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Fubini's theorem for double Wiener integrals and the variance of the Brownian path

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ABSTRACT. — Using Fubini's theorem for double Wiener integrals, it is possible to show that certain quadratic functionals of Brownian motion have the same law. This is applied to the variance of the Brownian path, which has the same law as the integral of the square of the Brownian bridge.

Key words : Fubini theorem, double Wiener integrals, Brownian path.

RÉSUMÉ. — A l'aide d'un théorème de Fubini pour des intégrales de Wiener doubles, on peut montrer que certaines fonctionnelles quadratiques du mouvement brownien ont même distribution. Ce résultat est appliqué à la variance de la trajectoire brownienne, qui a même loi que l'intégrale du carré du pont brownien.

Classification A.M.S. : Primary: 60 J 65, 60 J 55. Secondary: 60 H 05, 60 G 44.

1. INTRODUCTION

(1.1) We denote by $\xi \times \eta$ the quantity $\mathrm{Im}(\bar{\xi}\eta)$, for $\xi, \eta \in \mathbb{C}$. Let $(Z_s = X_s + i Y_s, s \geqq 0)$ be the complex valued Brownian motion, with $Z_0 = 0$, and let $G = \int_0^1 ds\, Z_s$ be its centre of gravity over the time interval $[0, 1]$.

Duplantier [3] obtained the distribution of

$$\mathscr{A}_G \equiv \int_0^1 (Z_s - G) \times dZ_s,$$

via the computation of its characteristic function:

$$(1\,a) \qquad E[\exp(i\lambda \mathscr{A}_G)] = \frac{4}{\cosh\lambda + 3(\sinh\lambda)/\lambda}.$$

This formula (1 *a*) may be derived simply (*see* Yor [13]) from the celebrated Lévy formula (*see* Lévy [7]; (1950)):

$$(1\,b) \qquad E[\exp(i\lambda \mathscr{A}) \mid Z_1 = z] = \frac{\lambda}{\sinh\lambda} \exp - \frac{|z|^2}{2}(\lambda \coth\lambda - 1),$$

where $\mathscr{A} = \int_0^1 Z_s \times dZ_s$.

It has been remarked that:

$$(1\,c) \qquad E[\exp(i\lambda \mathscr{A}) \mid Z_1 = z] = E\left[\exp\left(-\frac{\lambda^2}{2}\mathscr{V}\right) \middle| \, |Z_1| = |z|\right],$$

where

$$\mathscr{V} = \int_0^1 ds\, |Z_s|^2,$$

and this identity (1 *c*) may be used as a step towards the proof of (1 *b*), reducing this proof to a computation involving only the radial (Bessel) process $(|Z_s|, s \leqq 1)$ (*see* Williams [11] and Yor [12] for details).

(1.2) Following the publication of [13], the second author (M. Y.) was then asked by K. Jansons (personal communication; September 1989), following the computation made in [5], whether the computation of the Laplace transform of:

$$\mathscr{V}_G \equiv \int_0^1 ds\, |Z_s - G|^2,$$

might also follow from Lévy's formula (1 *b*). Actually, P. Malliavin has been interested for quite some time in the distribution of $\mathscr{V}_G$ (*see* [8]) and had also asked M. Y. in 1980 to compute its Laplace transform. However,

in December 1989, we came across the work of Chiang, Chow and Lee [2] from which it follows in particular that:

$$(1\,d) \qquad \mathscr{V}_G \overset{(\text{law})}{=} \int_0^1 ds\,|\tilde{Z}_s|^2,$$

where $(\tilde{Z}_s,\ s \leqq 1)$ is a standard, complex-valued, Brownian bridge. One of the proofs given in [2] is particularly simple, and provides an affirmative answer to K. Jansons and P. Malliavin's questions; here is this proof: the next formula $(1\,e)$ follows from Lévy's formula $(1\,b)$ by integrating with respect to the law of Z_1:

$$(1\,e) \qquad \mathrm{E}\left[\exp\left(i\lambda \int_0^1 (z+Z_s)\times dZ_s\right)\right] = \frac{1}{\cosh\lambda}\exp - \frac{|z|^2}{2}(\lambda\tanh\lambda).$$

The left-hand side of $(1\,e)$ is also equal to:

$$\mathrm{E}\left[\exp\left(-\frac{\lambda^2}{2}\int_0^1 ds\,|z+Z_s|^2\right)\right]$$

(the reader may also check that these formulae are in agreement with the more general formula $(2\,k)$, in the case $d=2$ and $\alpha=0$, in Pitman-Yor [10]) and integrating both the latter expression and the right-hand side of $(1\,e)$ with respect to the Lebesgue measure dz on $\mathbb{C}\simeq\mathbb{R}^2$, one obtains:

$$(1f) \qquad \mathrm{E}\left[\exp\left(-\frac{\lambda^2}{2}\mathscr{V}_G\right)\right] = \frac{\lambda}{\sinh\lambda},$$

from which, put together with $(1\,b)$ and $(1\,c)$, one deduces $(1\,d)$.

However, in the second paragraph of the present paper, yet another quite elementary proof of $(1\,d)$ is given, which essentially follows from Fubini's theorem. This argument allows us to derive a general extension of $(1\,d)$.

(1.3) We take this opportunity to mention two estimates on the Laplace transform of $\mathscr{V}_G$ which were made, on one hand by P. Malliavin [8], p. 328, and, on the other hand, by Ikeda-Watanabe [4], Lemma 8.6 (this has become Lemma 10.6 in the second edition of [4]).

From $(1f)$, we have:

$$(1f') \qquad \mathrm{E}\left[\exp\left(-2\lambda^2\mathscr{V}_G\right)\right] = \frac{2\lambda}{\sinh(2\lambda)} \equiv \left(\frac{\lambda}{\sinh\lambda}\right)\frac{1}{\cosh\lambda},$$

and it seems interesting to remark that P. Malliavin majorizes the left-hand side of $(1f')$ by $\dfrac{1}{\cosh\lambda}$, whereas Ikeda-Watanabe's majorization is $\dfrac{\lambda}{\sinh\lambda}$.

