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Vol. 26 n° 3, 1990, p. 461-488 *Probabilités et Statistiques*

On reflecting Brownian motion – a weak convergence approach*

by

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ABSTRACT. — Consider a d-dimensional domain D that has finite Lebesgue measure. We define a certain sequence of stationary diffusion processes with drifts that tend to infinity at the boundary in such a way as to keep the sample paths in D. We prove that this sequence is tight and any limit process is a continuous stationary Markov process in $\overline{D}$ that can identified with the stationary reflecting Brownian motion defined by Fukushima using the Dirichlet form that is proportional to $\int_D |\nabla g|^2 dx, g \in H^1(D)$. Futhermore, under a mild condition on the boundary of D, which is easily satisfied when D is a Lipschitz domain, we show that this process has a Skorokhod-like semimartingale representation.

RÉSUMÉ. — Soit D un domaine de $\mathbb{R}^d$, de volume fini. Nous introduisons une suite de processus de diffusion stationnaires, dont les drifts explosent au voisinage de la frontière de façon que les trajectoires restent dans D.

Key words : Reflecting Brownian motion, normal reflection, stationary symmetric Markov process, semimartingale, Skorokhod representation, Dirichlet form.

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La suite des lois de ces processus est tendue et toute limite faible est un processus de Markov stationnaire dans $\overline{D}$, qu'on peut identifier au mouvement brownien réfléchi dans $\overline{D}$ défini par Fukushima à l'aide de la forme de Dirichlet proportionnelle à $\int_D |\nabla g|^2 dx, g \in H^1(D)$.

De plus, sous une petite hypothèse supplémentaire sur la frontière de D, satisfaite quand D est un domaine lipschitzien, nous montrons que ce processus a une représentation comme semi-martingale de type Skorokhod.

I. INTRODUCTION

Given a domain D in $\mathbb{R}^d$ that has finite Lebesgue measure, Fukushima [6,7], has defined a stationary reflecting Brownian motion in D using the Dirichlet form

$$\mathcal{E}(g,g) = \frac{\gamma}{2}\int_D |\nabla g|^2 dx, \quad \text{for} \quad g \in H^1(D), \tag{1.1}$$

where $H^1(D) = \{g \in L^2(D) : \nabla g \in L^2(D)\}$, $\gamma = 1/m(D)$, and $m(D)$ is the Lebesgue measure of D. This process is a Markov process. In order to obtain an associated strong Markov process, in the case of a bounded D, Fukushima [6] compactified D with the Kuramochi (an analogue for the Neumann problem of the Martin compactification for the Dirichlet problem). Recently, Bass and Hsu [1, 2] have shown that for a bounded Lipschitz domain, the Kuramochi boundary is the same as the Euclidean boundary and the stationary reflecting Brownian motion has a Skorokhod representation of the following form on some filtered probability space :

$$X_t = X_0 + W_t + \int_0^t n(X_s)dL_s \in \overline{D} \quad \text{for all} \quad t \geq 0, \tag{1.2}$$

where W is a Brownian motion martingale, n is the inward normal vector to the boundary ∂D of the domain D, and L is a one-dimensional non-decreasing continuous adapted process that increases only when X is on ∂D :

$$\int_0^t 1_{\partial D}(X_s)dL_s = L_t \quad \text{for all} \quad t \geq 0.$$

The fact that n is only defined a.e. with respect to surface measure on the boundary does not matter here because the "local time" L does not

charge the set times that the process is at such points. By analogy with the one-dimensional case, (1.2) is often referred to as a Skorokhod equation for reflecting Brownian motion.

For smooth domains there are a number of ways of defining (normally) reflecting Brownian motion [16, 10, 13, 15, 9], besides that given by Fukushima [6,7]. For bounded C^3 domains, these methods are all applicable and yield the same strong Markov process. One approach is to use (1.2) as a starting point. Tanaka [16] first proved that when D is a bounded convex domain, given a Brownian motion W and initial position $X(0) = x \in \overline{D}$, there is a unique solution of (1.2) that is adapted to W. Subsequently, Lions and Sznitman [10] and Saisho [13] proved a similar result for bounded C^1 domains satisfying a uniform exterior sphere condition. It readily follows form Itô's formula that the process so generated yields a solution of a submartingale problem of the form used by Stroock and Varadhan [15] to characterize reflected diffusions in C^2 domains. Alternatively, one can take a more analytical approach and solve the heat equation for the transition density, when the domain is C^3 [9]. As a consequence of these various means of definition, quite a lot is known about reflecting Brownian motions in smooth domains. On the other hand, only recently, with the results of Bass and Hsu [1,2] for bounded Lipschitz domains, have some sample path properties of reflected Brownian motions in non-smooth domains been obtained. Indeed, for domains less smooth than Lipschitz, virtually nothing is known about sample path behavior. Moreover, there has been little discussion about methods for approximating reflected Brownian motions in non-smooth domains by more familiar diffusion processes.

