ANNALES DE L'I. H. P., SECTION B

STEVEN N. EVANS

Trapping a measure-valued Markov branching process conditioned on non-extinction

Annales de l'I. H. P., section B, tome 27, n° 2 (1991), p. 215-220
<http://www.numdam.org/item?id=AIHPB_1991__27_2_215_0>

Ann. Inst. Henri Poincaré,

Vol. 27, n° 2, 1991, p. 215-220. *Probabilités et Statistiques*

Trapping a measure-valued Markov branching process conditioned on non-extinction (*)

by

Steven N. EVANS

Department of Statistics,
University of California,
Berkeley, California 94720, U.S.A.

ABSTRACT. — We consider a class of measure-valued Markov branching processes that are constructed as "superprocesses" over some underlying Markov process. We condition such a process to stay away from the zero measure forever. We show that if there is a set of traps that will eventually catch the underlying process then the mass of the conditioned superprocess will eventually be concentrated on a single one of these traps. Moreover, the distribution of the point of eventual concentration is the same as that of the final destination for the underlying process.

Key words : Branching process, measure-valued, trap.

RÉSUMÉ. — Nous étudions une classe de processus de branchement markoviens à valeurs mesures définis comme « superprocessus » d'un processus de Markov sous-jacent. Nous conditionnons le processus de telle manière qu'il n'atteigne jamais la mesure nulle. Dans le cas où il existe un ensemble d'états pièges qui absorbent le processus de Markov avec probabilité un, nous montrons que la masse du superprocessus conditionné se concentrera en un temps fini sur un seul de ces états pièges, la loi du point de concentration étant la même que la loi de la destination finale du processus de Markov sous-jacent.

Classification A.M.S. : Primary: 60 G 57, 60 J 80. Secondary: 60 J 25.

(*) Research supported in part by an N.S.F. grant.

1. INTRODUCTION AND STATEMENT OF THE RESULT

We begin by recalling a special case of the *superprocess* construction given in [4].

Suppose that E is a topological Lusin space and $\mathscr{E}$ is the Borel σ-field of E. Let $\xi=(\Omega, \mathscr{F}, \mathscr{F}_t, \theta_t, \xi_t, P^x)$ be a Borel right Markov process with state space $(E, \mathscr{E})$ and semigroup (P_t). Assume that $P_t 1=1$.

Let $\varphi : [0, \infty[\to \mathbb{R}$ be given by

$$\varphi(\lambda)=-c\lambda^2+\int_0^\infty n(du)(1-e^{-\lambda u}-\lambda u),$$

where $c \geqq 0$ and $\int_0^\infty n(du)(u \vee u^2)<\infty$. For each bounded, non-negative, $\mathscr{E}$-measurable function f the integral equation

$$v_t(x)=P_t f(x)+\int_0^t P_s(x, \varphi(v_{t-s}))\,ds, \qquad t \geqq 0, \quad x \in E,$$

has a unique solution which we denote by $(t, x) \mapsto V_t f(x)$. Write M(E) for the space of finite measures on E equipped with the weak topology and write $\mathscr{M}(E)$ for the corresponding Borel σ-field. There exists a unique Markov kernel (Q_t) on $(M(E), \mathscr{M}(E))$ with Laplace functionals

$$\int Q_t(\mu, d\nu)\exp(-\nu(f))=\exp(-\mu V_t f)$$

for all $\mu \in M(E)$, $t \geqq 0$ and bounded, non-negative, $\mathscr{E}$-measurable functions f. Moreover, there is right Markov process $X=(W, \mathscr{G}, \mathscr{G}_t, \Theta_t, X_t, \mathbb{P}^\mu)$ with state space $(M(E), \mathscr{M}(E))$ and semigroup (Q_t). The process X is called the (ξ, φ)-superprocess. We refer the reader to [4] for a representative bibliography of the literature concerning such processes.

