde{B}(C)$,

$$M(C, E-C)_i^j \quad \text{and} \quad \sum_{j \in B(C)} M(C, E-C)_i^j$$

are adjacent to $q(C)/|F(C)| \mathcal{J}(H(C))$.

Comment. – We compare $M(C, E-C)_{i, m}^{j, n}$ with

$$\prod_{k=m+1}^{n-1} \left(1 - \frac{q(C)}{|F(C)|} e^{-H(C)/T_k}\right) \frac{q(C, j)}{|F(C)|} e^{-H(C)/T_n}. \tag{98}$$

General remark. – The property of being of class $\mathcal{P}_k$, $k = 1, \ldots, 5$, is independent of the initial distribution $\mathcal{L}_0$.

THEOREM 2.25. – *Let* (E, U, $\mathcal{L}_0$, q) *be an energy landscape with communications and initial distribution. Any cycle* C *of* $\mathcal{C}(E, U)$ *is of class* $\mathcal{P}_k$, $k = 1, \ldots, 5$.

Comment. – Theorem 2.25 characterizes the jumps from a cycle C at low temperatures. The estimates are uniform with respect to the cooling schedules in a L_∞ neighbourhood of zero. The convergence of the law of the system knowing that it stays in C towards the equidistributed law on the fundamental states of C is characterized as well as the concentration of this law on F (C).

We have chosen to dedicate a full section to the first step of the proof of theorem 2.25, which consists of proposition 4.5, may be the most interesting result of this paper.

As an illustration of theorem 2.25 we study the convergence of annealing algorithms, that is the second and third questions of the introduction, and bring some complement to Hajek's results.

3. CONVERGENCE OF ANNEALING ALGORITHMS

DEFINITION 3.1. – *Let* $(E, U, \mathcal{L}_0, q, T, X)$ *be an annealing algorithm. Let* C *be a cycle of* $\mathcal{C}(E, U)$. *We will say that the exit time from* C *is almost surely finite if, for any* $m \in \mathbb{N}$, *we have*

$$\sum_{n=m+1}^{+\infty} P(\tau(C, m)=n)=1. \tag{99}$$

DEFINITION 3.2. – *Let* $(E, U, \mathcal{L}_0, q, T, X)$ *be an annealing algorithm. Let* μ *be a probability distribution on* E. *We will say that the law of* X_n *converges to* μ *from any starting conditions if for any* $m \in \mathbb{N}$, *for any* $i, j \in E$, *we have*

$$\lim_{n \to +\infty} P(X_n=j \mid X_m=i)=\mu(j). \tag{100}$$

THEOREM 3.3. – *Let* $(E, U, \mathcal{L}_0, q, T, X)$ *be an annealing algorithm.*

1. *Let* C *be a cycle of* $\mathcal{C}(E, U)$. *The exit time from* C *is almost surely finite if and only if*

$$\sum_{n=1}^{+\infty} \exp(-H(C)/T_n)=+\infty. \tag{101}$$

2. *The probability of* $X_n \in F(E)$ *tends to one when n tends to infinity if and only if*

$$\lim_{n\to+\infty} T_n = 0 \tag{102}$$

and the exit time from any cycle $C \in \mathscr{C}''(E, U)$ *is almost surely finite. According to the first alinea this last condition is itself equivalent to* $T \in \mathscr{G}(+\infty, H''(E))$.

3. *The law of* X_n *converges to* μ_0 *from any starting conditions if and only if*

$$\lim_{n\to+\infty} T_n = 0 \tag{103}$$

and the exit time from any cycle of $\mathscr{C}'(E, U)$ *is almost surely finite – equivalently* $T \in \mathscr{G}(+\infty, H'(E))$.

Remarks:

• Points 1 and 2 are proved in Hajek [11] by different means. Point 3 is a complement to Hajek's results.

• If $1/T_n = A \ln n + B$ condition 2 is fulfilled when $A \leqq 1/H''(E)$ and condition 3 is fulfilled when $A \leqq 1/H'(E)$.

Proof. – The fact that the condition given in the first alinea of theorem 3.3 is sufficient is a simple consequence of the fact that any cycle C is of class $\mathscr{P}_5$. Indeed this implies that

$$\begin{aligned} P(\tau(C, m) > n) &\leqq 1 - (1 - e^{-\alpha/T_{m+1}}) \sum_{k=m+1}^{n} P(\tau(C, m) \geqq k) \frac{q(C)}{|F(C)|} e^{-H(C)/T_k} \\ &\leqq 1 - (1 - e^{-\alpha/T_{m+1}}) P(\tau(C, m) > n) \frac{q(C)}{|F(C)|} \sum_{k=m+1}^{n} e^{-H(C)/T_k}, \end{aligned} \tag{104}$$

as soon as $q(C)/|F(C)| \sum_{k=m+1}^{n} e^{-H(C)/T_k} > e^{-\alpha/T_{m+1}}$. Hence for any n such that

$$2 e^{-\alpha/T_{m+1}} \geqq \frac{q(C)}{|F(C)|} \sum_{k=m+1}^{n} e^{-H(C)/T_k} > e^{-\alpha/T_{m+1}} \tag{105}$$

we have

$$P(\tau(C, m) > n) \leqq \prod_{k=m+1}^{n} \left(1 - (1 - 4 e^{-\alpha/T_{m+1}}) \frac{q(C)}{|F(C)|} e^{-H(C)/T_k}\right). \tag{106}$$

Now we can apply the Markov property to see that it holds for any n such that

$$\frac{q(C)}{|F(C)|}\sum_{k=m+1}^{n} e^{-H(C)/T_k} > e^{-\alpha/T_{m+1}}.$$

(Let us remark that it would have been sufficient to use $\mathscr{P}_2$ instead of the more precise $\mathscr{P}_5$.)

The condition in the first part is sufficient because any cycle C is of class $\mathscr{P}_4$. In order to see that it is a necessary condition, let us suppose it is not fulfilled. Then we have

$$\lim_{n\to+\infty} T_n = 0. \tag{107}$$

The case when $T_n = 0$ for n large enough is trivial, thus we can assume that it is not so. We can find a subcycle $\tilde{C}$ of C such that T is in $\mathscr{G}(+\infty, H'(\tilde{C}))$, but not in $\mathscr{G}(+\infty, H(\tilde{C}))$. Then there exist positive constants T_0, d such that in the annealing framework $\mathscr{G}(T_0, H'(\tilde{C}))$ for any g in $F(\tilde{C})$ and any j in $B(\tilde{C})$, $M(\tilde{C}-\{g\}, E-\tilde{C})_g^j$ is of class $\mathscr{E}((U_j-U_{\tilde{C}})^+, q(\tilde{C}, j), H'(\tilde{C}), d)$. Using the right part of equation (87) with $r=2$ we see that, for any m such that $T_m \leqq T_0$ we have, for any $j\in B(\tilde{C})$:

$$\begin{aligned} M(\tilde{C}, E-\tilde{C})_{g,m}^{j,m\to} &= \sum_{k\geqq m} P(X_k=g, \tau(\tilde{C}, m)>k \mid X_m=g)\, M(\tilde{C}-\{g\}, E-\tilde{C})_{g,k}^{j,k\to} \\ &\leqq 2\,q(\tilde{C}, j) \sum_{k=m+1}^{+\infty} \exp(-H(\tilde{C})/T_k). \end{aligned} \tag{108}$$

We can then choose some m_0 such that

$$\sum_{j\in B(\tilde{C}} q(\tilde{C}, j) \sum_{k=m+1}^{+\infty} \exp(-H(\tilde{C})/T_k) \leqq \frac{1}{4}.$$

It is an easy exercise to see that for m large enough, for any i in C, we have:

$$P(X_m\in\tilde{C}, \tau(C, 0)>m \mid X_0=i)>0.$$

From these three equations we deduce that, for any $n\in\mathbb{N}$:

$$P(\tau(C, n)=+\infty \mid X_n=i)>0.$$

This ends the proof of the first alinea of theorem 3.3.

Proof of theorem 3.3.2.

LEMMA 3.4. – *If*

$$\lim_{n\to+\infty} T_n = T_\infty > 0, \tag{109}$$

then

$$\lim_{n\to+\infty} (P(X_n = i) - \mu_{T_n}(i)) = 0, \tag{110}$$

hence

$$\lim_{n\to+\infty} P(X_n = i) = \mu_{T_\infty}(i). \tag{111}$$

Proof of the lemma. – Putting $\mu_k = \mu_{T_k}$, $k \geqq 1$ and $\mu_0 = \mathscr{L}_0$, we have

$$P(X_n = i) = \left(\mu_0 \prod_{k=1}^{n} p_{T_k}\right)(i). \tag{112}$$

Using the fact that μ_k is p_{T_k} invariant, we find the following equality:

$$\mu_0 \prod_{k=1}^{n} p_{T_k} - \mu_n = \sum_{k=1}^{n} (\mu_{k-1} - \mu_k) \prod_{s=k}^{n} p_{T_s} \tag{113}$$

Using the fact that q is irreducible, it is easy to establish that there exist constants $K>0$ and $\lambda \in]0, 1[$ such that for any measure ρ on E such that $\rho(E)=0$, we have

$$\left\| \rho \prod_{k=m+1}^{n} p_{T_k} \right\| \leqq K \lambda^{n-m} \| \rho \|, \tag{114}$$

where

$$\| \rho \| = \sum_{i \in E} | \rho(i) | = | \rho | (E). \tag{115}$$

Thus we have

$$\sum_{i \in E} | P(X_n = i) - \mu_{T_n}(i) | \leqq \sum_{k=1}^{n} \| \mu_k - \mu_{k-1} \| K \lambda^{n-k}. \tag{116}$$

As

$$\lim_{n\to+\infty} \| \mu_n - \mu_{n-1} \| = 0, \tag{117}$$

we have

$$\lim_{n\to+\infty} \| \mathscr{L}_n - \mu_n \| = 0, \tag{118}$$

where $\mathscr{L}_n$ is the law of X_n.

End of the proof of lemma 3.4.

This proves that $\lim_{n \to +\infty} T_n = 0$ is a necessay condition in parts 2 and 3 of theorem 3.3.

Suppose now that there exists $C \in \mathscr{C}''(E, U)$ such that

$$\sum_n \exp(-H(C)/T_n) < +\infty. \tag{119}$$

It is easy to deduce from the fact that q is irreducible that there exist constants $n_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that for any $i \in E$ we have

$$P(X_{n_0} \in C \mid X_0 = i) > \varepsilon. \tag{120}$$

Let

$$\eta = \min_{i \in C} P(\tau(C, n_0) = +\infty \mid X_{n_0} = i), \tag{121}$$

then for all $n > n_0$

$$P(X_n \in C) > \varepsilon\eta. \tag{122}$$

As $C \cap F(E) = \varnothing$, this proves that

$$\limsup_{n \to +\infty} P(X_n \in F(E)) \leqq 1 - \eta\varepsilon. \tag{123}$$

The "if" part of theorem 3.3.2 is a straightforward consequence of the fact that every cycle is of class $\mathscr{P}_4$.

Proof of theorem 3.3.3. – The "if" part is a consequence of the fact that every cycle is of class $\mathscr{P}_3$.

As for the "only if" part, let us consider a cooling schedule T such that

$$\sum_n \exp(-H'(E)/T_n) < +\infty. \tag{124}$$

We can find a cycle $C \in \mathscr{C}'(E, U, q)$ such that

$$\sum_n \exp(-H(C)/T_n) < +\infty \tag{125}$$

and

$$\sum_n \exp(-H'(C)/T_n) = +\infty. \tag{126}$$

Thus applying property $\mathscr{P}_2$, we can prove that

$$P(\tau(C, n) = +\infty \mid X_n = i) > 0, \qquad i \in C. \tag{127}$$

Hence

$$\lim_{n \to +\infty} P(\tau(C, n) = +\infty \mid X_n = i) = 1, \qquad i \in C. \tag{128}$$

Consider $n \in \mathbb{N}$ and $i \in C$ such that

$$P(\tau(C, n) = +\infty \mid X_n = i) > \mu_0(F(E) \cap C). \tag{129}$$

For any $m>n$ we have

$$P(X_m \in F(E)-C \mid X_n=i) \leqq 1-P(\tau(C, n)=+\infty \mid X_n=i)<\mu_0(F(E)-C). \quad (130)$$

Hence the law of X_n cannot converge to μ_0 from the starting condition $X_n=i$.

This ends the proof of theorem 3.3.

4. BEHAVIOUR OF ANNEALING IN RESTRICTED DOMAINS

As we have already explained, we have to study the jumps from $C-\{f\}$ into $E-C$, that is the jumps from a subdomain of E which is not a cycle.

Within the logic of the proof of theorem 2.25 this is part of the induction step leading to $\mathscr{P}_1$. But it is also generalizing the study of the jumps to a general subdomain. As such it deserves a full section.

We want to know how the system jumps out of some subdomain A of the states space E. For that purpose it is not necessary to know exactly the starting point i. Indeed if i_1 and i_2 belong to the same subcycle C of A, jumping out of C, and hence out of A, does not depend at low temperatures on whether the starting point was i_1 or i_2. Hence it is enough to be interested in the jumps between components of the partition of A into cycles. Roughly speaking, we can consider in this study each of these components as a symbolic state and estimate the transitions between these symbolic states. We will assume that A is a strict subdomain of E and represent the complement of A in E by an abstract absorbing state O. If the temperature does not tend to zero too fast the process on the abstract states is absorbed in O with probability one. Proposition 4.5 gives sharp estimates for this process.

The process on the abstract states can be decomposed into two parts: the jumps and the dwelling times in the abstract states. The probability of the jumps depends only on the communication matrix q and the decomposition of A into cycles, thus it is independent of the temperatures (provided that they are low and provided that we consider only the most probable jumps). The dwelling times are of the form $\prod_{k=m+1}^{n} (1-a e^{-H/T_k})$.

Many envents can be expressed in terms of jumps from subdomains of the states space. Hence proposition 4.5 can serve as a tool for many applications. Some of them will be given in a forthcoming paper. For instance we see that if there are sequences $u_n < v_n$ such that $\lim 1/T_{v_n} - 1/T_{u_n} = 0$ and $\lim \sum_{k=u_n+1}^{v_n} e^{-H(A)/T_k} = +\infty$, H(A) being the depth of the deepest subcycle of A, jumping out of A will occur at almost constant temperature. Hence removing from E the deepest attractors of the system (*i.e.* the bottoms of the deepest cycles), we see that we can deduce an equivalent of the law of the system everywhere from an equivalent on these attractors for convergent cooling schedules of the type $1/T_n = A \ln n + B$.

DEFINITION 4.1. — *Let* (E, U, q) *be an energy landscape. For any subset* A *of* E, *we define the maximal partition of* A *to be the partition of* A *into its maximal subcycles. Note that the maximal partition of a cycle is the trivial partition reduced to one class. Our notation for the maximal partition of* A *will be* $\mathcal{M}(A)$.

We will generalize the notion of depth to any subset of E.

DEFINITION 4.2. — *Let* (E, U, q) *be an energy landscape. Let* A *be a subset of* E, *and let* $\mathcal{A}$ *be the maximal partition of* A. *We put*

$$H(A) = \max_{G \in \mathcal{A}} H(G). \tag{131}$$

DEFINITION 4.3. — *Let* (E, U, q) *be an energy landscape. Let* A *be a subset of* E. *We associate with* A *an "abstract" states space* $\mathcal{M}_A$ *which we construct by adjoining to the maximal partition* $\mathcal{A}$ *of* A *an "external" point* O; *the states in* $\mathcal{A}$ *will be called the "internal states". We associate with* A *a Markov kernel* v_A *which we call the communication kernel associated with* A. *This kernel is defined on* $\mathcal{M}_A$ *by*

$$\left.\begin{array}{c} v_A(G_1, G_2) = q(G_1, \tilde{B}(G_1) \cap G_2)/q(G_1), \\ G_1, G_2 \in \mathcal{A}, \qquad G_1 \neq G_2 \end{array}\right\} \tag{132}$$

$$v_A(G, O) = q(G, \tilde{B}(G) \cap (E - A))/q(G), \qquad G \in \mathcal{A}. \tag{133}$$

and

$$v_A(O, O) = 1, \qquad v_A(O, G) = v_A(G, G) = 0, \qquad G \in \mathcal{A}. \tag{134}$$

We call the homogeneous Markov chain Y_A *on* $\mathcal{M}_A$ *defined by* v_A *the communication chain associated with* A. *We will denote the potential of* v_A

by

$$k_A = \sum_{n=0}^{+\infty} v_A^n, \tag{135}$$

we will also put for technical purpose

$$k_A^* = \sum_{n=1}^{+\infty} v_A^n = k_A - I, \tag{136}$$

LEMMA 4.4. – *Let* (E, U, q) *be an energy landscape. Let* A *be a strict subset of* E. *For the associated communication chain* Y_A, *the external state* O *is reachable from any internal state* G.

The reason is that the internal states are *maximal* subcycles of A.

Proof. – Let G be an internal state. Let $\mathscr{R}(G)$ be the set of $S \in \mathscr{A}$ which are reachable from G. Let

$$\lambda = \inf_{S \in \mathscr{R}(G)} H(S) + U(S). \tag{137}$$

Let

$$C = \bigcup_{S \in \mathscr{R}(G) : H(S) + U(S) = \lambda} \tag{138}$$

If C is a cycle then $C \in \mathscr{R}(G)$ and $\tilde{B}(C) \cap \mathscr{R}(G) = \varnothing$, thus O is reachable from G. If C is not a cycle, then there is $i \in E - C$ such that $i \in B(C)$ and $U_i \leqq \lambda$. We cannot have $i \in \bigcup_{S \in \mathscr{R}(G)} S$, because this would imply $i \in C$, thus we cannot have $i \in A$, thus O is reachable from G.

End of the proof of lemma 4.4.

Let us consider the communication Markov kernel v_A associated with A. The maximal partition $\mathscr{A}$ of A is preordered by the relation

$$B_1 \underline{<} B_2 \quad \text{iff} \quad \sup_{n \in \mathbb{N}} v_A^n(B_1, B_2) > 0, \qquad B_1, B_2 \in \mathscr{A}. \tag{139}$$

We deduce from this the equivalence relation on $\mathscr{A}$

$$B_1 \sim B_2 \quad \text{iff} \quad B_1 \underline{<} B_2 \text{ and } B_2 \underline{<} B_1. \tag{140}$$

This equivalence relation determines a partition $\mathscr{A}'$ of $\mathscr{A}$ on which an order relation is induced by the pre-order relation $\underline{<}$ of $\mathscr{A}$. Notice that in general $\mathscr{A}'$ is not totally ordered. Notice also that $\mathscr{F} \in \mathscr{A}'$ are sets of sets of A. We can get down to the level of subsets of A by putting:

$$\mathscr{P} = \{ \bigcup_{G \in \mathscr{F}} G : \mathscr{F} \in \mathscr{A}' \}. \tag{141}$$

The order $\leq$ on $\mathscr{A}'$ canonically induces an order on $\mathscr{P}$ for which we will use the same notation. For any $D \in \mathscr{P}$, the maximal partition of D is a subset of the maximal partition of A, and the communication Markov kernels ν_D and ν_A have the same restriction to $\mathscr{M}(D)$.

For any $D \in \mathscr{P}$, for any G, $G' \in \mathscr{M}(D)$, we have

$$H(G)+U(G)=H(G')+U(G'). \tag{142}$$

We will call $\lambda(D)$ this common value.

The proof is the following. – If $\tilde{B}(G) \cap G' \neq \varnothing$ then

$$H(G)+U(G) \geqq H(G')+U(G'),$$

moreover, for any G, $G' \in \mathscr{M}(D)$, there are chains $G_1, \ldots, G_s$ and $G'_1, \ldots, G'_{s'}$ of elements $\mathscr{M}(D)$ such that $G_1 = G'_{s'} = G$, $G_s = G'_1 = G'$ and $\tilde{B}(G_k) \cap G_{k+1} \neq \varnothing$, as well as $\tilde{B}(G'_k) \cap G'_{k+1} \neq \varnothing$.

End of the proof.

Hence for any G, $G' \in \mathscr{M}(D)$, we have

$$\tilde{B}(G) \cap G' = B(G) \cap G', \tag{143}$$

and

$$q(G)\,\nu_A(G, G') = q(G')\,\nu_A(G', G). \tag{144}$$

In the same way, if G, $G' \in \mathscr{A}$ with $G \leq G'$, then

$$\tilde{B}(G) \cap G' = B(G) \cap G'. \tag{145}$$

It is time now to give the main proposition of this section:

PROPOSITION 4.5. – *Let* (E, U, q) *be an energy landscape. Let* A *be any strict subset of* E. *Let* $\mathscr{A}$ *be the maximal partition of* A. *Assume that any* $G \in \mathscr{A}$ *is of class* $\mathscr{P}_2$, *then there exist positive constants* T_0, *and* α *such that in the annealing framework* $(E, U, q, \mathscr{L}_0, \mathscr{G}(T_0, H(A)), \mathscr{X})$, *for any* G, $G' \in \mathscr{A}$, *for any* $i \in G$, $j \in G'$, *there are constants* $H(G, G') \geqq 0$ *and* $\gamma(i, j) \geqq 0$ *such that the GTK* $M(A, G')^{j, n}_{i, m}$ [*resp.* $\exp(-H(A)/T_{m+1})\,M(A, G')^{j, n}_{i, m}$] *is of class* $\mathscr{E}^r(H(G, G'), \gamma(i, j), H(A), \alpha)$ [*resp. of class*

$$\mathscr{E}^l(H(G, G')+H(A), \gamma(i, j), H(A), \alpha)].$$

Moreover, if $k^*_A(G, G') > 0$ *then* $H(G, G') = 0$ *and for any* $i \in G$

$$\sum_{j \in G'} \gamma(i, j) = k^*_A(G, G'), \tag{146}$$

and if $G \subset D \in \mathscr{P}$ *and* $G' \subset D' \in \mathscr{P}$ *then*

$$H(G, G') = \inf_{(D_1, \ldots, D_r)} \sum_{k=1}^{r-1} (\lambda(D_{k+1}) - \lambda(D_k))^+, \tag{147}$$

where the infimum is taken over the sequences of length $r \geqq 2$ *such that* $D_{k+1} \neq D_k$ *and such that there are* $i \in D_k$ *and* $j \in D_{k+1}$ *with* $q(i, j) > 0$.

Remark. — When the induction will be completed, we will know that any cycle is of class $\mathscr{P}_2$.

Proof of proposition 4.5. — We begin by proving proposition 4.5 for the sets $D \in \mathscr{P}$.

LEMMA 4.6. — *For any* $D \in \mathscr{P}$, *proposition* 4.5 *is true with* A *replaced by* D.

In other words, we are going to prove proposition 4.5 first in the case when $H(C) + U(C)$ does not depend on $C \in \mathscr{M}(A)$.

Let us prove lemma 4.6.

Proof of lemma 4.6. — Let us consider $D \in \mathscr{P}$. The maximal partition $\mathscr{D}$ of D is a subset of the maximal partition $\mathscr{A}$ of A and is an equivalence class for the equivalence relation $\sim$ of equation (140). Let us assume that any $G \in \mathscr{D}$ is of class $\mathscr{P}_2$. Let us notice that for any $G_1, G_2 \in \mathscr{D}$ we have

$$H(G_1) + U(G_1) = H(G_2) + U(G_2). \qquad (148)$$

Let us call $\lambda(D)$ this common value. We can assume that $\operatorname{Card} \mathscr{D} \geqq 2$, otherwise proposition 4.5 is trivial for D (it tells nothing more than $\mathscr{P}_2$). Assuming that $\operatorname{Card} \mathscr{D} \geqq 2$, we see that for any $S, S' \in \mathscr{D}$, $k_D(S, S') > 0$ (because $S < S'$ and $S' \overline{<} S$). Let us notice also that $M(S, S')_i^{S'} = 0$, $i \in S$ as soon as $\nu_D(S, S') = 0$ [because $\tilde{B}(S) \cap S' = B(S) \cap S'$].

