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Superprocesses and projective limits of branching Markov process

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ABSTRACT. — We present an a. s. limit theorem to construct Watanabe-Dawson "superprocesses" starting from a projective system of branching Markov processes.

RÉSUMÉ. — Cet article présente un théorème de limite presque sûre permettant d'obtenir des super-processus de Watanabe-Dawson à partir d'un système projectif de processus de branchement markoviens.

Branching Markov processes are generally considered as measure valued processes. However, as stressed by Neveu [N1], even in the case of the simplest Galton-Watson processes, the genealogy of the points can be also usefully described and a formalization in terms of trees of paths can be given as explained in the following. We will consider a family of such processes indexed by $\tau \in \mathbb{R}^+$. The paths start at a point x with a number of points following a Poisson law of mean $\psi\tau^{-1/\beta}$, and points are moving as independent Markov processes of semigroup P_t and branching at exponential times of expectation τ according to a reproduction law $\{q\}$

defined by the generating function :

$$\hat{q}(u)=\sum q_k u^k=\frac{1}{1+\beta}(1-u)^{1+\beta}+u,$$

β being a parameter in]0, 1].

Denote $\tilde{Z}_t^{(\tau)}$ the induced multiplet-valued process and $\tilde{Z}_{t,s}^{(\tau)}$ the multiplet of the ancestors of $\tilde{Z}_t^{(\tau)}$ at time $s<t$. One can check, as in [N2] that $(\tilde{Z}_{t+h,t}^{(\tau)}, t\in\mathbb{R}^+)$ and $(\tilde{Z}_t^{(\tau+h)}, t\in\mathbb{R}^+)$ have the same distributions.

Taking a projective limit, we can define the processes simultaneously in such a way that the above identities hold almost surely, for all τ and $h>0$.

By a martingale argument, it appears that the measure $\psi^{-1}\varepsilon^{1/\beta}\tilde{Z}_{t-\varepsilon}^{\varepsilon}P_\varepsilon$ converges a. s. towards a measure μ_t as $\varepsilon\downarrow 0$, and that $(\mu_t, t>0)$ is a Markov process.

If $U_t f(x)=-\operatorname{Log}(E(e^{-\mu_t(f)}))$, we have

$$U_t f=P_t f-\frac{\psi^{-\beta}}{1+\beta}\int_0^t P_s((U_{t-s}(f))^{1+\beta})\,ds$$

We recover a class of the measure valued process defined by Watanabe and Dawson, called superprocesses by Dynkin and studied by many authors : *see* [W], [Da], [D], [F1], [I], [P], [LG], [R].

When $\beta=1$, this construction is closely related to the one given by Le Gall in [LG1], using the Brownian excursion to generate the branching system *cf.* [NP], [LG2].

These results have been announced in [LJ].

Remarks : (*a*) Considering a general semi-group P_t instead of, for example, diffusions, has some drawbacks (measurability problems need more care) but also advantages. It allows to apply the theory to path-valued processes.

(*b*) The three first paragraphs introduce basic notions and technical (mostly measure-theoretical) preliminaries.

(*c*) Proofs of intuitively obvious properties have been only briefly sketched.

1. PATHS WITH EXPONENTIAL LIFETIME

Let E be a Polish space, with $\mathscr{E}$ its Borel σ-field. Let $P_t(x, dy)$ be a semi-group of markovian kernels on E. Let F be the space of mappings ω from a finite interval $]S(\omega), T(\omega)]$ into E. Denote X_t its coordinates $(\omega(t) = X_t(\omega))$. Let $\mathscr{F}^0$ be the σ-field generated by the sets $\{T \geqq t > S\} \cap \{X_t \in A\}$, $(t \in \mathbb{R}\ \mathscr{A} \in \mathscr{E})$ and set $\mathscr{F} = \mathscr{F}^0 \vee \{X_T^{-1}(A), A \in \mathscr{E}\}$. T and S are $\mathscr{F}^0$-measurable. The restriction of $\mathscr{F}$ to

$$E^{]s, t]} = T^{-1}(t) \cap S^{-1}(s) \qquad \text{is} \quad \sigma(X_u, s < u \leqq t).$$

Kolmogorov's theorem allows to construct a unique system of probabilities $\mathbb{P}_{x, s, t}$ on $E^{]s, t]}$, $\sigma(X_u, u \in]s, t])$ such that :

$$\mathbb{P}_{x, s, t}(\cap \{X_{u_i} \in A_i\}) = P_{u_i - s} 1_{A_1} \dots P_{u_n - u_{n-1}} 1_{A_n}(x)$$

for $s < u_1 \leqq \dots \leqq u_n \leqq t$ and $A_i \in \mathscr{E}$.

Using a generating semi algebra (*cf.* [N3], § I-6) of $\mathscr{F}$, composed with sets of the form $\{S < t_1\} \cap \{t_n \leqq T\} \cap \bigcap_i \{X_{t_i} \in A_i\} \cap \{X_T \in B\}$ with $t_1 < t_2 < \dots < t_n$, A_i and B belonging to $\mathscr{E}$, and the monotone class theorem, it appears that :

(1) *For any* $A \in \mathscr{F}$, $P_{x, s, t}(A \cap E^{]s, t]})$ *is a measurable function of* (x, s, t).

For any parameter $\tau > 0$, we can define a probability $\mathbb{P}^\tau_{x, s}$ on $(F, \mathscr{F})$, carried by $\{S = s\}$, by setting

$$\mathbb{P}^\tau_{x, s}(A) = \frac{1}{\tau} \int_0^\infty e^{-t/\tau} \mathbb{P}_{x, s, s+t}(A \cap E^{]s, s+t]}) dt.$$

For any $\omega \in F$ and $u \in]S(\omega), +\infty[$ $(]-\infty, T(\omega)[)$, define $k_u(\omega)$, $(\theta_u \omega)$ to be the restriction of ω to $]S(\omega), T(\omega) \wedge u]$, $(S(\omega) \vee u, T(\omega)])$.

For $0 \leqq h < T(\omega) - S(\omega)$, let $e_h(\omega)$ be the restriction of ω to $]S(\omega), T(\omega) - h]$, *i. e.* $k_{T(\omega) - h}\omega$.

The following measurability properties are easy to check :

(2) *For any* $A \in \mathscr{F}$ *and* $u \in \mathbb{R}$, $k_u^{-1}(A)$ *and* $\theta_u^{-1}(A)$ *belong to* $\mathscr{F}$.

(3) *For any* $B \in \mathscr{F}^0$ *and* $h \in \mathscr{R}^+$, $e_h^{-1}(B)$ *belongs to* $\mathscr{F}^0$.