Independently of this, our interest in these questions was further aroused by the computation, made at Cambridge by T. Chan and C. Rogers (Personal communication of K. Jansons [5]) of the joint Laplace-Fourier transform of the law of $(\mathscr{V}_G, G, Z_1)$. This prompted us to work out a complement to the development of Brownian motion along the Legendre polynomials orthogonal basis of $L^2[0, 1]$), and its relationship to Lévy's formula (1 *b*), as done originally by Biane and Yor ([1 *a*], [1 *b*]).

This complement consists in obtaining a simultaneous orthogonal development of ξ and Z, two independent 2-dimensional Brownian motions, and to represent $\int_0^1 d\xi_s . Z_s$ in terms of this development. This is done precisely in paragraph 3, where we also derive another simultaneous orthogonal development of ξ and Z, and express $\int_0^1 d\xi_s \times Z_s$ in terms of this development. In both cases, the Legendre polynomials again play a fundamental role.

(1.4) In paragraph 4, we return to our extensions – presented in paragraph 2 – of the identity in law (1 *d*). We show that those identities in law can also be derived by remarking that certain pairs of operators on $L^2([0, 1])$ have the same eigenvalues, thereby connecting our work with the time-honoured diagonalization procedure, which goes back to Lévy's original paper [7] and has been used again and again in most of the computations of the laws of quadratic Brownian functionals (examples already quoted above are [1 *b*], [2], [3], but see also Krée [6] and McAonghusa and Pulé [9] for other examples in the recent literature).

2. SOME IDENTITIES IN LAW BETWEEN BROWNIAN QUADRATIC FUNCTIONALS

(2.1) The identity in law (1 *d*), and, more generally, a large class of identities in law between two Brownian quadratic functionals, are immediate consequences of the following Fubini type identity between double Wiener integrals: let $\varphi: [0, 1]^2 \to \mathbb{R}$ be a deterministic function such that

$$\int_0^1 du \int_0^1 ds\, \varphi^2(u, s) < \infty.$$

Then, the almost sure identity holds:

$$\int_0^1 d\xi_u \int_0^1 dZ_s\, \varphi(u, s) = \int_0^1 dZ_s \int_0^1 d\xi_u\, \varphi(u, s), \tag{2 a}$$

where, here, for simplicity, ξ and Z denote two independent one-dimensional Brownian motions, starting from 0.

From $(2a)$ and the independence of ξ and Z, we now deduce the general identity in law:

$$(2b)\qquad \int_0^1 du\left(\int_0^1 d\mathrm{B}_s\,\varphi(u,s)\right)^2 \overset{(\mathrm{law})}{=} \int_0^1 du\left(\int_0^1 d\mathrm{B}_s\,\varphi(s,u)\right)^2,$$

where $(\mathrm{B}_s,\ s\leqq 1)$ denotes here again a 1-dimensional Brownian motion.

Remark. — The identity $(2a)$ and the identity in law $(2b)$ extend obviously to a pair of d-dimensional Brownian motions ξ and Z on one hand, and a d-dimensional Brownian motion B on the other hand, provided the products $d\xi_u . dZ_s$ and $dZ_s . d\xi_u$ in $(2a)$ are scalar products, and the squares in $(2b)$ are squares of the euclidean norm. □

Applying the identity $(2b)$ to some particular integrands φ, we obtain the following

PROPOSITION 1. — 1. *Let $f:[0,1]\to\mathbb{R}$ be a C^1-function. Then, we have:*

$$\int_0^1 du\left(\mathrm{B}_u-\int_0^1 ds f'(s)\,\mathrm{B}_s\right)^2 \overset{(\mathrm{law})}{=} \int_0^1 du\,[(1-f(1)+f(u))\,\mathrm{B}_1-\mathrm{B}_u]^2$$
$$\overset{(\mathrm{law})}{=} \int_0^1 du\,[\mathrm{B}_1\,(f(1-u)-f(1))+\mathrm{B}_u]^2.$$

In particular, if $f(1)=1$, we obtain:

$$(2c')\qquad \int_0^1 du\left(\mathrm{B}_u-\int_0^1 ds f'(s)\,\mathrm{B}_s\right)^2 \overset{(\mathrm{law})}{=} \int_0^1 du\,(\mathrm{B}_u-f(u)\,\mathrm{B}_1)^2$$

2. *Let $a\in\mathbb{R}$. Then, we have:*

$$(2c'')\qquad \int_0^1 du\,(\mathrm{B}_u-a\,\mathrm{G})^2 \overset{(\mathrm{law})}{=} \int_0^1 du\,(\mathrm{B}_u-au\,\mathrm{B}_1)^2$$
$$\overset{(\mathrm{law})}{=} \int_0^1 du\,[((1-a)+au)\,\mathrm{B}_1-\mathrm{B}_u]^2.$$

Proof. — $(2c')$ immediately follows from $(2c)$, and $(2c'')$ is obtained from $(2c)$ by taking $f(u)=au$.

It remains to prove $(2c)$. This follows from $(2b)$, when it is applied to:

$$\varphi(u,s)=1_{(s\leqq u)}-(f(1)-f(s)).$$

Indeed, with this particular φ, we obtain, on one hand:

$$\int_0^1 d\mathrm{B}_s\,\varphi(u,s)=\mathrm{B}_u-\left(f(1)\,\mathrm{B}_1-\int_0^1 f(s)\,d\mathrm{B}_s\right)=\mathrm{B}_u-\int_0^1 f'(s)\,\mathrm{B}_s\,ds,$$

whereas:

$$\int_0^1 dB_s\, \varphi(s, u) = (1 - f(1) + f(u))\, B_1 - B_u.$$

The first identity in law in (2 *c*) now appears as a consequence of (2 *b*). The second identity in law in (2 *c*) follows from the first by using:

$$(B_u,\ u \leqq 1) \overset{(\text{law})}{=} (B_1 - B_{1-u};\ u \leqq 1).$$

(2.2) We now strive to obtain another extension of the identity in law (1 *d*), which corresponds roughly to the computations of T. Chan and C. Rogers [*see* (1.3) above] and, more importantly, shall put us on the right road to the infinite dimensional extension which we develop in paragraph 3.