In this paper we suppose D is a domain in $\mathbb{R}^d$ that has finite Lebesgue measure. We use weak convergence to construct a stationary symmetric Markov process that behaves like Brownian motion in D and whose paths are confined to the Euclidean closure $\overline{D}$ of D. The approximating processes are diffusions with drifts that tend to infinity at the boundary in such a way as to confine the sample paths to D. We show that our process is the same as the stationary (normally) reflecting Brownian motion defined by Fukushima [6,7] using the Dirichlet form (1.1). Furthermore, under a mild regularity condition on the boundary, that is easily verified in the case when D is a Lipschitz domain, we show that our process has a Skorokhod-like semimartingale representation. Thus, we show how one can approximate Fukushima's stationary reflecting Brownian motions by more familiar diffusion processes, and we obtain a semimartingale decomposition of these reflecting Brownian motions under more general conditions than the Lipschitz conditions of Bass and Hsu. As one might expect with a weakening of the smoothness of the boundary, our form for the bounded variation term is less precise than that in (1.2) (see (4.5)).

Although our results provide some new information about reflected Brownian motions in non-smooth domains, a variety of open problems remain. In particular, the reader should note that we only study the *stationary* reflecting Brownian motion as a Markov process on $\overline{D}$, and do not address the question of whether it has an associated strong Markov process there. This is connected to the problem of whether the Kuramochi boundary is the same as the Euclidean boundary of D. Another interesting problem is that of finding necessary and sufficient conditions for existence and uniqueness of solutions to a suitable submartingale problem for reflecting Brownian motions in non-smooth domains. Even in the case of Lipschitz domains this question is unresolved.

II. STEIN'S REGULARIZED DISTANCE FUNCTION

Denote by m the Lebesgue measure on $\mathbb{R}^d$. Throughout this paper we will assume D is a domain in $\mathbb{R}^d$ such that $0 < m(D) < \infty$. Denote by $\theta(x)$ the distance of x from ∂D. For our discussion, we shall need a regularized distance function whose existence is guaranteed by the following lemma.

LEMMA 2.1 : *There exists a function $\delta(x) = \delta(x, \partial D)$ defined for $x \in D$ such that*

(i) $c_1\theta(x) \leq \delta(x) \leq c_2\theta(x)$ *for all* $x \in D$, *and*

(ii) δ *is* C^∞ *in* D *for any multi-index* β, *the* β*-th derivative* $\delta^{(\beta)}$ *satisfies the following inequality :*

$$|\delta^{(\beta)}(x)| \leq b_\beta(\theta(x))^{1-|\beta|} \quad \text{for all} \quad x \in D.$$

The constants b_β, c_1 *and* c_2 *are independent of* D.

Proof : This lemma is given in Stein [14, p. 171]. We outline its proof in the appendix of this paper, because we use some of the details to obtain Lemma 2.2 below. □

A refinement of Lemma 2.1, stated as Lemma 2.2 below, is needed in Section 4. It is proved under the following condition, which involves the cubes $Q_j \in \Gamma$ introduced in the appendix.

Condition (C.1) : Suppose there is a countable collection $\{B_n, n = 1, 2, \ldots\}$ of open sets in $\mathbb{R}^d$ satisfying the following two conditions.

$$\overline{D} \subset \bigcup_n B_n, \tag{2.1}$$

and for each n,

$$\limsup_{a \to \infty} \{2^\alpha m(B_n^\alpha)\} < \infty, \tag{2.2}$$

where for each integer α, B_n^α denotes the union of all those cubes Q_j of lengh $2^{-\alpha}$ that intersect B_n.

Before introducing Lemma 2.2, we give a sufficient condition for (C.1) to hold. For each n and α, let

$$D_n^\alpha = \{x \in B_n \cap D : \operatorname{dist}(x, \partial D) \leq 2^{-\alpha}\}.$$

Now, according to Stein's construction, the cubes of side lengh $2^{-\alpha}$ exist only in the region where the distance to ∂D is between $c2^{-\alpha}$ and $c2^{2-\alpha}$, where $c = 2\sqrt{d}$. Let $k \geq 0$ such that $2^2 c \leq 2^k$. Then, $B_n^\alpha \subset D_n^{\alpha-k}$ for all $\alpha > k$, and condition (C.1) will hold if for each n,

$$\limsup_{\alpha\to\infty}\{2^\alpha m(D_n^\alpha)\} < \infty. \tag{2.3}$$

If α is allowed to run through all real numbers, rather than just the integers, the lim sup constituting the left member of (2.3) defines the $(d-1)$-dimensional *upper Minkowski content* of $B_n \cap \partial D$ (see [5, pp. 273-275]). This is known to be finite when D is a Lipschitz domain [5, p. 274]. Indeed, (2.3) holds for more general domains as we will now show.