If ω and ω′ belong to F and $S(\omega') = T(\omega)$, the path $\omega\omega'$ is defined on $]S(\omega), T(\omega')]$ by : $\omega\omega' = \omega$ on $]S(\omega), T(\omega)]$, $\omega\omega' = \omega'$ on $]S(\omega'), T(\omega')]$.

(4) *For any* $A \in \mathscr{F}$, $\{(\omega, \omega'), \omega\omega' \in \mathscr{A}\}$ *is* $\mathscr{F}_0 \otimes \mathscr{F}$ *measurable.*

The following properties can be easily checked.

P1 (Markov property).

For any A, $B \in \mathscr{F}$, $t > s \in \mathbb{R}$, $x \in E$,

$$\mathbb{P}^\tau_{x, s}(k_t^{-1}(A) \cap \theta_t^{-1}(B)) = \mathbb{E}^\tau_{x, s}(k_t^{-1}(A)\ \mathbb{P}^\tau_{X_t, t}(\theta_t^{-1} B))$$

P2 *For any* $A \in \mathscr{F}^0$ *and* $h>0$, $\mathbb{P}^{\tau}_{x,s}(e_h^{-1}(A)) = e^{-h/\tau}\,\mathbb{P}^{\tau}_{x,s}(A)$

P3 *For any* $A \in \mathscr{F}$ *and* $\tau_1, \tau_2 > 0$,

$$\int \{\omega\omega' \in A\}\, \mathbb{P}^{\tau_1}_{x,s}(d\omega)\, \mathbb{P}^{\tau_2}_{X_T(\omega),\, T(\omega)}(d\omega') = \mathbb{P}^{\tau_1+\tau_2}_{x,s}(A).$$

2. BRANCHING PATHS

Let U be the space of empty or finite sequences of positive integers, *i. e.* "words". There is a natural composition law on U as well as a lexicographical order induced by the order on $\mathbb{N} - \{0\}$. Define the mapping π from $U - \{\varnothing\}$ onto U by $\pi(j_1 \ldots j_n) = j_1 \ldots j_{n-1}$. Set $|j_1 \ldots j_n| = n$, and $|\varnothing| = 0$, to define the lenght of a word.

Following Neveu [N1], we define trees as sets of words δ containing $\varnothing$ and such that if $uj \in \delta$, for $j \in \mathbb{N} - \{0\}$ and $u \in U$, then $u \in \delta$ and also $ui \in \delta$ for any $i \leq j$.

Define the height $|\delta|$ of a tree as the length of the longest word it contains. If δ is a tree, and $u \in \delta$, then $T_u\delta = \{v \in U,\ uv \in \delta\}$ is a tree.

Set $\upsilon(\delta) = \sup(j \in \mathbb{N} - \{0\}, j \in \delta)$ if $\delta \neq \{\varnothing\}$, and $\upsilon(\{\varnothing\}) = 0$.

Let $\hat{F}$ be the space of branching paths $\alpha = (\alpha_0, \xi_\alpha)$ defined by a tree α_0 and a mapping ξ_α from α_0 into F, such that $T(\xi_\alpha(\pi(u))) = S(\xi_\alpha(u))$ for any $u \in \alpha_0 - \varnothing$.

F is imbedded in $\hat{F}$ in a canonical way. If $\omega \in F$, we identify it with the element ω' of $\hat{F}$ such that $\omega'_0 = \{\varnothing\}$ and $\xi_{\omega'}(\varnothing) = \omega$.

We set $\upsilon(\alpha) = \upsilon(\alpha_0)$, $|\alpha| = |\alpha_0|$.

If $u \in \alpha_0$, $T_u\alpha$ is defined by $(T_u\alpha)_0 = T_u(\alpha_0)$ and $\xi_{T_u\alpha}(v) = \xi_\alpha(uv)$.

Set $T_\alpha(u) = T(\xi_\alpha(u))$, $X_\alpha(u) = X_T(\xi_\alpha(u))$ and $S_\alpha(u) = S(\xi_\alpha(u))$ for any $u \in \alpha_0$. We define also $\zeta(\alpha) = \sup_{u \in \alpha_0} T_\alpha(u)$, $\sigma(\alpha) = S_\alpha(\varnothing)$.

Let $\hat{\mathscr{F}}$ be the smallest σ-algebra on $\hat{F}$ containing the sets

$$\Lambda_{u,A} = \{\alpha \in \hat{F},\ u \in \pi(\alpha_0) \text{ and } X_\alpha(u) \in A\},$$

for all $u \in U$, $A \in \mathscr{E}$ and

$$\Gamma_{u,B} = \{\alpha \in \hat{F},\ u \in \alpha_0 \text{ and } \xi_\alpha(u) \in B\},$$

for all $u \in U$, $B \in \mathscr{F}^0$.

(5) *If* $\Xi \in \hat{\mathscr{F}}$, $\Xi^{(i)} = \{\alpha,\ i \in \alpha_0$ *and* $T_i\alpha \in \Xi\} \in \hat{\mathscr{F}}$.

(If $\Xi = \Lambda_{u,A}$, $\Xi^{(i)} = \Lambda_{iu,A}$ etc.)

Set $\hat{\mathscr{F}}_0 = \{\{\alpha, \xi_\alpha(\varnothing) \in A\}\ A \in \mathscr{F}^0\}$, and define by induction the σ-algebra $\hat{\mathscr{F}}_{(n)}$ generated by subsets of $\hat{F}$ of the form :

$$\left\{\{\alpha, \upsilon(\alpha)=0 \text{ and } \xi_\alpha(\varnothing)\in A_0\} \cup \bigcup_{i=1}^{\infty} \{\alpha, \upsilon(\alpha)=i, \xi_\alpha(\varnothing)\in A_i \text{ and } T_j\alpha \in \Xi_{i,j} \text{ for } j \leqq i\},\ A_0 \in \mathscr{F}^0,\ A_i \in \mathscr{F},\ \Xi_{i,j} \in \hat{\mathscr{F}}_{(n+1)}\right\}.$$

($\hat{\mathscr{F}}_{(n)}$ describes the history until the n-th generation).