Let us consider G_1 and $G_2 \in \mathscr{D}$.

For any integer $k \geqq 2$ let us call $\mathscr{P}_k$ the set of k-tuples $(S_1, \ldots, S_k) \in \mathscr{D}^k$ such that $S_1 = G_1$ and $S_k = G_2$ and $\nu_D(S_l, S_{l+1}) > 0$, $l = l, \ldots, k-1$. For any $i \in G_1, j \in G_2$ we have

$$M(D, G_2)_i^j = \sum_{k=2} \sum_{(S_1, \ldots, S_k) \in \mathscr{P}_k} \{M(S_1, S_2) \ldots M(S_{k-1}, S_k)\}_i^j. \qquad (149)$$

Let us write down in full length what means the assumption that any $S \in \mathscr{D}$ is of class $\mathscr{P}_2$. There are positive constants T_0, α and a such that for any $S, S' \in \mathscr{D}$, $S \neq S'$, for any $i \in S$, any $j \in S'$, in the cooling framework $\mathscr{G}(T_0, H(\mathscr{G}(S)))$ — hence in $\mathscr{G}(T_0, H(D))$ — there are RKIs $Q_1(i, j)$ and $Q_2(i, j)$ of class $\mathscr{D}^r(H(D), a, e^{-\alpha/T_1})$ such that

$$(1 - e^{-\alpha/T_1}) Q_1(i, j) q(S, j)/q(S) \leqq M(S, S')_i^j$$
$$\leqq (1 + e^{-\alpha/T_1}) Q_2(i, j) q(S, j)/q(S), \qquad (150)$$

and there are LKIs $Q_3(i, j)$ and $Q_4(i, j)$ of class $\mathscr{D}^l(H(D), a, e^{-\alpha/T_1})$ such that

$$\begin{aligned}(1-e^{-\alpha/T_1})\, Q_3(i, j)^n_m \exp(-H(S)/T_n)\, q(S, j)/q(S) \\ \leqq \exp(-H(S)/T_{m+1})\, M(S, S')^{j, n}_{i, m} \\ \leqq (1+e^{-\alpha/T_1})\, Q_4(i, j)^n_m \exp(-H(S)/T_{m+1})\, q(S, j)/q(S). \quad (151)\end{aligned}$$

Let us examine first the kernel $M(D, G_2)^E_i$. We have

$$\left.\begin{aligned} M(D, G_2)^{G_2}_i \leqq \sum_{k=2}^{+\infty} \sum_{(S_1, \ldots, S_k)\in \mathscr{P}_k} \sum_{(i_1, i_k)\in\{i\}\times S_2\times \ldots S_k} \\ (1+e^{-\alpha/T_1})^{k-1} \prod_{l=1}^{k-1} q(S_l, i_{l+1})/q(S_l) \prod_{l=1}^{k-1} Q_r(i_l, i_{l+1}) \\ r=2,4. \end{aligned}\right\} \quad (152)$$

Let us call $\tilde{Q}_r$, $r=2,4$ the right member of this inequality. We have

$$\begin{aligned}(\tilde{Q}_4)^n_{\leftarrow n} = (\tilde{Q}_2)^{m\rightarrow}_m \\ = \sum_{k=2}^{+\infty} (1+e^{-\alpha/T_1})^{k-1} \sum_{(S_1, \ldots, S_k)\in\mathscr{P}_k} \prod_{l=1}^{k-1} q(S_l, S_{l+1})/q(S_l) \\ = \sum_{k=2}^{+\infty} (1+e^{-\alpha/T_1})^{k-1} \nu_D^{k-1}(G_1, G_2). \quad (153)\end{aligned}$$

But there exist positive constants K and μ, with $0<\mu<1$ such that

$$\nu_D^k(G_1, G_2) \leqq K\mu^k, \quad (154)$$

because the external state O is reachable from any internal state and is absorbing. Hence

$$\begin{aligned}\sum_{k=2}^{+\infty} ((1+e^{-\alpha/T_1})^{k-1}-1)\, \nu_D^{k-1}(G_1, G_2) \\ \leqq \sum_{k=2}^{+\infty} ((1+e^{-\alpha/T_1})^{k-1}-1)\, K\mu^{k-1} \quad (155) \\ = \frac{K\mu e^{-\alpha/T_1}}{(1-(1-e^{-\alpha/T_1})\mu)(1-\mu)}. \quad (156)\end{aligned}$$

Hence there are positive constants T_0 and K such that in the cooling framework $\mathscr{G}(T_0, H(D))$ we have

$$(\tilde{Q}_2)^{m\rightarrow}_m \leqq k^*_D(G_1, G_2) + K e^{-\alpha/T_1} \quad (157)$$

and

$$(\tilde{Q}_4)^n_{\leftarrow n} \leqq k^*_D(G_1, G_2) + K e^{-\alpha/T_1} \quad (158)$$

Let us prove now that there are positive constants T_0, a and b such that in the cooling framework $\mathscr{G}(T_0, H(D))$, the kernels $\tilde{Q}_2$ and $\tilde{Q}_4$ are of class $\mathscr{D}(H(D), a, b)$. We have to show that for any $m, n \in \mathbb{N}$

$$(\tilde{Q}_2)_m^{n\rightarrow} \leqq (1+b) \sum_{k=m+1}^{n-1} (1 - a e^{-H(D)/T_k}) \tag{159}$$

and

$$(\tilde{Q}_4)_{\leftarrow m}^{n} \leqq (1+b) \sum_{k=m+1}^{n-1} (1 - a e^{-H(D)/T_k}). \tag{160}$$

From lemma 6.4 and from equation (152) we deduce that

$$(\tilde{Q})_m^{n\rightarrow} \leqq \sum_{k=2} \sum_{(S_1, \ldots, S_k) \in \mathscr{P}_k} (1+e^{-\alpha/T_1})^{k-1} \times \nu_D^{k-1}(G_1, G_2) \{ Z_r(H(D), a, e^{-\alpha/T_1})^{k-1} \}_m^{n\rightarrow} \tag{161}$$

and that

$$(\tilde{Q}_4)_{\leftarrow m}^{n} \leqq \sum_{k=2} \sum_{(S_1, \ldots, S_k) \in \mathscr{P}_k} (1+e^{-\alpha/T_1})^{k-1} \times \nu_D^{k-1}(G_1, G_2) \{ Z_l(H(D), a, e^{-\alpha/T_1})^{k-1} \}_{\leftarrow m}^{n} \tag{162}$$

where Z is the maximal RKI in the first equation and the maximal LKI in the second one.

According to equation (154), and changing $k-1$ into k we have

$$(\tilde{Q}_2)_m^{n\rightarrow} \leqq \sum_{k=1}^{+\infty} K((1+e^{-\alpha/T_1})\mu)^k \{ Z(H(D), a, e^{-\alpha/T_1})^k \}_m^{n\rightarrow} \tag{163}$$

and

$$(\tilde{Q}_4)_{\leftarrow m}^{n} \leqq \sum_{k=1}^{+\infty} K((1+e^{-\alpha/T_1})\mu)^k \{ Z(H(D), a, e^{-\alpha/T_1})^k \}_{\leftarrow m}^{n} \tag{164}$$

We have now, putting Z for $Z(H(D), a, e^{-\alpha/T_1})$,

$$\{ Z^k \}_m^{n\rightarrow} = \{ Z^{k-1} \}_m^{n\rightarrow} + \sum_{l=m}^{n-1} \{ Z^{k-1} \}_m^{l} Z_l^{n\rightarrow}, \tag{165}$$

$$\left[\text{resp.} \quad \{ Z^k \}_{\leftarrow m}^{n} = \{ Z^{k-1} \}_{\leftarrow m}^{n} + \sum_{l=m+1}^{n} Z_{\leftarrow m}^{l} \{ Z^{k-1} \}_l^{n} \right] \tag{166}$$

hence

$$\{Z^k\}_m^{n\rightarrow}=\sum_{s=1}^{k} B_s, \tag{167}$$

with

$$B_s=\sum_{l=m}^{n-1}\{Z^{s-1}\}_m^l Z_l^{n\rightarrow}, \qquad s\geqq 2 \tag{168}$$

and

$$B_1=Z_m^{n\rightarrow} \tag{169}$$

[resp.

$$\{Z^k\}_{\leftarrow m}^{n}=\sum_{s=1}^{k} B_s, \tag{170}$$

with

$$B_s=\sum_{l=m}^{n-1} Z_{\leftarrow m}^l\{Z^{s-1}\}_l^n, \ s\geqq 2 \tag{171}$$

and

$$B_1=Z_{\leftarrow m}^n\Big]. \tag{172}$$

Hence, integrating by parts the left member of equation (163) we get

$$(\tilde{Q}_2)_m^{n\rightarrow}\leqq K\sum_{k=1}^{+\infty} B_k\sum_{s=k}^{+\infty}((1+e^{-\alpha/T_1})\mu)^s, \tag{173}$$

or

$$(\tilde{Q}_2)_m^{n\rightarrow}\leqq K\sum_{k=1}^{+\infty} B_k\frac{((1+e^{-\alpha/T_1})\mu)^k}{1-(1+e^{-\alpha/T_1})\mu}. \tag{174}$$

[resp.

$$(\tilde{Q}_4)_{\leftarrow m}^n\leqq K\sum_{k=1}^{+\infty} B_k\frac{((1-e^{-\alpha/T_1})\mu)^k}{1-(1+e^{-\alpha/T_1})\mu}\Bigg]. \tag{175}$$

But

$B_k\leqq(1+e^{-\alpha/T_1})^k$

$$\times \sum_{(l_1, \ldots, l_{k-1}),\, m<l_1<\ldots<l_{k-1}<n} \frac{a^{k-1} \exp(-\mathrm{H}(\mathrm{D})(\mathrm{T}_{l_1}^{-1}+\ldots+\mathrm{T}_{l_{k-1}}^{-1}))}{(1-a e^{-\mathrm{H}(\mathrm{D})/\mathrm{T}_{l_1}}) \ldots (1-e^{-\mathrm{H}(\mathrm{D})/\mathrm{T}_{l_{k-1}}})}$$

$$\times \prod_{s=m+1}^{n-1} (1-a e^{-\mathrm{H}(\mathrm{D})/\mathrm{T}_s})$$

$$\leq \frac{(1+e^{-\alpha/\mathrm{T}_1})^k}{(1-a e^{-\mathrm{H}(\mathrm{D})/\mathrm{T}_1})^{k-1}} \left(\sum_{s=m+1}^{n-1} e^{-\mathrm{H}(\mathrm{D})/\mathrm{T}_s} \right)^{k-1}$$

$$\times \frac{a^{k-1}}{(k-1)!} \prod_{s=m+1}^{n-1} (1-a e^{-\mathrm{H}(\mathrm{D})/\mathrm{T}_s}). \quad (176)$$

(The same expression holds in the case of left and right kernels.)

Thus equation (174) gives

$$(\tilde{\mathrm{Q}}_2)_m^{n\rightarrow}, (\tilde{\mathrm{Q}}_4)_{\leftarrow m}^{n} \leq \mathrm{K} \frac{(1+e^{-\alpha/\mathrm{T}_1})^2 \mu}{1-(1+e^{-\alpha/\mathrm{T}_1}) \mu}$$

$$\times \exp\left(\frac{(1+e^{-\alpha/\mathrm{T}_1})^2 \mu a}{(1-a e^{-\mathrm{H}(\mathrm{D})/\mathrm{T}_1})} \sum_{s=m+1}^{n-1} e^{-\mathrm{H}(\mathrm{D})/\mathrm{T}_s} \right)$$

$$\times \prod_{s=m+1}^{n-1} (1-a e^{-\mathrm{H}(\mathrm{D})/\mathrm{T}_s}). \quad (177)$$

Hence there is a positive constant K′ such that for a suitable T_0, in the cooling framework $\mathscr{G}(\mathrm{T}_0, \mathrm{H}(\mathrm{D}))$ we have

$$(\tilde{\mathrm{Q}}_2)_m^{n\rightarrow}, (\tilde{\mathrm{Q}}_4)_{\leftarrow m}^{n} \leq \mathrm{K}' \exp\left(-a \left(1 - \mu \frac{(1+e^{-\alpha/\mathrm{T}_1})^2}{1-a e^{-\mathrm{H}(\mathrm{D})/\mathrm{T}_1}} \right) \right.$$

$$\left. \times \sum_{s=m+1}^{n-1} e^{-\mathrm{H}(\mathrm{D})/\mathrm{T}_s} \right). \quad (178)$$

Hence we can find positive constants T_0 and $a'<a$ such that in the cooling framework $\mathscr{G}(\mathrm{T}_0, \mathrm{H}(\mathrm{D}))$ we have

$$(\tilde{\mathrm{Q}}_2)_m^{n\rightarrow}, (\tilde{\mathrm{Q}}_4)_{\leftarrow m}^{n} \leq \mathrm{K}' \prod_{s=m+1}^{n-1} (1-a' e^{-\mathrm{H}(\mathrm{D})/\mathrm{T}_s}). \quad (179)$$

Thus in $\mathscr{G}(\mathrm{T}_0, \mathrm{H}(\mathrm{D}))$ the kernels $\tilde{\mathrm{Q}}_r$, $r=2$, 4 are of class $\mathscr{D}(\mathrm{H}(\mathrm{D}), a', \mathrm{K}'-1)$. From this fact and equation (157) we deduce that there exist positive constants T_0, α, a and b such that in the cooling framework $\mathscr{G}(\mathrm{T}_0, \mathrm{H}((\mathrm{D}))$, there are a RKI Q_2 and a LKI Q_4 of class $\mathscr{D}(\mathrm{H}(\mathrm{D}), a, b)$ such that

$$\mathrm{M}(\mathrm{D}, \mathrm{G}_2)_{i, m}^{\mathrm{E}, n} \leq (1-e^{-\alpha/\mathrm{T}_1}) k_{\mathrm{D}}^{*}(\mathrm{G}_1, \mathrm{G}_2) (\mathrm{Q}_r)_m^n, \qquad r=2,4. \quad (180)$$

We will now seek a lower bound for $M(G_1, G_2)_i^{G_2}$, $i \in G_1$. According to equations (149) and (150), we have

$$\tilde{Q}_1 \leqq M(G_1, G_2)_i^{G_2} \tag{181}$$

with

$$\tilde{Q}_1 = \sum_{k=2}^{+\infty} \sum_{(S_1, \ldots, S_k) \in \mathcal{P}_k} \sum_{(i_1, \ldots, i_k) \in \{i\} \times S_2 \times \ldots \times S_k} (1 - e^{-\alpha/T_1})^{k-1} \times \prod_{l=1}^{k-1} q(S_l, i_{l+1})/q(S_l) \prod_{l=1}^{k-1} Q_1(i_l, i_{l+1}). \tag{182}$$

As $(\tilde{Q}_1)_m^n \leqq (\tilde{Q}_2)_m^n$, we see that there exist positive constants T_0, a and b such that in the cooling framework $\mathcal{G}(T_0, H(D))$, $\tilde{Q}_1$ is of class $\mathcal{D}(H(D), a, b)$. Moreover

$$(\tilde{Q}_1)_m^{n \rightarrow} = \sum_{k=1}^{+\infty} (1 - e^{-\alpha/T_1})^k \nu_D^k(G_1, G_2) \tag{183}$$

and

$$\sum_{k=1}^{+\infty} (1 - (1 - e^{-\alpha/T_1})^k) \nu_D^k(G_1, G_2) \leqq \frac{K \mu e^{-\alpha/T_1}}{(1-\mu)(1-(1-e^{-\alpha/T_1})\mu}. \tag{184}$$

[Compare with equations (153) and (156)].

Hence there are positive constants T_0 and K such that in the cooling framework $\mathcal{G}(T_0, H((D))$ we have

$$(\tilde{Q}_1)_m^{m \rightarrow} \geqq k^*(G_1, G_2) - K e^{-\alpha/T_1}. \tag{185}$$

Hence there are positive T_0, α, a and b such that in the cooling framework $\mathcal{G}(T_0, H(D))$ there is a RKI Q_1 of class $\mathcal{D}(H(D), a, b)$ such that

$$(1 - e^{-\alpha/T_1}) k^*(G_1, G_2) (Q_1)_m^n \leqq M(G_1, G_2)_m^n \tag{186}$$

Let us study the case of left kernels now. We have, according to equation (149) and (151)

$$\sum_{k=1}^{+\infty} \sum_{(S_1, \ldots, S_{k+1}) \in \mathcal{P}_{k+1}} \sum_{(i_1, \ldots, i_{k+1}) \in \{i\} \times S_2 \times \ldots \times S_{k+1}} \times \sum_{m = m_1 \leqq \ldots \leqq m_{k+1} = n} (1 - e^{-\alpha/T_1})^k \times \prod_{l=1}^{k} q(S_l, i_{l+1})/q(S_l) \prod_{l=1}^{k} Q_3(i_l, i_{l+1})_{m_l}^{m_l+1} \prod_{l=1}^{k} \exp(-H(S_l)(T_{m_l+1}^{-1} - T_{m_l+1}^{-1})) \leqq M(G_1, G_2)_{i, m}^{G_2, n}. \tag{187}$$

Let us notice now that

$$\exp(-H(S_l)(T_{m_{l+1}}^{-1}-T_{m_l+1}^{-1}))\geqq\exp(-H(D)(T_{m_{l+1}}^{-1}-T_{m_l+1}^{-1})) \quad (188)$$

hence

$$\prod_{l=1}^{k}\exp(-H(S_l)(T_{m_{l+1}}^{-1}-T_{m_l+1}^{-1}))\geqq\exp(H(D)(T_n^{-1}-T_{m+1}^{-1})). \quad (189)$$

Coming back to equation (187) we see that

$$(\tilde{Q}_3)_m^n\exp(-H(D)/T_n)\leqq\exp(-H(D)/T_{m+1})\,M(G_1, G_2)_m^n, \quad (190)$$

with

$$\tilde{Q}_3=\sum_{k=1}^{+\infty}\sum_{(S_1, \ldots, S_{k+1})\in\mathscr{P}_{k+1}}\sum_{(i_1, \ldots, i_{k+1})\in\{i\}\times S_2\times\ldots\times S_{k+1}}(1-e^{-\alpha/T_1})^k \times\prod_{l=1}^{k}q(S_l, i_{l+1})/q(S_l)\prod_{l=1}^{k}Q_3(i_l, i_{l+1}). \quad (191)$$

Following the same route as for Q_4, we deduce from this equation that there are positive constants T_0, α, a and b such that in the cooling framework $\mathscr{G}(T_0, H(D))$ there is a LKI Q_3 of class $\mathscr{D}^l(H(D), a, b)$ such that

$$(1-e^{-\alpha/T_1})\,k^*(G_1, G_2)(Q_3)_m^n\exp(-H(D)/T_n) \leqq M(D, G_2)_{i,m}^{E,n}\exp(-H(D)/T_{m+1}). \quad (192)$$

The estimations for kernels $M(D, G_2)_i^j$, $i\in G_1$, $j\in G_2$, are easy to derive from those for kernels $M(G_1, G_2)_i^E$ and the equation

$$M(D, G_2)_i^j=\sum_{S\in\mathscr{M}(D)}\{M(D, S)\,M(S, G_2)\}_i^j+M(G_1, G_2)_i^j. \quad (193)$$

More precisely, we see that $M(D, G_2)_i^j$ is of class

$$\mathscr{E}^r(0, \sum_{S\in\mathscr{M}(D)}k_D(G_1, S)\,q(S, j), H(D), \alpha) \quad (194)$$

for some positive α in some framework $\mathscr{G}(T_0, H(A))$.

End of the proof of lemma 4.6.

For any $\mathscr{F}\in\mathscr{A}'$, let us put

$$D=\bigcup_{G\in\mathscr{F}}G \quad (195)$$

and let us consider the potential kernel

DEFINITION 4.7:

$$k_{A,D}=\sum_{n=0}^{+\infty}(I(\mathscr{F})\,\nu_A)^n, \quad (196)$$

where

$$I(\mathscr{F})(G) = \begin{cases} 1 & \text{if } G \in \mathscr{F} \\ 0 & \text{otherwise} \end{cases} \tag{197}$$

DEFINITION 4.8. – *We will say that two disjoint subsets* D *and* D′ *of* E *communicate if there exist* $i \in D$, $j \in D'$ *such that* $q(i, j) > 0$. *Now let* D, $D' \in \mathscr{P}$ *and let* $\mathscr{F}$ *and* $\mathscr{F}'$ *be the associated elements of* $\mathscr{A}'$. *We will write* $D -| D'$ *and say that* D′ *is a successor of* D *if* $D \leq D'$, $D \neq D'$ *and* D *and* D′ *communicate. Notice that in this case* $\{i\} \in \mathscr{A}$, $j \in \tilde{B}(\{i\})$ *and for any* $G \in \mathscr{F}$ *and any* $G' \in \mathscr{F}'$ *we have* $k_{A, D}(G, G') > 0$.

The following step of the proof is

LEMMA 4.9. – *For distinct* D, $D' \in \mathscr{P}$ *such that there exist* $i \in D$, $j \in D'$ *with* $q(i, j) > 0$, *for any* $i \in D$, *any* $j \in D'$ *there exist positive constants* $\gamma(i, j)$, T_0 *and* α *such that in* $\mathscr{G}(T_0, H(A))$ *the kernel* $M(D, D')_i^j$ [*resp.* $\exp(-H(A)/T_{m+1})\, M(D, D')_{i,\,m}^{j,\,n}$] *is of class* $\mathscr{E}^r((\lambda(D') - \lambda(D))^+, \gamma(i, j), H(A), \alpha)$ [*resp.* $\mathscr{E}^l((\lambda(D') - \lambda(D))^+ + H(A), \gamma(i, j), H(A), \alpha)$]. *Moreover if* $G \in \mathscr{M}(D)$, $G' \in \mathscr{M}(D')$ *and* $\lambda(D') < \lambda(D)$ *then for any* $i \in G$

$$\sum_{j \in G'} \gamma(i, j) = k_{A, D}(G, G'). \tag{198}$$

Proof. – We have according to proposition 2.10, for any $i \in G$, $j \in G'$,

$$M(D, D')_i^j = \sum_{\tilde{G} \in \mathscr{M}(D)} \{M(D, \tilde{G})\, M(\tilde{G}, G')\}_i^j + M(G, G')_i^j. \tag{199}$$

According to lemma 4.6 we know that $M(D, \tilde{G})_i^{\tilde{G}}$ is of class $\mathscr{E}^r(0, k_D^*(G, \tilde{G}), H(D), b)$ in the framework $\mathscr{G}(T_0, H(D))$ for suitable positive constants T_0 and b [let us insist on the fact that $k_D^*(G, \tilde{G}) > 0$]. Moreover $M(G, G')_i^j$ is of class $\mathscr{E}^r((\lambda(D') - \lambda(D))^+, q(G, j), H(D), b)$ when it is non null [which is the case for at least one $G \in \mathscr{M}(D)$ from our hypothesis]. We conclude with the help of composition lemmas 6.8 and 6.10 (*cf.* appendix). The proof with left classes is of the same kind and is left to the reader.

End of the proof of lemma 4.9.

LEMMA 4.10. – *With the same hypothesis as in proposition* 4.5, *let us consider* D *and* $D' \in \mathscr{P}$ *such that* D *and* D′ *are not comparable for the relation* $\leq$, *then* $M(D, D') = 0$.

Hence $M(D, D') \neq 0$ *if and only if either* $D -| D'$ or $D' -| D$.

Proof. – Let us consider D and $D' \in \mathscr{P}$. Let us consider the associated $\mathscr{F}$ and $\mathscr{F}' \in \mathscr{A}'$. Let us consider $G \in \mathscr{F}$ and $G' \in \mathscr{F}'$. Assume that there is $i \in G$ and $j \in G'$ such that $q(i, j) > 0$. As G and G′ are cycles, one of them is reduced to one point, thus either $\{i\}=G\subset B(G')$ or $\{j\}=G'\subset\tilde{B}(G)$. Hence either $G\underline{<}G'$ or $G'\underline{<}G$, and $\mathscr{F}$ and $\mathscr{F}'$ are comparable. Thus if $\mathscr{F}$ and $\mathscr{F}'$ are not comparable, then $M(D, D')=0$.