Let $-\infty < a_i < b_i < \infty$ for $i = 1, \ldots, d-1$, and define $S = \prod_{i=1}^{d-1}[a_i, b_i)$. Suppose h is continuous real-valued function defined on $\overline{S} = \prod_{i=1}^{d-1}[a_i, b_i] \subset \mathbb{R}^{d-1}$. For each fixed positive integer m and $x \in S$, there are unique indices $j(i) \in \{0, 1, \ldots, m-1\}$, $i = 1, \ldots, d-1$, such that

$$x \in S(x) \equiv \prod_{i=1}^{d-1}[a_i + j(i)m^{-1}(b_i - a_i), a_i + (j(i)+1)m^{-1}(b_i - a_i)).$$

Define upper and lower "step" functions for h as follows

$$\overline{h}_m(x) = \sup_{y\in S(x)} h(y) \quad \text{and} \quad \underline{h}_m(x) = \inf_{y\in S(x)} \quad \text{for all} \quad x \in S.$$

Further define

$$\overline{H}_m = \sum\sum_{i=1}^{d-1}\left\{\prod_{\substack{k=1\\k\neq i}}^{d-1}(m^{-1}(b_k - a_k))|\overline{h}_m(x_{j(1),\ldots,j(d-1)}) - \overline{h}_m(x_{j(1),\ldots,j(i)-1,\ldots,j(d-1)})|\right\},$$

where the first sum is over all $(d-1)$-tuples $(j(1),\dots,j(d-1))$ for $j(i)$ taking values in $\{1,\dots,m-1\}$, $i=1,\dots,d-1$; and $x_{j(1),\dots,j(d-1)}$ is a point in $\mathbb{R}^{d-1}$ such that its kth coordinate is given by $a_k+j(k)m^{-1}(b_k-a_k)$ and $x_{j(1),\dots,j(i)-1,\dots,j(d-1)}$ is defined similarly except that its ith coordinate is $a_i+(j(i)-1)m^{-1}(b_i-a_i)$. The quantity $\overline{H}_m$ is the total surface area of the vertical faces that lie above $\prod_{i=1}^{d-1}(a_i,b_i)$ and join the steps of $\overline{h}_m$. Consider the following two conditions on h :

(C.2a) $$\sup_m \overline{H}_m < \infty$$

and

$$\int_S (\overline{h}_m(x)-\underline{h}_m(x))dx \le m^{-1}C \quad \text{for all} \quad m,$$

where C is a constant independent of m.

Remark : When $d=2$, conditions (C.2a)-(C.2b) are satisfied if and only if h is of bounded variation.

DEFINITION : We say ∂D is *locally summable* if there is a covering of $\overline{D}$ by a countable collection of balls $\{B_n, n=1,2,\dots,\}$ such that whenever $B_n\cap\partial D\neq\emptyset$, there is a Cartesian coordinate system $y=(y_1,\dots,y_d)$ with origin at the center of B_n and a continuous real-valued function h_n defined on some cell $\prod_{i=1}^{d-1}[a_i^{(n)},b_i^{(n)}]$ such that conditions (C.2a) and (C.2b) hold for $h=h_n$ and

$$B_n\cap D=\{y\in B_n : y_d>h_n(y_{|d}), y_{|d}\in S_n\},$$

where $y_{|d}=(y_1,\dots,y_{d-1})$ and $S_n=\prod_{i=1}^{d-1}[a_i^{(n)},b_i^{(n)})$.

Remark : If D is a Lipschitz domain, *i.e.*, D is locally representable as the region lying on one side of the graph of a Lipschitz function, then ∂D is locally summable. For in this case, condition (C.2a) holds because for each n, $\sup_m \overline{H}_{n,m}\le c_1C_nm_{d-1}(S_n)$, where $\overline{H}_{n,m}$ denotes $\overline{H}_m$ for h_n, $c_1=2\sum_{i=1}^{d-1}\Big(\sum_{k=1}^{d-1}(b_k^{(n)}-a_k^{(n)})^2\Big)^{\frac{1}{2}}/(b_i^{(n)}-a_i^{(n)})$, C_n is the Lipschitz constant for h_n, and $m_{d-1}(S_n)$ denotes the $(d-1)$-dimensional Lebesgue measure of S_n. In addition, (C.2b) holds because

$$\int_{S_n}(\overline{h}_{n,m}(x)-\underline{h}_{n,m}(x))dx\le 2c_2m^{-1}C_nm_{d-1}(S_n),$$

where $c_2 = \Big(\sum_{i=1}^{d-1}(b_i^{(n)} - a_i^{(n)})^2\Big)^{\frac{1}{2}}$ and $\overline{h}_{n,m}, \underline{h}_{n,m}$ are the mth upper and lower step functions for the function h_n.