By (5), $\hat{\mathscr{F}}_{(n)} \subseteqq \hat{\mathscr{F}}$ for every n. Moreover $\Lambda_{u,A} \in \hat{\mathscr{F}}_{(|u|+1)}$ and $\Gamma_{u,B} \in \hat{\mathscr{F}}_{(|u|)}$. Hence

(6) *$\hat{\mathscr{F}}$ is the σ-algebra generated by the algebra $\bigcup\limits^{\infty} \hat{\mathscr{F}}_{(n)}$.*

Let q be a probability on $\mathbb{N}$ with $\sum\limits_0^\infty k\, q(k) < \infty$. We shall extend the probabilities $\mathbb{P}^\tau_{x,s}$ to $\hat{F}$ as follows :

PROPOSITION 1. — *There exist a unique system of probabilities $\mathbb{P}^{\tau, q}_{x,s}$ on $(\hat{F}, \hat{\mathscr{F}})$ such that :*

(*a*) *For any $\Xi \in \hat{\mathscr{F}}$, $\mathbb{P}^{\tau,q}_{x,s}(\Xi)$ is measurable in (x, s).*

(*b*) *$\mathbb{P}^{\tau,q}_{x,s}$ induces $\mathbb{P}^\tau_{x,s}$ on $(F, \mathscr{F}^0)$.*

(*c*) *For any $A \in \mathscr{F}^0$, $B \in \mathscr{E}$, $n \in \mathbb{N} - \{0\}$, $\Xi_1 \ldots \Xi_m \in \hat{\mathscr{F}}_{(n)}$,*

$$\mathbb{P}^{\tau,q}_{x,s}(\{\alpha, \xi_\alpha(\varnothing)\in A, X_\alpha(\varnothing)\in B, \upsilon(\alpha)=m \textit{ and } T_i\alpha \in \Xi_i \textit{ for } i \leqq m\}) = q(m)\, \mathbb{E}^\tau_{x,s}\left(1_A 1_{\{X_T \in B\}} \prod_{i=1}^m \mathbb{E}^{\tau,q}_{X_T, T}(\Xi_i)\right).$$

Proof. $\mathbb{P}^{\tau,q}_{x,s}$ can be defined by induction on each $\hat{\mathscr{F}}_{(n)}$ [by (*c*)]. The σ-additivity of the projective limit follows from the existence of a compact class (*cf.* [N3], § I-6) generated by sets of the form

$$\{u \in \pi(\alpha_0)\} \cap \{X_\alpha(u) \in K\},$$

$$\{u \in \alpha_0\} \cap \{a \leqq S_\alpha(u) \leqq b\} \cap \{c \leqq T_\alpha(u) \leqq d\} \cap \bigcap_{i=1}^n \{X_{t_i}(\xi_\alpha(u)) \in K_i\}$$

with $K, K_i \in K(E)$; where $K(E)$ denotes the class of compact subsets of E.

N. B. — Set, for any $n \in \mathbb{N}$, $z_n = \{u \in \alpha_0, |u| = n\}$.

z_n is a Galton Watson process of reproduction law q. Hence, if $\sum k\, q(k) \leqq 1$, α_0 is almost surely finite for any (x, s, τ).

Similary set $|Z_t| = \{u \in \alpha_0, S_\alpha(u) \leqq t < T_\alpha(u)\}$. Under $\mathbb{P}^{\tau, q}_{x, s}$, $|Z_{t+s}|$ is a continuous time Galton Watson process with reproduction law q and time constant τ.

We extend k_t to $\{\alpha, \sigma(\alpha) < t\}$ as follows : $(k_t\alpha)_0 = \{u \in \alpha_0, S_\alpha(u) < t\}$ and $\xi_{k_t\alpha} = k_t \xi_\alpha$.

To extend θ_t, set $\hat{Z}_t(\alpha) = \{v \in \alpha_0, t \in [S_\alpha(v), T_\alpha(v)[\}$.

If $u \in \hat{Z}_t(\alpha)$, define $\theta_{t,u}(\alpha)$ as follows :

$(\theta_{t,u}\alpha)_0 = T_u \alpha_0$ and, if $\beta = \theta_{t,u}\alpha$, we define $\xi_\beta(\varnothing) = \theta_t \xi_\alpha(u)$ and $\xi_\xi(v) = \xi_\alpha(uv)$ for any $v \in T_u \alpha_0 - \varnothing$.

Measurability properties

(7) *For any* $\Xi \in \hat{\mathscr{F}}$, $k_t^{-1}(\Xi)$ *is* $\hat{\mathscr{F}}$ *measurable.*

Indeed : $k_t^{-1} \Lambda_{u, A} = \Lambda_{u, A} \cap \{T_\alpha(u) < t\}$ [since $u \in \pi(k_t\alpha)_0$ if $T_\alpha(u) < t$]

$$k_t^{-1}(\Gamma_{u, B}) = \Gamma_{u, k_t^{-1}(B)} \in \hat{\mathscr{F}}.$$

Set $\hat{\mathscr{F}}_t = \{k_t^{-1}\Xi, \Xi \in \hat{\mathscr{F}}\}$.

(8) *For any* $u \in U$ *and* $\Xi \in \hat{\mathscr{F}}$, $\{\alpha \in \hat{\mathscr{F}}, u \in \hat{Z}_t(\alpha)$ *and* $\theta_{t, u}(\alpha) \in \Xi\}$ *belongs to* $\hat{\mathscr{F}}$.

(Indeed for example $\{u \in \hat{Z}_t\} \cap \theta_{t, u}^{-1}(\Gamma_{v, B}) = \{u \in \hat{Z}_t\} \cap \Gamma_{uv, B}$ if $v \neq \varnothing$, and $\{u \in \hat{Z}_t\} \cap \theta_{t, u}^{-1} \Gamma_{\varnothing, B} = \{u \in \hat{Z}_t\} \cap \Gamma_{u, \theta_t^{-1}(B)}$.)

For $u \in \hat{Z}_t$, set $X(t, u) = X(\pi(u))$ if $t = S(u)$ and $X(t, u) = X_t(\xi(u))$ if $t > S(u)$. Then, we have

PROPOSITION 2. — *For any* $s < t$, $u_1, u_2 \ldots u_n$ *and* $\Xi_1, \ldots, \Xi_n \in \hat{\mathscr{F}}$,

$$\mathbb{E}^{\tau, q}_{x, s}\left(\bigcap_{i=1}^{n} \{u_i \in \hat{Z}_t\} \cap \theta_{t, u_i}^{-1}(\Xi_i) \,|\, \hat{\mathscr{F}}_t\right) = \prod_{i=1}^{n} 1_{\{u_i \in \hat{Z}_t\}} \mathbb{E}^{\tau, q}_{X(t, u_i), t}(\Xi_i).$$

Sketch of proof. First one shows that for any $A \in \mathscr{F}$ and $\Xi \in \hat{\mathscr{F}}$,

$$\mathbb{P}^{\tau, q}_{x, s}(\{\xi(\varnothing) \in k_t^{-1}(A)\} \cap \theta_{t, \varnothing}^{-1}(\Xi) \cap \{t < T(\varnothing)\}) = \mathbb{E}^{\tau}_{x, s}(k_t^{-1}(A) \cap \{t < T\} \, E^{T, q}_{X_t, t}(\Xi))$$

by using the Markov property P1.