End of the proof of lemma 4.10.

Let $D_1, D_2\in\mathscr{P}$, let $G_1\in\mathscr{M}(D_1)$ and $G_2\in\mathscr{M}(D_2)$. Consider $i\in G_1$ and $j\in G_2$. According to proposition 2.11

$$M(A, G_2)_i^j=\sum_{k=1}^{+\infty}\sum_{S_1, \ldots, S_k\in\mathscr{P}, S_1=D_1}\{M(S_1, S_2)M(S_2, S_3)\times\ldots \times M(S_{k-1}, S_k)M(S_k, G_2)\}_i^j. \quad (200)$$

Let us call $\mathscr{P}_k$ the set of sequences $(S_1, S_2, \ldots, S_k)\in\mathscr{P}^k$ of lenth k such that, $S_1=D_1$, $M(S_l, S_{l+1})\neq 0$, $l=1, \ldots, k-1$ and $M(S_k, G_2)\neq 0$ and such that moreover

$$\sum_{l=1}^{k}(\lambda(S_{k+1})-\lambda(S_k))^+=H(D_1, D_2) \quad (201)$$

with the convention that $S_{k+1}=D_2$. [The definition of $H(D_1, D_2)$ is that of the end of proposition 4.5.]

As $\mathscr{P}$ is finite and $(\lambda(S_{l+1})-\lambda(S_l))^+=0$ if and only if $S_l\leq S_{l+1}$, we see that $\mathscr{P}_k$ is empty for any large enough k.

More precisely we can remark that if $(S_1, \ldots, S_k)\in\mathscr{P}_k$ all the $S_1, \ldots S_k$ must be distinct from one another [otherwise there is a loop that we could suppress to obtain an other sequence of weight $H(D_1, D_2)$, but the weight of a loop cannot be 0, $\leq$ being an order relation, hence we get a contradiction]. Thus $\mathscr{P}_k=0$ for $k>\mathrm{Card}(\mathscr{P})$.

Let us call $\mathscr{P}^{(k)}$ the set of sequences of length k $(S_1, \ldots, S_k)\in\mathscr{P}^k$ such that $S_1=D_1$, $M(S_l, S_{l+1})\neq 0$, $l=1, \ldots, k-1$, $M(S_k, G_2)\neq 0$ and $(S_1, \ldots, S_k)\notin\mathscr{P}_k$. Let us put $L=\mathrm{Card}(\mathscr{P})$. For any sequence $(S_1, \ldots, S_k)\in\mathscr{P}^{(k)}$ we have, putting $S_{k+1}=D_2$, for some positive constant α,

$$\sum_{l=1}^{k}(\lambda(S_{l+1})-\lambda(S_l))^+\geqq H(D_1, D_2)+\alpha\sup\left\{1, \left[\frac{k-1}{L}\right]\right\}, \quad (202)$$

where $[x]$ is the integer par of x.

We can decompose equation (200) into

$$M(A, G_2)_i^{G_2}=W+B \quad (203)$$

with

$$\left.\begin{aligned} W &= \sum_{k=1}^{+\infty} \sum_{(S_1, \ldots, S_k) \in \mathscr{P}_k} \{ M(S_1, S_2) \ldots M(S_k, G_2) \}_i^j, \\ B &= \sum_{k=1}^{+\infty} \sum_{(S_1, \ldots, S_k) \in \mathscr{P}^{(k)}} \{ M(S_1, S_2) \ldots M(S_k, G_2) \}_i^j. \end{aligned}\right\} \tag{204}$$

We will prove first that for some positive constants T_0 and α, the kernel B is of class $\mathscr{E}_-(H(D_1, D_2)+\alpha, H(A))$ in the cooling framework $\mathscr{G}(T_0, H(A))$.

From our preceding remarks we see that there are for any i, j belonging to different cycles of $\mathscr{M}(A)$ RKIs $Q_1(i, j)$ and LKIs $Q_2(i, j)$ of class $\mathscr{D}(H(A), a, b)$ in some $\mathscr{G}(T_0, H(A))$ such that

$$\left.\begin{aligned} B_m^n \leq & \sum_{k=1}^{+\infty} \sum_{(S_1, \ldots, S_k) \in \mathscr{P}^{(k)}} \sum_{(i_1, \ldots, i_{k+1}) \in \{i\} \times S_2 \times \ldots \times S_k \times G_2} \\ & \times K^k \left\{ \prod_{l=1}^{k} Q_r(i_l, i_{l+1}) \right\}_m^n \prod_{l=1}^{k} \exp(-(\lambda(S_{l+1}) - \lambda(S_l))^+ / T_{m+1}) \\ & \leq \{ \tilde{Q}_r \}_m^n \exp(-(H(D_1, D_2)+\alpha)/T_{m+1}), \\ & r = 1,2. \end{aligned}\right\} \tag{205}$$

with

$$\left.\begin{aligned} \tilde{Q}_r = & \sum_{k=1}^{+\infty} \sum_{(S_1, \ldots, S_k) \in \mathscr{P}^{(k)}} \sum_{(i_1, \ldots, i_{k+1}) \in \{i\} \times S_2 \times \ldots \times S_k \times G_2} \\ & \times K^k \left\{ \prod_{l=1}^{k} Q_r(i_l, i_{l+1}) \right\}_m^n \exp\left(-\alpha\left(0 \vee \left[\frac{k-L-1}{L} \right] \right) \Big/ T_{m+1} \right) \\ & r = 1,2. \end{aligned}\right\} \tag{206}$$

Let us prove now a lemma about kernels $\tilde{Q}_r$, $r = 1,2$.

LEMMA 4.11. – *There are positive constants* T_0, a, K' *such that in the cooling framework* $\mathscr{G}(T_0, H(A))$ *the kernels* $\tilde{Q}_r$, $= 1,2$ *are such that*

$$(\tilde{Q}_1)_m^{n\rightarrow} \leq K' \prod_{l=m+1}^{n-1} (1 - a e^{-H(A)/T_l}) \tag{207}$$

and

$$(\tilde{Q}_2)_{\leftarrow m}^{n} \leq K' \prod_{l=m+1}^{n-1} (1 - a e^{-H(A)/T_l}). \tag{208}$$

Proof. – We have

$$(\tilde{Q}_1)_m^{n\to} \leqq \sum_{k=1}^{+\infty} K^k \operatorname{Card}(\mathscr{P}^{(k)}) \{ Z_r(H(A), a, b)^k \}_m^{n\to} \times \exp\left(-\left(0 \vee \left[\frac{k-L-1}{L}\right]\right) \alpha \Big/ T_1\right) \quad (209)$$

$$\left[\text{resp. } (\tilde{Q}_2)_{\leftarrow m}^{n} \leqq \sum_{k=1}^{+\infty} K^k \operatorname{Card}(\mathscr{P}^{(k)}) \{ Z_l(H(A), a, b)^k \}_{\leftarrow m}^{n} \times \exp\left(-\left(0 \vee \left[\frac{k-L-1}{L}\right]\right) \alpha \Big/ T_1\right)\right]. \quad (210)$$

Let us remark that

$$\operatorname{Card}(\mathscr{P}^{(k)}) \leqq \operatorname{Card}(\mathscr{P})^k, \quad (211)$$

hence abreviating Z (H (A), *a*, *b*) by Z, putting $K' = K \operatorname{Card}(\mathscr{P})$ and

$$B_k = \sum_{s=m+1}^{n-1} \{ Z^{k-1} \}_m^l Z_l^{n\to}, \qquad k \geqq 2 \quad (212)$$

$$B_1 = Z_m^{n\to} \quad (213)$$

$$\left[\text{resp. } B_k = \sum_{s=m+1}^{n-1} Z_{\leftarrow m}^l \{ Z^{k-1} \}_l^n, \; k \geqq 2, \quad (214)\right.$$

$$\left. B_1 = Z_{\leftarrow m}^n \right] \quad (215)$$

and integrating by parts equation (209) [resp. equation (210)], we have

$$(\tilde{Q}_1)_m^{n\to} \leqq \sum_{k=1}^{+\infty} B_k \sum_{l=k}^{+\infty} K'^l \exp\left(-\left(0 \vee \frac{l-2L-1}{L}\right) \alpha \Big/ T_1\right). \quad (216)$$

$$\left[\text{resp. } (\tilde{Q}_2)_{\leftarrow m}^n \leqq \sum_{k=1}^{+\infty} B_k \sum_{l=k}^{+\infty} K'^l \exp\left(-\left(0 \vee \frac{l-2L-1}{L}\right) \alpha \Big/ T_1\right)\right]. \quad (217)$$

There are positive constants T_0 and K_3 such that in the cooling framework $\mathscr{F}(T_0)$

$$K_3 \geqq \sum_{l=1}^{2L} K'^l + \frac{K'^{2L+1}}{1 - K' \exp(-\alpha/(LT_1))}. \quad (218)$$

Hence there is a positive constant T_0 such that in $\mathscr{G}(T_0, H(A))$ we have $(\tilde{Q}_1)_m^{n\rightarrow}$ [resp. $(\tilde{Q}_2)^n_{\leftarrow m}$]

$$\leq \sum_{k=1}^{2L} K_3 B_k - \left(1 - K' \exp\left(-\frac{\alpha}{LT_1}\right)\right)^{-1} \times \sum_{k=2L+1}^{+\infty} K'^k B_k \exp\left(-\frac{(k-2L-1)\alpha}{LT_1}\right) = A_1 + A_2. \quad (219)$$

Let us examine A_2 first. We have in $\mathscr{G}(T_0, H(A))$ for a suitable positive T_0,

$$B_k \leq \frac{(1+b)^k}{(1-ae^{-H(A)/T_1})^{k-1}} \left(\sum_{s=m+1}^{n-1} \exp(-H(A)/T_s)\right)^{k-1} \frac{a^{k-1}}{(k-1)!} \prod_{s=m+1}^{n-1} (1-ae^{-H(A)/T_s}). \quad (220)$$

[*cf.* equation (176) in the appendix. The expression is the same for right and left kernels.]

Hence there are positive constants T_0, K_3 and K_4 such that in the cooling framework $\mathscr{G}(T_0, H(A))$

$$K_3 \geqq (1+b)^{2L+1} a^{2L} (1-ae^{-H(A)/T_1})^{-2L} \left(1 - K' \exp\left(-\frac{\alpha}{LT_1}\right)\right)^{-1} \quad (221)$$

$$K_4 \geqq K'(1+b)(1-ae^{-H(A)/T_1})^{-1} a \quad (222)$$

(223)

and consequently

$$A_2 \leqq K_3 \exp\left(K_4 \exp\left(-\frac{\alpha}{LT_1}\right)\left(\sum_{s=m+1}^{n-1} e^{-H(A)/T_s}\right)\right) \times \prod_{s=m+1}^{n-1} (1-ae^{-H(A)/T_s}\bigg). \quad (224)$$

Let us examine A_1 now. Due to equation (220) there is a polynome P with constant coefficients and positive constants T_0, a such that in $\mathscr{G}(T_0, H(A))$

$$A_1 \leqq P\left(\sum_{s=m+1}^{n-1} e^{-H(A)/T_s}\right) \prod_{s=m+1}^{n-1} (1-ae^{-H(A)/T_s}). \quad (225)$$

The function $x \mapsto P(x) \exp(-ax/2)$ is bounded on $\mathbb{R}_+$ hence there is a positive constant K_5 such that

$$P(x) \leqq K_5 \exp(ax/2). \qquad x \geqq 0, \quad (226)$$

Hence

$$A_1 \leqq K_5 \exp\left(\frac{a}{2}\sum_{s=m+1}^{n-1} e^{-H(A)/T_s}\right)\prod_{s=m+1}^{n-1}(1-a\,e^{-H(A)/T_s}). \tag{227}$$

In vew of equations (224) and (227) (which hold both in the case of right and left kernels) we see that there are positive constants T_0, a_2 and K_6 such that in the cooling framework $\mathscr{G}(T_0, H(A))$ we have

$$(\tilde{Q}_1)_m^{n\rightarrow}\,[\text{resp. }(\tilde{Q}_2)_{\leftarrow m}^{n}] \leqq K_6 \prod_{s=m+1}^{n-1}(1-a_2\,e^{-H(A)/T_s}). \tag{228}$$

End of the proof of lemma 4.11.

We deduce from lemma 4.11 that there are positive constants T_0, K, a and b such that in the cooling framework $\mathscr{G}(T_0, H(A))$ there is a RKI Q_1 and a LKI Q_2 of class $\mathscr{D}(H(A), a, b)$ such that

$$(\tilde{Q}_s)_m^n \leqq K\,(Q_s)_m^n, \qquad s=1, 2. \tag{229}$$

Hence

LEMMA 4.12. – *There are positive constants* T_0 *and* α *such that in the cooling framework* $\mathscr{G}(T_0, H(A))$ *the GTK* B *of equation* (204) *is of class* $\mathscr{E}(H(D_1, D_2)+\alpha, H(A))$.

As the sum in the definition of W is finite, we see by applying the composition lemmas and lemma 4.9 that there are positive constants T_0, α such that in $\mathscr{G}(T_0, H(A))$ the kernel W is of class $\mathscr{E}^r(H(D_1, D_2), \delta(i, j), H(A), \alpha)$ [resp. $\exp(-H(A)/T_{m+1})\,W_m^n$ is of class

$$\mathscr{E}^l(H(D_1, D_2)+H(A), \delta(i, j), H(A), \alpha)]$$

with

$$\delta(i, j)=\sum_{k=1}^{+\infty}\ \sum_{(S_1, \ldots, S_k)\in\mathscr{P}^{(k)}}\ \times \sum_{(i_1, \ldots, i_{k+1})\in\{i\}\times S_2\times\ldots S_k\times G_2}\ \sum_{l=1}^{k}\gamma(i_l, i_{l+1}). \tag{230}$$

In case $H(D_1, D_2)=0$, or equivalently $D_1 \leq D_2$, or equivalently $k_A^*(G_1, G_2)>0$, we deduce from the fact that for any S, $S'\in\mathscr{P}$ such that $S \mathrel{-\!\!|} S'$

$$\sum_{j\in G'}\gamma(i, j)=k_{A, S}(G, G'), \qquad i\in G\in\mathscr{M}(S), \quad G'\in\mathscr{M}(S'), \tag{231}$$

that

$$\sum_{j\in G_2}\delta(i, j)=k_A^*(G_1, G_2), \qquad i\in G_1. \tag{232}$$

[Indeed in this case for any $(S_1, \ldots, S_k) \in \mathscr{P}_k$, for any $l=1, \ldots, k-1$, $S_l \dashv S_{l+1}$ and either $S_k \dashv D_2$ or $S_k = D_2$.]

End of the proof of proposition 4.5.

5. PROOFS OF LARGE DEVIATIONS ESTIMATES

The aim of this section is to end the proof of theorem 2.25.

Proof of theorem 2.25. – Let us consider some cycle C. We may assume by induction that theorem 2.25 is true for any strict subcycle of C.

We are going to prove first that C is of class $\mathscr{P}_1$.

Let f be some state of F (C) and let j be in B (C). Let $A = C - \{f\}$, and let $\mathscr{A}$ be the maximal partition of A. Let $(G_s)_{s=1, \ldots, r}$ be the natural partition of C (the partition of C into its maximal strict subcycles). We will assume that

$$H(G_1) \geqq H(G_2) \geqq \ldots \geqq H(G_r), \tag{233}$$

and consequently that

$$U(G_1) \leqq U(G_2) \leqq \ldots \leqq U(G_r). \tag{234}$$

We will also assume that $f \in G_1$. It is easy to see that $G_s \in \mathscr{A}$ for $s=2, \ldots, r$. Let us put $G_1^* = G_1 - \{f\}$. According to proposition 2.9 we have

$$M(A, E-C) = M(G_1^*, E-C) + M(G_1^*, C-G_1) M(A, E-C) \tag{235}$$

By induction we may assume that G_1 is of class $\mathscr{P}_1$. Thus there are positive constants T_0, d, such that in the annealing framework $(E, U, \mathscr{L}_0, \mathscr{G}(T_0, H'(C)), d)$, for any $i \in B(G_1)$, $M(G_1^*, E-G_1)_f^i$ is of class $\mathscr{E}((U_i - U_f), q(G_1, i), H'(G_1), d)$.

We need the following lemma:

LEMMA 5.1. – *With the above notations, we have for any* G_s, $s=2, \ldots, r$

$$\sum_{t=2, r} q(G_1, G_t) k_A(G_t, G_s) = q(G_s) \tag{236}$$

Proof. – Let us put $\tilde{A} = C - G_1$ and let $\tilde{\mathscr{A}}$ be the maximal partition of $\tilde{A}$, that is $(G_s)_{s=2, \ldots, r}$. We have for any G, G′ in $\tilde{\mathscr{A}}$

$$k_{\tilde{A}}(G, G') = k_A(G, G') \tag{237}$$

because v_A and $v_{\tilde{A}}$ coincide on $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{A}}$ is not v_A reachable from any state of $\mathscr{A}-\tilde{\mathscr{A}}$. Then we can identify the external state of $\mathscr{M}_{\tilde{A}}$ with G_1. Consider on $\mathscr{M}_{\tilde{A}}$ the Markov kernel

$$v(G, G')=q(G, G')/q(G), \qquad G, G'\in\mathscr{M}_{\tilde{A}}, \tag{238}$$

with the above mentioned identification of the external state. Then v coincide with v_A on $\tilde{\mathscr{A}}$, because

$$B(G_s)\cap G_t=\tilde{B}(G_s)\cap G_t, \qquad s, t=1, \ldots, r. \tag{239}$$

The measure $q(G)$ on $\tilde{\mathscr{A}}$ is v invariant. Let Z be the Markov chain associated with v. It is easy to see that Z is recurrent and irreducible. Then it is well known that, putting

$$\sigma=\inf\{n>0 : Z_n=G_1\}, \tag{240}$$

we have

$$\left.\begin{array}{c}\sum_n P(Z_n=G_s, \sigma>n\,|\,Z_0=G_1)=q(G_s)/q(G_1), \\ s=2, \ldots, r.\end{array}\right\} \tag{241}$$

Equation (236) is the exact translation of equation (241) with some changes in the notations.

End of the proof of lemma 5.1.

According to proposition 2.10 we have:

$$\begin{aligned}M(G_1^*, C-G_1)M(A, E-C)=&\sum_{s\geqq 2} M(G_1^*, G_s)\\ &\times\left(M(G_s, E-C)+\sum_{t=2}^{r} M(A, G_t)M(G_t, E-C)\right)+N,\end{aligned} \tag{242}$$

with

$$N=\sum_{s\geqq 2}\ \sum_{G\in\mathscr{A}-\tilde{\mathscr{A}}} M(G_1^*, G_s)M(A, G)M(G, E-C). \tag{243}$$

There are positive constants T_0 and d such that, in the annealing framework

$$(E, U, q, \mathscr{L}_0, \mathscr{G}(T_0, H'(C)), \mathscr{X}), \tag{244}$$

- for any $s\in[2, r]$, $M(G_1^*, G_s)_f^{G_s}$ is of class

$$\mathscr{E}((U(G_s)+H(G_s)-U_f), q(G_1, G_s), H'(C), d); \tag{245}$$

- for any $G\in\mathscr{A}$, for any $s\in[2, r]$, for any $i\in G_s$, according to proposition 4.5,

$$\exp(-H(A)/T_{m+1})M(A, G)_{i,m}^{G,n} \tag{246}$$

is of class $\mathscr{E}(\mathrm{H}(\mathrm{A}), k_{\mathrm{A}}^{*}(\mathrm{G}_s, \mathrm{G}), \mathrm{H}'(\mathrm{C}), d)$;

- for any $\mathrm{G} \in \mathscr{A}$, for any $i \in \mathrm{G}$ and any $j \in \mathrm{B}(\mathrm{G}) \cap (\mathrm{E}-\mathrm{G})$,

$$\exp(-\mathrm{H}(\mathrm{G})/\mathrm{T}_{m+1})\, \mathrm{M}(\mathrm{G}, \mathrm{E}-\mathrm{C})_{i,\, m}^{j,\, n} \tag{247}$$

is of class

$$\mathscr{E}(\mathrm{U}_j - \mathrm{U}(\mathrm{G}), q(\mathrm{G}, j)/q(\mathrm{G}), \mathrm{H}'(\mathrm{C}), d). \tag{248}$$

Applying lemma 6.8 to $\mathrm{M}(\mathrm{G}_1^*, \mathrm{G}_s)_f$, $\mathrm{M}(\mathrm{A}, \mathrm{G}_t)$ and again to

$$(\mathrm{M}(\mathrm{G}_1^*, \mathrm{G}_s)\, \mathrm{M}(\mathrm{A}, \mathrm{G}_t))_f, \qquad \mathrm{M}(\mathrm{G}_t, \mathrm{E}-\mathrm{C})^j, \tag{249}$$

with $j \in \mathrm{B}(\mathrm{C})$, we find that there is a positive constant b such that, for each couple $(s, t) \in [2, r]^2$, the kernel

$$(\mathrm{M}(\mathrm{G}_1^*, \mathrm{G}_s)\, \mathrm{M}(\mathrm{A}, \mathrm{G}_t)\, \mathrm{M}(\mathrm{G}_t, \mathrm{E}-\mathrm{C}))_f^j \tag{250}$$

is of class

$$\mathscr{E}(\mathrm{U}_j - \mathrm{U}_f, q(\mathrm{G}_1, \mathrm{G}_s)\, k_{\mathrm{A}}^{*}(\mathrm{G}_s, \mathrm{G}_t)\, q(\mathrm{G}_t, j)/q(\mathrm{G}_t), \mathrm{H}'(\mathrm{C}), b). \tag{251}$$

In the same way we see that

$$\mathrm{M}(\mathrm{G}_1^*, \mathrm{G}_s)\, \mathrm{M}(\mathrm{A}, \mathrm{G})\, \mathrm{M}(\mathrm{G}, \mathrm{E}-\mathrm{C})_f^j \tag{252}$$

is of class

$$\mathscr{E}((\mathrm{U}_j - \mathrm{U}(\mathrm{G}) - \mathrm{H}(\mathrm{G}) + \mathrm{H}(\mathrm{G}_1)), c', \mathrm{H}'(\mathrm{C}), b). \tag{253}$$

The expression of c' is unimportant, the remarkable point is that

$$\begin{aligned} &\mathrm{U}_j - \mathrm{U}(\mathrm{G}) - \mathrm{H}(\mathrm{G}) + \mathrm{H}(\mathrm{G}_1) \\ &\qquad = \mathrm{U}_j - \mathrm{U}_f + (\mathrm{H}(\mathrm{G}_1) + \mathrm{U}(\mathrm{G}_1) - \mathrm{H}(\mathrm{G}) - \mathrm{U}(\mathrm{G})) \end{aligned} \tag{254}$$

and that, G being a strict subcycle of G_1

$$\mathrm{H}(\mathrm{G}_1) + \mathrm{U}(\mathrm{G}_1) - \mathrm{H}(\mathrm{G}) - \mathrm{U}(\mathrm{G}) > 0, \tag{255}$$

consequently

$$\mathrm{U}_j - \mathrm{U}(\mathrm{G}) - \mathrm{H}(\mathrm{G}) + \mathrm{H}(\mathrm{G}_1) > \mathrm{U}_j - \mathrm{U}_f. \tag{256}$$

Hence coming back to equation (242), we deduce, according to lemma 6.10, that $[\mathrm{M}(\mathrm{G}_1^*, \mathrm{C}-\mathrm{G}_1)\, \mathrm{M}(\mathrm{A}, \mathrm{E}-\mathrm{C})]_f^j$ is of class

$$\mathscr{E}(\mathrm{U}_j - \mathrm{U}_f, c, \mathrm{H}'(\mathrm{C}), b)$$

where

$$c = \sum_{s,\, t \geq 2} q(\mathrm{G}_1, \mathrm{G}_s)\, k_{\mathrm{A}}^{*}(\mathrm{G}_s, \mathrm{G}_t)\, q(\mathrm{G}_t, j)/q(\mathrm{G}_t). \tag{257}$$

Summing in s and using lemma 5.1 we get that

$$c = \sum_{t \geq 2} q(\mathrm{G}_t, j). \tag{258}$$

Coming back to equation (235), we conclude that $M(C-\{f\}, E-C)_f^j$ is of class

$$\mathscr{E}(U_j - U_f, q(C, j), H'(C), d), \tag{259}$$

for some constant d.