PROPOSITION 1.2 : *Suppose ∂D is locally summable; then condition (C.1) holds.*

Proof : Let the sets B_n be as in the definition of locally summability. By the above discussion, it suffices to verify condition (2.3) for each of the balls B_n. For the remainder of the proof we will fix n and suppress the index n in the notations B_n, h_n, S_n, etc. We define $\varepsilon = \inf_{i=1}^{d-1} |b_i - a_i|$. When $x \in B \cap D$ satisfies $\operatorname{dist}(x, \partial D) < m^{-1}\varepsilon$, there are three possibilities. Letting $x = (x_1, \ldots, x_d)$ be the local coordinate representation of x in B, we have at least one of the following.

(i) $x_{|d} \in S$ and $\underline{h}_m(x_{|d}) \leq x_d \leq \overline{h}_m(x_{|d})$,

(ii) for $G_m \equiv \{y \in \mathbb{R}^d : y_{|d} \in S, y_d > \overline{h}_m(y_{|d})\}$ and $\partial G_m^o \equiv \{y \in \partial G_m : y_{|d} \in \prod_{i=1}^{d-1}(a_i, b_i)\}$, we have $x \in G_m$ and $\operatorname{dist}(x, \partial G_m^o) < m^{-1}\varepsilon$,

(iii) $x_{|d} \in S$ and $\operatorname{dist}(x, \partial B) < m^{-1}\varepsilon$.

If we denote by κ the surface of the ball B, then

$$
\begin{aligned}
&m(\{x \in B \cap D : d(x, \partial D) < m^{-1}\varepsilon\}) \\
&\leq m(\{x \in B : x_{|d} \in S, \underline{h}_m(x_{|d}) \leq x_d \leq \overline{h}_m(x_{|d}\}) \\
&\qquad + m(\{x \in B : x_{|d} \in S, \operatorname{dist}(x, \partial B) < m^{-1}\varepsilon\}) \\
&\qquad + m(\{x \in B : x \in G_m, \operatorname{dist}(x, \partial G_m^o) < m^{-1}\varepsilon\}) \\
&\leq \int_S (\overline{h}_m(y_{|d}) - \underline{h}_m(y_{|d}))dy_{|d} + m^{-1}\varepsilon\kappa \\
&\qquad + m^{-1}\varepsilon\Big(3^{d-2}\overline{H}_m + 3^{d-1}2\prod_{i=1}^{d-1}(b_i - a_i)\Big).
\end{aligned} \tag{2.4}
$$

For the last inequality we have used the following estimate.

$$
\begin{aligned}
&m(\{x \in B : x \in G_m, \operatorname{dist}(x, \partial G_m^o) < m^{-1}\varepsilon\}) \\
&\qquad \leq m^{-1}\varepsilon\Big(3^{d-2}\overline{H}_m + 3^{d-1}2\prod_{i=1}^{d-1}(b_i - a_i)\Big),
\end{aligned} \tag{2.5}
$$

which is based on the following observation. For $j(i) \in \{0, 1, \ldots, m-1\}$, $i = 1, \ldots, d-1$, let

$$Q_{j(1),\ldots,j(d-1)} = \left\{ \prod_{i=1}^{d-1} [a_i + j(i)m^{-1}(b_i - a_i), \right.$$
$$\left. a_i + (j(i)+1)m^{-1}(b_i - a_i)) \right\} \otimes (-\infty, +\infty),$$

and let $\overline{Q}_{j(1),\ldots,j(d-1)}$ denote its closure in $\mathbb{R}^d$. Then the set

$$\{x \in B : x \in G_m, \operatorname{dist}(x, \partial G_m^o) < m^{-1}\varepsilon\}$$

is covered by the union over all $(d-1)$-tuples $(j(1), \ldots, j(d-1))$ of the following sets

$$\{x \in B : x \in G_m, \operatorname{dist}(x, \partial G_m^o \cap \overline{Q}_{j(1),\ldots,j(d-1)}) < m^{-1}\varepsilon\}. \tag{2.6}$$

Since each $\overline{Q}_{j(1),\ldots,j(d-1)}$ is a column with a base-length of $m^{-1}(b_i - a_i)$ in the ith coordinate direction, the set (2.6) is contained in a column with base-length in the ith coordinate direction of $3m^{-1}(b_i - a_i)$, $i = 1, \ldots, d-1$. More precisely, for $i = 1, \ldots, d-1$, the points in G_m that are distance less than $m^{-1}\varepsilon$ to the vertical face

$$\partial G_m^o \cap \{x \in \overline{Q}_{j(i),\ldots,j(d-1)} : x_i = a_i + j(i)m^{-1}(b_i - a_i)\},$$

are contained for $j(i) \neq 0$ in a set with volume

$$m^{-1}\varepsilon \prod_{\substack{k=1 \\ k\neq i}}^{d-1} \{3m^{-1}(b_k - a_k)\} |\overline{h}_m(x_{j(1),\ldots,j(d-1)}) \tag{2.7}$$
$$-\overline{h}_m(x_{j(1),\ldots,j(i)-1,\ldots,j(d-1)})| + 2m^{-2}\varepsilon(b_i - a_i) \prod_{\substack{k=1 \\ k\neq i}}^{d-1} (3m^{-1}(b_k - a_k)).$$