Starting from any initial measure μ the process X "dies out" $\mathbb{P}^\mu$-almost surely; that is, $X_t=0$ for all t sufficiently large. One can consider what happens if the process is started off at $\mu \in M(E)\setminus\{0\}$ and "conditioned to stay alive forever"; that is, one conditions on $X_T \neq 0$ and looks for a limit as $T \to \infty$. Combining details from [5] (for the case when ξ is a Feller process on $\mathbb{R}^d$) and [2] (for the case when $n \equiv 0$); we see that the result of this procedure is a right Markov process $(\tilde{W}, \tilde{\mathscr{G}}, \tilde{\mathscr{G}}_t, \tilde{\Theta}_t, \tilde{X}_t, \tilde{\mathbb{P}}^\mu)$ with state space $(\tilde{M}(E), \tilde{\mathscr{M}}(E))$, where $\tilde{M}(E)=M(E)\setminus\{0\}$ and $\tilde{\mathscr{M}}(E)$ is the trace of $\mathscr{M}(E)$ on $\tilde{M}(E)$. This process has the semigroup $(\tilde{Q}_t)$ which is that of the Doob h-transform of X using the function $h(\nu)=\nu(1)$; that is,

$$\begin{aligned}\tilde{Q}_t F(\mu)&=\tilde{\mathbb{P}}^\mu F(\tilde{X}_t)\\&=\mu(1)^{-1}\,\mathbb{P}^\mu[F(X_t)X_t(1)].\end{aligned}$$

Besides the two papers mentioned above, this conditioned process has also been studied in [3], where the entrance space is identified when $n \equiv 0$ and ξ is Feller.

When $n \equiv 0$ it was shown in [2] that under suitable conditions, which include the convergence of (P_t) to some unique invariant probability measure ν, the distribution of $t^{-1} \tilde{X}_t$ converges to that of $Z\nu$ as $t \to \infty$, where Z is some strictly positive, real random variable. In this paper we investigate what happens if (P_t) has more than one invariant measure in the special case when the extremal invariant measures are point masses – in other words , when the process ξ can end up being caught in one of a number of trap states. Let $K \equiv \{ x \in E : P_t 1_{\{x\}}(x) = 1, \forall t \geqq 0 \}$ be the set of traps for (P_t). Write supp ν for the closed support of $\nu \in M(E)$. Our result is the following.

THEOREM. – *Suppose that* $\mu \in \tilde{M}(E)$ *is such that* $\lim_{t \to \infty} \mu P_t 1_{E \setminus K} = 0$. *Then* $\tilde{\mathbb{P}}^{\mu}$*-almost surely there exists* $\kappa \in K$ *such that* $\operatorname{supp} \tilde{X}_t = \{ \kappa \}$ *for all t sufficiently large. The distribution of* κ *is given by*

$$\tilde{\mathbb{P}}^{\mu}(\kappa \in A) = \lim_{t \to \infty} \mu P_t 1_A / \mu(1).$$

We give the proof of the theorem in section 2 after a number of preliminary results.

2. PRELIMINARY RESULTS AND PROOF OF THE THEOREM

For the sake of completeness, we begin with the following simple observation.

LEMMA 1. – *Suppose that* $A \in \mathscr{E}$ *is a subset of* K. *Then* $1_{E \setminus A}$ *is an excessive function for* (P_t).

Proof. – It is obvious that $P_t 1_{E \setminus A} \leqq 1_{E \setminus A}$ for all $t \geqq 0$, so we need only show that $P_t 1_{E \setminus A}(x) \uparrow 1_{E \setminus A}(x)$ as $t \downarrow 0$ for all $x \in E$. This is clear for $x \in A$. Consider the case $x \notin A$ and suppose, to the contrary, that $\lim_{t \downarrow 0} P_t 1_{E \setminus A}(x) < 1$ [equivalently, $\lim_{t \downarrow 0} P_t 1_A(x) > 0$]. We may define a finite measure γ by

$$\gamma(B) = \lim_{t \downarrow 0} P_t 1_{B \cap K}(x) = \inf_{t \downarrow 0} P_t 1_{B \cap K}(x), \qquad B \in \mathscr{E}.$$

By our assumption, $\gamma(A) > 0$, so there is a compact set $F \subset A$ such that $\gamma(F) > 0$. This, however, would contradict the right-continuity of the paths of ξ. □

LEMMA 2. — *Let f be a bounded, $\mathscr{E}$-measured function. Then*

$$\tilde{\mathbb{P}}^{\mu}[\tilde{X}_t(f)/\tilde{X}_t(1)] = (\mu P_t f)/\mu(1)$$

for $\mu \in \tilde{M}(E)$.

Proof. — Applying Proposition 2.7 of [4] we have

$$\begin{aligned}\tilde{\mathbb{P}}^{\mu}[\tilde{X}_t(f)/\tilde{X}_t(1)] &= \mu(1)^{-1}\,\mathbb{P}^{\mu}[X_t(f)]\\ &= (\mu P_t f)/\mu(1). \quad \square\end{aligned}$$

LEMMA 3. — *Suppose that* $A \in \mathscr{E}$ *is a subset of* K *and* $\nu \in \tilde{M}(E)$.