Then, one proves the Proposition by induction on the "degree" $|u_1| + |u_2| + \ldots + |u_n|$, using the branching property c in proposition 1 to reduce the degree after choosing test set, in $\hat{\mathscr{F}}_t$ of the form $k_t^{-1}(\Lambda)$, with $\Lambda \in \hat{\mathscr{F}}_{(n)}$ for some n.

N. B. A stronger version of this proposition has been proved in [C], when P_t is the heat semigroup.

3. GENERATING FUNCTIONAL

For any positive bounded measure μ on E and $f \in \mathscr{E}^+$, $0 < f(x) \leqq 1$ for every x, set :

$$f^\mu = \exp\left(\int \mathrm{Log} f \, d\mu\right).$$

Define $Z_t(\alpha) = \left(\sum_{u \in \hat{Z}_t(\alpha)} \varepsilon_{X(t, u)}\right)$.

The law of Z_t is determined by the generating functional

$$\Phi^\tau_{t-s}(f)(x) = \mathrm{E}^{\tau, q}_{x, s}(f^{Z_t}) \qquad (t > s).$$

Applying proposition 1 (*c*), we have

$$\Phi^\tau_t(f) = e^{-t/\tau} \mathrm{P}_t \mathrm{f} + \frac{1}{\tau}\int_0^t e^{-s/\tau} \mathrm{P}_s \hat{q}(\Phi^\tau_{t-s}(f)) ds \tag{9}$$

$\hat{q}$ being the generating function of q.
(9) can be written in the equivalent form

$$\Phi^\tau_t(f) = \mathrm{P}_t \mathrm{f} + \frac{1}{\tau}\int_0^t P_s [\hat{q}(\Phi^\tau_{t-s}(f)) - \Phi^\tau_{t-s}(f)] \, ds \tag{10}$$

Indeed, assuming (9), (10) is equivalent to the identity

$$(1 - e^{-t/\tau}) \mathrm{P}_t \mathrm{f} + \frac{1}{\tau}\int_0^t (1 - e^{-s/\tau}) \mathrm{P}_s \hat{q}(\Phi^\tau_{t-s}(f)) \, ds = \frac{1}{\tau}\int_0^t (\mathrm{P}_{t-s}(\Phi^\tau_s(f)) \, ds$$

Replacing $\Phi^\tau_s(\mathrm{f})$ by its expression in (9) in the second member, we get the first one by performing an integration.

The semigroup property

$$\Phi^\tau_{t+s} = \Phi^\tau_t \circ \Phi^\tau_s \tag{11}$$

follows from the branching markov property (proposition 2).

For any f, $0 < f \leqq 1$, equation (10) has a unique solution $\Phi^\tau_t(f)$ verifying

$$\Phi^\tau_0(f) = f, \qquad 0 \leqq \Phi^\tau_t(f) \leqq 1.$$

[Set $a_t = \Phi^{(1)}_t(f) - \Phi^{(2)}_t(f)$. Then

$$a_t = \int_0^t (\mathrm{P}_s(\hat{\mathrm{q}}(\Phi^{(1)}_{t-s} - \hat{q}(\Phi^{(2)}_{t-s}) \, ds$$

and

$$\| a_t \|_\infty \leqq \frac{1}{\tau} \int_0^t \| a_s \|_\infty (1 + \sum k\, q(k))\, ds,$$

hence $a_t = 0$ $\rbrack$.

The semigroup property follows also from the uniqueness.

(Set $\Phi_t^{(s)} = \Phi_t$ for $t \leqq s$ and $\Phi_t^{(s)} = \Phi_{t-s} \circ \Phi_s$ for $t \geqq s$, and check $\Phi_t^{(s)}$ verifies the same integral equation as Φ_t.)

Note also that for any bounded $g \in \mathscr{E}^+$, and $t > s$,

$$\mathbb{E}_{x,s}^{\tau,q}\left(\int g\, dZ_t\right) = -\frac{d}{d\varepsilon} \Phi_{t-s}^\tau (e^{-\varepsilon g}) \big|_{\varepsilon = 0}.$$

Differentiating (10) and solving the resulting linear integral equation yields

(12) $$\mathbb{E}_{x,s}^{\tau,q}\left(\int g\, dZ_t\right) = \exp\left(\frac{t-s}{\tau} \hat{q}'(1) - \frac{t-s}{\tau}\right) P_t g.$$

Note also that $\mathbb{P}_{x,s}^{\tau,q}(\zeta < t) = \Phi_{t-s}^\tau (0+)$.

In the following we shall be particulary interested in the laws of reproduction q_β, $\beta \in]0, 1]$defined by the generating functions

$$\hat{q}_\beta(u) = \frac{1}{1+\beta}(1-u)^{1+\beta} + u, \qquad u \in [0, 1] \quad (cf.\ [Z], [H]).$$

Then equation (10) can be solved for $f = u$ (const.) to yield

$$\Phi_t^\tau(u) = 1 - \frac{1-u}{(1 + t\tau^{-1}(1-u)^\beta)^{1/\beta}}.$$

In particular $\Phi_t^\tau(0+) = 1 - \left(\dfrac{\tau}{\tau + t}\right)^{1/\beta}$.

4. ERASURE

e_h extends to $\{\zeta - \sigma > h\}$ as follows :

$$(e_h \alpha)_0 = \{ u \in \alpha_0,\ \zeta(\alpha) - h > S_\alpha(u) \}$$

and $\xi_{e_h \alpha}(u) = k_{\zeta(\alpha) - h}\, \xi_\alpha(u)$ for any $u \in (e_h \alpha)_0$.

(13) *For any* $\Xi \in \hat{\mathscr{F}}$, $e_h^{-1}(\Xi) \in \hat{\mathscr{F}}$.

Indeed

$$e_h^{-1}\Lambda_{u,\mathrm{A}}=\Lambda_{u,\mathrm{A}}\cap\{\zeta-h>\mathrm{T}_\alpha(u)\}$$
$$e_h^{-1}\Gamma_{u,\mathrm{B}}=\{\zeta-h>\mathrm{S}_\alpha(u)\}\cap\Gamma_{u,k_{\zeta-h}^{-1}\mathrm{B}},$$

where $\Gamma_{u,k_{\zeta-h}^{-1}\mathrm{B}}=\{\zeta-h\geqq t\}\cap\Gamma_{u,\mathrm{B}}\in\hat{\mathscr{F}}$.