For $j(i) = 0$, $\partial G_m^o \cap \{x \in \overline{Q}_{j(1),\ldots,j(d-1)} : x_i = a_i\} = \emptyset$ by the definition of ∂G_m^o. The last term in (2.7) is part of the volume

$$2m^{-1}\varepsilon(3m^{-1})^{d-1} \prod_{k=1}^{d-1} (b_k - a_k),$$

which includes the volume of all points that are distance less than $m^{-1}\varepsilon$ from the horizontal face of $\partial G_m^o \cap \overline{Q}_{j(1),\ldots,j(d-1)}$. The latter is true even

for $j(i) = 0$. Summing over $i = 1, \ldots, d-1$, and then over all $(d-1)$-tuples $(j(1), \ldots, j(d-1))$, we deduce that the Lebesgue measure of the union of the sets in (2.6) is bounded by

$$\sum_{(j(1),\ldots,j(d-1))} \left\{ 2m^{-1}\varepsilon(3m^{-1})^{d-1} \prod_{k=1}^{d-1}(b_k - a_k) \right.$$
$$+m^{-1}\varepsilon(3m^{-1})^{d-2} \sum_{i=1}^{d-1} \prod_{\substack{k=1 \\ k\neq i}}^{d-1}(b_k - a_k)|\overline{h}_m(x_{j(1),\ldots,j(d-1)})$$
$$\left. -\overline{h}_m(x_{j(1),\ldots,j(i)-1,\ldots,j(d)-1})| \right\},$$

and then (2.5) follows.

To complete the verification of (2.3), let β be a positive integer such that $2^{-\beta} < \varepsilon$. Then

$$2^{\alpha}m(\{x \in B \cap D : \operatorname{dist}(x, \partial D) \leq 2^{\alpha}\})$$
$$\leq 2^{\beta}2^{\alpha-\beta}m(\{x \in B \cap D : \operatorname{dist}(x, \partial D) < 2^{-(\alpha-\beta)}\varepsilon\}).$$

Letting $m = 2^{(\alpha-\beta)} \to \infty$ in (2.4), we conclude that (2.3) holds. □

LEMMA 2.2 : *Suppose condition (C.1) holds. Then for any fixed B_n, there is a finite constant C, depending on n, such that for each $i \in \{1, \ldots, d\}$,*

$$\int_{B_n \cap D} \left| \frac{\partial}{\partial x_i} q(\delta(x)) \right| dx \leq C,$$

for any monotone function q defined on $[0, \infty)$ that is C^1 on $(0, \infty)$ and satisfies $q(0) = 0$ and $q(\infty) \equiv \lim_{x\to\infty} q(x) = 1$.

Proof: (i) We first recall a fact from real analysis. If ξ is a C^1-function on some interval I and ρ is a monotone C^1-function on $\xi(I)$, then

$$\int_I \left| \frac{d\rho(\xi(t))}{dt} \right| dt = \left| \int_{\xi(I)} N(s) d\rho(s) \right|,$$

where N is the Banach indicatrix of ξ, *i.e.* for each $s \in \xi(I), N(s)$ is the number of $t \in I$ for which $\xi(t) = s$, (see for example, Hewitt and Stromberg [8, p. 270-271]).

(ii) In the following discussion, we will systematically adopt the notation introduced in the appendix. Thus the regularized distance is given by

$$\delta(x) = \sum_j \operatorname{diam}(Q_j)\phi_j(x),$$

where the Q_j are cubes that divide D in a certain way and the functions ϕ_j are C^∞ approximations to 1_{Q_j}. We first estimate

$$\int_{Q_j} \left| \frac{\partial}{\partial x_i} q(\delta(x)) \right| dx.$$

From the appendix, $\phi_k(x) \neq 0$ in Q_j iff Q_k touches Q_j. Suppose the length of Q_j is $2^{-\alpha}$. Then the cubes which touch Q_j are of lengths between $2^{-\alpha-2}$ and $2^{-\alpha+2}$. So the number of cubes which touch Q_j is bounded by a constant c that depends on the dimension d, but does not depend on α. Then, by the special form of the function ϕ used to define the ϕ_k, the Banach indicatrix of $\delta(\cdot)$ in Q_j will not be bigger than $2c$. Therefore, by (i), we have

$$\int 1_{Q_j}(x) \left| \frac{\partial}{\partial x_i} q(\delta(x)) \right| dx_i \leq 2c|q(2^{-\alpha+2}) - q(2^{-\alpha-2})|.$$

Integrating out over the x_k, $k \neq i$, we obtain

$$\int_{Q_j} \left| \frac{\partial}{\partial x_i} q(\delta x)) \right| dx \leq 2c2^{-(d-1)\alpha}|q(2^{-\alpha+2}) - q(2^{-\alpha-2})|.$$