In this paragraph, ξ and Z are two independent 2-dimensional Brownian motions starting from 0. We recall some notation from [13]: consider the decomposition

$$Z_s = \tilde{Z}_s + s Z_1 \quad (s \leqq 1); \tag{2 d}$$

$(\tilde{Z}_s,\ s \leqq 1)$ is a Brownian bridge which is independent of Z_1. Using the decomposition (2 *d*), we develop $\mathscr{A} \equiv \int_0^1 Z_s \times dZ_s$, and we find:

$$\mathscr{A} = \tilde{\mathscr{A}} + Z_1 \times \beta \qquad \text{where} \quad \tilde{\mathscr{A}} = \int_0^1 \tilde{Z}_s \times d\tilde{Z}_s, \tag{2 e}$$

and

$$\beta = \int_0^1 (s\, d\tilde{Z}_s - \tilde{Z}_s\, ds) = \int_0^1 (s\, dZ_s - Z_s\, ds).$$

An integration by parts shows that:

$$\beta = Z_1 - 2 \int_0^1 ds\, Z_s = -2 \int_0^1 ds\, (Z_s - s Z_1)$$

so that, with our previous notation for the barycentre, we obtain:

$$\beta = Z_1 - 2G = -2\tilde{G}. \tag{2 f}$$

The rest of this paragraph is devoted to showing the following identity in law:

$$(\tilde{\mathscr{A}},\ \beta,\ Z_1) \overset{(\text{law})}{=} \left(\int_0^1 d\xi_s . (Z_s - G),\ G,\ \xi_1 \right). \tag{2 g}$$

To prove (2 *g*), we shall use some independence properties between certain r.v's, together with the following almost sure identity, which is a particular

case of $(2a)$:

$$(2h)\qquad \int_0^1 d\xi_s.(Z_s-G)=-\int_0^1 dZ_s.\tilde{\xi}_s=\int_0^1 d\tilde{\xi}_s.Z_s$$

(the last equality is obtained by integration by parts). We now remark that, on the left-hand side of $(2g)$, the pair $(\tilde{\mathscr{A}}, \beta)$ is independent of Z_1, since, from $(2f)$, $\beta=-2\tilde{G}$ is measurable with respect to $\tilde{Z}$. Likewise, on the right-hand side of $(2g)$, we see that the variable ξ_1 is independent from the pair $\left(\int_0^1 d\xi_s.(Z_s-G), G\right)$, by using the identity $(2h)$. Hence, in order to prove $(2g)$, it is now sufficient to prove

$$(2i)\qquad (\tilde{\mathscr{A}}, \beta)\overset{(\text{law})}{=}\left(\int_0^1 d\xi_s.(Z_s-G), G\right).$$

We shall derive this identity in law from:

$$(2j)\qquad (\mathscr{A}, Z_1)\overset{(\text{law})}{=}\left(\int_0^1 d\xi_s.Z_s, \xi_1\right),$$

the proof of which we postpone.

Assuming for the moment that $(2j)$ holds, we derive the following equality, for every $\lambda\in\mathbb{C}$, and $z\in\mathbb{C}$:

$$(2j')\qquad E[\exp(i\lambda\mathscr{A})\,|\,Z_1=z]=E\left[\exp i\lambda\int_0^1 d\xi_s.Z_s\,|\,\xi_1=z\right].$$

We deduce from $(2e)$ that the left-hand side of $(2j')$ equals:

$$(2k)\qquad E[\exp i\lambda(\tilde{\mathscr{A}}+z\times\beta)],$$

while the right-hand side of $(2j')$ is equal to:

$$(2l)\qquad E\left[\exp\left(i\lambda\int_0^1 d\xi_s.(Z_s-G)+i\lambda z.G\right)\right],$$

thanks to the independence of the right-hand side of $(2i)$ and ξ_1. The equality in law $(2i)$ now follows from the equality of $(2k)$ and $(2l)$ and the invariance by rotation of the law of the Brownian bridge $\tilde{Z}$.

It now remains to prove $(2j)$. By linearity, it is sufficient to show that for any $z\in\mathbb{C}$:

$$(2m)\qquad \int_0^1 (z+Z_s)\times dZ_s\overset{(\text{law})}{=}\int_0^1 (z+Z_s).d\xi_s.$$

This follows from the fact that each of the processes:

$$\left(\int_0^t \frac{(z+Z_s)\times dZ_s}{|z+Z_s|},\ t\geqq 0\right)\quad\text{and}\quad\left(\int_0^t \frac{(z+Z_s).d\xi_s}{|z+Z_s|},\ t\geqq 0\right)$$

is a real-valued Brownian motion, which is independent of $(|z+Z_s|, s \geqq 0)$. The proof of $(2g)$ is now complete.

(2.3) In paragraph 3, we shall give an infinite dimensional extension of the identity in law $(2g)$. However, before doing so, it may be of some interest to compare, for any $\alpha \in \mathbb{R}$, the characteristic function of

$$\mathscr{A}^{(\alpha)} \equiv \int_0^1 (Z_s - \alpha G) \times dZ_s,$$

and the Laplace transform of $\mathscr{V}^{(\alpha)} \equiv \int_0^1 |Z_s - \alpha G|^2 ds$, thus showing that the identity:

$$(1c') \qquad E[\exp(i\lambda \mathscr{A})] = E\left[\exp\left(-\frac{\lambda^2}{2}\mathscr{V}\right)\right]$$

which is a consequence of $(1c)$ has an adequate, but non-trivial, extension for any $\alpha \in \mathbb{R}$. We prove the following

PROPOSITION 2. — *Let* Z *and* ξ *be two independent* 2-*dimensional Brownian motions starting from* 0. *We define:* $\mathscr{S}^{(\alpha)} = \int_0^1 d\xi_s . (Z_s - \alpha G)$. *Then, we have:*

$$(2n1) \qquad \mathscr{A}^{(\alpha)} \overset{(\text{law})}{=} \mathscr{S}^{(\alpha/2)}$$

and, consequently, for every $\lambda \in \mathbb{R}$:

$$(2n2) \quad E[\exp(i\lambda \mathscr{A}^{(\alpha)})] = E\left[\exp\left(-\frac{\lambda^2}{2}\mathscr{V}^{(\alpha/2)}\right)\right]$$
$$= \left[\left(1-\frac{\alpha}{2}\right)^2 \cosh\lambda + \frac{\alpha}{2}\left(2-\frac{\alpha}{2}\right)\frac{\sinh\lambda}{\lambda}\right]^{-1}.$$