PROPOSITION 3. — *For any* $\Xi\in\hat{\mathscr{F}}$,

$$\mathbb{P}_{x,s}^{\tau,q}(e_h^{-1}(\Xi))=\lambda_h\,\mathbb{P}_{x,s}^{\tau,q^{(h)}}(\Xi)$$

with

$$\lambda_h=\mathbb{P}_{x,s}^{\tau}(\zeta>s+h)=1-\Phi_h^{\tau}(0+)$$

and

$$q^{(h)}(k)=\frac{1}{1-\lambda_h}\left(\sum_{l\geqq k}\mathrm{q}(l)\binom{l}{k}\left(1-\lambda_h\right)^k\lambda_h^{l-k}-\lambda_h\,\delta_{0k}\right).$$

The proof given by Neveu in [N2], p. 106-107, can be easily generalized to this context, using the caracterization of $\mathbb{P}_{x,0}^{\tau,q}$ given in proposition 1 and property P2 to prove the identity for $\Xi\in\hat{\mathscr{F}}_{(n)}$, by induction on n.

To start the recurrence, we need to compute $\mathbb{P}_{x,s}^{\tau,q}(e_h^{-1}\{\xi\,\varnothing)\in\mathrm{A}\})$ for $\mathrm{A}\in\mathscr{F}_0$.

We show that for $k>0$,

$$\mathbb{P}_{x,s}^{\tau,q}(e_h^{-1}(\{\xi(\varnothing)\in\mathrm{A}\}\cap\{\upsilon=k\}))=\sum_{l\leqq k}\mathrm{q}(l)\binom{l}{k}(1-\lambda_h)^{l-k}\lambda_h^{l}\,\mathbb{P}_{x,s}^{\tau}(\mathrm{A}).$$

Then, if μ denotes the law of ζ-σ under $\mathbb{P}_{.,\,.}^{\tau,\,q}$,

$$\mathbb{P}_{x,s}^{\tau,q}(e_h^{-1}(\{\xi_\varnothing\in\mathrm{A}\}\cap\{\upsilon=0\}))$$
$$=\sum q(l)\int_0^h d_\mu(z_1)\ldots\int_0^h d_\mu(z_l)\,\mathbb{E}_{x,s}^{\tau}(e_{h-max\,z_i}^{-1}(\mathrm{A}))$$
$$=\text{some constant})\ \mathbb{P}_{x,s}^{\tau}(\mathrm{A})\quad\text{by } \mathrm{P}_2.$$

It follows that

$$\mathbb{P}_{x,s}^{\tau,q}(e_n^{-1}\{\xi(\varnothing)\in\mathrm{A}\})=\lambda_h\,\mathbb{P}_{x,s}^{\tau,q}(\mathrm{A})$$

and

$$\mathbb{P}_{x,s}^{\tau,q}(e_h^{-1}(\{\xi(\varnothing)\in\mathrm{A}\}\cap\{\upsilon=0\}))=\lambda_h\,\mathrm{q}^{(h)}(0)\,\mathbb{P}_{x,s}^{\tau}(\mathrm{A}).$$

Then one proves the recurrence on events of the form

$$\{\xi(\varnothing)\in\mathrm{A}\}\cap\{\upsilon=k\}\cap\bigcap_{i=1}^{k}\mathrm{T}_i^{-1}(\Xi_i)$$

with $k>0$, $\mathrm{A}\in\mathscr{F}_0$ and $\Xi_i\in\hat{\mathscr{F}}_{(n)}$.

In particular, if $q=q_\beta$,

$$1-\lambda_h=\left(\frac{\tau}{\tau+h}\right)^{1/\beta} \quad \text{and} \quad \hat{q}_\beta^{(h)}(u)=\frac{\tau}{\tau+h}\,\frac{(1-u)^{1+\beta}}{1+\beta}+u.$$

Set $Z_{t,s}=Z_s\circ e_{t-s}$ for $t>s$. Then

$$Z_{t,s}(\alpha)=\sum_{u\in\hat{Z}_s(\alpha)}\varepsilon_{X(s,u)}\,1_{\{\zeta(\theta_{s,u}\alpha)\geqq t\}}.$$

Hence for any $t_1<t_2<\ldots<t_n, f_1, \ldots, f_n\in\mathscr{E}^+$,

$$\prod_{i=1}^n f_i^{Z_{t_n},t_i}=\prod_{u\in\hat{Z}_{t_1}}\left[f_1(X(t_1,u))\,\theta_{t_1,u}^{-1}\left(1_{\{\zeta\geqq t_2\}}\prod_{i=2}^n f_i^{Z_{t_n},t_i}\right)+\theta_{t_1,u}^{-1}(1_{\{\zeta\leqq t_2\}})\right].$$

By the branching Markov property, it follows that for $t_0<t_1$

$$\mathbb{E}_{x,t_0}^{\tau,q}\left[\prod_{i=1}^n(f_i^{Z_{t_n},t_i})\right]=\Phi_{t_1-t_0}R_{t_2-t_1}^{f_1}\ldots R_{t_n-t_{n-1}}^{f_{n-1}}(f_n)(x)$$

with $R_s^f(g)=f(\Phi_s(g)-\Phi_s(0+)+\Phi_s(0+)$.

Hence, it is easy to check that, for $h>0$

$$\mathbb{E}_{x,0}^{\tau,q}(f^{Z_t}\,|\,\sigma(Z_{t,s}, s<t-h))=\left[\frac{\Phi_h(f)-\Phi_h(0+)}{1-\Phi_h(0+)}\right]^{Z_{t,t-h}}$$

This formula says that the conditional law of Z_t given $Z_{t,s}=\mu$ is identical to the conditional law of Z_{t-s} given $Z_0=\mu$ and that each point of μ survives until $t-s$. In particular

$$(14)\qquad \mathbb{E}_{x,0}^{\tau,q}\left(\int g\,dZ_t\,|\,\sigma(Z_{t,s}, s<t-h)\right)=\frac{1}{1-\Phi_h^\tau(0+)}\int P_h g\,dZ_{t,t-h}$$

5. REDUCTION

We define a reduction operation R on trees, by induction on the height. It amounts to extending the lifetime of particles whose offspring is only one particle. $R(\delta)$ will be defined together with a mapping N_δ from $R(\delta)$

into $\mathbb{N}$, as follows :

– $R(\{\varnothing\})=(\{\varnothing\})$, $N_{\{\varnothing\}}(\varnothing)=0$;

– if $v\in R(\delta)$, $N_\delta(v)=\mathit{inf}(m, v\,1^m\in\delta$ and $\upsilon(T_{v1^m}\delta)\neq 1)$ (1) and $vi\in R(\delta)$ if and only if $N(v)<\infty$ and $i\leq\upsilon(T_{v1\,N_\delta(v)}\delta)$.