(iii) Now, let B_n^α be the union of all cubes Q_j that interest B_n and are of length $2^{-\alpha}$. The total number of such cubes is $2^{d\alpha}m(B_n^\alpha)$. Thus,

$$\begin{aligned}
\int_{B_n \cap D} \left| \frac{\partial}{\partial x_i} q(\delta(x)) \right| dx &\leq \sum_\alpha \int_{B_n^\alpha} \left| \frac{\partial}{\partial x_i} q(\delta(x)) \right| dx \\
&\leq \sum_\alpha 2^{d\alpha} m(B_n^\alpha) 2c 2^{-(d-1)\alpha} |q(2^{-\alpha+2}) - q(2^{-\alpha-2})| \\
&= 2c \sum_\alpha 2^\alpha m(B_n^\alpha) |q(2^{-\alpha+2}) - q(2^{-\alpha-2})|.
\end{aligned}$$

Since condition (C.1) holds, it follows that

$$\int_{B_n \cap D} \left| \frac{\partial}{\partial x_i} q(\delta(x)) \right| dx \leq 2C'|q(\infty) - q(0)| < \infty,$$

for some constant C' that depend on n and d. □

III. EXISTENCE THEOREM

For all $x \in D$, define

$$f_n(x) = \exp\{(n\delta(x))^{-1}\},$$

and for each fixed n, consider the stochastic differential equation in D :

$$dX^{(n)}(t) = -\frac{1}{2}\nabla f_n(X^{(n)}(t))dt + dW(t), \tag{3.1}$$

where $W = \{W(t), t \geq 0\}$ is a d-dimensional Brownian motion. For each n, define a probability measure $P^{(n)}$ on D by

$$P^{(n)}(dx) = \gamma_n \exp\{-f_n(x)\}dx, \tag{3.2}$$

where

$$\gamma_n = \left(\int_D \exp\{-f_n(x)\}dx\right)^{-1}.$$

Then we have

$$\begin{aligned} &\int_D |\nabla f_n(x)|^2 P^{(n)}(dx) \\ &= \gamma_n \int_D n^{-2}|\nabla\delta(x)|^2\delta^{-4}(x)\exp\{2(n\delta(x))^{-1} - \exp\{(n\delta(x))^{-1}\}\}dx \\ &\leq \gamma_n b_1^2 \int_D n^{-2}\delta^{-4}(x)\exp\{2(n\delta(x))^{-1} - \exp\{(n\delta(x))^{-1}\}\}dx \\ &< \infty. \end{aligned} \tag{3.3}$$

Here we used property (ii) of Lemma 2.1 to get the bound on the gradient of δ; and we used the facts that (a) the integrand is bounded in x, because of the rapid decay of $\exp\{-\exp((n\delta(x))^{-1})\}$ as $x \to \partial D$, and (b) D has finite area, to conclude that the last integral is finite. It follows from this "finite energy condition" that for each n there is a unique solution $X^{(n)}$ of (3.1) with initial measure $P^{(n)}$, and $X^{(n)}$ is a stationary symmetric Markov process (see Carlen [4] or Zheng [17]). Moreover, $X^{(n)}$ does not reach ∂D (see Zheng [18]). Let $\mathcal{B}(D)$ denote the Borel σ-field on D. It follows from (3.3), in a similar manner to Lemma 3.3 below, that the semigroup $\{P_t^{(n)}, t \geq 0\}$ on $L^2(D, \mathcal{B}(D), P^{(n)})$ associated with $X^{(n)}$ is strongly continuous.

Since we will only be concerned with *stationary* Markov processes, from hereon we restrict our attention to processes defined on the time interval [0,1]. Now, $X^{(n)}$ is a stationary symmetric Markov process whose

infinitesimal generator on smooth functions is given by $\frac{1}{2}\Delta - \frac{1}{2}\nabla f_n \cdot \nabla$, and so it follows as in Lyons and Zheng [11, p. 251-252] that

$$X^{(n)}(t) = X^{(n)}(0) + \frac{1}{2}N^{(n)}(t) - \frac{1}{2}(\overline{N}^{(n)}(1) - \overline{N}^{(n)}(1-t)) \qquad \forall\, t \in [0,1],$$

where $N^{(n)} = \{N^{(n)}(t), 0 \le t \le 1\}$ is a Brownian motion martingale with respect to the forward filtration generated by $\{X^{(n)}(t), 0 \le t \le 1\}$ and $\overline{N}^{(n)} = \{\overline{N}^{(n)}(t), 0 \le t \le 1\}$ is a Brownian motion martingale with respect to the backward filtration generated by $\{X^{(n)}(1-t), 0 \le t \le 1\}$.

Notation : For each $k \ge 1$, let $C([0,1], \mathbb{R}^k)$ denote the space of continuous $\mathbb{R}^k$-valued functions defined on [0,1], and, unless indicated otherwise, consider $C([0,1], \mathbb{R}^k)$ to be endowed with the topology of uniform convergence.