The proof for $M(C^*, E-C)_i^j$ can be done in the same way as the preceding estimation, and is left to the reader.

End of the proof that C *is of class* $\mathscr{P}_1$.

We will now prove that C is of class $\mathscr{P}_3$ and $\mathscr{P}_4$. We will deduce from it that C is of class $\mathscr{P}_5$ and eventually that it is of class $\mathscr{P}_2$.

Proof that C *is of class* $\mathscr{P}_4$.

We still consiser the natural partition $(G_s)_{s=1, \dots, r}$ of C indexed in such a way that

$$U(G_1) = U(G_2) = \dots = U(G_s) < U(G_{s+1}) \leqq U(G_{s+2}) \leqq \dots U(G_r). \tag{260}$$

Thus we call s the number of components of the natural partition of C which are intersecting F(C).

Remark. – *If* $H(G_1) > 0$, *then* $s < r$. *Indeed* $r = 1$ *is impossible, hence there is in the natural partition of* C *a cycle of null depth.*

We are going to prove the following lemma:

LEMMA 5.2. – *For any positive constant* $\lambda \in (0, 1)$, *there are positive constants* T_0 *and* α *such that, in the annealing framework*

$$(E, U, q, \mathscr{L}_0, \mathscr{G}(T_0, H(G_{s+1})), X),$$

putting for any $T_{1/2}$ *such that* $T_0 \geqq T_{1/2} \geqq T_1$

$$N = N(H(G_{s+1}), T_{1/2}, \lambda(H(G_1) - H(G_{s+1})), 0), \tag{261}$$

we have, for any $i \in C$,

$$P\left(\tau(C, 0) > N, X_N \in \bigcup_{k=1}^{s} G_k \,\middle|\, X_0 = i\right) \geqq 1 - \exp(-\alpha/T_{1/2}). \tag{262}$$

Proof. – Let us fix λ and put

$$A = \bigcup_{k=s+1}^{r} G_k,$$

then $(G_k)_{k=s+1, \dots, r}$ is the maximal partition of A.

The proof rely on the fact that we are getting out from A faster than from $C-A$. We will establish to lemmas:

LEMMA 5.3. – *With the notations of lemma* 5.2, *for any* $\lambda \in (0, 1)$, *there are positive constants* T_0 *and* α *such that in the cooling framework*

$\mathcal{G}(T_0, H(G_{s+1}))$, *for any* $i \in A$, *for any* $T_{1/2} \in [T_1, T_0]$ *we have for the corresponding*

$$N = N(H(G_{s+1}), T_{1/2}, \lambda(H(G_1) - H(G_{s+1})), 0), \tag{263}$$

$$\sum_{k=1}^{N} M(A, C-A)_{i, m}^{C-A, k} \geqq 1 - \exp(-\alpha/T_{1/2}). \tag{264}$$

Proof. – Using proposition 2.10 and proposition 4.5 we see easily that there exist positive constants T_0, d such that in the annealing framework $(E, U, q, \mathcal{L}_0, \mathcal{G}(T_0, H(G_{s+1})), \mathcal{X})$, for any $i \in A$, $M(A, C-A)_{i_1}^{C-A}$ is of class $\mathcal{E}^r(0, 1, H(G_{s+1}, d))$. Hence there exist positive constants c, d and a RKIQ of class $\mathcal{D}(H(G_{s+1}), c, d)$ such that in the cooling framework $\mathcal{G}(T_0, H(G_{s+1}))$ for any $k \in \mathbb{N}$

$$M(A, C-A)_{i, 0}^{C-A, k} \geqq (1 - e^{-\alpha/T_1}) Q_0^n. \tag{265}$$

Thus

$$\begin{aligned}\sum_{k=1}^{N} & M(A, C-A)_{i, 0}^{C-A, k} \\ & \geqq (1 - e^{-\alpha/T_1}) \left(1 - (1+d) \prod_{k=1}^{N} (1 - c \exp(-H(G_{s+1})/T_k))\right) \\ & \geqq (1 - e^{-\alpha/T_1}) \left(1 - (1+d) \exp\left(-c \sum_{k=1}^{N} e^{-H(G_{s+1})/T_k}\right)\right) \\ & \geqq (1 - e^{-\alpha/T_{1/2}}) (1 - (1+d) \exp(-c\, e^{\lambda(H(G_1) - H(G_{s+1}))/T_{1/2}})).\end{aligned} \tag{266}$$

End of the proof of lemma 5.3.

LEMMA 5.4. – *Let* G *be a cycle of class* $\mathcal{P}_1$. *There are positive constants* T_0, α *such that in the cooling schedule* $\mathcal{G}(T_0, H'(G))$ *for any* $i \in G$ *we have:*

$$P(\tau(G, m) \leqq n \mid X_m = i) \leqq e^{-\alpha/T_1} + 2q(G) \sum_{k=m+1}^{n} \exp(-H(G)/T_k). \tag{267}$$

Remark. – As a consequence for any $k = 1, \ldots, s$, any $i \in G_k$, any $m = 0, \ldots, N$, we have

$$\begin{aligned}P(\tau(G_k, m) > N \mid X_0 = i) & \\ & \geqq 1 - e^{-\alpha/T_{1/2}} - 2q(G_k) e^{-\lambda(H(G_1) - H(G_{s+1}))/T_{1/2}}.\end{aligned} \tag{268}$$

Hence there is a positive constant α such that

$$P(\tau(G_k, m) > N \mid X_0 = i) \geqq 1 - \exp(-\alpha/T_{1/2}) \tag{269}$$

Proof of lemma 5.4. – Let f be some state in $F(G)$ and let $G^* = G - \{f\}$. The idea of the proof is to introduce the last time when X visits f before leaving G. Using the assumption that G is of class $\mathscr{P}_1$, we find positive constants T_0, α such that in the cooling framework $\mathscr{G}(T_0, H'(G))$ the following estimation holds:

$$\begin{aligned}P(\tau(G, m)\leqq n\,|\,X_0=i) &= \sum_{k=m+1}^{n} M(G^*, E-G)_{i, m}^{E-G, k} \\ &+ \sum_{k=m+1}^{n-1} \sum_{l=k}^{n} P(\tau(G, m)>k, X_k=f\,|\,X_0=i)\, M(G^*, E-G)_{f, k}^{E-G, l} \\ &\leqq M(G^*, E-G)_{i, m}^{E-G, m\rightarrow} + \sum_{k=m}^{n-1} M(G^*, E-G)_{f, k}^{E-G, k\rightarrow} \\ &\leqq e^{-\alpha/T_1} + 2\,q(G) \sum_{k=m+1}^{n} \exp(-H(G)/T_k). \qquad (270)\end{aligned}$$

End of the proof of lemma 5.4.

Let us resume now the proof of lemma 5.2:

There exists $T_0>0$ such that in the cooling framework $\mathscr{G}(T_0, H(G_{s+1}))$ we have for any $T_{1/2}\in[T_1, T_0]$ and any $i\in C$,

$$\begin{aligned}P(\tau(C, 0)>N, X_N\in C-A\,|\,X_0=i) &\geqq \sum_{n=1}^{N} \sum_{k=1}^{r} \sum_{j\in G_k} M(A, C-A)_{i, 0}^{j, n}\, P(\tau(G_k, n)>N\,|\,X_n=j) \\ &\geqq \sum_{n=1}^{N} M(A, C-A)_{i, 0}^{C-A, n}(1-\exp(-\alpha/T_{1/2})) \\ &\geqq (1-\exp(-\alpha/T_{1/2}))^2. \qquad (271)\end{aligned}$$

End of the proof of lemma 5.2.

The following step towards the fact that C is of class $\mathscr{P}_4$ is

LEMMA 5.5. – *For any* $\lambda\in(0, 1)$ *there are positive constants* T_0, α *such that in the cooling framework* $\mathscr{G}(T_0, H''(C))$, *for any* $T_{1/2}\in[T_1, T_0]$ *the subset of* C

$$\bigcup_{k=1}^{s} G_k \qquad (272)$$

is a concentration set of class $\mathscr{O}(H''(C), T_{1/2}, \lambda(H(C)-H''(C)), \alpha)$.

Proof. – Let us put as above

$$A=\bigcup_{k=s+1}^{r} G_k.$$

Let T_0 and α be chosen as in lemma 5.2 and let n be some integer such that

$$n > N(H''(C), T_{1/2}, \lambda(H(C)-H''(C)), 0).$$

Then

$$\sum_{k=0}^{n} \exp(-H''(C)/T_k) \geqq \exp\left(\lambda \frac{H(C)-H''(C)}{T_{1/2}}\right) \tag{273}$$

hence

$$\sum_{k=0}^{n} \exp(-H(G_{s+1})/T_k) \geqq \exp\left(\lambda \frac{H(C)-H''(C)}{T_{1/2}} + \frac{H''(C)-H(G_{s+1})}{T_{1/2}}\right) \geqq \exp\left(\lambda \frac{H(C)-H(G_{s+1})}{T_{1/2}}\right). \tag{274}$$

Thus we can find $m_1 < n$ such that

$$\exp\left(\lambda \frac{H(G_1)-H(G_{s+1})}{T_{1/2}}\right) \leqq \sum_{k=m_1+1}^{n} \exp(-H(G_{s+1})/T_k) \leqq 2\exp\left(\lambda \frac{H(G_1)-H(G_{s+1})}{T_{1/2}}\right) \tag{275}$$

Let us put

$$m_2 = N(H(G_{s+1}), T_{1/2}, \lambda(H(G_1)-H(G_{s+1})), m_1). \tag{276}$$

Lowering the values of T_0 and α if necessary we can assume that lemma 5.4 holds with these values for any $G_k, k=1, \ldots, s$.

From now and up to the end of the proof of lemma 5.5, let us work in the cooling framework $\mathcal{G}(T_0, H''(C))$.

Then, for any $j \in C-A$ we have

$$P(X_n \in C-A, \tau(C, m_2) > n \mid X_{m_2} = j) \geqq 1 - \exp(-\alpha/T_{1/2}), \tag{277}$$

according to lemma 5.4 and equation (275). Applying lemma 5.2 to the cooling schedule $(T_{m_1+n})_{n \in \mathbb{N}^*}$, which is in $\mathcal{G}(T_0, H''(C))$ if T is, we see that

$$P(X_{m_2} \in C-A, \tau(C, m_1) > m_2 \mid X_{m_1} = i) \geqq 1 - \exp(-\alpha/T_{1/2}). \tag{278}$$

Hence in the cooling framework $\mathcal{G}(T_0, H''(C))$ we have, for any $i \in C$

$$
\begin{aligned}
P(X_n \in (C-A), \tau(C, 0) > n \mid X_0 = i) \\
&\geqq \sum_{j_1 \in C} \sum_{j_2 \in C-A} P(X_{m_1} = j_1, \tau(C, 0) > m_1 \mid X_0 = i) \\
&\quad \times P(X_{m_2} = j_2, \tau(C, m_1) > m_2 \mid X_{m_1} = j_1) \\
&\quad \times P(X_n \in C-A, \tau(C, m_2) > n \mid X_{m_2} = j_2) \\
&\geqq (1 - \exp(-\alpha/T_{1/2})) \sum_{j_1 \in C} P(X_{m_1} = j_1, \tau(C, 0) > m_1 \mid X_0 = i) \\
&\quad \times P(X_{m_2} \in C-A, \tau(C, m_1) > m_2 \mid X_{m_1} = j_1) \\
&\geqq (1 - \exp(-\alpha s T_{1/2}))^2 P(\tau(C, 0) > m_1 \mid X_0 = i) \\
&\geqq (1 - \exp(-\alpha/T_{1/2}))^2 P(\tau(C, 0) > n \mid X_0 = i). \quad (279)
\end{aligned}
$$

End of the proof of lemma 5.5.

We are able to prove that C is of class $\mathcal{P}_4$.

Let T_0 and α be chosen such that both lemma 5.4 and lemma 5.5 work (it is possible since if any of these two lemmas is true for some values of T_0 and α, then it is true for any lower values of these parameters). Moreover, lowering if necessary the values of α and T_0, we can assume by induction hypothesis, that within the cooling framework $\mathcal{G}(T_0, H''(C))$, for any $k = 1, \ldots, s$, $F(G_k)$ is a concentration set of class

$$
\mathcal{O}(H''(G_k), T_0, \lambda(H(G_k) - H''(G_k)), \alpha).
$$

Let us work in the cooling framework $\mathcal{G}(T_0, H''(C))$. Let $T_{1/2}$ be such that $T_0 \geqq T_{1/2} \geqq T_1$. Let $n \in \mathbb{N}$ be chosen such that

$$
n \geqq N(H''(C), T_{1/2}, \lambda(H(C) - H''(C)), 0). \quad (280)
$$

Let

$$
m_1 = N\left(H''(C), T_{1/2}, \frac{\lambda}{2}(H(C) - H''(C)), 0\right). \quad (281)
$$

Then we have for any $k = 1, \ldots, s$

$$
n \geqq N\left(H''(G_k), T_{1/2}, \frac{\lambda}{2}(H(G_k) - H''(G_k)), m_1\right), \quad (282)
$$

because

$$\sum_{k=m_1+1}^{n} \exp(-H''(G_k)/T_k)$$

$$\geqq \exp((H''(C)-H''(G_k))/T_{1/2}) \sum_{k=m_1+1}^{n} \exp(-H''(C)/T_k)$$

$$\geqq \exp((H''(C)-H''(G_k))/T_{1/2}) (\exp(\lambda(H(C)-H''(C))/T_{1/2}) - 2\exp\left(\frac{\lambda}{2}(H(C)-H''(C))/T_{1/2}\right)$$

$$\geqq \exp((H''(C)-H''(G_k))/T_{1/2}) \exp\left(\frac{\lambda}{2}(H(C)-H''(C))/T_{1/2}\right)$$

$$\geqq \exp\left(\frac{\lambda}{2}(H(C)-H''(G_k))/T_{1/2}\right)$$

$$\geqq \exp\left(\frac{\lambda}{2}(H(G_k)-H''(G_k))/T_{1/2}\right). \quad (283)$$

Let $H''=\max_{1\leqq k\leqq s} H''(G_k)$, we can find $m<n$ such that

$$\exp(\lambda(H(G_1)-H'')/T_{1/2}) \leqq \sum_{k=m+1}^{n} \exp(-H''/T_k) \leqq 2\exp(\lambda(H(G_1)-H'')/T_{1/2}) \quad (284)$$

with this choice of m we have for any $k=1, \ldots, s$, for any $j\in G_k$, according to lemma 5.4

$$P(\tau(G_k, m)>n \mid X_m=j) \geqq 1-\exp(-\alpha/T_{1/2}). \quad (285)$$

Hence for any $i\in C$

$$P(X_n\in F(C), \tau(C, 0)>n \mid X_0=i)$$

$$\geqq \sum_{k=1}^{s} \sum_{j\in G_k} P(X_m=j, \tau(C, 0)>m \mid X_0=i) \times P(X_n\in F(G_k), \tau(G_k, m)>n \mid X_m=j)$$

$$\geqq \sum_{k=1}^{s} \sum_{j\in G_k} P(X_m=j, \tau(C, 0)>m \mid X_0=i) \times P(\tau(G_k, m)>n \mid X_m=j)(1-\exp(-\alpha/T_{1/2}))$$

$$\geqq P(X_m\in C-A, \tau(C, 0)>m \mid X_0=i)(1-\exp(-\alpha/T_{1/2}))^2$$

$$\geqq P(\tau(C, 0)>m \mid X_0=i)(1-\exp(-\alpha/T_{1/2}))^3$$

$$\geqq P(\tau(C, 0)>n \mid X_0=i)(1-\exp(-\alpha/T_{1/2}))^3. \quad (286)$$

End of the proof that C is of class $\mathscr{P}_4$.

Proof that C *is of class* $\mathscr{P}_3$.

The first step is contained in the following proposition

PROPOSITION 5.6. – *For any* $\lambda \in 0, 1)$, *there are positive constants* T_0, α *such that in the cooling framework* $\mathscr{G}(T_0, H'(C))$, *for any* $T_{1/2} \in [T_1, T_0]$, *for any* $k=1, \ldots, s$ *the subcycle* G_k *is a concentration set of* C *of class* $\mathscr{O}(H'(C), T_{1/2}, \lambda(H(C)-H'(C)), \alpha)$.

Proof. – We can assume that $s \geqq 2$, otherwise proposition 5.6 reduces to lemma 5.5.

We are going to do some kind of rescaling in the time variable by introducing the following sequence of times:

By induction hypothesis , we know that for any $k=1, \ldots, s$, G_k is of class $\mathscr{P}_5$, thus there is a positive constant α, such that for any k, $l=1, \ldots, r$ the kernel $M(G_k, G_l)_i^{G_l}$ is α adjacent to

$$q(G_k, G_l) \,|\, F(G_k) \,|^{-1} \mathscr{I}(H(G_k)).$$

DEFINITION 5.7. – *On the abstract state space* $\{1, \ldots, r\}$, *we consider the Markov kernel* $\tilde{\mathscr{K}}$ *defined by*

$$\tilde{\mathscr{K}}(k, k') = q(G_k, G_{k'})/q(G_k) \tag{287}$$

We call Y *the associated Markov chain and consider the stopping time*

$$\sigma = \inf\{n>0 : Y_n \leqq s\}. \tag{288}$$

With the help of σ *we define the Markov kernel on* $\{1, \ldots, r\}$:

$$\mathscr{K}(k, k') = P(Y_\sigma = k' \,|\, Y_0 = k). \tag{289}$$

DEFINITION 5.8. – *Let* $\mathscr{F}$ *be some cooling framework. We define* $V(H, T_0, \alpha, m)$ *to be the sequence* $(v_n)_{n \in \mathbb{N}}$ *characterized by* $v_0 = m$ *and*

$$v_{k+1} = N(H, T_0, -\alpha, v_k), \qquad k \geqq 1. \tag{290}$$

We will put $V(H, T_0, \alpha)$ *for* $V(H, T_0, \alpha, 0)$.

LEMMA 5.9. – *If* $H(G_1)>0$, *for any* $k, k'=1, \ldots, s$ *such that* $k \neq k'$, *for any* $i \in G_k$ *the kernel*

$$M(G_k, G_{k'})_i^{G_{k'}} \tag{291}$$

is null.

Proof. – If $B(G_k) \cap G_{k'} \neq \varnothing$, then either G_k or $G_{k'}$ is of null depth.

End of the proof of lemma 5.9.

LEMMA 5.10. – *There are positive constants* T_0, α *such that in the cooling framework* $\mathscr{G}(T_0, H'(C))$, *for any* $k, k' \in [1, s]$ *such that* $k \neq k'$, *for* *any* $i \in G_k$, *if* $\mathcal{K}(k, k') > 0$, *the kernel*

$$R = (M(G_k, G_{k'}) + M(G_k, A) M(A, G_{k'}))_i^{G_{k'}} \tag{292}$$

is adjacent to

$$q(G_k) \,|\, F(G_k) \,|^{-1} \mathcal{I}(H(G_1)$$

and satisfies

$$|\, R_m^{m \to} - \mathcal{K}(k, k') \,| \leqq e^{-\alpha/T_{m+1}}. \tag{293}$$

On the other hand, if $\mathcal{K}(k, k') = 0$, *then* $R = 0$.

Proof. – Let us examine first the case when $H(G_1) = 0$. Then $H(G_k) = 0$, $k = 1, \ldots, r$, hence all the G_k, $k = 1, \ldots, r$ are singletons $-G_k = \{ g_k \}-$ and $A = \varnothing$, thus $M(G_k, A) M(A, G_{k'}) = 0$ and

$$M(G_k, G_{k'})_{g_k, m}^{g_{k'}, n} = \prod_{l=m+1}^{n-1} p_{T_l}(g_k, g_k) p_{T_n}(g_k, g_{k'}). \tag{294}$$

We conclude that the lemma is true in this case.

Let us assume now that $H(G_1) > 0$, then according to lemma 5.9 $M(G_k, G_{k'})_i^{G_{k'}} = 0$.

The fact that $M(G_k, A) M(A, G_{k'})_i = 0$ when $\mathcal{K}(k, k') = 0$ is a consequence of the fact that for any l, $l' = 1, \ldots, r$, $M(G_l, G_{l'}) = 0$ when $P(Y = l' \,|\, Y_0 = l) = 0$ as is easily seen from equation (239).

Let us assume that $\mathcal{K}(k, k') \neq 0$.

We have according to proposition 2.10

$$\begin{aligned} M(G_k, A) M(A, G_{k'}) &= \sum_{l=s+1}^{r} M(G_k, G_l) M(A, G_{k'}) \\ &= M(G_k, G_l) \left(M(G_l, G_{k'}) + \sum_{l'=s+1}^{r} M(A, G_{l'}) M(G_{l'}, G_{k'}) \right). \end{aligned} \tag{295}$$

When it is not null the kernel $M(G_k, G_l)$ is adjacent to

$$q(G_k) \,|\, F(G_k) \,|^{-1} \mathcal{I}(H(G_1))$$

and

$$\left| M(G_k, G_l)_m^{m \to} - \frac{q(G_k, G_l)}{q(G_k)} \right| \leqq e^{-\alpha/T_{m+1}} \tag{296}$$

for some positive constant α. There exists a positive constant a such that the kernel

$$M(G_l, G_{k'}) + \sum_{l'=s+1}^{r} M(A, G_{l'}) M(G_{l'}, G_{k'}) \tag{297}$$

is of class

$$\mathscr{E}^r(0, \mathscr{K}(l, k'), \mathrm{H}(\mathrm{G}_{s+1}), a).$$

We deduce from lemma 6.11 and equation (295) that

$$\left| \{ \mathrm{M}(\mathrm{G}_k, \mathrm{A}) \mathrm{M}(\mathrm{A}, \mathrm{G}_{k'}) \}_m^{m \to} - \sum_{l=s+1}^{r} \frac{q(\mathrm{G}_k, \mathrm{G}_l)}{q(\mathrm{G}_k)} \mathscr{K}(l, k') \right| \leqq e^{-\alpha/\mathrm{T}_{m+1}} \quad (298)$$

and that the kernel $\mathrm{M}(\mathrm{G}_k, \mathrm{A}) \mathrm{M}(\mathrm{A}, \mathrm{G}_{k'})$ is adjacent to

$$|\mathrm{F}(\mathrm{G}_k)|^{-1} q(\mathrm{G}_k) \mathscr{I}(\mathrm{H}(\mathrm{G}_1)). \quad (299)$$

Noticing that

$$\mathscr{K}(k, k') = |\mathrm{F}(\mathrm{G}_k)|^{-1} \sum_{l=s+1}^{r} q(\mathrm{G}_k, \mathrm{G}_l) \mathscr{K}(l, k'), \quad (300)$$

ends the proof of lemma 5.10.

Let us choose ν such that

$$\nu(\mathrm{H}(\mathrm{G}_1) - \mathrm{H}(\mathrm{G}_{s+1})) \leqq \alpha. \quad (301)$$

Let us call $(u_n)_{n \in \mathbb{N}}$ the sequence $\mathrm{V}(\mathrm{H}(\mathrm{G}_1), \mathrm{T}_{1/2}, \nu(\mathrm{H}(\mathrm{G}_1) - \mathrm{H}(\mathrm{G}_{s+1}))/\mathrm{T}_{1/2})$.