In the following, *we assume* $q(1)\neq 1$ so that $N(v)<\infty$ a. s. when $v\in R(\alpha_0)$.

Set

$$\hat{F}_0=\{\alpha\in\hat{\mathscr{F}},\ N(v)<\infty \text{ for any } v\in R(\alpha_0)\}.$$

$\hat{F}_0\in\hat{\mathscr{F}}$, is stable under k_t, $\theta_{t,u}$ etc. and has full measure under all $P^{\tau,q}_{x,s}$. Hence all the results we have obtained are still valid on $\hat{F}_0$. From now on, we shall work on $\hat{F}_0$ only.

R can be extended to $\hat{F}_0$ as follows : if $\alpha\in\hat{F}_0$, $(R\alpha)_0=R\alpha_0$ and $\xi_{R\alpha}(v)=\prod_{i=0}^{N(v)}\xi_\alpha(v\,1^i)$ for any $v\in R\alpha_0$ (where the product refers to the concatenation of paths).

PROPOSITION 4. – $R\,\mathbb{P}^{\tau,q}_{x,s}=\mathbb{P}^{\tau_R,q_R}_{x,s}$ *with*

$$\tau_R=\frac{\tau}{1-q(1)} \quad \textit{and} \quad q_R(k)=\frac{1}{1-q(1)}(q(k)-q(1)\,\delta_1(k)).$$

Sketch of proof. Note first that for any $A\in\mathscr{F}$,

$$\mathbb{E}^{\tau,q}_{x,s}\left(\prod_{i=0}^{N(\varnothing)}\xi(1^i)\in A\right)=\mathbb{P}^{\tau_R}_{x,s}(A) \quad \text{by P3.}$$

Then prove that $\mathbb{P}^{\tau,q}_{x,s}(R^{-1}(\Xi))=\mathbb{P}^{T_R,q_R}_{x,s}(\Xi)$ for Ξ in $\hat{\mathscr{F}}_{(n)}$ by induction on n.

COROLLARY. – *For any* $\Xi\in\hat{\mathscr{F}}$,

$$\mathbb{P}^{\tau,q_\beta}_{x,s}(R^{-1}\circ e_h^{-1}(\Xi))=\left(\frac{\tau}{\tau+h}\right)^{1/\beta}\mathbb{P}^{h+\tau,q_\beta}_{x,s}(\Xi).$$

[An easy computation shows $(q_\beta^{(h)})_R=q_\beta$.]

(1) 1^m means 1 written m times.

6. SUPERPROCESSES

Let μ be a probability on $(E,\mathscr{E})$, ψ and ε be positive parameters. Set $\mathbb{P}_\mu^\varepsilon = \int \mu(dx)\, \mathbb{P}_{x,0}^{\varepsilon; q_\beta}$ (β will be fixed in the following). Let π_μ^ε be a Poisson point process in $\hat{F}_0$, with intensity $\psi\varepsilon^{-1/\beta}\,\mathbb{P}_\mu^\varepsilon$. For any $F \in \hat{\mathscr{F}}^+$,

(15) $$\operatorname{Log}(E(e^{-\int F(\alpha)\pi^\varepsilon\mu(d\alpha)})) = \psi\varepsilon^{-1/\beta}\,\mathbb{E}_\mu^\varepsilon(e^{-F}-1).$$

π_μ^ε describes the evolution of a population starting at time 0 with a Poisson distribution of intensity $\psi\varepsilon^{-1/\beta}\mu$, and with reproduction law q_β. Each particle has an exponential lifetime of average ε and moves independently of the others according to the law determined by the semigroup P_t.

From the corollary to proposition 4, it follows that $R \circ e_h \pi_\mu^\varepsilon$ and $\pi_\mu^{\varepsilon+h}$ have the same distribution.

Taking a sequence $\varepsilon_n \downarrow 0$ and applying the Ionescu-Tulcea theorem, it appears one can define the processes π_μ^ε simultaneously, so that

(16) $$R \circ e_h \pi_\mu^\varepsilon = \pi_\mu^{\varepsilon+h} \quad \text{for any } \varepsilon, h \in \mathbb{R}^+ - \{0\}.$$

Set $\tilde{Z}_t^{\varepsilon,\mu} = \int Z_t(\alpha)\,\pi_\mu^\varepsilon(d\alpha)$. (For fixed ε, it describes the position of the population living at time t.)

Similary, set $\tilde{Z}_{t,s}^{\varepsilon,\mu} = \int Z_{t,s}(\alpha)\,\pi_\mu^\varepsilon(d\alpha)$. (It describes the population at times s whose offspring shall be alive at time t.)

By (16), we have for any $t > h > 0$,

(17) $$\tilde{Z}_{t,t-h}^{\varepsilon,\mu} = \tilde{Z}_{t-h}^{\varepsilon+h,\mu}.$$

Hence, from identity (14), for any bounded function f, and $0 < \eta < \varepsilon$, $E(<\tilde{Z}_{t-\eta}^{\eta,\mu}, P_\eta f> \,\|\, \sigma(\tilde{Z}_{t-\gamma}^{\gamma,\mu}, \gamma \geq \varepsilon))$ can be computed and is equal to

$$(1-\Phi_{\varepsilon-\eta}^{\eta}(0+))^{-1} <\tilde{Z}_{t-\varepsilon}^{\varepsilon,\mu}, P_\varepsilon f>.$$

$\left(\text{If } \mu \text{ is a measure and } f \text{ a function, } \langle \mu, f\rangle = \int f\, d\mu.\right)$

$\Big[$One uses the fact that for any σ-field $\mathscr{B} \subseteq \hat{\mathscr{F}}$, and bounded measurable function F, $E\left(\int F\, d\pi^\varepsilon \mid \sigma(\pi^\varepsilon(\Xi), \Xi \in \mathscr{B})\right) = \pi^\varepsilon(\mathbb{E}_\mu^\varepsilon(F \mid \mathscr{B}))$.$\Big]$

Moreover $(1-\Phi_{\varepsilon-\eta}^{\eta}(0+)) = \left(\dfrac{\eta}{\varepsilon}\right)^{1/\beta}$. Hence, we have, for f positive,

(18) *For any sequence* $\varepsilon_n \downarrow 0$, $Y_t^{\varepsilon_n, \mu}(f) = \varepsilon_n^{1/\beta} \psi^{-1} < \tilde{Z}_{t-\varepsilon_n}^{\varepsilon_n, \mu}, P_{\varepsilon_n} f >$ *is a positive martingale which has to converge a. s. for all f towards a limit* $Y_t^{\mu}(f)$, *clearly independent of the sequence* ε_n.