Proof. — On one hand, we deduce from formula $(2e)$ that:

$$\mathscr{A}^{(\alpha)} = \tilde{\mathscr{A}} + \left(1-\frac{\alpha}{2}\right) Z_1 \times \beta.$$

On the other hand, we have, from the definition of $\mathscr{S}^{(\alpha)}$:

$$\mathscr{S}^{(\alpha)} = \int_0^1 d\xi_s . (Z_s - G) + (1-\alpha)\,\xi_1 . G,$$

and we deduce from $(2g)$ that:

$$\mathscr{S}^{(\alpha)} \overset{(\text{law})}{=} \tilde{\mathscr{A}} + (1-\alpha)\,\beta . Z_1 \overset{(\text{law})}{=} \tilde{\mathscr{A}} + (1-\alpha)\, Z_1 \times \beta.$$

Comparing this with the above expression of $\mathscr{A}^{(\alpha)}$, we obtain $(2n1)$, and $(2n2)$ follows, with the help of formula (2.8) in [13], for the explicit computation in terms of cosh and sinh. □

3. A SIMULTANEOUS ORTHOGONAL DECOMPOSITION OF TWO INDEPENDENT BROWNIAN MOTIONS

(3.1) The decomposition (2*e*) $\mathscr{A} = \tilde{\mathscr{A}} + Z_1 \times \beta$ arises as the first step in a series development of $\mathscr{A}$, which is obtained when developing $(Z_s, s \leqq 1)$ along the Legendre polynomials orthogonal basis of $L^2[0, 1]$. More precisely, we have the following

THEOREM 3 (Biane and Yor [1 *a*], [1 *b*]). — *Let* $P_n(t) \equiv \frac{1}{2^n n!} \frac{d^n}{dt^n}((t^2-1)^n)$ *be the sequence of Legendre polynomials.*

Consider the orthogonal decomposition of Brownian motion:

$$Z_t = \sum_{p=0}^{\infty} \left((2p+1) \int_0^t ds\, P_p(2s-1) \right) \beta_p \quad (t \leqq 1),$$

where:

$$\beta_p = \int_0^1 dZ_s\, P_p(2s-1).$$

Then:

(i) *The stochastic area* $\mathscr{A}$ *may be represented as:*

$$\mathscr{A} = \sum_{p=0}^{\infty} \beta_p \times \beta_{p+1} \equiv \mathscr{A}_p + \sum_{k=0}^{p-1} \beta_k \times \beta_{k+1}, \tag{3a}$$

where the convergence holds both in L^2 *and a.s.*

(ii) *For any* $p \in \mathbb{N}$, *we have:*

$$E[\exp(i\lambda \mathscr{A}_p) \mid \beta_p = m] = h_p(\lambda) \exp\left(-\frac{|m|^2}{2} k_p(\lambda) \right),$$

where:

$$h_p(\lambda) = \frac{\lambda^\nu}{2^\nu \Gamma(\nu+1) I_\nu(\lambda)}; \qquad k_p(\lambda) = \lambda \frac{I_{\nu+1}}{I_\nu}(\lambda); \qquad \nu = p + \frac{1}{2}.$$

(3.2) We insist that the identity (2*e*) is the first step in the decomposition (3*a*), since $\beta_0 = Z_1$, and $\beta_1 = \beta$.

We took this remark, put together with the identity in law (2*g*) as an indication for, possibly, the existence of an interesting decomposition of $\int_0^1 d\xi_s \cdot Z_s$.

In fact, the rest of this paragraph shall be devoted to proving the following companion to Theorem 3.

THEOREM 4. — *Let ξ and Z be two $\mathbb{R}^2$-valued independent Brownian motions, starting from* 0.

Consider the two orthogonal decompositions:

$$\xi(s)=\sum_{k=0}^{\infty} v_{2k+1}(s)\gamma_{2k} \qquad \text{and} \qquad Z(s)=\sum_{k=0}^{\infty} v_{2k+2}(s)\gamma_{2k+1}$$

where:

$$(3\,b)\quad \begin{cases} v_{k+1}(s)=(2k+1)\displaystyle\int_0^s du\,P_k(1-u) & \text{if } k\equiv 0,1 \pmod 4 \\ v_{k+1}(s)=-(2k+1)\displaystyle\int_0^s du\,P_k(1-u) & \text{if } k\equiv 2,3 \pmod 4 \end{cases}$$

$$\gamma_{2k}=(-1)^k\int_0^1 P_{2k}(1-s)\,d\xi_s, \qquad \gamma_{2k+1}=(-1)^k\int_0^1 P_{2k+1}(1-s)\,dZ_s$$

and $(P_p, p\in\mathbb{N})$ *is the sequence of Legendre polynomials.*

Define:

$$\xi_{2p}(s)=\sum_{k=p}^{\infty} v_{2k+1}(s)\gamma_{2k} \qquad \text{and} \qquad Z_{2p+1}(s)=\sum_{k=p}^{\infty} v_{2k+2}(s)\gamma_{2k+1}$$

$$\mathcal{S}_{2p}=\int_0^1 d\xi_{2p}(s).Z_{2p+1}(s) \qquad \text{and} \qquad \mathcal{S}_{2p+1}=\int_0^1 d\xi_{2p+2}(s).Z_{2p+1}(s).$$

Then, the stochastic integral $\mathcal{S}_0$ *may be represented as:*

$$\mathcal{S}_0=\sum_{k=0}^{\infty}\gamma_k\cdot\gamma_{k+1},$$

and, for every p, we have:

$$(\mathcal{A}_p, \beta_p, \ldots, \beta_0)\overset{(\text{law})}{=}(\mathcal{S}_p, \gamma_p, \ldots, \gamma_0),$$

or equivalently: $(\beta_k;\ 0\leqq k<\infty)\overset{(\text{law})}{=}(\gamma_k; 0\leqq k<\infty)$.

Finally, the following formula holds:

$$E[\exp(i\lambda\mathcal{S}_p)\,|\,\gamma_p=m]=h_p(\lambda)\exp\left(-\frac{|m|^2}{2}k_p(\lambda)\right),$$

where h_p *and* k_p *have the same meaning as in Theorem* 3.