LEMMA 5.11. — *With the above notations we have:*

$$\sum_{k=1}^{u_n} \exp(-\mathrm{H}(\mathrm{C})/\mathrm{T}_k) \leqq 2n \exp(-(\mathrm{H}(\mathrm{C}) - \mathrm{H}(\mathrm{G}_1) + \nu(\mathrm{H}(\mathrm{G}_1) - \mathrm{H}(\mathrm{G}_{s+1})))/\mathrm{T}_{1/2}). \quad (302)$$

Proof of lemma 5.11. — We have

$$\begin{aligned} \sum_{k=1}^{u_n} \exp(-\mathrm{H}(\mathrm{C})/\mathrm{T}_k) &\leqq \exp(-(\mathrm{H}(\mathrm{C}) - \mathrm{H}(\mathrm{G}_1))/\mathrm{T}_{1/2}) \sum_{k=1}^{n_n} \exp(-\mathrm{H}(\mathrm{G}_1)/\mathrm{T}_k) \\ &\quad \exp(-(\mathrm{H}(\mathrm{C}) - \mathrm{H}(\mathrm{G}_1))/\mathrm{T}_{1/2}) \\ &\quad \times \sum_{k=0}^{n-1} \left[\left(\sum_{l=u_k+1}^{u_{k+1}-1} \exp(-\mathrm{H}(\mathrm{G}_1)/\mathrm{T}_l) \right) + \exp(-\mathrm{H}(\mathrm{G}_1)/\mathrm{T}_{u_{k+1}}) \right] \\ &\leqq n \exp(-(\mathrm{H}(\mathrm{C}) - \mathrm{H}(\mathrm{G}_1))/\mathrm{T}_{1/2}) \\ &\quad \times (\exp(-\nu(\mathrm{H}(\mathrm{G}_1) - \mathrm{H}(\mathrm{G}_{s+1}))/\mathrm{T}_{1/2}) + \exp(-\mathrm{H}(\mathrm{G}_1)/\mathrm{T}_{1/2})). \end{aligned} \quad (303)$$

End of the proof of lemma 5.11.

DEFINITION 5.12. — *We define the rescaled chain on* $C\cup\{\Delta\}$, *where* Δ *is some abstract "external state", by*

$$Z_n=\begin{cases} X_{u_n} & \text{if } \tau(C,0)>u_n \\ \Delta & \text{otherwize.}\end{cases} \tag{304}$$

This rescaled chain satisfies the following lemma:

LEMMA 5.13. — *There exist positive constants* T_0, α *and* γ *such that in the cooling schedule* $\mathscr{G}(T_0, H(G_1))$ *for any* $k, k'\in\{1,\dots,s\}$ *such that* $k\neq k'$, *for any* $i\in G_k$, *for any* $n\in\mathbb{N}$, *we have*

$$\left|\frac{P(Z_{n+1}\in G_{k'}\mid Z_n=i)}{\mathscr{K}(k,k')\,q(G_k)\,|F(G_k)|^{-1}\exp(-\nu(H(G_1)-H(G_{s+1}))/T_{1/2})}-1\right| \leq \exp(-\alpha/T_{1/2}) \tag{305}$$

if $\mathscr{K}(k,k')\neq 0$ *and*

$$P(Z_{n+1}\in G_{k'}\mid Z_n=i)\leq \exp(-\nu(H(G_1)-H(G_{s+1})+\gamma)/T_{1/2}) \tag{306}$$

if $\mathscr{K}(k,k')=0$; *and for any* $k=1,\dots,s$, *for any* $i\in G_k$,

$$\left|\frac{1-P(Z_{n+1}\in G_k\mid Z_n=i)}{q(G_k)\,|F(G_k)|^{-1}\exp(-\nu(H(G_1)-H(G_{s+1}))/T_{1/2})}-1\right| \leq \exp(-\alpha/T_{1/2}). \tag{307}$$

Proof of lemma 5.13.
Let us recall that

$$A=\bigcup_{k=s+1}^{r} G_k. \tag{308}$$

For any $k, k'=1,\dots,s$, $k\neq k'$, we have for any $i\in G_k$

$$P(Z_{n+1}\in G_{k'}\mid Z_n=i) = \sum_{l=u_n+1}^{u_n+1}\sum_{j\in G_{k'}} M(G, G_{k'})^{j,l}_{i,u_n} P(\tau(G_{k'}, l)>u_{n+1}\mid X_l=j) \tag{309}$$

We are going to use repeatedly the decomposition formula of proposition 2.9. We can decompose $M(C, G_{k'})$ into

$$M(C, G_{k'})=M(A\cup G_k, G_{k'}) + M(A\cup G_k, C-(A\cup G_k))\,M(C, G_{k'}), \tag{310}$$

then we can decompose $M(A\cup G_k, G_{k'})$ into

$$M(A\cup G_k, G_{k'})=M(G_k, G_{k'})+M(G_k, A)\,M(A\cup G_k, G_{k'}), \tag{311}$$

and into

$$M(A\cup G_k, G_{k'})=M(A, G_{k'})+M(A, G_k)\,M(A\cup G_k, G_{k'}). \quad (312)$$

Substituting each of these equations into the preceding one, we get

$$M(C, G_{k'})=M(G_k, G_{k'})+M(G_k, A)\,M(A, G_{k'})+R, \quad (313)$$

with

$$R=M(A\cup G_k, C-(A\cup G_k))\,M(C, G_{k'}) \\ +M(G_k, A)\,M(A, G_k)\,M(A\cup G_k, G_{k'}). \quad (314)$$

There are positive constants T_0, α such that in the cooling framework $\mathscr{G}(T_0, H(G_{s+1}))$, for any $i\in G_k$, for any, $N\in\mathbb{N}$, $m<N$, we have

$$\sum_{n=m+1}^{N} M(A\cup G_k, C-(A\cup G_k))_{i,m}^{E,n}\leqq \sum_{n=m+1}^{N} M(G_k, E-G_k)_{i,m}^{E,n} \\ \leqq \exp(-\alpha/T_1)+2q(G_k)\sum_{n=m+1}^{N}\exp(-H(G_1)/T_n). \quad (315)$$

In the same way, for any $j\in C-A$ there is $l\in[1, s]$ such that $j\in G_l$ and

$$\sum_{n=m+1}^{N} M(C, G_{k'})_{j,m}^{E,n}\leqq \sum_{n=m+1}^{N} M(G_l, E-G_l)_{j,m}^{E,n} \\ \leqq \exp(-\alpha/T_1)+2q(G_l)\sum_{n=m+1}^{N}\exp(-H(G_1)/T_n). \quad (316)$$

Hence there is a constant K such that

$$\sum_{l=u_n+1}^{u_n+1} M(A\cup G_k, C-(A\cup G_k))\,M(C, G_{k'})_{i,u_n}^{E,l} \\ \leqq\left(\exp(-\alpha/T_1)+K\sum_{l=u_n}^{u_n+1}\exp(-H(G_1)/T_l)\right)^2 \\ \leqq(\exp(-\alpha/T_1)+2K\exp(-\nu(H(G_1)-H(G_{s+1}))/T_{1/2}))^2. \quad (318)$$

Lowering if necessary the value of ν we can assume that

$$\alpha>\nu(H(G_1)-H(G_{s+1})), \quad (319)$$

hence we get that the above upper bound is itself bounded by

$$K'\exp(-2\nu(H(G_1)-H(G_{s+1}))/T_{1/2}), \quad (320)$$

where K′ is a constant. We can in the same way find a positive constant K such that

$$\sum_{l=u_n+1}^{u_n+1} M(G_k, A)\, M(A, G_k)\, M(A \cup G_k, G_{k'})_{i,\, u_n}^{G_{k'},\, l}$$
$$\leq K \exp(-2\nu (H(G_1) - H(G_{s+1}))/T_{1/2}). \quad (321)$$

Hence we get that there is a constant K such that in the cooling schedule $\mathcal{G}(T_0, H'(C))$ we have

$$\sum_{l=u_n+1}^{u_n+1} R_{i,\, u_n}^{G_{k'},\, l} \leq K \exp(-2\nu (H(G_1) - H(G_{s+1}))/T_{1/2}). \quad (322)$$

There remain to examine the case when $k=k'$, that is to estimate

$$P(Z_{n+1} \in G_k \mid Z_n = i), \qquad i \in G_k, \quad (323)$$

the lower bound is given by the fact G_k is of class $\mathcal{P}_5$ and the upper bound is given by substracting from 1 the sum of the lower bounds obtained for $k' \neq k$.

End of the proof of lemma 5.13.

DEFINITION 5.14. — *We will get rid of the states* G_k, $k=s+1, \ldots, r$ *by putting*

$$\sigma = \inf\{n>0 : Z_n \in A \cup \Delta\} \quad (324)$$

and

$$\tilde{Z}_n = \begin{Bmatrix} Z_n & \textit{if} \quad \sigma > n \\ \Delta & \textit{otherwise} \end{Bmatrix} \quad (325)$$

DEFINITION 5.15. — *We can choose a positive constant* χ *such that, putting*

$$\tilde{Q}(k, k') = \chi\, \mathcal{K}(k, k')\, q(G_k)\, |F(G_k)|^{-1}, \qquad k, k' \in [1, s], \quad k \neq k', \quad (326)$$

and

$$\tilde{Q}(k, k) = 1 - \sum_{k'=1}^{s} \tilde{Q}(k, k'), \qquad k = 1, \ldots, s, \quad (327)$$

$\tilde{Q}$ *is a positive Markov matrix.*

Let us put

$$W = \tilde{Q} - I, \quad (328)$$

and for any fixed $T_{1/2}$

$$Q = I + \exp(-\nu(H(G_1) - H(G_{s+1}))/T_{1/2})\,\chi^{-1}W. \tag{329}$$

Choose some $i \in C - A$ and put

$$\rho_n(k) = P(\tilde{Z}_n \in G_k \mid \tilde{Z}_0 = i), \qquad n \in \mathbb{N}. \tag{330}$$

Let us put

$$R_n(k, k') = \sum_{j \in G_k} (P(\tilde{Z}_n \in G_{k'} \mid \tilde{Z}_{n-1} = j) - Q(k, k')) \times \frac{P(\tilde{Z}_{n-1} = j \mid \tilde{Z}_0 = i)}{P(\tilde{Z}_{n-1} \in G_k \mid \tilde{Z}_0 = i)}. \tag{331}$$

We have

$$\rho_n(k') = \sum_{k=1}^{s} \rho_{n-1}(k)(Q + R_n)(k, k'). \tag{332}$$

Hence putting

$$\mu(k) = \frac{|F(G_k)|}{|F(C)|}, \qquad k = 1, \ldots, s, \tag{333}$$

we have

$$\rho_n - \mu = (\rho_0 - \mu) Q^n + \sum_{l=1}^{n} \rho_{l-1} R_l Q^{n-l}. \tag{334}$$

LEMMA 5.16. — *There are positive constants* α, a *and* b *such that in the cooling schedule* $\mathscr{G}(T_0, H(G_1))$ *of lemma* 5.13, *with the above notations,*

$$|\rho_n - \mu| \leq b \exp(-an\, e^{-\nu(H(G_1) - H(G_{s+1}))/T_{1/2}}) + n \exp(-(\alpha + \nu(H(G_1) - H(G_{s+1})))/T_{1/2}). \tag{335}$$

[*Let us recall that* ν *is defined by equation* (301).]

Proof of lemma 5.16.

According to the Perron-Frobenius theorem, the spectrum of W, $\operatorname{sp} W = \{0, \lambda_1, \ldots, \lambda_{t-1}\}$ satisfies:

$$|1 + \lambda_k| \leq 1, \qquad k = 1, \ldots, t-1, \tag{336}$$

$$\lambda_k \neq 0, \qquad k = 1, \ldots, t-1 \tag{337}$$

and 0 is a simple root of the characteristic polynomial of W.

Let us consider $\lambda \in \operatorname{sp}(W)$, $\lambda \neq 0$. Let us write

$$\lambda = |\lambda| (\cos\theta + i \sin\theta). \tag{338}$$

We have $\cos\theta<0$ and $|\lambda|\leqq -2\cos\theta$. For any ε such that $0<\varepsilon<1/4$ we have

$$|1+\varepsilon\lambda|^2=1+2\varepsilon|\lambda|\cos\theta\left(1+\frac{\varepsilon|\lambda|}{2\cos\theta}\right) \quad (339)$$

hence

$$(1+2\varepsilon|\lambda|\cos\theta)\leqq|1+\varepsilon\lambda|^2\leqq(1+\varepsilon|\lambda|\cos\theta) \quad (340)$$

hence

$$(1+2\varepsilon|\lambda|\cos\theta)\leqq|1+\varepsilon\lambda|\leqq\left(1+\varepsilon\frac{|\lambda|\cos\theta}{2}\right). \quad (341)$$

Let us consider the Jordan decomposition of W on $\mathbb{C}$:

$$W=\sum_{k=1}^{r}(\lambda_k P_k+N_k) \quad (342)$$

where P_k is the projection on the kth caracteristic space E_k, of dimension r_k, and where N_k is nilpotent of degree lower or equal to r_k. Let us note P_0 the projection on $\mathbb{C}\mu$ with direction $\bigoplus_{k=1}^{r} E_k$. [We choose to identify a matrix M of $\mathcal{M}_s(\mathbb{C})$ with the endomorphism of $\mathbb{C}^s$

$$x\mapsto x'M=\sum_{i=1}^{s} x_i M_{i,j}.]$$

It is easy to deduce the decomposition of $Q=I+\varepsilon W$ from the decomposition of W, we get

$$Q=P_0+\sum_{k=1}^{r}(1+\varepsilon\lambda_k)P_k+\varepsilon N_k. \quad (343)$$

We can identify $\bigoplus_{k=1}^{r} E_k$ as the kernel of the linear form $x\mapsto\sum_{i=1}^{s} x_i$. As a matter of fact, if $\rho\in\mathbb{R}^n$ and $\rho'\mathbf{1}=0$, then $\lim_{n\to+\infty}\rho' Q^n=0$, thus ρ is in $\bigoplus_{k=1}^{r} E_k$. Hence

$$\rho' P_0=\rho'\mathbf{1}\times\mu. \quad (344)$$

and

$$|\rho' Q^n|\leqq|\rho'\mathbf{1}|+|(\rho'-\rho'\mathbf{1}\mu)Q^n| \quad (345)$$

It is thus enough to estimate $|\nu Q^n|$ for $\nu \mathbf{1}=0$, that is for $\nu \in \bigoplus_{k=1}^{r} E_k$. We have:

$$\nu Q^n = \nu \sum_{k=1}^{r} [(1+\varepsilon\lambda_k) P_k + \varepsilon N_k]^n$$
$$= \nu \sum_{k=1}^{r} (1+\varepsilon\lambda_k)^n \sum_{l=0}^{r_k - 1} \binom{n}{l} \left(\frac{\varepsilon}{1+\varepsilon\lambda_k}\right)^l N_k^l P_k. \quad (346)$$

Let us put

$$a = \inf_{k=1,\ldots,r} -\frac{|\lambda_k| \cos\theta_k}{2},$$
$$b = \sup_{k=1,\ldots,r} -2|\lambda_k| \cos\theta_k,$$

and

$$c = \sup_{k=1,\ldots,r} |N_k|$$

where

$$|M| = \sup_{\rho, |\rho|=1} |\rho' M| \qquad \text{and} \qquad R = \sup_{k=1,\ldots,r} r_k - 1.$$

From equation (341) we deduce that

$$|\nu Q^n| \leqq |\nu| (1-a\varepsilon)^n r \sum_{l=0}^{R} \binom{n}{l} \left(\frac{\varepsilon c}{1-b\varepsilon}\right)^l. \quad (347)$$

But

$$\binom{n}{l} \leqq \frac{n^l}{l!} \qquad \text{and} \qquad \left(\frac{\varepsilon cn}{1-b\varepsilon}\right)^l \leqq \sup\left(1, \left(\frac{\varepsilon cn}{1-b\varepsilon}\right)^R\right), \quad (348)$$

hence

$$|\nu Q^n| \leqq |\nu| (1-a\varepsilon)^n r e \sup\left(1, \left(\frac{\varepsilon cn}{1-b\varepsilon}\right)^R\right). \quad (349)$$

It is not hard to deduce from this that there are positive constants a, b, such that for any measure ρ on E such that $\rho(E)=0$ we have

$$|\rho Q^n| \leqq |\rho| b (1 - a e^{-\nu (H(G_1) - H(G_{s+1}))/T_{1/2}})^n$$
$$\leqq |\rho| b \exp(-an\, e^{-\nu (H(G_1) - H(G_{s+1}))/T_{1/2}}). \quad (350)$$

This inequality combined to equation (334) and lemma 5.13

ends the proof of lemma 5.16.

We will now establish a lemma of the same kind as lemma 5.2.

LEMMA 5.17. — *For any positive constant* λ *there are positive constants* T_0 *and* α *such that in the cooling framework* $\mathscr{G}(T_0, H(G_1))$, *for any* $T_{1/2} \in [T_1, T_0]$ *there is* $N \in \mathbb{N}$ *such that*

$$N \leqq N(H(G_1), T_{1/2}, \lambda(H(C)-H(G_1)), 0) \tag{351}$$

and for any $i \in (C-A)$,

$$\left| P(\tau(C,0)>N, X_N \in G_k \mid X_0 = i) - \frac{|F(G_k)|}{|F(C)|} \right| \leqq \exp(-\alpha/T_{1/2}) \tag{352}$$

Proof of lemma 5.17.

Let us put

$$n = [e^{\{(\alpha/2)+\nu(H(G_1)-H(G_{s+1}))\}/T_{1/2}}] \tag{353}$$

in the inequality of lemma 5.16. We get

$$|\rho_n - \mu| \leqq b \exp(-a e^{\alpha/(2T_{1/2})}) + e^{-\alpha/(2T_{1/2})} \tag{354}$$

hence for T_0 small enough and $T_{1/2} \leqq T_0$,

$$|\rho_n - \mu| \leqq 2 e^{-\alpha/(2T_{1/2})}, \tag{355}$$

thus

$$P(\tau(C,0)>u_n, X_{u_n} \in G_k \mid X_0 = i) \geqq \mu(k) - 2 e^{-\alpha/(2T_{1/2})}, \tag{356}$$

and

$$\begin{aligned} P(\tau(C,0)>u_n, X_{u_n} \in G_k \mid X_0 = i) &\leqq 1 - \sum_{k',\, k \neq k'} P(\tau(C,0)>u_n, X_{u_n} \in G_{k'} \mid X_0 = i) \\ &\leqq \mu(k) + 2 s e^{-\alpha/(2T_{1/2})}. \end{aligned} \tag{357}$$

Moreover we can assume that

$$\frac{\alpha}{2} \leqq \frac{\lambda}{2}(H(C)-H(G_1)) \tag{358}$$

by lowering its value if necessary, from which we deduce that

$$N = u_n \leqq N(H(G_1), T_{1/2}, \lambda(H(C)-H(G_1)), 0). \tag{359}$$

End of the proof of lemma 5.17.

From lemma 5.17 and lemma 5.2 we deduce proposition 5.6.

From proposition 5.6 and the induction assumption that for every $k=1, \ldots, r$ the cycle G_k is of class $\mathscr{P}_3$ we deduce that C is of class $\mathscr{P}_3$ as we had deduced that it is of class $\mathscr{P}_4$ from lemma 5.2 and lemma 5.5.

End of the proof that C *is of class* $\mathscr{P}_3$.

Proof that C *is of class* $\mathscr{P}_5$.

We will establish first the weaker proposition

PROPOSITION 5.18. – *There are positive constants* T_0 *and* α *such that in the cooling framework* $\mathscr{G}(T_0, H(C))$, *putting*

$$N = N(H(C), T_1, -\alpha, 0) \tag{360}$$

we have for any $i \in C$ *and any* $j \in \tilde{B}(C)$

$$\left| \frac{\sum_{n=1}^{N} M(C, E-C)_{i,0}^{j,n}}{\sum_{n=1}^{N} q(C,j)\,|F(C)|^{-1}\exp(-H(C)/T_n)} - 1 \right| \leqq \exp(-\alpha/T_1), \tag{361}$$

and

$$\left| \frac{\sum_{n=1}^{N} M(C, E-C)_{i,0}^{E,n}}{\sum_{n=1}^{N} q(C)\,|F(C)|^{-1}\exp(-H(C)/T_1)} - 1 \right| \leqq \exp(-\alpha/T_1). \tag{362}$$

Proof. – Let us fix $i \in C$ and $f \in F(C)$. Let us put $C^* = C - \{f\}$. As we have established that C is of class $\mathscr{P}_1$, we know that there exist positive constants T_0, a, b and α such that in the cooling framework $\mathscr{G}(T_0, H'(C))$ there are RKIs (resp. LKIs) Q_1 and Q_2 of class $\mathscr{D}(H'(C), a, b)$ such that

$$q(C,j)(1-e^{-\alpha/T_1})(Q_1)_k^l \leqq M(C^*, E-C)_{f,k}^{j,l} \leqq q(C,j)(1+e^{-\alpha/T_1})(Q_2)_k^l. \tag{363}$$

Moreover

$$(M(C, E-C) - M(C^*, E-C))_{i,0}^{j,n} = \sum_{k=0}^{n-1} P(X_k = f, \tau(C,0) > k \mid X_0 = i)\, M(C^*, E-C)_{f,k}^{j,n}. \tag{364}$$

There is a positive constant β such that $\{f\}$ is a concentration subset of C of class

$$\mathscr{C}(H'(C), T_0, (H(C)-H'(C))/3, \beta). \tag{365}$$

Let us put

$$\gamma = \min(\alpha, \beta, (H(C)-H'(C))), \tag{366}$$

$$N = N(H(C), T_1, -\gamma/4, 0), \tag{367}$$

and

$$R = N(H'(C), T_1, (H(C) - H'(C))/2, 0). \tag{368}$$

LEMMA 5.19. – *We have in the cooling framework* $\mathcal{G}(T_0, \underline{H}(C))$

$$\left| \frac{\sum_{k=R}^{N} \sum_{n=k+1}^{N} P(X_k = f, \tau(C, 0) > k \mid X_0 = i)\, M(C^*, E - C)_{f,k}^{j,n}}{\sum_{k=R}^{N} q(C, j) \mid F(C) \mid^{-1} \exp(-H(C)/T_k)} - 1 \right|$$
$$\leqq \exp(-\gamma/(6T_1)). \tag{369}$$

Proof of lemma 5.19. – We have

$$\sum_{k=R}^{N} \sum_{n=k+1}^{N} P(X_k = f, \tau(C, 0) > k \mid X_0 = i)\, M(C^*, E - C)_{f,k}^{j,n}$$
$$\leqq \sum_{k=R}^{N} \left(\frac{1}{\mid F(C) \mid} + e^{-\beta/T_1} \right) M(C^*, E - C)_{f,k}^{j,k \to}$$
$$\leqq \sum_{k=R}^{N} (\mid F(C) \mid^{-1} + e^{-\beta/T_1})\, q(C, j)(1 + e^{-\alpha/T_1})\, e^{-H(C)/T_k}$$
$$\leqq \sum_{k=R}^{N} \mid F(C) \mid^{-1} q(C, j)\, e^{-H(C)/T_k} (1 + \mid F(C) \mid e^{-\gamma/T_1})(1 + e^{-\gamma/T_1})$$
$$\leqq \sum_{k=R}^{N} \mid F(C) \mid^{-1} q(C, j)\, e^{-H(C)/T_k} (1 + e^{-\gamma/(2T_1)}) \tag{370}$$

for T_1 small enough.