Since the individual laws of $N^{(n)}$ and $\overline{N}^{(n)}$ on $C([0,1], \mathbb{R}^d)$ are fixed as n varies, they are trivially tight. Moreover, the law $P^{(n)}(dx)$ of $X^{(n)}(0)$ converges weakly to $P(dx) = \gamma 1_D(x)dx$, where $\gamma = 1/m(D)$. It follows from the form of the tightness criterion in Billingsley [3, p. 55], which can be applied component by component, that the joint laws of $\{(N^{(n)}, \overline{N}^{(n)}, X^{(n)}(0))\}$ are tight on $C([0,1], \mathbb{R}^{2d}) \times \mathbb{R}^d$. Suppose $(N, \overline{N}, X(0))$ is a weak limit point of this sequence. Define

$$X(t) = X(0) + \frac{1}{2}N(t) - \frac{1}{2}(\overline{N}(1) - \overline{N}(1-t)). \tag{3.4}$$

Then it follows that $(N^{(n)}, \overline{N}^{(n)}, X^{(n)})$ converges weakly along a subsequence to $(N, \overline{N}, X)$, and consequently N is a martingale with respect to the forward filtration generated by $\{X(t) : 0 \le t \le 1\}$, $\overline{N}$ is a martingale with respect to the backward filtration generated by $\{X(1-t) : 0 \le t \le 1\}$, and $X(0)$ has the uniform distribution on D. We can verify that, for each fixed n, $X^{(n)}$ has the following two properties :

(3.5) $\{X^{(n)}(t) : 0 \le t \le 1\}$ and $\{X^{(n)}(1-t) : 0 \le t \le 1\}$ have the same distribution,

(3.6) $\{X^{(n)}(t)\}_{0 \le t \le 1-\delta}$ and $\{X^{(n)}(t+\delta)\}_{0 \le t \le 1-\delta}$ have the same distribution.

By convergence of the finite dimensional distributions, it follows that these properties also hold for any weak limit point X of the $X^{(n)}$. Moreover, since the paths of $X^{(n)}$ are all in D, it follows that X has paths in $\overline{D}$.

Let $C_c^\infty(D)$ denote the space of infinitely differentiable functions that have compact support in D.

LEMMA 3.1. *Suppose X is a weak limit point of the sequence $\{X^{(n)}\}$ and $g \in C_c^\infty(D)$. Then,*

$$\left\{ g(X_t) - \frac{1}{2}\int_0^t \Delta g(X_s)ds, t \geq 0 \right\}, \tag{3.7}$$

is martingale relative to the filtration generated by X.

Proof: From Itô's formula we have each n,

$$g(X_t^{(n)}) - \frac{1}{2}\int_0^t \Big(\Delta g(X_s^{(n)}) - \nabla f_n(X_s^{(n)}) \cdot \nabla g(X_s^{(n)})\Big)ds \tag{3.8}$$

is a martingale relative to the filtration generated by $X^{(n)}$. Since $\nabla f_n \to 0$ uniformly on each compact subset of D, the integral $\int_0^t \nabla f_n(X_s^{(n)}) \cdot \nabla g(X_s^{(n)})ds$ in (3.8) tends to zero uniformly on each bounded time interval as $n \to \infty$ and so can be neglected as far as martingale properties of the weak limit are concerned. It is not difficult to show that the martingale property of the remaining terms is preserved in the weak limit. □

We will now show that any limit point X of the sequence $\{X^{(n)}\}$ is a Markov process. For this, it suffices to show that the finite dimensional distributions of X are determined by a semigroup. To simplify notation, we will assume $X^{(n)}$ converges weakly to X. Since X_0 is uniformly distributed on D, there is one-to-one correspondence between $L^2(D) \equiv L^2(D, \mathcal{B}(D), \gamma dx)$ and $L^2(\Omega, \sigma(X_0), dP)$ where $\sigma(X_0)$ is the σ-field generated by X_0, and X is defined on the space Ω with probability measure P. Thus for any $t \in [0,1]$, we can define a bounded linear operator P_t on $L^2(D)$ by

$$P_t g(X_0) = E[g(X_t) \mid X_0] \quad \forall g \in L^2(D).$$

Note that although X has paths in $\overline{D}$, at any fixed time t, $X(t) \in D$ almost surely, by stationarity. Thus, there is no inconsistency in defining P_t on $L^2(D)$. From (3.5) and (3.6) for X, we deduce P_t is symmetric :

$$\int_D f P_t g dx = \int_D g P_t f dx \quad \text{for all} \quad f, g \in L^2(D).$$

As a first step towards proving that $\{P_t, t \in [0,1]\}$ is a semigroup, we prove the following.