(3.3) To prove Theorem 4, we now mimick the proof of Theorem 3, given in Biane-Yor [1 *a*].

We first define a sequence of Gaussian processes $(\eta_p(t),\ t\leqq 1)_{p\in\mathbb{N}}$, Gaussian variables γ_p, via the recurrence formula (3 *c*) below.

Since for the even (resp.: odd) indices, η_{2p} (resp.: η_{2p+1}) is measurable with respect to ξ, resp.: Z, we shall write:

$$\eta_{2p}(t) \equiv \xi_{2p}(t) \text{ and } \eta_{2p+1}(t) \equiv Z_{2p+1}(t).$$

These recurrence formulae are:

$$(3\,c)\quad \begin{cases} \xi_{2p}(s) = \xi_{2p+2}(s) + v_{2p+1}(s)\,\gamma_{2p}, \qquad \gamma_{2p} = \displaystyle\int_0^1 v_{2p}(s)\,d\xi_{2p}(s) \\ Z_{2p+1}(s) = Z_{2p+3}(s) + v_{2p+2}(s)\,\gamma_{2p+1}, \\ \qquad\qquad \gamma_{2p+1} = \displaystyle\int_0^1 Z_{2p+1}(s)\,dv_{2p+1}(s) \end{cases}$$

with the initial conditions: $\xi_0(s) = \xi_s$ and $Z_1(s) = Z_s$, and the additional requirement that, for every s, $\eta_{k+2}(s)$ is orthogonal to γ_k. We define:

$$\mathcal{S}_{2p} = \int_0^1 d\xi_{2p}(s) \,.\, Z_{2p+1}(s) \qquad \text{and} \qquad \mathcal{S}_{2p+1} = \int_0^1 d\xi_{2p+2}(s) \,.\, Z_{2p+1}(s).$$

Now, from $(3\,c)$, we deduce:

$$(3\,d)\qquad \mathcal{S}_k = \mathcal{S}_{k+1} + \gamma_k \cdot \gamma_{k+1}.$$

In order to prove the theorem completely, it now remains to show that, for any k, γ_k and β_k have the same covariance, that is, thanks to [1 *a*]:

$$\lambda_k \overset{\text{def}}{=} \frac{1}{2} E[|\gamma_k|^2] = \frac{1}{2} E[|\beta_k|^2] = \frac{1}{2k+1},$$

and, moreover, to identify the function v_k, that is, more precisely, to show the formulae $(3\,b)$.

From now on, we shall, as we may, assume that ξ and Z are two real-valued, independent, Brownian motions. We shall proceed in 5 steps.

Step 1. — *v_k is absolutely continuous, and the sequence:* $(v'_{2p}, p \geqq 0)$, *resp.:* $(v'_{2p+1}, p \geqq 0)$ *is an orthogonal sequence in* $L^2[0, 1]$.

Proof. — γ_{2p+1} admits the Wiener representation:

$$\gamma_{2p+1} = \int_0^1 \varphi_p(s)\,dZ_s, \quad \text{for some function } \varphi_p \text{ in } L^2[0, 1].$$

Since the variables $(\gamma_{2p+1}, p \geqq 0)$ are orthogonal, the functions φ_p are also orthogonal in $L^2[0, 1]$.

From the orthogonal development $(3\,c)$, we deduce:

$$Z_1(s) = Z(s) = Z_{2p+3}(s) + \sum_{k=0}^{p} v_{2k+2}(s)\,\gamma_{2k+1},$$

hence:

$$v_{2p+2}(s)\,\mathrm{E}[\gamma_{2p+1}^2]=\mathrm{E}[\gamma_{2p+1}Z_s]=\int_0^s \varphi_p(u)\,du,$$

and consequently:

$$v'_{2p+2}(s)=\frac{1}{\mathrm{E}[\gamma_{2p+1}^2]}\varphi_p(s).$$

The proof for the odd integers is identical.

Step 2. — We prove the *following relations*: for $k\geqq 2$,

$$(3e)\qquad \begin{aligned} &\text{(i)}\quad \int_0^1 dv_k(s)\,v_{k+1}(s)=(-1)^{k+1};\\ &\text{(ii)}\quad \int_0^1 dv_k(s)\,v_{k+2p+1}(s)=0\quad (p>0)\end{aligned}$$

which may be written, using integration by parts, as:

$$(3e')\qquad \begin{aligned} &\text{(i)}\quad \int_0^1 v_k(s)\,dv_{k+1}(s)=(-1)^{k};\\ &\text{(ii)}\quad \int_0^1 v_k(s)\,dv_{k+2p+1}(s)=0\quad (p>0)\end{aligned}$$

using:

$$(3f)\qquad v_{2p+1}(1)=0,\qquad \textit{for}\quad p>0.$$

The last relation follows from the orthogonality of γ_{2p} $(p\geqq 0)$ and $\gamma_0=\xi(1)$.

Proof of (3 *e*). — Note that γ_{2p}, as defined in (3 *c*), can be written as

$$\gamma_{2p}=-\int_0^1 \xi_{2p}(s)\,dv_{2p}(s),\qquad \text{for}\quad p>0,\quad \text{since}\ \xi_{2p}(1)=0.$$

Thus, for $k\geqq 1$, we have:

$$(3g)\qquad \gamma_k=(-1)^{k+1}\int_0^1 \eta_k(s)\,dv_k(s).$$

Now, $\eta_k(s)=\eta_{k+2q+2}(s)+\sum_{p=0}^{q} v_{k+2p+1}(s)\,\gamma_{k+2p}$, therefore:

$$\gamma_k=(-1)^{k+1}\int_0^1\left(\eta_{k+2q+2}(s)+\sum_{p=0}^{q} v_{k+2p+1}(s)\,\gamma_{k+2p}\right)dv_k(s).$$

Finally, (3 *e*) follows from the orthogonality of γ_k, $\gamma_{k+2},\ldots,\gamma_{k+2q}$, and η_{k+2q+2}. □

Step 3. – We compute the two *covariances of the processes* ξ_{2p} and Z_{2p+1}. From the formulae:

$$\xi_0(s)=\xi_{2k}(s)+\sum_{p=0}^{k-1} v_{2p+1}(s)\,\gamma_{2p} \quad \text{and} \quad Z_1(s)=Z_{2k+1}(s)+\sum_{p=0}^{k-1} v_{2p+2}(s)\,\gamma_{2p+1},$$

we deduce:

$$(3h) \qquad E[\xi_{2k}(s)\,\xi_{2k}(t)]=t\wedge s-\sum_{p=0}^{k-1} v_{2p+1}(s)\,v_{2p+1}(t)\,\lambda_{2p}$$

$$(3i) \qquad E[Z_{2k+1}(s)\,Z_{2k+1}(t)]=t\wedge s-\sum_{p=0}^{k-1} v_{2p+2}(s)\,v_{2p+2}(t)\,\lambda_{2p+1},$$

where: $\lambda_k=E[\gamma_k^2]$.