Let us put $R = R_1$ and

$$R_2 = N(H'(C), T_1, (H(C) - H'(C))/2, 0), \tag{371}$$

we have

$$\sum_{k=R}^{N} \sum_{n=k+1}^{N} P(X_k = f, \tau(C, 0) > k \mid X_0 = i)\, M(C^*, E - C)_{f,k}^{j,n}$$
$$\geqq (\mid F(C) \mid^{-1} - e^{-\beta/T_1})\, P(\tau(C, 0) > N \mid X_0 = i)$$
$$\times \sum_{n=R_2}^{N} \sum_{k=R}^{n} M(C^*, E - C)_{f,k}^{j,n}$$
$$\geqq (\mid F(C) \mid^{-1} - e^{-\beta/T_1}) \left(1 - e^{-\alpha/T_1} - 2q(C) \sum_{l=1}^{N} e^{-H(C)/T_l} \right)$$

$$\times \sum_{n=R_2}^{N} (1-e^{-\alpha/T_1})\, q(C,j)\, e^{-H(C)/T_n}\left(1-(1+b)\prod_{l=R_1}^{R_2-1}(1-a e^{-H'(C)/T_l})\right)$$

$$\geqq \sum_{n=R_2}^{N} |F(C)|^{-1} q(C,j)$$

$$\times e^{-H(C)/T_n}(1-|F(C)|^{-1} e^{-\beta/T_1})(1-4q(C)e^{-\gamma/(4T_1)})$$

$$\times (1-e^{-\alpha/T_1})(1-(1+b)\exp(-a e^{(H(C)-H'(C))/(2T_1)})$$

$$\geqq \sum_{n=R_2}^{N} |F(C)|^{-1} q(C,j)\, e^{-H(C)/T_n}(1-e^{\gamma/(5T_1)}), \quad (372)$$

but

$$\sum_{n=R_1}^{R_2-1} q(C,j)\,|F(C)|^{-1} e^{-H(C)/T_n}$$

$$\leqq 2q(C,j)\,|F(C)|^{-1} e^{-(H(C)-H'(C))/(2T_1)} \quad (373)$$

and

$$\sum_{n=R}^{N} q(C,j)\,|F(C)|^{-1} e^{-H(C)/T_n}$$

$$\geqq (e^{-\gamma/(4T_1)} q(C,j)\,|F(C)|^{-1} - 2e^{-(H(C)-H'(C))/(2T_1)})$$

$$\geqq \frac{1}{2} e^{-\gamma/(4T_1)} q(C,j)\,|F(C)|^{-1}. \quad (374)$$

Hence

$$\frac{\displaystyle\sum_{n=R_1}^{R_2-1} q(C,j)\,|F(C)|^{-1} e^{-H(C)/T_n}}{\displaystyle\sum_{n=R_1}^{N} q(C,j)\,|F(C)|^{-1} e^{-H(C)/T_n}} \leqq 4e^{-(H(C)-H'(C))/(2T_1)+\gamma/(4T_1)}$$

$$\leqq 4e^{-(H(C)-H'(C))/(4T_1)} \leqq 4e^{-\gamma/(4T_1)}. \quad (375)$$

End of the proof of lemma 5.19.

Now we have

$$\sum_{k=1}^{R-1} \sum_{n=k+1}^{N} P(X_k=f, \tau(C,0)>k \mid X_0=i)\, M(C^*, E-C)_{f,k}^{j,n}$$

$$\leqq \sum_{k=1}^{R=1} M(C^*, E-C)_{f,k}^{j,k\rightarrow}$$

$$\leqq 2q(C,j) \sum_{k=1}^{R-1} \exp(-H(C)/T_k)$$

$$\leqq 2q(C,j)\exp(-(H(C)-H'(C))/(2T_1)) \quad (376)$$

Hence

$$\frac{\displaystyle\sum_{k=1}^{R-1}\sum_{n=k+1}^{N} P(X_k=f, \tau(C,0)>k \mid X_0=i)\, M(C^*, E-C)_{f,k}^{j,n}}{\displaystyle\sum_{k=1}^{N} \exp(-H(C)/T_k)} \leqq e^{-(H(C)-H'(C))/(4T_1)}. \quad (377)$$

We can make the same calculations for $M(C, E-C)_i^E$, substituting j with E and $q(C, j)$ with $q(C)$.

End of the proof of proposition 5.18.

LEMMA 5.20. — *For any* $i \in C$, *for any* $m \in \mathbb{N}$ *we have*

$$P(\tau(C,m) < +\infty \mid X_m = i) = 1. \quad (378)$$

Consequently

$$P(\tau(C,m) \geqq n \mid X_m = i) = M(C, E-C)_{i,m}^{E,n\,\rightarrow}. \quad (379)$$

Proof. — According to proposition 5.18

$$P(\tau(C,m) \leqq N(H(C), T_1, -\alpha, m) \mid X_m = i) \geqq \frac{q(C)}{2\,|F(C)|} e^{-\alpha/T_1}. \quad (380)$$

Let us put $(u_n)_{n\in\mathbb{N}} = V(H(C), T_1, \alpha, m)$. We have

$$\begin{aligned} P(\tau(C,m) > u_{n+1} \mid X_m = i) &= \sum_{j\in C} P(\tau(C,m) > u_n, X_{u_n} = j \mid X_m = i) \\ &\quad \times P(\tau(C,u_n) > u_{n+1} \mid X_{u_n} = j) \\ &\leqq P(\tau(C,m) > u_n \mid X_m = i)\left(1 - \frac{q(C)}{2\,|F(C)|} e^{-\alpha/T_1}\right). \end{aligned} \quad (381)$$

Hence

$$P(\tau(C,m) > u_n \mid X_m = i) \leqq \left(1 - \frac{q(C)}{2\,|F(C)|} e^{-\alpha/T_1}\right)^n. \quad (382)$$

End of the proof of lemma 5.20.

Continuation of the proof that C *is of class* $\mathcal{P}_5$.

Let T_0, α be as in proposition 5.18. Let $(u_n)_{n\in\mathbb{N}}$ be $V(H(C), T_1, \alpha)$. Let us put

$$N = N(H(C), T_1, -\alpha/2, 0). \quad (383)$$

For any $n > \mathrm{N}$ let k be defined by

$$u_k \leqq n \leqq u_{k+1}. \tag{384}$$

Writing

$$\left.\begin{aligned} &\mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{i,\,0}^{j,\,n} \\ &\quad = \sum_{j_1 \in \mathrm{C}} \mathrm{P}(\tau(\mathrm{C}, 0) > u_m, \mathrm{X}_{u_m} = j_1 \mid \mathrm{X}_0 = i)\, \mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{j_1,\,u_m}^{j,\,n}, \\ &\qquad\qquad n > u_m, \end{aligned}\right\} \tag{385}$$

we deduce from proposition 5.18 and lemma 5.20 that for any $l \leqq k+1$ we have

$$\left| \frac{\displaystyle\sum_{m=u_l+1}^{u_{l+1}} \mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{i,\,0}^{j,\,m}}{\displaystyle\sum_{m=u_l+1}^{u_{l+1}} \mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{i,\,0}^{\mathrm{E},\,u_l \rightarrow} q(\mathrm{C}, j) \mid \mathrm{F}(\mathrm{C}) \mid^{-1} \exp(-\mathrm{H}(\mathrm{C})/\mathrm{T}_m)} - 1 \right| \leqq \exp(-\alpha/\mathrm{T}_1). \tag{386}$$

Moreover

$$\left| \frac{\mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{i,\,0}^{\mathrm{E},\,u_l \rightarrow}}{\mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{i,\,0}^{\mathrm{E},\,m \rightarrow}} - 1 \right| \leqq e^{-\alpha/\mathrm{T}_1}, \qquad u_l < m \leqq u_{l+1}. \tag{387}$$

Hence

$$\left| \frac{\displaystyle\sum_{m=1}^{u_k} \mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{i,\,0}^{j,\,m}}{\displaystyle\sum_{m=1}^{u_k} \mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{i,\,0}^{\mathrm{E},\,m \rightarrow} q(\mathrm{C}, j) \mid \mathrm{F}(\mathrm{C}) \mid^{-1} \exp(-\mathrm{H}(\mathrm{C})/\mathrm{T}_m)} - 1 \right| \leqq \exp(-\alpha/\mathrm{T}_1), \tag{388}$$

and

$$\frac{\displaystyle\sum_{m=u_k+1}^{u_{k+1}} \mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{i,\,0}^{\mathrm{E},\,m \rightarrow} \exp(-\mathrm{H}(\mathrm{C})/\mathrm{T}_m)}{\displaystyle\sum_{m=1}^{u_k} \mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{i,\,0}^{\mathrm{E},\,m \rightarrow} \exp(-\mathrm{H}(\mathrm{C})/\mathrm{T}_m)} \leqq 2 \exp(-\alpha/(2\,\mathrm{T}_1)). \tag{389}$$

We deduce from this that

$$\left| \frac{\sum_{m=1}^{n} \mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{i,0}^{j,m}}{\sum_{m=1}^{n} \mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{i,0}^{\mathrm{E},m\to} q(\mathrm{C}, j) \,|\, \mathrm{F}(\mathrm{C}) \,|^{-1} \exp(-\mathrm{H}(\mathrm{C})/\mathrm{T}_m)} - 1 \right| \leqq 3 \exp(-\alpha/(2\,\mathrm{T}_1)). \quad (390)$$

In the same way we have

$$\left| \frac{\sum_{m=1}^{n} \mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{i,0}^{\mathrm{E},m}}{\sum_{m=1}^{n} \mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{i,0}^{\mathrm{E},m\to} q(\mathrm{C}) \,|\, \mathrm{F}(\mathrm{C}) \,|^{-1} \exp(-\mathrm{H}(\mathrm{C})/\mathrm{T}_m)} - 1 \right| = 3 \exp(-\alpha/(2\,\mathrm{T}_1)). \quad (391)$$

Hence

$$\left| \frac{\mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{i,0}^{j,m\to}}{\mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{i,0}^{\mathrm{E},m\to}} - \frac{q(\mathrm{C}, j)}{q(\mathrm{C})} \right| \leqq 9 \exp(-\alpha/(2\,\mathrm{T}_1)), \quad (392)$$

hence

$$\left| \frac{\sum_{m=1}^{n} \mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{i,0}^{j,m}}{\sum_{m=1}^{n} \mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{i,0}^{j,m\to} q(\mathrm{C}) \,|\, \mathrm{F}(\mathrm{C}) \,|^{-1} \exp(-\mathrm{H}(\mathrm{C})/\mathrm{T}_k)} - 1 \right| \leqq 15 \exp(-\alpha/(2\,\mathrm{T}_1)). \quad (393)$$

(The number 15 is of course somehow arbitrary.)

End of the proof that C *is of class* $\mathscr{P}_5$.

Proof that C *is of class* $\mathscr{P}_2$.

LEMMA 5.21. — *There are positive constants* T_0 *and* α *such that in the cooling framework* $\mathscr{G}(\mathrm{T}_0, \mathrm{H}(\mathrm{C}))$ *we have for any* $i \in \mathrm{C}$, *putting* $\varepsilon = e^{-\alpha/\mathrm{T}_1}$,

$$(1-\varepsilon) \prod_{l=m+1}^{n-1} (1-(1+\varepsilon) \frac{q(\mathrm{C})}{|\mathrm{F}(\mathrm{C})|} \exp(-\mathrm{H}(\mathrm{C})/\mathrm{T}_l)) \leqq \mathrm{P}(\tau(\mathrm{C}, m) \geqq n \,|\, \mathrm{X}_m = i) \leqq (1+\varepsilon) \prod_{l=m+1}^{n-1} \left(1-(1-\varepsilon) \frac{q(\mathrm{C})}{|\mathrm{F}(\mathrm{C})|} \exp(-\mathrm{H}(\mathrm{C})/\mathrm{T}_l)\right), \quad (394)$$

Proof of lemma 5.21. – Let us put $(u_n)_{n\in\mathbb{N}}=\mathrm{V}(\mathrm{H}(\mathrm{C}),\mathrm{T}_1,\alpha,m)$. We have

$$\mathrm{P}(\tau(\mathrm{C},m)\geqq n\,|\,\mathrm{X}_m=i)=\mathrm{M}(\mathrm{C},\mathrm{E}-\mathrm{C})_{i,m}^{\mathrm{E},n\to}. \tag{395}$$

Hence we deduce from equation (386) that

$$\left|\frac{\mathrm{P}(\tau(\mathrm{C},m)\geqq u_l\,|\,\mathrm{X}_m=i)-\mathrm{P}(\tau(\mathrm{C},m)\geqq u_{l+1}\,|\,\mathrm{X}_m=i)}{\mathrm{P}(\tau(\mathrm{C},m)\geqq u_l\,|\,\mathrm{X}_m=i)(q(\mathrm{C})/|\mathrm{F}(\mathrm{C})|)\sum_{k=u_l+1}^{u_{l+1}}e^{-\mathrm{H}(\mathrm{D})/\mathrm{T}_k}}-1\right| \leqq e^{-\alpha/\mathrm{T}_1}. \tag{396}$$

It is an elementary calculation to deduce from equation (396) that

$$\begin{aligned}\prod_{k=1}^{l}\left(1-(1+e^{-\alpha/\mathrm{T}_1})\sum_{k=u_l+1}^{u_{l+1}}\frac{q(\mathrm{C})}{|\mathrm{F}(\mathrm{C})|}e^{-\mathrm{H}(\mathrm{C})/\mathrm{T}_k}\right)&\\ \leqq \mathrm{P}(\tau(\mathrm{C},m)\geqq u_l\,|\,\mathrm{X}_m=i)&\\ \leqq \prod_{k=1}^{l}\left(1-(1-e^{-\alpha/\mathrm{T}_1})\sum_{k=u_l+1}^{u_{l+1}}\frac{q(\mathrm{C})}{|\mathrm{F}(\mathrm{C})|}e^{-\mathrm{H}(\mathrm{C})/\mathrm{T}_k}\right).&\end{aligned} \tag{397}$$

Let us notice that for $x_1,\ldots,x_n\in[0,1]$ such that $\sum_{l=1}^{n}x_l\leqq 1/2$ we have

$$\begin{aligned}(1-\sum_l x_l)&=\exp\{\ln(1-\sum_l x_l)\}\\ &\geqq \exp\{-\sum_l x_l(1+\sum_l x_l)\}\geqq\prod_{l=1}^{n}(1-(1+\sum_l x_l)x_l)\end{aligned} \tag{398}$$

and

$$(1-\sum_l x_l)\leqq 1-\sum_l x_l\prod_{k=1}^{l-1}(1-x_k)=\prod_{l=1}^{n}(1-x_l). \tag{399}$$

From equation (397), (398) and (399), we deduce that there are positive constants T_0 and α_1 such that in the cooling framework $\mathscr{G}(\mathrm{T}_0,\mathrm{H}(\mathrm{C}))$, we have for any $i\in\mathrm{C}$:

$$\begin{aligned}\prod_{k=m+1}^{u_l}\left(1-(1+e^{-\alpha_1/\mathrm{T}_1})\frac{q(\mathrm{C})}{|\mathrm{F}(\mathrm{C})|}e^{-\mathrm{H}(\mathrm{C})/\mathrm{T}_k}\right)&\\ \leqq \mathrm{P}(\tau(\mathrm{C},m)>u_l\,|\,\mathrm{X}_m=i)&\\ \leqq\prod_{k=m+1}^{u_l}\left(1-(1-e^{-\alpha_1/\mathrm{T}_1})\frac{q(\mathrm{C})}{|\mathrm{F}(\mathrm{C})|}e^{-\mathrm{H}(\mathrm{C})/\mathrm{T}_k}\right).&\end{aligned} \tag{400}$$

Let us consider now $n \geqq m$ and let l be such that

$$u_l < n \leqq u_{l+1}; \tag{401}$$

we have

$$P(\tau(C,m) > u_{l+1} \mid X_m = i) \leqq P(\tau(C,m) > n \mid X_m = i) \leqq P(\tau(C,m) > u_l \mid X_m = i), \tag{402}$$

hence ther are positive constants T_0 and α_2 such that in $\mathcal{G}(T_0, H(C))$ we have

$$\begin{aligned}
(1-e^{-\alpha_2/T_1}) \prod_{k=m+1}^{n-1} \left(1-(1+e^{-\alpha_2/T_1}) \frac{q(C)}{|F(C)|} e^{-H(C)/T_k}\right) \\
\leqq \prod_{k=m+1}^{u_{l+1}} \left(1-(1+e^{-\alpha_1/T_1}) \frac{q(C)}{|F(C)|} e^{-H(C)/T_k}\right) \\
\leqq P(\tau(C,m) > n \mid X_m = i) \\
\leqq \prod_{k=m+1}^{u_l} \left(1-(1-e^{-\alpha_1/T_1}) \frac{q(C)}{|F(C)|} e^{-H(C)/T_k}\right) \\
\leqq (1+e^{-\alpha_2/T_1}) \prod_{k=m+1}^{n-1} \left(1-(1-e^{-\alpha_2/T_1}) \frac{q(C)}{|F(C)|} e^{-H(C)/T_k}\right).
\end{aligned} \tag{403}$$

End of the proof of lemma 5.21.

Continuation of the proof that C *is of class* $\mathcal{P}_2$.

We have, for any $f \in F(C)$, putting $C^* = C - \{f\}$,

$$M(C, E-C)_{i,m}^{j,n} = M(C^*, E-C)_{i,m}^{j,n} + \sum_{k=m}^{n-1} P(X_k = f, \tau(C,m) > k \mid X_m = i) \times M(C^*, E-C)_{f,k}^{j,n}. \tag{404}$$

Let us put

$$G_m^n = P(X_k = f, \tau(C,m) > k \mid X_m = i) \exp\left(-\frac{H(C)}{T_{k+1}}\right) \tag{405}$$

and

$$R_m^n = \exp(H(C)/T_{k+1}) M(C^*, E-C)_{f,k}^{j,n}. \tag{406}$$

We will need the following lemma:

LEMMA 5.22. — *There are positive constants* T_0 *and* α_1 *such that in the cooling framework* $\mathcal{G}(T_0, H(C))$, *the increasing kernel* G *is of class*

$\mathscr{E}^r(0, 1/q(C), H(C), e^{-\alpha_1/T_1})$, *and the* $KI\exp(-H(C)/T_{m+1})G_m^n$ *is of class* $\mathscr{E}^l(H(C), 1/q(C), H(C), e^{-\alpha_1})$.

Proof. – Let us put

$$\tilde{G}_m^n = \frac{1}{|F(C)|} P(\tau(C, m) > n \mid X_m = i) \exp(-H(C)/T_{n+1}). \tag{407}$$

It is easy to deduce from lemma 5.21 that $\tilde{G}$ is of class $\mathscr{E}^r(0, 1/q(C), e^{-\alpha/T_1})$ for some positive α in a suitable framework $\mathscr{G}(T_0, H(C))$ and that in the same framework $\exp(-H(C)/T_{m+1})\tilde{G}_m^n$ is of class $\mathscr{E}^l(H(C), 1/q(C), H(C), e^{-\alpha/T_1})$.

Let us notice now that there are positive constants T_0 and α_2 such that f is concentration set of class $\mathscr{O}\left(H'(C), T_1, \frac{H(C)-H'(C)}{2}, \alpha\right)$. Hence, putting

$$R = N\left(H'(C), \frac{H(C)-H'(C)}{2}, m\right) \tag{408}$$

we have for any m and any $k \geqq R$

$$\left|\frac{G_m^n}{\tilde{G}_m^n} - 1\right| \leqq e^{-\alpha_2/T_1}. \tag{409}$$

Moreover

$$\sum_{k=m}^{R-1} G_m^k, \sum_{k=m}^{R-1} \tilde{G}_m^k \leqq \sum_{k=m+1}^{R} \exp\left(-\frac{H(C)}{T_k}\right) \leqq \exp\left(-\frac{H(C)-H'(C)}{2T_1}\right). \tag{410}$$

Hence, putting $\alpha = (H(C)-H'(C))/2$, there are positive constants α_1, α_2, T_0, such that

$$\begin{aligned} G_m^{n\to} &\leqq \sum_{k=R}^{+\infty} \tilde{G}_m^k (1 + e^{-\alpha_1/T_1}) + e^{-\alpha/T_1} \\ &\leqq \frac{1}{q(C)}(1 + e^{-\alpha_1/T_1}) + e^{-\alpha/T_1} \\ &\leqq \frac{1}{q(C)}(1 + e^{-\alpha_2/T_1}). \end{aligned} \tag{411}$$

For $n \leqq R$

$$G_m^n \leqq \frac{1}{q(C)}(1 + e^{-\alpha_2/T_1}) \prod_{k=m+1}^{n-1} (1 - a e^{-H(C)/T_k}) \tag{412}$$

and for $n \geqq \mathrm{R}$

$$G_m^n \leqq (1+e^{-\alpha_3/T_1})\,\tilde{G}_m^{n\rightarrow} \leqq (1+e^{-\alpha_4/T_1}) \prod_{k=m+1}^{n-1} (1-a\,e^{-H(C)/T_k}). \quad (413)$$

In the same way

$$\begin{aligned} G_m^{n\rightarrow} &\geqq \sum_{k=R}^{+\infty} \tilde{G}_m^k (1-e^{-\alpha/T_1}) \\ &\geqq \sum_{k=m}^{+\infty} \tilde{G}_m^k (1-e^{-\alpha_1/T_1}) - e^{-\alpha_1/T_1}(1-e^{-\alpha_1/T_1}) \\ &\geqq \frac{1}{q(C)}(1-e^{-\alpha_2/T_1})(1-e^{-\alpha_1/T_1}) - e^{-\alpha_1/T_1}(1-e^{-\alpha_1/T_1}) \\ &\geqq \frac{1}{q(C)}(1-e^{-\alpha_3/T_1}). \quad (414) \end{aligned}$$

From equations (411), (412), (413) and (414) we deduce that G_m^n is of class $\mathscr{E}^r(0, 1/q(C), H(C), \alpha)$ for some $\alpha > 0$ in a suitable cooling framework $\mathscr{G}(T_0, H(C))$.

Let $n \in \mathbb{N}$ be fixed and define

$$R = \sup\left\{ m < n \,\middle|\, \sum_{k=m}^{n-1} e^{-H'(C)/T_k} \geqq e^{-(H(C)-H')/(2\,T_1)} \right\}, \quad (415)$$

we have

$$\begin{aligned} \sum_{m=-\infty}^{n} G_{\leftarrow n}^n &\leqq e^{-\alpha/T_1} + \sum_{k=-\infty}^{n} \tilde{G}_k^n (1+e^{-\alpha_1/T_1}) \\ &\leqq e^{-\alpha/T_1} + q(C)^{-1}(1+e^{-\alpha_2/T_1})(1-e^{-\alpha_1/T_1}) \\ &\leqq q(C)^{-1}(1+e^{-\alpha_3/T_1}) \quad (416) \end{aligned}$$

and

$$\begin{aligned} &\sum_{m=-\infty}^{n} e^{-H(C)/T_{m+1}}\, G_m^n\, e^{H(C)/T_n} \\ &\qquad \geqq \sum_{k=-\infty}^{R} \tilde{G}_k^n (1-e^{-\alpha_1/T_1})\, e^{H(C)/T_{k+1}}\, e^{H(C)/T_n} \\ &\qquad \geqq q(C)^{-1}(1-e^{-\alpha_4/T_1})\, e^{H(C)/T_n}. \quad (417) \end{aligned}$$

For $m \geqq R$ we have the equivalent of equation (412):

$$G_{\leftarrow m}^n \leqq q(C)^{-1}(1+e^{-\alpha_2/T_1}) \prod_{k=m+1}^{n-1} (1-a\,e^{-H(C)/T_k}), \quad (418)$$

for $m<\mathrm{R}$

$$\mathrm{G}^n_{\leftarrow m} \leqq \tilde{\mathrm{G}}^n_{\leftarrow m}(1+e^{-\alpha_3/\mathrm{T}_1}) \leqq (1+e^{-\alpha_4/\mathrm{T}_1}) \prod_{k=m+1}^{n-1} (1-a\,e^{-\mathrm{H}(\mathrm{C})/\mathrm{T}_k}). \quad (419)$$

Hence there are positive constants T_0, α such that in $\mathscr{G}(\mathrm{T}_0, \mathrm{H}(\mathrm{C}))$ the kernel $\exp(\mathrm{H}(\mathrm{C})/\mathrm{T}_{m+1})\,\mathrm{G}^n_m$ is of class $\mathscr{E}^l(\mathrm{H}(\mathrm{C}), q(\mathrm{C})^{-1}, \mathrm{H}(\mathrm{C}), \alpha)$.