Using a dense countable $\mathbb{Q}$-vector subspace V of C (E), we define a random measure μ_t such that

$$<\mu_t, f> = Y_t^{\mu}(f) \quad \text{a. s. for any } f \text{ in } C(E).$$

Since $|\mathbb{E}(\mu_t(f))| \leq \mu P_t(|f|)$, the a. s. equality extends to any bounded measurable function f.

THEOREM. — μ_t *is a Markov process whose law is determined by the semigroup* Q_t *verifying :*

$$-\operatorname{Log}\int Q_t(\mu, d\upsilon)\exp\left(-\int f\, d\upsilon\right) = \int U_t(f)\, d\mu$$

for any positive bounded measurable function f, where $U_t(f)$ *is uniquely determined by the equation*

$$U_t f = P_t f - \frac{\psi^{-\beta}}{1+\beta}\int_0^t P_s(U_{t-s}(f)^{1+\beta})\, ds.$$

Proof. For any bounded f in $\mathscr{E}^+$, set

$$U_t(\mu, f) = -\operatorname{Log} E(\exp(-Y_t^{\mu}(f))$$
$$U_t^{\varepsilon}(\mu, f) = -\operatorname{Log} E(\exp(-Y_t^{\varepsilon, \mu}(f)).$$

By (15), we have, $U_t^{\varepsilon}(\mu, f) = -\frac{1}{\varepsilon'}\mathbb{E}_{\mu}^{\varepsilon}(e^{-\varepsilon' Z_{t-\varepsilon} P_{\varepsilon} f} - 1)$, where $\varepsilon' = \varepsilon^{1/\beta}\psi^{-1}$.

Hence $U_t^{\varepsilon}(\mu, f) \leq \|f\|_{\infty}$ for all t, ε and more generally

$$|U_t^{\varepsilon}(f) - U_t^{\varepsilon}(g)| \leq \|f - g\|_{\infty}.$$

Also, if we set $U_t^{\varepsilon}(f)(x) = U_t^{\varepsilon}(\varepsilon_x, f)$, we have

$$U_t^{\varepsilon}(\mu, f) = \int \mu(dx)\, U_t^{\varepsilon}(f)(x),$$

and

$$U_t^{\varepsilon}(f) = \frac{1}{\varepsilon'}(1 - \Phi_{t-\varepsilon}^{\varepsilon}(e^{-\varepsilon' P_{\varepsilon} f})). \tag{19}$$

By (18) since e^{-x} is convex, $U_t^{\varepsilon}(\mu, f)$ decreases with ε. By dominated convergence its limit has to be $U_t(\mu, f)$.

From (19) and (10) applied to $\Phi^{\varepsilon}_{t-\varepsilon}$ and $q=q_{\beta}$,

$$U^{\varepsilon}_t(f)=\frac{1}{\varepsilon'}P_{t-\varepsilon}(1-e^{-\varepsilon' P_{\varepsilon} f})-\frac{\psi^{-\beta}}{1+\beta}\int_0^{t-\varepsilon}P_s(U^{\varepsilon}_{t-s}(f)^{1+\beta})\,ds.$$

Set $U_t(f)(x)=U_t(\varepsilon_x, f)$.

Since, for any ε_0 and $\varepsilon<\varepsilon_0$,

$$\int_0^{t-\varepsilon_0}P_s(U^{\varepsilon}_{t-s}(f)^{1+\beta})\,ds+\|f\|_{\infty}^{1+\beta}\varepsilon_0 \geqq \int_0^{t-\varepsilon}P_s(U^{\varepsilon}_{t-s}(f)^{1+\beta})\,ds \geqq \int_0^{t-\varepsilon_0}P_s(U^{\varepsilon}_{t-s}(f)^{1+\beta})\,ds$$

we obtain $U_t(f)=P_t f-\frac{\psi^{-\beta}}{1+\beta}\int_0^t P_s(U_{t-s}(f)^{1+\beta})\,ds$ by dominated convergence.

Uniqueness can be shown in the same way as for equation (10).

We also have the semigroup property $U_{t+s}=U_t\circ U_s$.

We still have to show that μ_t is a Markov process, which is equivalent to prove that

$$-\operatorname{Log}E\left(\exp\left(-\sum_{i=1}^{n}Y_{t_i}(f_i)\right)\right)=U_{t_1}(f_1+U_{t_2-t_1}(f_2+\ldots))$$

for $0<t_1<\ldots<t_n$ and bounded $f_i\in\mathscr{E}^+$.

The first member equals

$$-\lim_{\varepsilon\downarrow 0}\operatorname{Log}E\left(\exp\left(-\varepsilon'\int\sum\langle Z_{t_i-\varepsilon}(\alpha),\,P_{\varepsilon}f_i\rangle\,\pi^{\varepsilon}_{\mu}(d\alpha)\right)\right)$$

$$=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon'}(1-\Phi^{\varepsilon}_{t-\varepsilon}(e^{-\varepsilon' P_{\varepsilon}f_1}\,\Phi^{\varepsilon}_{t_2-t_1}(e^{-\varepsilon' P_{\varepsilon}f_2}\ldots)))$$

$$=\lim_{\varepsilon\downarrow 0}\tilde{U}^{\varepsilon}_{t_1-\varepsilon}(P_{\varepsilon}f_1+V^{\varepsilon}_{t_2-t_1}(P_{\varepsilon}f_2+\ldots)))=(*)$$

with

$$V^{\varepsilon}_t(f)=-\frac{1}{\varepsilon'}\log(\Phi^{\varepsilon}_t(e^{-\varepsilon' f}))$$

and

$$\tilde{U}^{\varepsilon}_t(f)=\frac{1}{\varepsilon'}\mathbb{E}^{\varepsilon}_{\cdot}(1-e^{-\varepsilon' Z_t(f)})=\frac{1}{\varepsilon'}(1-\Phi^{\varepsilon}_t(e^{-\varepsilon' f}))$$

(note that $U_t^\varepsilon = \tilde{U}_{t-\varepsilon}^\varepsilon \circ P_\varepsilon$). We can replace V_t^ε. by $\tilde{U}_t^\varepsilon$. Indeed,

$$\| V_t^\varepsilon(f) - \tilde{U}_t^\varepsilon(f) \|_\infty \leqq \frac{1}{2\varepsilon'} \| 1 - \Phi_t^\varepsilon(e^{-\varepsilon' f}) \|_\infty^2 \leqq \frac{\varepsilon' \|f\|_\infty^2}{2}$$

when $\varepsilon' \|f\|_\infty < \frac{1}{2}$.