Step 4. – *Recurrence formulae for the functions* v_p. Let $k\geqq 2$; we have:

$$\begin{aligned}\lambda_{2k}\,v_{2k+1}(t)&=E[\xi_{2k}(t)\,\gamma_{2k}]=-\int_0^1 E[\xi_{2k}(s)\,\xi_{2k}(t)]\,dv_{2k}(s)\\ &=-\int_0^1 dv_{2k}(s)\left(t\wedge s-\sum_{p=0}^{k-1} v_{2p+1}(s)\,v_{2p+1}(t)\,\lambda_{2p}\right), \quad \text{by } (3h)\\ &=-\int_0^1 s\,dv_{2k}(s)+t\,(v_{2k}(t)-v_{2k}(1))-\lambda_{2k-2}\,v_{2k-1}(t)+t\int_0^1 s\,dv_{2k}(s),\end{aligned}$$

using $(3e)$ and $v_1(s)=s$.

Hence, we obtain:

$$(3j) \qquad \lambda_{2k}\,v_{2k+1}(t)=\int_0^t v_{2k}(s)\,ds-\lambda_{2k-2}\,v_{2k-1}(t),$$

using $\int_0^1 v_{2k}(s)\,ds=0$, which follows from the orthogonality of $\gamma_1=G$, and γ_{2k-1}, for $k\geqq 2$.

Formula $(3j)$ is also valid for $k=1$, using $\int_0^1 v_2(s)\,ds=1$. For $k\geqq 1$, we also show in a similar way:

$$(3k) \qquad \lambda_{2k+1}\,v_{2k+2}(t)=-\int_0^t v_{2k+1}(s)\,ds-\lambda_{2k-1}\,v_{2k}(t).$$

For $k=0$, we have:

$$\lambda_1\,v_2(t)=\int_0^1 (t\wedge s)\,dv_1(s)=-\int_0^t v_1(s)\,ds+t.$$

These formulae show, by a recurrence argument, that v_p is a polynomial of degree p, for $p \geqq 1$.

Step 5. — *Conclusion.* We define a sequence of polynomials $(\tilde{v}_k, k \geqq 1)$ by:

$$\tilde{v}_k(s) = v_k(s), \qquad \text{if} \quad k \equiv 1, 2 \pmod 4,$$
$$\tilde{v}_k(s) = -v_k(s), \qquad \text{if} \quad k \equiv 0, 3 \pmod 4.$$

Then, $(\tilde{v}'_{2k}, k \geqq 1)$, resp.: $(\tilde{v}'_{2k+1}, k \geqq 0)$, is an orthogonal family of $L^2[0, 1]$.

Now, formulae $(3j)$ and $(3k)$ can be written, in terms of $\tilde{v}$, as one formula:

$$\lambda_p \tilde{v}_{p+1}(s) = -\int_0^s \tilde{v}_p(u)\, du + \lambda_{p-2} \tilde{v}_{p-1}(s) \qquad (p \geqq 2)$$

and

$$\lambda_1 \tilde{v}_2(s) = -\int_0^s \tilde{v}_1(u)\, du + \tilde{v}_1(s).$$

From the classical recurrence formula between Legendre polynomials:

$$P'_{p+1} = (2p+1) P_p + P'_{p-1},$$

it is easy to deduce that $\lambda_p = \dfrac{1}{2p+1}$ and, finally:

$$(3b') \qquad \tilde{v}'_{p+1}(t) = (2p+1) P_p(1-t),$$

which gives the desired conclusion $(3b)$. □

(3.4) Starting from the identity in law

$$(2g') \qquad (\tilde{\mathscr{A}}, \beta, Z_1) \overset{(\text{law})}{=} \left(\int_0^1 d\xi_s \times (Z_s - G), G, \xi_1 \right)$$

instead of $(2g)$, where the only difference lies in the right-hand sides, with the scalar product . in $(2g)$ replacing the exterior product $\times$ in $(2g')$, we obtain the following Theorem.

THEOREM 5. — *Let* ξ *and* Z^* *be two* $\mathbb{R}^2$*-valued independent Brownian motions, starting from* 0.

Consider the two orthogonal decompositions:

$$\xi(s) = \sum_{k=0}^{\infty} \tilde{v}_{2k+1}(s) \tilde{\gamma}_{2k} \qquad \textit{and} \qquad Z^*(s) = \sum_{k=0}^{\infty} \tilde{v}_{2k+2}(s) \tilde{\gamma}_{2k+1},$$

where:

$$\tilde{v}_{k+1}(s) = (2k+1) \int_0^s P_{2k}(1-u)\, du,$$

$$\tilde{\gamma}_{2k}=\int_0^1 \mathrm{P}_{2k}(1-s)\,d\xi_s \qquad \textit{and} \qquad \tilde{\gamma}_{2k+1}=\int_0^1 \mathrm{P}_{2k+1}(1-s)\,dZ_s^*.$$

Define:

$$\tilde{\xi}_{2p}(s)=\sum_{k=p}^{\infty} \tilde{v}_{2k+1}(s)\,\tilde{\gamma}_{2k} \qquad \textit{and} \qquad \tilde{Z}_{2p+1}^*(s)=\sum_{k=p}^{\infty} \tilde{v}_{2k+2}(s)\,\tilde{\gamma}_{2k+1}.$$

$$\tilde{\mathscr{S}}_{2p}=\int_0^1 d\tilde{\xi}_{2p}(s)\times \tilde{Z}_{2p+1}^*(s), \qquad \tilde{\mathscr{S}}_{2p+1}=\int_0^1 d\tilde{\xi}_{2p+2}(s)\times \tilde{Z}_{2p+1}^*(s).$$