(i) *The paths of* $t \mapsto \tilde{X}_t(E \setminus A)$ *are cadlag* $\tilde{\mathbb{P}}^{\nu}$*-almost surely.*

(ii) *If* $\nu(E \setminus A) = 0$ *then* $\tilde{\mathbb{P}}^{\nu}$*-almost surely* $\tilde{X}_t(E \setminus A) = 0$ *for all* $t \geq 0$.

Proof. — (i) For each $T \geq 0$ the law of $\{\tilde{X}_t : 0 \leq t \leq T\}$ under $\tilde{\mathbb{P}}^{\nu}$ is absolutely continuous with respect to that of $\{X_t : 0 \leq t \leq T\}$ under $\mathbb{P}^{\nu}$, so it suffices to prove the statement with $\tilde{X}$ replaced by X and $\tilde{\mathbb{P}}^{\nu}$ replaced by $\mathbb{P}^{\nu}$. This is clear from our Lemma 1 and Theorem 3.5 of [4], which states $t \mapsto X_t(f)$ is $\mathbb{P}^{\eta}$-almost surely cadlag for all $\eta \in M(E)$ when $t \mapsto f(\xi_t)$ is almost surely cadlag. (Alternatively, one can also use Theorem 2.20 of [4], which states that $t \mapsto X_t$ is cadlag $\mathbb{P}^{\eta}$-almost surely if we retopologise M(E) with the relative topology inherited from M($\bar{E}$), where M($\bar{E}$) is the space of finite measures on $\bar{E}$, the Ray-Knight compactification of E, equipped with the weak* Ray topology.)

(ii) Given part (i), the result follows immediately from Lemma 2. $\square$

LEMMA 4. — *Suppose that* $A \in \mathscr{E}$ *is a subset of* K *and* $\nu \in \tilde{M}(E)$. *Then*

$$\liminf_{t \to \infty} \tilde{\mathbb{P}}^{\nu}(\tilde{X}_t(E \setminus A) = 0) \geq \nu(A)/\nu(E).$$

Proof. — Define measures $\alpha, \beta \in M(E)$ by $\alpha(.) = \nu(. \cap A)$ and $\beta(.) = \nu(. \cap (E \setminus A))$. On some probability space $(\Sigma, \mathscr{A}, \hat{\mathbb{P}})$ we may construct two independent M(E)-valued processes Y and Z such that the law of Y (respectively, Z) is that of X under $\mathbb{P}^{\alpha}$ (respectively, $\mathbb{P}^{\beta}$). It follows from the form of the Laplace functionals of (Q_t) that the law of Y + Z is that of X under $\mathbb{P}^{\nu}$. Now from the analogue of Lemma 3 (ii) for X we have

$$\begin{aligned}\tilde{\mathbb{P}}^{\nu}[\tilde{X}_t(E \setminus A) = 0] &= \nu(E)^{-1}\,\hat{\mathbb{P}}[\{[Y_t + Z_t](E \setminus A) = 0\}\,[Y_t + Z_t](E)]\\ &= \nu(E)^{-1}\,\hat{\mathbb{P}}[\{Z_t(E \setminus A) = 0\}\,[Y_t + Z_t](E)]\\ &\geq \nu(E)^{-1}\,\hat{\mathbb{P}}[Z_t(E \setminus A) = 0]\,\hat{\mathbb{P}}[Y_t(E)]\\ &= \nu(E)^{-1}\,\mathbb{P}^{\beta}[X_t(E \setminus A) = 0]\,\nu(A)\\ &\to \nu(A)/\nu(E) \quad \text{as } t \to \infty,\end{aligned}$$

since X dies out almost surely. $\square$

LEMMA 5. — *Suppose that* $\nu \in \tilde{M}(E)$ *is such that* $\nu(E \setminus K) = 0$. *Then for* $t > 0$, supp $\tilde{X}_t$ *is a finite subset of* K, $\tilde{\mathbb{P}}^{\nu}$*-almost surely.*