End of the proof of lemma 5.22.

We deduce from the composition lemmas that there are positive constants T_0 and b such that in the cooling framework $\mathscr{G}(\mathrm{T}_0, \mathrm{H}(\mathrm{C}))$ the kernel $\mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})^{j,\,n}_{i,\,m}$ is of class

$$\mathscr{E}^r(0, q(\mathrm{C}, j)/q(\mathrm{C}), \mathrm{H}(\mathrm{C}), b),$$

the kernel $\mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})^{\mathrm{E},\,n}_{i,\,m}$ is of class $\mathscr{E}^r(0, 1, \mathrm{H}(\mathrm{C}), b)$, the kernel

$$\mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})^{j,\,n}_{i,\,m}\exp(-\mathrm{H}(\mathrm{C})/\mathrm{T}_{m+1})$$

is of class

$$\mathscr{E}^l(\mathrm{H}(\mathrm{C}), q(\mathrm{C}, j)/q(\mathrm{C}), \mathrm{H}(\mathrm{C}), b)$$

and the kernel $\mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})^{\mathrm{E},\,n}_{i,\,m}\exp(-\mathrm{H}(\mathrm{C})/\mathrm{T}_{m+1})$ is of class $\mathscr{E}^l(\mathrm{H}(\mathrm{C}), 1, \mathrm{H}(\mathrm{C}), b)$.

Moreover there are positive constants T_0, a, b, d and α such that in $\mathscr{G}(\mathrm{T}_0, \mathrm{H}(\mathrm{C}))$

$$\begin{aligned}\mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})^{\mathrm{E},\,n\,\to}_{i,\,m} &\leqq \mathrm{M}(\mathrm{C}^*, \mathrm{E}-\mathrm{C})^{\mathrm{E},\,n\,\to}_{i,\,m} + \{\mathrm{GR}\}^{n\,\to}_m \\ &\leqq (1+b)\prod_{k=m+1}^{n-1}(1-a\,e^{-\mathrm{H}'(\mathrm{C})/\mathrm{T}_k}) \\ &\quad + \{\mathrm{Z}(\mathrm{H}(\mathrm{C}), a, e^{-\alpha/\mathrm{T}_1})\,\mathrm{Z}(\mathrm{H}'(\mathrm{C}), a, b)\}^{n\,\to}_m. \quad (420)\end{aligned}$$

From lemma 6.6 and lemma 6.5 we deduce that there are positive constants T_0 and α such that in $\mathscr{G}(\mathrm{T}_0, \mathrm{H}(\mathrm{C}))$ the kernel $\mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})^{\mathrm{E},\,n}_{i,\,m}$ is of class $\mathscr{D}(\mathrm{H}(\mathrm{C}), a/3, e^{-\alpha/\mathrm{T}_1})$. (*Remark:* $a/3$ is of course not sharp.) Hence $\mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})^{j,\,n}_{i,\,m}$, $j\in\mathrm{B}(\mathrm{C})$, is of class $\mathscr{E}^r(0, q(\mathrm{C}, j)/q(\mathrm{C}), \mathrm{H}(\mathrm{C}), \alpha)$.

Let us examine now the case of left classes. We have

$$\begin{aligned}\mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})^{j,\,n}_{i,\,m}&\exp(-\mathrm{H}(\mathrm{C})/\mathrm{T}_{m+1}) \\ &= \exp(-\mathrm{H}(\mathrm{C})/\mathrm{T}_{m+1})\,\mathrm{M}(\mathrm{C}^*, \mathrm{E}-\mathrm{C})^{j,\,n}_{i,\,m} \\ &\quad + \sum_k (e^{-\mathrm{H}(\mathrm{C})/\mathrm{T}_{m+1}}\mathrm{G}^k_m)\,\mathrm{R}^n_k \quad (421)\end{aligned}$$

There are positive constants T_0, α, a, b and γ such that in $\mathscr{G}(\mathrm{T}_0, \mathrm{H}(\mathrm{C}))$ there are LKIs Q_1, Q_2, Q_3 and Q_4 of class $\mathscr{D}(\mathrm{H}'(\mathrm{C}), a, b)$ and Q_5, Q_6 of class $\mathscr{D}(\mathrm{H}(\mathrm{C}), a, e^{-\alpha/\mathrm{T}_1})$ such that

$$\begin{aligned}q(\mathrm{C}, j)(1-e^{-\alpha/\mathrm{T}_1})\,\mathrm{Q}_1\,&e^{-\mathrm{H}(\mathrm{C})/\mathrm{T}_n} \\ &\leqq e^{\mathrm{H}(\mathrm{C})/\mathrm{T}_{m+1}}\,\mathrm{M}(\mathrm{C}^*, \mathrm{E}-\mathrm{C})^{j,\,n}_{i,\,m} \\ &\leqq q(\mathrm{C}, j)(1+e^{-\alpha/\mathrm{T}_1})\,\mathrm{Q}_2\,e^{-\mathrm{H}(\mathrm{C})/\mathrm{T}_{m+1}}, \quad (422)\end{aligned}$$

$$q(\mathrm{C})^{-1}(1-e^{-\alpha/\mathrm{T}_1})\,\mathrm{Q}_5\,e^{-\mathrm{H}(\mathrm{C})/\mathrm{T}_k} \leqq e^{-\mathrm{H}(\mathrm{C})/\mathrm{T}_{m+1}}\,\mathrm{G}_m^k$$
$$\leqq q(\mathrm{C})^{-1}(1+e^{-\alpha/\mathrm{T}_1})\,\mathrm{Q}_6\,e^{-\mathrm{H}(\mathrm{C})/\mathrm{T}_{m+1}}, \quad (423)$$
$$q(\mathrm{C}, j)(1-e^{-\alpha/\mathrm{T}_1})\,\mathrm{Q}_3 \leqq \mathrm{R}_k^n \leqq q(\mathrm{C}, j)(1+e^{-\alpha/\mathrm{T}_1})\,\mathrm{Q}_4. \quad (424)$$

Hence there is $\alpha_1 > 0$ such that

$$\frac{q(\mathrm{C}, j)}{q(\mathrm{C})}(1-e^{-\alpha/\mathrm{T}_1})\,e^{-\mathrm{H}(\mathrm{C})/\mathrm{T}_n}\left\{\mathrm{Q}_5\,\mathrm{Q}_3\right\}_m^n$$
$$\leqq e^{-\mathrm{H}(\mathrm{C})/\mathrm{T}_{m+1}}\left\{\mathrm{GR}\right\}_m^n$$
$$\leqq \frac{q(\mathrm{C}, j)}{q(\mathrm{C})}(1+e^{-\alpha/\mathrm{T}_1})\,e^{-\mathrm{H}(\mathrm{C})/\mathrm{T}_{m+1}}\left\{\mathrm{Q}_6\,\mathrm{Q}_4\right\}_m^n. \quad (425)$$

According to lemma 6.6 $\mathrm{Q}_5\,\mathrm{Q}_3$ and $\mathrm{Q}_6\,\mathrm{Q}_4$ are of class $\mathscr{D}_l(\mathrm{H}(\mathrm{C}), a, e^{-\alpha/\mathrm{T}_1})$ for suitable positive constants a and α, hence we deduce from equation (421) and lemma 6.10 that

$$\mathrm{M}(\mathrm{C}, \mathrm{E}-\mathrm{C})_{i,\,m}^{j,\,n}\,e^{-\mathrm{H}(\mathrm{C})/\mathrm{T}_{m+1}}$$

is of class $\mathscr{E}^l(\mathrm{H}(\mathrm{C}), q(\mathrm{C}, j)/q(\mathrm{C}), \mathrm{H}(\mathrm{C}), e^{-\alpha/\mathrm{T}_1})$, for suitable $\alpha > 0$ in a suitable framework of type $\mathscr{G}(\mathrm{T}_0, \mathrm{H}(\mathrm{C}))$.

End of the proof that C *is of class* $\mathscr{P}_2$.

End of the proof of theorem 2.25.

CONCLUSION

The proofs given in this paper are quite elaborate. Anyhow we feel that the results were worth spending some efforts. The underlying ideas are simple and proposition 4.5 gives a clear understanding of the behaviour of the system at low temperatures.

The computations of precise multiplicative constants, involving the flow of the communication kernel q through the boundary of subsets of the states space E, is a first step towards a study of the influence of the size of E on the rate of convergence.

If we had not asked for so much precision on the constants, we could have made shorter proofs. This will be done in a forthcoming paper entitled "Rough Large Deviation Estimates" [3].

Applications of the present "Sharp Estimates" will also be given in a forthcoming paper.

6. APPENDIX

6.1. Comparison lemmas

We need some comparison lemmas concerning the tails $Q_m^{n\rightarrow}$ and $Q_{\leftarrow m}^{n}$ of KIs. Here they are:

LEMMA 6.1. — *For any couple of RKIs* Q_1 *and* Q_2 *such that*

$$(Q_1)_m^{n\rightarrow} \leqq (Q_2)_m^{n\rightarrow}, \qquad m \leqq n, \tag{426}$$

and any non-decreasing bounded function $f : \mathbb{Z} \rightarrow \mathbb{R}$, *for any* $m \in \mathbb{Z}$, *we have:*

$$\sum_n (Q_1)_m^n f(n) \leqq \sum_n (Q_2)_m^n f(n). \tag{427}$$

For any couple of LKIs Q_1 *and* Q_2 *such that*

$$(Q_1)_{\leftarrow m}^{n} \leqq (Q_2)_{\leftarrow m}^{n}, \qquad m \leqq n, \tag{428}$$

and any non-increasing bounded function $f : \mathbb{Z} \rightarrow \mathbb{R}$, *for any* $n \in \mathbb{Z}$, *we have:*

$$\sum_m (Q_1)_m^n f(m) \leqq \sum_m (Q_2)_m^n f(m). \tag{429}$$

LEMMA 6.2. — *For any RKI* Q *such that, for any fixed* n, $m \mapsto Q_m^{n\rightarrow}$ *is non-decreasing, for any non-decreasing bounded function* $f : \mathbb{Z} \rightarrow \mathbb{R}$,

$$m \mapsto \sum_n Q_m^n f(n) \tag{430}$$

is a bounded non-decreasing function.

For any LKI Q *such that, for any fixed* m, $n \mapsto Q_{\leftarrow m}^{n}$ *is non-increasing, for any non-increasing bounded function* $f : \mathbb{Z} \rightarrow \mathbb{R}$,

$$n \mapsto \sum_m Q_m^n f(m) \tag{431}$$

is a bounded non-increasing function.

Proof of lemmas 6.1 *and* 6.2. — The proofs are given for RKIs, the case of LKIs is left to reader. The method is integration by parts. Moreover lemma 6.2 is a consequence of lemma 6.1. To prove lemma 6.1 we write:

$$\sum_n (Q_1)^n_m f(n) = \sum_n \{ (Q_1)^{n-1\rightarrow}_m - (Q_1)^{n\rightarrow}_m \} f(n)$$
$$= f(m) + \sum_n (Q_1)^{n\rightarrow}_m (f(n+1) - f(n))$$
$$\leqq f(m) + \sum_n (Q_2)^{n\rightarrow}_m (f(n+1) - f(n))$$
$$= \sum_n (Q_2)^n_m f(n). \quad (432)$$

We can see that lemma 6.2 is a consequence of lemma 6.1 by putting $Q_1 = Q$ and $(Q_2)^n_m = Q^n_{m+1}$.

End of the proof of lemmas 6.1 *and* 6.2.

Let us introduce now a useful notation.

DEFINITION 6.3. — *Let* S, *a*, *b be positive constants. Let* T_0 *be such that* $a \exp(-S/T_0) < 1$. *In the simple cooling framework* $\mathscr{F}(T_0)$, *we define the maximal kernel of class* $\mathscr{D}^r(S, a, b)$ [resp. $\mathscr{D}^l(s, a, b)$] *to be the RKI* [*resp. LKI*] $Z(S, a, b)$ *characterized by*

$$K^n_m = \left[(1+b) \prod_{k=m+1}^{n-1} (1 - a e^{-S/T_k}) \right] \wedge 1 \quad (433)$$

and

$$Z_r(S, a, b) = K^n_m - K^{n+1}_m \quad (434)$$

[*resp.*

$$Z_l(S, a, b) = K^n_m - K^n_{m-1}]. \quad (435)$$

Let us notice that

$$Z_r(S, a, b)^{n\rightarrow}_m = K^n_m \quad (436)$$

[*resp.*

$$Z_l(S, a, b)^n_{\leftarrow m} = K^n_m]. \quad (437)$$

We draw the following conclusion from lemma 6.1:

LEMMA 6.4. — *Let* $\mathscr{F}$ *be a cooling framework, and let* S, *a*, *b be positive constants. Let* $Q_1, Q_2, \ldots, Q_s$ *be RKIs* [*resp. LKIs*] *of class* $\mathscr{D}(S, a, b)$. *We have*

$$\{Q_1 Q_2 \ldots Q_s\}^{n\rightarrow}_m \leqq \{Z_r(S, a, b)^s\}^{n\rightarrow}_m, \quad (438)$$

[*resp.*

$$\{Q_1 Q_2 \ldots Q_s\}^n_{\leftarrow m} \leqq \{Z_l(S, a, b)^r\}^n_{\leftarrow m}] \quad (439)$$

where s is a power in $Z(S, a, b)^s$.

Proof. — We can assume by induction that the lemma is true for $s-1$ kernels (it is trivial for one kernel). Then we have

$$\begin{aligned}\{Q_1 \dots Q_s\}_m^{n\rightarrow} &= \sum_k \{Q_1 \dots Q_{s-1}\}_m^k (Q_s)_k^{n\rightarrow} \\ &\leqq \sum_k \{Q_1 \dots Q_{s-1}\}_m^k Z(S, a, b)_k^{n\rightarrow} \\ &\leqq \sum_k \{Z(S, a, b)^{s-1}\}_m^k Z(S, a, b)_k^{n\rightarrow} = \{Z(S, a, b)^s\}_m^{n\rightarrow} \end{aligned} \tag{440}$$

[resp.

$$\begin{aligned}\{Q_1 \dots Q_s\}_{\leftarrow m}^{n} &= \sum_k (Q_1)_{\leftarrow m}^k \{Q_2 \dots Q_s\}_k^n \\ &\leqq \sum_k Z(S, a, b)_{\leftarrow m}^k \{Q_2 \dots Q_s\}_k^n \\ &\leqq \sum_k Z(S, a, b)_{\leftarrow m}^k \{Z(S, a, b)^{s-1}\}_k^n = \{Z(S, a, b)^s\}_{\leftarrow m}^n]. \end{aligned} \tag{441}$$

The second inequality is a consequence of lemma 6.1 since

$$k \mapsto Z(S, a, b)_k^{n\rightarrow} \tag{442}$$

is non-decreasing [resp.

$$k \mapsto Z(S, a, b)_{\leftarrow m}^k \tag{443}$$

is non-increasing].

End of the proof of lemma 6.4.

LEMMA 6.5. — *Let* $(E, U, q, \mathscr{L}_0, \mathscr{G}(\tilde{T}_0, D), \mathscr{X})$ *be an annealing framework, let* D, *a*, *b be positive constants and let* Q_m^n *be a RKI* [*resp. LKI*] *of class* $\mathscr{D}_r(D, a, b)$ [*resp.* $\mathscr{D}_l(D, a, b)$]. *Let* D′ *a positive constant such that* $D' > D$. *There are positive constants* α, T_0 *such that in* $\mathscr{G}(T_0, D')$ *the kernel* Q *is of class* $\mathscr{D}_r(D', a, e^{-\alpha/T_1})$ [*resp.* $\mathscr{D}_l(D', a, e^{-\alpha/T_1})$].

Proof. — *Case of a RKI:*

$$Q_m^{n\rightarrow} \leqq \left((1+b) \prod_{k=m+1}^{n-1} (1 - a^{-D/T_k}) \wedge 1 \right). \tag{444}$$

Let us put $R = N\left(D, T_1, \dfrac{D'-D}{2}, m\right)$. We have

$$\sum_{k=m+1}^{R-1} e^{-D'/T_k} \leqq \exp\left(-\frac{D'-D}{2T_1}\right) \tag{445}$$

and

$$(1+b)\prod_{k=m+1}^{R-1}(1-a\,e^{-D/T_k})\leqq(1+b)\exp\left(-a\,e^{(D'-D)/(3\,T_1)}\right)$$
$$\leqq 1 \tag{446}$$

in a suitable framework $\mathcal{G}(T_0, D')$.

Let us put $\alpha=\dfrac{D'-D}{4T_1}$, then for $n\leqq R-1$,

$$(1+e^{-\alpha/T_1})\prod_{k=m+1}^{n-1}(1-a\,e^{-D'/T_k})$$
$$\geqq 1\geqq\left((1+b)\prod_{k=m+1}^{n-1}(1-a\,e^{-D/T_k})\wedge 1\right), \tag{447}$$

and for $n\geqq R$

$$(1+b)\prod_{k=m+1}^{n-1}(1-a\,e^{-D/T_k})$$
$$=\left((1+b)\prod_{k=m+1}^{R-1}(1-a\,e^{-D/T_k})\right)\prod_{k=R}^{n-1}(1-a\,e^{-D/T_k})$$
$$\leqq(1+e^{-\alpha/T_1})\prod_{k=m+1}^{R-1}(1-a\,e^{-D'/T_k})\prod_{k=R}^{n-1}(1-a\,e^{-D'/T_k}). \tag{448}$$

Case of a LKI:

$$Q^n_{\leftarrow m}\leqq\left((1+b)\prod_{k=m+1}^{n-1}(1-a\,e^{-D/T_k})\wedge 1\right) \tag{449}$$

Let us put

$$R=\sup\left\{k<n\,\Big|\,\sum_{l=k}^{n-1}\exp-D/T_l\geqq e^{(D'-D)/(2T_1)}\right\}, \tag{450}$$

then

$$\sum_{k=R+1}^{n-1}e^{-D'/T_k}\leqq e^{-(D'-D)/2T_1} \tag{451}$$

and

$$(1+b)\prod_{k=R+1}^{n-1}(1-a\,e^{-D/T_k})\leqq(1+b)\exp\left(-a\exp\left(\frac{D'-D}{3T_1}\right)\right)\leqq 1 \tag{452}$$

in a suitable framework $\mathcal{G}(T_0, D')$.

The end of the proof is just a transposition of the case of a RKI and is left to the reader.

End of the proof of lemma 6.5.

6.2. Composition lemmas

Now we give some lemmas about the composition of KIs:

LEMMA 6.6. – *For any positive constants* T_0, H, H′, a, b, α *such that* $H' < H$, *for any annealing framework of type* $(E, U, q, \mathcal{L}_0, \mathcal{G}(T_0, H), \mathcal{X})$, *for any increasing KIs* Q, S *such that* Q *is of class* $\mathcal{D}_r(H, a, e^{-\alpha/T_1})$ [*resp.* $\mathcal{D}_l(H, a, e^{-\alpha/T_1})$] *and* S *is of class* $\mathcal{D}_r(H', a, b)$ [*resp.* $\mathcal{D}_l(H', a, b)$] *there are positive constants* $\tilde{T}_0$, a_1, α_1 *such that in* $\mathcal{G}(\tilde{T}_0, H)$ *the kernel* QS *is of class* $\mathcal{D}(H, a_1, e^{-\alpha_1/T_1})$.

Proof. – *Case of RKIs:*
Putting

$$R = \sup\left\{ r < n \,\middle|\, \sum_{k=r+1}^{n-1} e^{-H'/T_k} \geqq e^{(H-H')/2T_1} \right\} \tag{453}$$

we have

$$\{ Z_r(H, a, e^{-\alpha/T_1}) Z_r(H', a, b) \}_m^{n\rightarrow} \leqq Z(H, a, e^{-\alpha/T_1})_m^{R\rightarrow} + \sum_{k=m+1}^{k-1} Z(H, a, e^{-\alpha/T_1})_m^k Z(H', a, b)_k^{n\rightarrow} \tag{454}$$

and

$$\begin{aligned} \sum_{k=m+1}^{R-1} & Z(H, a, e^{-\alpha/T_1})_m^k Z(H', a, b)_k^{n\rightarrow} \\ & \leqq (1+b) \prod_{k=R+1}^{n-1} (1 - a e^{-H'/T_k}) \\ & \quad \times \left(a \sum_{k=m+1}^{R-1} e^{-H/T_k} \right) \prod_{k=m+1}^{R-1} (1 - a e^{-H/T_k}) \\ & \leqq (1+b) \exp(-a e^{(H-H')/(2T_1)}) \exp\left(-\frac{a}{2} \sum_{k=m+1}^{R_1} e^{-H/T_k} \right) \end{aligned}$$

$$\leqq (1+b)\exp(-a\,e^{(\mathrm{H}-\mathrm{H}')/(2\mathrm{T}_1)})\exp\left(-\frac{a}{2}\sum_{k=m+1}^{n-1} e^{-\mathrm{H}/\mathrm{T}_k}\right)$$

$$\times \exp(a\,e^{-(\mathrm{H}-\mathrm{H}')/(2\mathrm{T}_1)}) \leqq e^{-\alpha/\mathrm{T}_1} \prod_{k=m+1}^{n-1}\left(1-\frac{a}{3}e^{-\mathrm{H}/\mathrm{T}_k}\right). \quad (455)$$

Case of LKIs. — We have

$$\{\mathrm{QS}\} \leqq \{\mathrm{Z}_l(\mathrm{H}, a, e^{-\alpha/\mathrm{T}_1})\,\mathrm{Z}_l(\mathrm{H}', a, b)\}_{\leftarrow m}^{n}. \quad (456)$$

Let m and n be fixed. For any $\mathrm{R} \leqq n$ we have

$$\{\mathrm{QS}\}_{\leftarrow m}^{n} \leqq \mathrm{Z}(\mathrm{H}', a, b)_{\leftarrow(\mathrm{R}+1)}^{n} + (1+e^{-\alpha/\mathrm{T}_1}) \prod_{k=m+1}^{\mathrm{R}} (1-a\,e^{-\mathrm{H}/\mathrm{T}_k}), \quad (457)$$

let us put

$$\mathrm{R} = \sup\left\{ r \leqq n \,\Big|\, \sum_{k=\mathrm{R}}^{n-1} e^{-\mathrm{H}'/\mathrm{T}_k} \geqq \frac{\mathrm{H}-\mathrm{H}'}{2\,\mathrm{T}_1} + \sum_{k=m+1}^{n-1} e^{-\mathrm{H}/\mathrm{T}_k} \right\}. \quad (458)$$

With this choice of R we have

$$\sum_{k=\mathrm{R}+1}^{n-1} e^{-\mathrm{H}/\mathrm{T}_k} \leqq e^{-(\mathrm{H}-\mathrm{H}')/\mathrm{T}_1} \sum_{k=\mathrm{R}+1}^{n-1} e^{-\mathrm{H}'/\mathrm{T}_k}$$

$$\leqq e^{-(\mathrm{H}-\mathrm{H}')/(2\mathrm{T}_1)} + e^{-(\mathrm{H}-\mathrm{H}')/\mathrm{T}_1} \sum_{k=m+1}^{n-1} e^{-\mathrm{H}/\mathrm{T}_k}, \quad (459)$$

hence

$$\prod_{k=m+1}^{\mathrm{R}} (1-a\,e^{-\mathrm{H}/\mathrm{T}_k}) \leqq \exp\left(-a\sum_{k=m+1}^{\mathrm{R}} e^{-\mathrm{H}/\mathrm{T}_k}\right)$$

$$\leqq \exp\left(-a\sum_{k=m+1}^{n-1} e^{-\mathrm{H}/\mathrm{T}_k}(1-e^{-(\mathrm{H}-\mathrm{H}')/\mathrm{T}_1}) + a\,e^{-(\mathrm{H}-\mathrm{H}')/(2\mathrm{T}_1)}\right)$$

$$\leqq (1+2\,a\,e^{-(\mathrm{H}-\mathrm{H}')/(2\mathrm{T}_1)}) \prod_{k=m+1}^{n-1} \exp(-a(1-e^{-(\mathrm{H}-\mathrm{H}')/\mathrm{T}_1})\,e^{-\mathrm{H}/\mathrm{T}_k})$$

$$\leqq (1+e^{-\alpha/\mathrm{T}_1}) \prod_{k=m+1}^{n-1}\left(1-\frac{a}{2}e^{-\mathrm{H}/\mathrm{T}_k}\right). \quad (460)$$

Moreover

$$Z_l(H', a, b)^n_{\leftarrow (R+1)} \leqq (1+b) \prod_{k=R}^{n-1} (1 - a e^{-H'/T_k})$$
$$\leqq (1+b) \exp\left(-a \frac{H-H'}{2T_1} \right) \exp\left(-a \sum_{k=m+1}^{n-1} e^{-H/T_k} \right)$$
$$\leqq e^{-\alpha/T_1} \prod_{k=m+1}^{n-1} \left(1 - \frac{a}{2} e^{-H/T_k} \right). \quad (461)$$

We deduce from these equations that there are positive constants T_0, α, a such that in $\mathcal{G}(T_0, H)$

$$\{ QS \}^n_{\leftarrow m} \leqq (1 + e^{-\alpha/T_1}) \prod_{k=m+1}^{n-1} (1 - a e^{-H/T_k}). \quad (462)$$

End of the proof of lemma 6.6.