Then, for every $m\in\mathbb{N}$, the stochastic integral $\tilde{\mathscr{S}}_m$ may be represented as:

$$\tilde{\mathscr{S}}_m=\sum_{k=m}^{\infty} \tilde{\gamma}_k\times\tilde{\gamma}_{k+1},$$

and we have:

$$(\mathscr{A}_m, \beta_m, \ldots, \beta_0)\overset{(\mathrm{law})}{=}(\tilde{\mathscr{S}}_m, \tilde{\gamma}_m, \ldots, \tilde{\gamma}_0),$$

or, equivalently:

$$(\beta_k;\ 0\leqq k<\infty)\overset{(\mathrm{law})}{=}(\tilde{\gamma}_k;\ 0\leqq k<\infty).$$

Proof. — We introduce the matrix $\mathrm{A}=\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$ which satisfies: $x.y=x\times \mathrm{A}\,y$, for x, $y\in\mathbb{R}^2$, and another Brownian motion Z, which we define by: $\mathrm{Z}^*=\mathrm{AZ}$. Using the notation introduced in Theorem 4 for the pair (ξ, Z), we find that:

$$\tilde{\gamma}_{2k}=(-1)^k\gamma_{2k}; \qquad \tilde{\gamma}_{2k+1}=(-1)^k\,\mathrm{A}\,\gamma_{2k+1}; \qquad \gamma_p\cdot\gamma_{p+1}=\tilde{\gamma}_p\times\tilde{\gamma}_{p+1};$$

$$\tilde{v}_{2k+1}(s)\,\tilde{\gamma}_{2k}=v_{2k+1}(s)\,\gamma_{2k}; \qquad \tilde{v}_{2k+2}(s)\,\tilde{\gamma}_{2k+1}=v_{2k+2}(s)\,\mathrm{A}\,\gamma_{2k+1},$$

and Theorem 5 now follows from Theorem 4 thanks to the above identities since, for every m, we have: $\tilde{\mathscr{S}}_m=\mathscr{S}_m$. □

4. AN EIGENVALUE INTERPRETATION OF THE ABOVE IDENTITIES IN LAW

(4.1) Our main aim in this paragraph is to show how the identity in law

$$(2b) \qquad \int_0^1 du\left(\int_0^1 d\mathrm{B}_s\,\varphi(u, s)\right)^2\overset{(\mathrm{law})}{=}\int_0^1 du\left(\int_0^1 d\mathrm{B}_s\,\varphi(s, u)\right)^2,$$

where φ is a general function of $\mathrm{L}^2([0, 1]^2, du\,ds)$ could also have been derived using the diagonalization procedure initiated by P. Lévy [7].

We first use Itô's formula to develop $\left(\int_0^t d\mathrm{B}_s\,\varphi(u, s)\right)^2$; we obtain:

$$(4a)\quad \left(\int_0^t d\mathrm{B}_s\,\varphi(u, s)\right)^2 = 2\int_0^t d\mathrm{B}_s\,\varphi(u, s)\int_0^s d\mathrm{B}_h\,\varphi(u, h) + \int_0^t ds\,\varphi^2(u, s).$$

Taking $t=1$, and integrating both sides of $(4a)$ with respect to du, we obtain the following *a.s.* identities:

$$(4b)\quad \begin{cases} 2\displaystyle\int_0^1 d\mathrm{B}_s \int_0^s d\mathrm{B}_h\,\mathrm{H}(s, h) = \int_0^1 du\left(\int_0^1 d\mathrm{B}_s\,\varphi(u, s)\right)^2 - \|\varphi\|_2^2 \\ \text{and} \\ 2\displaystyle\int_0^1 d\mathrm{B}_s \int_0^s d\mathrm{B}_h\,\mathrm{K}(s, h) = \int_0^1 du\left(\int_0^1 d\mathrm{B}_s\,\varphi(s, u)\right)^2 - \|\varphi\|_2^2 \end{cases}$$

where:

$$\mathrm{H}(s, h) = \int_0^1 du\,\varphi(u, s)\,\varphi(u, h), \qquad \mathrm{K}(s, h) = \int_0^1 du\,\varphi(s, u)\,\varphi(h, u)$$

and

$$\|\varphi\|_2^2 = \int_0^1 du \int_0^1 ds\,\varphi^2(u, s).$$

As a consequence of $(4b)$, the identity in law $(2b)$ is equivalent to:

$$(4c)\qquad \int_0^1 d\mathrm{B}_s \int_0^s d\mathrm{B}_h\,\mathrm{H}(s, h) \overset{(\mathrm{law})}{=} \int_0^1 d\mathrm{B}_s \int_0^s d\mathrm{B}_h\,\mathrm{K}(s, h).$$

(4.2) We now recall the main facts about Lévy's diagonalization procedure and apply them to prove $(4c)$.

To any function $a:(s, h) \to a(s, h)$ belonging to $\mathrm{L}^2([0, 1]^2, ds\,dh)$, one can associate an Hilbert-Schmidt operator A, which is defined by:

$$\mathrm{A}f(s) = \int_0^1 a(s, h) f(h)\,dh \qquad \text{for} \quad f \in \mathrm{L}^2([0, 1], dh).$$

We now remark that, if Φ denotes the Hilbert-Schmidt operator thus associated to φ, then $\Phi^*\Phi$, resp.: $\Phi\Phi^*$, is the trace-class (and, consequently, Hilbert-Schmidt) operator associated to H, resp.: K, where A^* denotes the adjoint of A.

The key to the proof of $(4c)$ is now that $\Phi^*\Phi$ and $\Phi\Phi^*$ have the same eigenvalues, with the same orders of multiplicity.

Indeed, with this fact in hand, it remains to use the following well-known arguments:

if a is a *symmetric* function of $\mathrm{L}^2([0, 1]^2)$, and if A is a trace-class operator, then A is diagonalizable in an orthonormal basis of eigenvectors

$(a_i)_{i\in\mathbb{N}}$ of $L^2([0,1])$, corresponding to the eigenvalues (μ_i), so that: $A a_i = \mu_i a_i$ $(i\in\mathbb{N})$.