Proof. – Again, it suffices to prove the result with $\tilde{\mathbb{P}}^{\nu}$ replaced by $\mathbb{P}^{\nu}$ and $\tilde{X}$ replaced by X. From the arguments of Proposition III.1.1 in [1] we see that for $t>0$, X_t has a finite cluster representation. That is, there exists a kernel $R_t(x, d\eta)$ such that $R_t(x, .)$ is concentrated on $M(E)\setminus\{0\}$, $R_t(x, M(E)\setminus\{0\})$ is finite and independent of x, and under $\mathbb{P}^{\nu}$ the random measure X_t has the same law as $\int \Pi(d\eta)\,\eta$, where Π is a Poisson random measure on $M(E)\setminus\{0\}$ with finite intensity $\int \nu(dx)\,R_t(x, d\eta)$. It is clear from Lemma 3 (ii), that if $k\in K$ then $R_t(k, .)$ is concentrated on the family of measures supported by $\{k\}$, and so the lemma follows. □

Proof of Theorem

We begin by showing that $\tilde{X}_t(E\setminus K)=0$ for all t sufficiently large, $\tilde{\mathbb{P}}^{\mu}$-almost surely. From our hypothesis and Lemmas 1 and 2 we see that $\tilde{X}_t(E\setminus K)/\tilde{X}_t(E)$ is a supermartingale converging $\tilde{\mathbb{P}}^{\mu}$-almost surely and in L^1 to 0. The claim now follows from an easy argument using Lemmas 3 and 4 and the strong Markov property.

Next, if we apply Lemma 5 and the strong Markov property we conclude that, $\tilde{\mathbb{P}}^{\mu}$-almost surely, supp $\tilde{X}_t$ is a finite subset of K for all t sufficiently large.

Let us now show that, $\tilde{\mathbb{P}}^{\mu}$-almost surely, there exists $\kappa\in K$ such that supp $\tilde{X}_t=\{\kappa\}$ for all t sufficiently large. From the above and the strong Markov property we may assume for the moment that supp μ is a finite subset of K, say $\{k_1, \ldots, k_m\}$. We find from Lemmas 3 and 4 that

$$\tilde{\mathbb{P}}^{\mu}(\tilde{X}_t(E\setminus\{k_j\})=0,\ \text{all } t \text{ suff. large})\geqq \mu(\{k_j\})/\mu(E),$$

and so

$$\tilde{\mathbb{P}}^{\mu}(\exists j : \tilde{X}_t(E\setminus\{k_j\})=0,\ \text{all } t \text{ suff. large})=1.$$

To finish the proof of the theorem, we need only determine the distribution of κ. It is clear from our hypothesis and Lemma 1 that $\lim_{t\to\infty} (\mu P_t 1_A)/\mu(1)$ exists for all $A\in\mathscr{E}$ and defines a probability measure concentrated on K. For any set $A\in\mathscr{E}$ observe that $1_A(\kappa)=\tilde{X}_t(A)/\tilde{X}_t(E)$ for all t sufficiently large, and so, by Lemma 2,

$$\begin{aligned}\tilde{\mathbb{P}}^{\mu}(\kappa\in A) &= \lim_{t\to\infty} \tilde{\mathbb{P}}^{\mu}(\tilde{X}_t(A)/\tilde{X}_t(E))\\ &= \lim_{t\to\infty} (\mu P_t 1_A)/\mu(1),\end{aligned}$$

as required. □

REFERENCES

[1] N. El Karoui and S. Roelly-Coppoletta, *Study of a General Class of Measure-Valued Branching Processes; a Lévy-Hinčin Representation*, Preprint, Université de Paris-VI, Laboratoire de Calcul des Probabilités, 1987.

[2] S. N. Evans and E. Perkins, Measure-Valued Markov Branching Processes Conditioned on Non-Extinction, *Israel J. Math.*, Vol. **71**, 1990, pp. 329-337.

[3] S. N. Evans, The Entrance Space of a Measure-Valued Markov Branching Process Conditioned on Non-Extinction, *Canadian Math. Bull.*, 1989 (to appear).

[4] P. J. Fitzsimmons, Construction and Regularity of Measure-Valued Markov Branching Processes, *Israel J. Math.*, Vol. **64**, 1988, pp. 337-361.

[5] S. Roelly-Coppoletta and A. Rouault Processus de Dawson-Watanabe conditionné par le futur lointain, *C.R. Acad. Sci. Paris*. T. **309**, Séries I, 1989, pp. 867-872.

(*Manuscript received March* 21, 1990;
revised form November 14, 1990.)