LEMMA 6.7. – *For any positive constants* T_0, D, *d*, H_1, H_2, *for any annealing framework of the type* (E, U, *q*, $\mathcal{L}_0$, $\mathcal{G}(T_0, D)$, $\mathcal{X}$), *for any increasing KIs* G, R *such that* G *is of class* $\mathcal{E}^r_-(H_1, D)$ [*resp. of class* $\mathcal{E}^l_-(H_1, D)$] *and* R *is of class* $\mathcal{E}^r_-(H_2, D)$ [*resp. of class* $\mathcal{E}^l_-(H_2, D)$] *the kernel* GR *is of class* $\mathcal{E}^r_-(H_1 + H_2, D)$ [*resp. of class* $\mathcal{E}^l_-(H_1 + H_2, D)$].

Proof. – Let us notice first that if two increasing KIs Q_1 and Q_2 are such that Q_2 is of class $\mathcal{E}^r_-(H, D)$ [resp. $\mathcal{E}^l_-(H, D)$] and if $(Q_1)^n_m \leqq (Q_2)^n_m$, $m, n \in \mathbb{Z}$, then Q_1 is of class $\mathcal{E}^r_-(H, D)$ [resp. of class $\mathcal{E}^l_-(H, D)$].

Let us notice also that a KI Q is of class $\mathcal{E}^r_-(H, D)$ [resp. $\mathcal{E}^l_-(H, D)$] if and only if, putting $(\tilde{Q})^n_m = Q^n_m \exp(H/T_{m+1})$, $\tilde{Q}$ is of class $\mathcal{E}^r_-(0, D)$ [resp. of class $\mathcal{E}^l_-(0, D)$].

With these two remarks in mind, we can restrict ourselves to the case when $H_1 = H_2 = 0$ in the following manner. Let us put

$$\tilde{G}^n_m = \exp(H_1/T_{m+1})\, G^n_m \quad (463)$$
$$\tilde{R}^n_m = \exp(H_2/T_{m+1})\, R^n_m, \quad (464)$$

then

$$\{ RG \}^n_m \exp((H_1 + H_2)/T_{m+1})$$
$$\leqq \sum_{l=m}^{n} G^l_m R^n_l \exp(H_1/T_{m+1}) \exp(H_2/T_{l+1})$$
$$= \{ \tilde{G}\tilde{R} \}^n_m \quad (465)$$

hence it is enough to prove that $\tilde{G}\tilde{R}$ is of class $\mathcal{E}^r_-(0, D)$ [resp. of class $\mathcal{E}^l_-(0, D)$], thus it is enough to prove the lemma when $H_1 = H_2 = 0$.

Let us point out now that, given positive constants T_0 and D, an increasing KI Q defined in the framework $\mathcal{G}(T_0, D)$ is of class $\mathcal{E}^r_-(0, D)$ [resp. of class $\mathcal{E}^l_-$ 0, D)] if and only if there are positive constants a and b such that it is of class $\mathcal{D}(D, a, b)$.

Hence, using lemma 6.4, we see that there are positive constants T_0, a and b such that in the cooling framework $\mathcal{G}(T_0, D)$ we have

$$\{GR\}_m^{n\to} \leqq \{Z_r(D, a, b)^2\}_m^{n\to} \tag{466}$$

$$[\text{resp. } \{GR\}^n_{\leftarrow m} \leqq \{Z_l(D, a, b)^2\}^n_{\leftarrow m}]. \tag{467}$$

But, putting Z for $Z_r(D, a, b)$ [resp. for $Z_l(D, a, b)$] we have

$$(Z^2)_m^{n\to} = Z_m^{n\to} + \sum_{l=m+1}^{n-1} Z_m^l Z_l^{n\to}, \tag{468}$$

$$\left[\text{resp. } (Z^2)^n_{\leftarrow m} = Z^n_{\leftarrow m} + \sum_{l=m+1}^{n-1} Z^l_{\leftarrow m} Z^n_l\right] \tag{469}$$

hence there is a positive T_0 such that in $\mathcal{G}(T_0, D)$

$$\begin{aligned}\{Z^2 - Z\}_m^{n\to} &\leqq (1+b)^2 \sum_{l=m+1}^{n-1} \frac{a\exp(-D/T_l)}{1 - a\exp(-D/T_l)} \\ &\quad \times \prod_{l=m+1}^{n-1} (1 - a e^{-D/T_l}) \\ &\leqq (1+b)^2 (1 - a e^{-D/T_1})^{-1} \sum_{l=m+1}^{n-1} a e^{-D/T_l} \prod_{l=m+1}^{n-1} (1 - a e^{-D/T_l}) \\ &\leqq 2(1+b)^2 \sum_{l=m+1}^{n-1} a e^{-D/T_l} \exp\left(-\sum_{l=m+1}^{n-1} a e^{-D/T_l}\right) \\ &\leqq 4e^{-1}(1+b)^2 \exp\left(-a/2 \sum_{l=m+1}^{n-1} e^{-D/T_l}\right) \\ &\leqq 4e^{-1}(1+b)^2 \prod_{l=m+1}^{n-1}\left(1 - \frac{a}{4} e^{-D/T_l}\right).\end{aligned} \tag{470}$$

$$\left[\text{resp. } \{Z^2 - Z\}^n_{\leftarrow m} \leqq (1+b)^2 \sum_{l=m+1}^{n-1} \frac{a\exp(-D/T_l)}{1 - a\exp(-D/T_l)} \prod_{l=m+1}^{n-1} (1 - a e^{-D/T_l}) \leqq \dots\right] \tag{471}$$

Thus there are positive constants T_0, a_2, b_2 such that $Z_r(D, a, b)^2$ [resp. $Z_l(D, a, b)^2$] is of class $\mathcal{D}(D, a_2, b_2)$ in the framework $\mathcal{G}(T_0, D)$. According to equation (466) the same is true of GR.

End of the proof of lemma 6.7.

LEMMA 6.8. — *For positive constants* T_0 *and* D, *consider the annealing framework* (E, U, q, $\mathscr{L}_0$, $\mathscr{G}(T_0, D)$, $\mathscr{X}$). *Let* d *be some positive constant and let* G, R *be some finite increasing KIs depending on the cooling schedule* $T \in \mathscr{G}(T_0, D)$. *Assume that* G *is of class* $\mathscr{E}^r(H_1, a_1, D, d)$ [*resp. of class* $\mathscr{E}^l(H_1, a_1, D, d)$] *and that* $\exp(-H_2/T_{m+1}) R_m^n$ *is of class* $\mathscr{E}^r(H_3, a_2, D, d)$ [resp. of class $\mathscr{E}^l(H_3, a_2, D, d)$]. *Then, if* $0 \leqq H_2 \leqq H_1$ *and* $H_2 \leqq H_3$, *there exists a positive constant* d' *such that the composed kernel* GR *is of class* $\mathscr{E}^r(H_1 - H_2 + H_3, a_1 a_2, D, d')$ [*resp. of class* $\mathscr{E}^l(H_1 - H_2 + H_3, a_1 a_2, D, d')$].

Proof. — Let us remark first that, given positive constants T_0, H, D, a, b, given a cooling framework $\mathscr{G}(T_0, D)$ and an increasing KI G defined in this framework, G is of class $\mathscr{E}^r(H, a, D, b)$ [resp. of class $\mathscr{E}^l(H, a, D, b)$] if and only if there are positive constants α, c such that the kernel $G_m^n e^{H/T_n}$ is of class $\mathscr{D}(D, c, b)$ and

$$\left.\begin{aligned} \sum_n G_m^n \exp(H/T_{m+1}) &\leqq a(1 + e^{-\alpha/T_1}), \\ \sum_n G_m^n \exp(H/T_n) &\geqq a(1 - e^{-\alpha/T_1}). \end{aligned}\right\} \tag{472}$$

$$[\text{resp. } \sum_m G_m^n \exp(H/T_{m+1}) \leqq a(1 + e^{-\alpha/T_1}),$$

$$\sum_m G_m^n \exp(H/T_n) \geqq a(1 - e^{-\alpha/T_1})]. \tag{473}$$

With this in mind, we put

$$\tilde{R}_m^n = R_m^n \exp(-H_2/T_{m+1}), \tag{474}$$

and we find that

$$\{GR\} \exp(-(H_1 + H_3 - H_2)/T_n) \leqq \{G\tilde{R}\}_m^n \exp((H_1 + H_3)/T_n). \tag{475}$$

But according to lemma 6.7 $G\tilde{R}$ is of class $\mathscr{E}^r_-(H_1 + H_3, D)$ [resp. $\mathscr{E}^l_-(H_1 + H_3, D)$], hence there are positive constants T_0, a_2, b such that in the framework $\mathscr{G}(T_0, D)$

$$\{GR\} \exp((H_1 + H_3 - H_2)/T_n)$$

is of class $\mathscr{D}(D, a_2, b_2)$.

Moreover

$$\begin{aligned} \{GR\}_m^{m \to} &= \sum_{n=m}^{+\infty} G_m^n R_n^{n \to} \\ &\leqq \sum_{n=m}^{+\infty} G_m^n \exp((H_2 - H_3)/T_{n+1})(1 + e^{-\alpha/T_1}) a_2 \\ &\leqq G_m^{m \to} \exp(-(H_3 - H_2)/T_{m+1}) a_2 (1 + e^{-\alpha/T_1}) \\ &\leqq a_1 a_2 (1 + e^{-\alpha/T_1})^2 \exp(-(H_1 + H_3 - H_2)/T_{m+1}). \end{aligned} \tag{476}$$

[resp.

$$\sum_{n=-\infty}^{m} \{GR\}_m^n e^{(H_1+H_3-H_2)/T_{m+1}}$$

$$= \sum_{m,\,l\,|\,m \leqq l \leqq n} e^{(H_1+H_3-H_2)/T_{m+1}} G_m^l R_l^n$$

$$\leqq \sum_{m,\,l\,|\,m \leqq l \leqq n} e^{H_1/T_{m+1}} G_m^l e^{(H_3-H_2)/T_{l+1}} R_l^n$$

$$\leqq \sum_{l\,|\,l \leqq n} a_1 (1+e^{-\alpha/T_1}) e^{(H_3-H_2)/T_{l+1}} R_l^n$$

$$\leqq a_1 a_2 (1+e^{-\alpha/T_1})^2]. \quad (477)$$

$$\sum_n \{GR\}_m^n e^{(H_1+H_3-H_2)/T_n}$$

$$\geqq \sum_{l,\,n\,|\,m \leqq l \leqq n} G_m^l e^{(H_1-H_2)/T_l} R_l^n e^{H_3/T_n}$$

$$\geqq \sum_{l,\,n\,|\,m \leqq l \leqq n} G_m^l e^{H_1/T_l} e^{-H_2/T_{l+1}} R_l^n e^{H_3/T_n}$$

$$\geqq \sum_{l=m}^{+\infty} G_m^l e^{H_1/T_l} a_2 (1-e^{-\alpha/T_1})$$

$$\geqq a_1 (1-e^{-\alpha/T_1}) a_2 (1-e^{-\alpha/T_1}). \quad (478)$$

$$\Bigg[\text{resp. } (GR)_{\leftarrow n}^n = \sum_{m=-\infty}^{n} G_{\leftarrow m}^m R_m^n$$

$$\geqq \sum_{m=-\infty}^{n} e^{-H_1/T_m} a_1 (1-e^{-\alpha/T_1}) R_m^n$$

$$\geqq \sum_{m=-\infty}^{n} a_1 (1-e^{-\alpha/T_1}) e^{-(H_1-H_2)/T_{m+1}} e^{-H_2/T_{m+1}} R_m^n$$

$$\geqq \sum_{m=-\infty}^{n} a_1 (1-e^{-\alpha/T_1}) e^{-(H_1-H_2)/T_n} e^{-H_2/T_{m+1}} R_m^n$$

$$\geqq a_1 a_2 (1-e^{-\alpha/T_1})^2 e^{-H_1-H_2+H_3)/T_n} \Bigg] \quad (479)$$

End of the proof of lemma 6.8.

We will also need a lemma on the sum of KIs of class $\mathscr{E}$:

LEMMA 6.9. – *For any positive constants* T_0, D, H_1, H_2, *a and d such that* $H_1 < H_2$, *for any annealing framework of the type* (E, U, *q*, $\mathscr{L}_0$, $\mathscr{G}(T_0, D)$, $\mathscr{X}$), *for any KIs* G *and* R *such that* G *is of class* $\mathscr{E}^r(H_1, a, D, d)$ [*resp. of class* $\mathscr{E}^l(H_1, a, D, d)$] *and* R *is of class* $\mathscr{E}^r_-(H_2, D)$ [*resp. of class* $\mathscr{E}^l_-(H_2, D)$] *the kernel* G + R *is of class* $\mathscr{E}^r(H_1, a, D, d)$ [*resp. of class* $\mathscr{E}^l(H_1, a, D, d)$].

Proof. – There are positive constants T_0, a_1, b_1, α, K such that in the cooling schedule $\mathcal{G}(T_0, D)$ there are RKIs Q_1, Q_2 and Q_3 and LKIs Q_4, Q_5 and Q_6 of class $\mathcal{D}(D, a_1, b_1)$ such that

$$\left.\begin{array}{c}(1-e^{-\alpha/T_1})\,a\,(Q_1)_m^n e^{-H_1/T_n} \leqq G_m^n \leqq (1+e^{-\alpha/T_1})\,a\,(Q_2)_m^n e^{-H_1/T_{m+1}}\\ (1-e^{-\alpha/T_1})\,a\,(Q_4)_m^n e^{-H_1/T_n} \leqq G_m^n \leqq (1+e^{-\alpha/T_1})\,a\,(Q_5)_m^n e^{-H_1/T_{m+1}}\\ R_m^n \leqq K\, e^{-H_2/T_{m+1}}\,(Q_3)_m^n\\ R_m^n \leqq K\, e^{-H_2/T_{m+1}}\,(Q_5)_m^n.\end{array}\right\} \quad (480)$$

Hence putting

$$\left.\begin{array}{l}(\tilde{Q}_2)_m^n = (a+K\,e^{-(H_2-H_1)/T_1})^{-1}\,(K\,e^{-(H_2-H_1)/T_1}\,(Q_3)_m^n + a\,(Q_2)_m^n)\\ (\tilde{Q}_5)_m^n = (a+K\,e^{-(H_2-H_1)/T_1})^{-1}(K\,e^{-(H_2-H_1)/T_1}\,(Q_6)_m^n + a\,(Q_5)_m^n),\end{array}\right\} \quad (481)$$

$\tilde{Q}_2$ is a RKI and $\tilde{Q}_5$ a LKI of class $\mathcal{D}(D, a_1, b_1)$ and we have

$$\left.\begin{array}{l}(Q_1)_m^n (1-e^{-\alpha/T_1})\,a\,e^{-H_1/T_n} \leqq (G+R)_m^n\\ \qquad \leqq (1+e^{-\alpha/T_1})\,(\tilde{Q}_2)_m^n\, a\,(1+K\,a^{-1}\,e^{-(H_2-H_1)/T_{m+1}}),\\ (Q_4)_m^n\, a\,e^{-H_1/T_n}(1-e^{-\alpha/T_1}) \leqq (G+R)_m^n\\ \leqq (\tilde{Q}_5)_m^n\, a\,e^{-H_1/T_{m+1}}\,(1+e^{-\alpha/T_1})\,(1+K\,a^{-1}\,e^{-(H_2-H_1)/T_1}).\end{array}\right\} \quad (482)$$

End of the proof of lemma 6.9.

LEMMA 6.10. – *For positive constants* T_0 *and* D, *consider the annealing framework* $(E, U, q, \mathcal{L}_0, \mathcal{G}(T_0, D), \mathcal{X})$. *Let* d, H *be positive constants and let* G, R *be two KIs. Assume that* G *is of class* $\mathcal{E}^r(H, a_1, D, d)$ [*resp.* $\mathcal{E}^l(H, a_1, D, d)$] *and that* R *is of class* $\mathcal{E}^r(H, a_2, D, d)$ [*resp.* $\mathcal{E}^l(H, a_2, D, d)$]. *Then* G+R *is of class* $\mathcal{E}^r(H, a_1+a_2, D, d)$ [*resp.* $\mathcal{E}^l(H, a_1+a_2, D, d)$].

The proof is analogue to that of lemma 6.9 and is left to the reader.

LEMMA 6.11. – *Let* $(E, U, q, \mathcal{L}_0)$ *be an energy landscape. Let* G, K *be two finite increasing KIs. Let* T_0, a, b, c *be positive constants, let* H *and* S *be such that* $0 \leqq S < H$. *Assume that in* $\mathcal{G}(T_0, H)$, G *is adjacent to* $c\,\mathcal{I}(H)$ *and that* K *is of class* $\mathcal{E}^r(0, a, S, b)$, *then the product* GK *is also adjacent to* $c\,\mathcal{I}(H)$.

Proof of lemma 6.11. – There is $\alpha > 0$ such that G is α-adjacent to $c\,\mathcal{I}(H)$. We can assume without loss of generality that $\alpha \leqq (H-S)/2$. For any $m \leqq n_1 \leqq n_3$ such that

$$n_3 \geqq N\left(H, T_{m+1}, -\frac{\alpha}{2}, n_1\right) \quad (483)$$

let us put

$$n_2=\sup\left\{k \,\middle|\, \sum_{l=k}^{n_3} e^{-S/T_l} \geqq e^{(H-S)/(2\,T_{m+1})}\right\}, \tag{484}$$

then for some $\beta>0$ and some $d>0$

$$a \sum_{k=n_1+1}^{n_2} G_m^k\left(1-(1+b)\prod_{l=n_2+1}^{n_3}(1-d e^{-S/T_l})\right)\left(1-\frac{1}{2}e^{-\beta/T_{m+1}}\right) \leqq \sum_{k=n_1+1}^{n_3}\{GK\}_m^k \leqq a \sum_{k=n_1+1}^{n_3} G_m^k(1+e^{-\beta/T_{m+1}}). \tag{485}$$

Hence as

$$\prod_{l=n_2+1}^{n_3}(1-d e^{-S/T_l}) \leqq \exp\left(-\frac{a}{2}e^{(H-S)/(2T_{m+1})}\right), \tag{486}$$

for T_0 small enough in $\mathscr{G}(T_0, H)$ we have

$$a(1-e^{-\beta/T_{m+1}})\sum_{k=n_1+1}^{n_2} G_m^k \leqq \sum_{k=n_1+1}^{n_3}\{GK\}_m^k \leqq a(1+e^{-\beta/T_{m+1}})\sum_{k=n_1+1}^{n_3} G_m^k. \tag{487}$$

Moreover

$$\sum_{l=n_2+1}^{n_3} e^{-H/T_l} \leqq e^{-(H-S)/(2T_{m+1})}, \tag{488}$$

hence

$$\sum_{l=n_1+1}^{n_2} e^{-H/T_l} \geqq e^{-\alpha/(2T_{m+1})} - e^{-(H-S)/(2T_{m+1})} \geqq e^{-\alpha/T_{m+1}}. \tag{489}$$

Hence $n_2 \geqq N(H, T_{m+1}, n_1, \alpha)$ and consequently for some $\beta>0$ in some $\mathscr{G}(T_0, H)$

$$\sum_{k=n_1+1}^{n_2} G_m^k \geqq (1-e^{-\beta/T_{m+1}})\sum_{k=n_1+1}^{n_2} G_m^{k\,\rightarrow}\, c\, e^{-H/T_k}. \tag{490}$$

But

$$\sum_{k=n_2+1}^{n_3} G_m^{k\,\rightarrow} e^{-H/T_k} \leqq G_m^{n_2\,\rightarrow}\sum_{k=n_2+1}^{n_3} e^{-H/T_k} \leqq G_m^{n_2\,\rightarrow} e^{-(H-S)/(2T_{m+1})}, \tag{491}$$

and

$$\sum_{k=n_1+1}^{n_2} G_m^{k\rightarrow} e^{-H/T_k} \geqq G_m^{n_2\rightarrow} \sum_{k=n_1+1}^{n_2} e^{-H/T_k}$$
$$\geqq G_m^{n_2\rightarrow} (e^{-\alpha/(2T_{m+1})} - e^{(H-S)/(2T_{m+1})}), \quad (492)$$

hence there exist $\beta_1 > 0$ and $\beta_2 > 0$ such that

$$\sum_{k=n_1+1}^{n_2} G_m^{k\rightarrow} e^{-H/T_k} \geqq (1 - e^{\beta_1/T_{m+1}}) \sum_{k=n_1+1}^{n_3} G_m^{k\rightarrow} e^{-H/T_k} \quad (493)$$

and such that

$$\sum_{k=n_1+1}^{n_2} G_m^k \geqq (1 - e^{-\beta_2/T_{m+1}}) c \sum_{k=n_1+1}^{n_3} G_m^{k\rightarrow} e^{-H/T_k}. \quad (494)$$

Hence for some positive T_0, β_1, in $\mathcal{G}(T_0, H)$,

$$a(1 - e^{-\beta_1/T_{m+1}}) \sum_{k=n_1+1}^{n_3} G_m^{k\rightarrow} c e^{-H/T_k}$$
$$\leqq \sum_{k=n_1+1}^{n_3} \{GK\}_m^k \leqq a(1 + e^{-\beta_1/T_{m+1}}) \sum_{k=n_1+1}^{n_3} G_m^{k\rightarrow} c e^{-H/T_k}. \quad (495)$$

Thus for some $\beta_2 > 0$

$$a(1 - e^{-\beta_2/T_{m+1}}) \sum_{k=n_1+1}^{n_3} G_m^k$$
$$\leqq \sum_{k=n_1+1}^{n_3} \{GK\}_m^k \leqq a(1 + e^{-\beta_2/T_{m+1}}) \sum_{k=n_1+1}^{n_3} G_m^k. \quad (496)$$

Letting n_3 go to infinity we get that for any $l \geqq m$

$$a(1 - e^{-\beta_2/T_{m+1}}) G_m^{l\rightarrow} \leqq \{GK\}_m^{l\rightarrow} \leqq a(1 + e^{-\beta_2/T_{m+1}}) G_m^{l\rightarrow}. \quad (497)$$

We conclude the proof from equations (495) and (497).

End of the proof of lemma 6.11.

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