The function a may be represented as:

$$a(s,h)=\sum_{j=1}^{\infty}\mu_j a_j(s)\,a_j(h) \qquad \text{in} \quad L^2([0,1]^2, ds\,dh),$$

and we have:

$$(4d)\quad X_a \overset{\text{def}}{=} \int_0^1 dB_s \int_0^s dB_h\, a(s,h) = \sum_j \mu_j \int_0^1 dB_s\, a_j(s) \int_0^s dB_h\, a_j(h)$$
$$= \frac{1}{2}\sum_j \mu_j \left\{ \left(\int_0^1 dB_s\, a_j(s) \right)^2 - 1 \right\} = \frac{1}{2}\left(\sum_j \mu_j N_j^2\text{-trace}(A) \right),$$

where $(N_j;\, j\in\mathbb{N}) \equiv \left(\int_0^1 dB_s\, a_j(s);\, j\in\mathbb{N} \right)$ is a sequence of $\mathcal{N}(0,1)$ independent random variables.

In our application, if we call (φ_i) the sequence (a_i) associated to $A=\Phi^*\Phi$, resp.: $(\tilde{\varphi}_i)$ the sequence $(\tilde{a}_i)$ associated to $\tilde{A}=\Phi\Phi^*$ ([1]), we obtain:

$$\text{trace}(\Phi^*\Phi) = \text{trace}(\Phi\Phi^*) = \|\varphi\|_2^2,$$

and, from $(4d)$:

$$(4e)\quad \begin{cases} \displaystyle\int_0^1 du \left(\int_0^1 dB_s\, \varphi(u,s) \right)^2 = \frac{1}{2}\sum_j \mu_j \left(\int_0^1 dB_s\, \varphi_j(s) \right)^2 \\ \text{and} \\ \displaystyle\int_0^1 du \left(\int_0^1 dB_s\, \varphi(s,u) \right)^2 = \frac{1}{2}\sum_j \mu_j \left(\int_0^1 dB_s\, \tilde{\varphi}_j(s) \right)^2, \end{cases}$$

which implies $(2b)$.

(4.2) We end this paragraph with a few more remarks concerning the identity in law $(2b)$.

(i) It is tempting, when considering the identity $(2b)$, to wonder whether a "polarized" version might also be true, that is: for any pair of functions φ and ψ belonging to $L^2([0,1]^2, ds\,du)$, does the following:

$$(4f?)\quad \int_0^1 du \left(\int_0^1 dB_s\, \varphi(u,s) \right) \left(\int_0^1 dB_s\, \psi(u,s) \right)$$
$$\overset{(\text{law})}{=} \int_0^1 du \left(\int_0^1 dB_s\, \varphi(s,u) \right) \left(\int_0^1 dB_s\, \psi(s,u) \right)$$

([1]) The tilda notation should not cause any confusion, since we use the notation θ^* for the transpose of the operator θ.

hold?

We now prove that, even for some very particular φ and ψ, $(4f?)$ may not be true.

Indeed, if we take:

$$\varphi(u, s) = a(u)\, b(s); \qquad \psi(u, s) = c(u)\, d(s),$$

and denote:

$$A = \int_0^1 dB_s\, a(s), \qquad B = \int_0^1 dB_s\, b(s),$$
$$C = \int_0^1 dB_s\, c(s), \qquad D = \int_0^1 dB_s\, d(s),$$

then, the left-hand side, resp.: the right-hand side, of $(4f?)$ is equal to: αBD, resp.: βAC, where:

$$\alpha = \int_0^1 du\, a(u)\, c(u) \qquad \text{and} \qquad \beta = \int_0^1 du\, b(u)\, d(u).$$

Hence, in this example, $(4f?)$ is true if, and only if:

$$(4g?) \qquad \alpha BD \overset{(\text{law})}{=} \beta AC.$$

Now, it is easy to show that $(4g?)$ holds if and only if:

$$(4h) \qquad \alpha^2 b^* d^* = \beta^2 a^* c^*,$$

where $a^* = E(A^2)$, $b^* = E(B^2)$, and so on...

(ii) We go back again to the beginning of our discussion of $(4f?)$; if $(4f?)$ were true, then by linearity in φ and ψ, the identity in law would hold jointly for both sides of $(4f?)$, considered as two families of random variables indexed by the pairs (φ, ψ).

In particular, going back to Proposition 1, the law of the *process*:

$$\left(\int_0^1 ds\, (B_s - aG)^2;\ a \in \mathbb{R}\right)$$

would then be identical to that of the *process*:

$$\left(\int_0^1 ds\, (B_s - as\, B_1)^2;\ a \in \mathbb{R}\right).$$

However, this is nonsense, since replacing a by $a' + 1$ and developing both squares in the above integrals, we would arrive at the identity in law between:

$$\left(\int_0^1 ds\, (B_s - G)^2;\ 0;\ G^2\right)$$

and:

$$\left(\int_0^1 ds\,(\mathrm{B}_s - s\,\mathrm{B}_1)^2;\ \left(\int_0^1 ds\,s\,\mathrm{B}_s\right)\mathrm{B}_1;\ \frac{1}{3}\mathrm{B}_1^2\right),$$

which is, of course, absurd as far as the middle terms are concerned. In the same vein, we remark that it is also easy to show the following: the two pairs

$$\left(\int_0^1 ds\,(\mathrm{B}_s - a\,\mathrm{G})^2;\ \int_0^1 ds\,(\mathrm{B}_s - b\,\mathrm{G})^2\right)$$

and

$$\left(\int_0^1 ds\,(\mathrm{B}_s - as\,\mathrm{B}_1)^2;\ \int_0^1 ds\,(\mathrm{B}_s - bs\,\mathrm{B}_1)^2\right)$$

are identical in law if, and only if: $a = b$.

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Note added in proof. — Since we submitted our manuscript in March 1990, we have received the preprint of the paper [14] by T. Chan, where the computation of the law of $\int_0^1 ds\,(\mathrm{B}_s - \mathrm{G})^2$ is obtained as a particular case of a more general computation for Gaussian processes.

A very recent work of J. J. Prat [15] also contains this computation, which plays an important role in the study of oscillating integrals.

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