Moreover, $|\tilde{U}_t^\varepsilon(f) - \tilde{U}_t^\varepsilon(g)| \leqq \|f-g\|$. Hence,

$$(*) = \lim_{\varepsilon \downarrow 0} \tilde{U}_{t_1-\varepsilon}^\varepsilon (P_\varepsilon f_1 + \tilde{U}_{t_2-t_1}^\varepsilon (P_\varepsilon f_2 + \ldots)).$$

Finally since

$$|\tilde{U}_\varepsilon^\varepsilon(f) - P_\varepsilon f| \leqq \frac{1}{\varepsilon' \varepsilon (1+\beta)} \int_0^\varepsilon P_s (1 - \Phi_{t-s}^\varepsilon (e^{-\varepsilon' f}))^{1+\beta} ds$$

$$+ P_\varepsilon \left| \frac{1-e^{-\varepsilon' f}}{\varepsilon'} - f \right| \leqq \frac{1}{\varepsilon\varepsilon' (1+\beta)} \left| \int_0^\varepsilon P_s (\varepsilon' \tilde{U}_{t-s}^\varepsilon (f))^{1+\beta} ds \right| + \frac{\varepsilon' \|f\|^2}{2}$$

$$\leqq \frac{\varepsilon \psi^{-\beta}}{1+\beta} \|f\|^{1+\beta} + \frac{\varepsilon' \|f\|^2}{2} \qquad \textit{and} \qquad \tilde{U}_{t_1-t_2}^\varepsilon = \tilde{U}_\varepsilon^\varepsilon \tilde{U}_{t_1-t_2-\varepsilon}^\varepsilon,$$

$$\begin{aligned} (*) &= \lim_{\varepsilon \downarrow 0} \tilde{U}_{t_1-\varepsilon}^\varepsilon (P_\varepsilon (f_1 + \tilde{U}_{t_2-t_1-\varepsilon}^\varepsilon (P_\varepsilon (f_2 + \ldots)) \\ &= \lim_{\varepsilon \downarrow 0} \tilde{U}_{t_1}^\varepsilon (f_1 + U_{t_2-t_1}^\varepsilon (f_2 + \ldots)) \\ &= U_{t_1} (f_1 + U_{t_2-t_1} (f_2 + \ldots))). \end{aligned}$$

Q.E.D.

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REFERENCES

[C] B. Chauvin, *Product Martingales and Stoppings Lines for Branching Brownian Motion* (to appear).

[Da] D. A. Dawson, The Critical Measure Diffusion Process, *Z. Wahrsch. Verw Gebiete*, Vol. **40**, pp. 125-245.

[D1] E. B. DYNKIN, Representation for Functionals of Superprocesses by Multiple Stochastic Integrals, with Application to Self Intersection Local Times, *Asterisque*, N° **157-158**, pp. 147-171.

[D2] E. B. DYNKIN, Superprocesses and Their Additive Functionals, *Trans. A.M.S.*, Vol. **314**, pp. 255-282.

[D3] E. B. DYNKIN, Three Classes of Infinite Dimensional Diffusions, *J. Funct. Anal.*, Vol. **86**, pp. 75-110.

[D4] E. B. DYNKIN, Branching Particle Systems and Superprocesses, *Ann. Prob.* (to appear).

[F] P. J. FITZSIMMONS, Construction and Regularity of Measure Valued Processes, *Israël J. Math.*, Vol. **64**, 1988, pp. 337-361.

[Fl] K. FLEISCHANN, Critical behaviour of Some Measure Valued Processes, *Math. Nachr.*, Vol. **135**, pp. 131-147.

[H] T. E. HARRIS, *The Theory of Branching Processes*, Springer Verlag, Berlin, 1963.

[I] I. ISCOE, On the Supports of Measure Valued Critical Branching Brownian Motions, *Ann. Prob.*, Vol. **16**, pp. 200-221.

[LG1] J. F. LE GALL, Brownian Excursions, Trees and Measure Valued Branching Processes, *Ann. Prob.* (to appear).

[LG2] J. F. LE GALL, Marches aléatoires, mouvement brownien et processus de branchement, *Séminaire de Probabilités*, XXIII; *Lect. Notes Math.*, Springer, Berlin, 1989.

[LJ] Y. LE JAN, Limites projectives de processes de branchement markoviens, *C. R. Acad. Sci. Paris*, T. **308**, Séries I, 1989.

[N1] J. NEVEU, Arbres et processus de Galton-Watson, *Ann. Inst. H. Poincaré*, Vol. **22**, pp. 199-207.

[N2] J. NEVEU, Erasing a Branching Tree. Supplement to *Adv. Appl. Prob.*, 1986, pp. 101-108.

[N3] J. NEVEU, *Mathematical Foundations of Probability Calculus*, Holdenday, 1965.

[N-P] J. NEVEU and J. W. PITMAN, The Branching Process in a Brownian Excursion, *Séminaire de Probabilités*, XXIII; *Lect. Notes Math.*, Springer, Berlin, 1989.

[P1] E. A. PERKINS, A Space-Time Property of a Class of Measure-Valued Branching Diffusions, *Trans. A.M.S.*, Vol. **305**, pp. 743-795.

[P2] E. A. PERKINS, The Hausdroff Measure of the Closed Support of Super Brownian Motion, *Ann. Inst. H. Poincaré* (*Probabilités et Statistiques*), Vol. **25**, pp. 205-224.

[R] S. ROELLY-COPPOLETTA, A Criterion of Convergence of Measure Valued Process : Application to Measure Branching Processes, *Stochastics*, Vol. **17**, pp. 43-65.

[W] S. WATANABE, A Limit Theorem of Branching Processes and Continuous State Branching Processes, *J. Math. Kyoto Univ.*, Vol. **8**, pp. 141-167.

[Z] V. M. ZOLATAREV, More Exact Statements of Several Theorems in the Theory of Branching Processes, *Theor. Prob. Appl.*, Vol. **2**, pp. 256-266.

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