LEMMA 3.2 : *For any* $g \in L^\infty(D) \equiv L^\infty(D, \mathcal{B}(D), \gamma dx)$,

$$\int_D |P_t^{(n)} g - P_t g|^2 dx \to 0 \quad \text{as} \quad n \to \infty.$$

Proof: Since $m(D) < \infty$, $L^\infty(D) \subset L^2(D)$ and the bounded continuous functions are dense in $L^2(D)$. Indeed, by the form of the density of $P^{(n)}$ with respect to Lebesgue measure, for any $g \in L^\infty(D)$ and $\varepsilon > 0$, there is a bounded continuous function f such that the distance between f and g is less than ε in the L^2-norms of all of the spaces $L^2(P^{(n)}) \equiv L^2(D, \mathcal{B}(D), P^{(n)})$ and $L^2(D)$ simultaneously. In addition we have the following facts : (i) as operators on their L^2-spaces and on $L^\infty(D)$, $P_t^{(n)}$ and P_t have norms equal to one, and (ii) the density $\gamma_n \exp(-f_n)$ of $P^{(n)}(dx)$ converges boundedly to γ as $n \to \infty$. Combining the above, we see that it suffices to prove the lemma for bounded continuous functions.

Suppose g is continuous and bounded. Assume $|g| \le C$, then $\{P_t^{(n)} g, n = 1, 2, \dots, \}$ and $P_t g$ are all bounded by C. Now,

$$\int_D |P_t^{(n)} g - P_t g|^2 dx$$
$$= \int_D \{(P_t^{(n)} g)^2 - 2(P_t^{(n)} g)(P_t g) + (P_t g)^2\} dx.$$

Let h be a bounded continuous function on D and let $\|h\|_\infty = \sup_{x \in D} |h(x)|$. Then,

$$\begin{aligned}
&\left| \int_D h P_t^{(n)} g dx - \int_D h P_t g dx \right| \\
&\le \left| \int_D h P_t^{(n)} g (1 - \gamma^{-1} \gamma_n e^{-f_n}) dx \right| + \left| \gamma^{-1} \int_D h P_t^{(n)} g P^{(n)}(dx) - \int_D h P_t g dx \right| \\
&\le C \|h\|_\infty \int_D |1 - \gamma^{-1} \gamma_n e^{-f_n}| dx + \gamma^{-1} \Big| E[h(X_0^{(n)}) g(X_t^{(n)})] \\
&\qquad\qquad - E[h(X_0) g(X_t)] \Big|.
\end{aligned}$$

The first integral in the last line above converges to zero as $n \to \infty$ since $\gamma^{-1} \gamma_n e^{-f_n}$ converges boudedly to 1. The last integral above also converges to zero because of the weak convergence of $X^{(n)}$ to X. Since the bounded continuous functions are dense in $L^2(D)$, and $P_t^{(n)} g$, $P_t g$ are bounded by C, it then follows from the above that $P_t^{(n)} g$ converges weakly in $L^2(D)$ to $P_t g$. Hence,

$$\lim_{n \to \infty} \int_D P_t^{(n)} g P_t g dx = \int_D (P_t g)^2 dx.$$

It remains to prove that

$$\lim_{n\to\infty}\int_D (P_t^{(n)}g)^2 dx = \int_D (P_t g)^2 dx.$$

For this, we first observe that since $X^{(n)}$ is a solution of the stochastic differential equation (3.1), for fixed $x \in D$ and $\varepsilon > 0$, by choosing $s > 0$ sufficiently small we can ensure that

$$P(X_s^{(n)} \in dy \mid X_0^{(n)} = x) = \phi(s,x,y)dy + \varepsilon_{s,x}^{(n)}(dy) \quad \text{on} \quad D,$$

where $\phi(s,x,\cdot)$ is the Gaussian density at time s of a Brownian motion that started from x at time zero, and $\varepsilon_{s,x}^{(n)}$ is a signed measure that satisfies $|\varepsilon_{s,x}^{(n)}|(D) < \varepsilon$ for all n. By the Markov property of $X^{(n)}$, for $t > s$ we have

$$\begin{aligned}(P_t^{(n)}g)(x) &= \int_D (P_{t-s}^{(n)}g)(y)P(X_s^{(n)} \in dy \mid X_0^{(n)} = x)\\ &= \int_D (P_{t-s}^{(n)}g)(y)\phi(s,x,y)dy + \int_D (P_{t-s}^{(n)}g)(y)\varepsilon_{s,x}^{(n)}(dy).\end{aligned}$$

The second term in the last line above is bounded by $C\varepsilon$, and since $P_{t-s}^{(n)}g$ converges weakly in $L^2(D)$ to $P_{t-s}g$, the first term in that line converges to

$$\int_D (P_{t-s}g)(y)\phi(s,x,y)dy$$

as $n \to \infty$. Thus, we have shown that

$$\limsup_{n\to\infty}\left|(P_t^{(n)}g)(x) - \int_D (P_{t-s}g)(y)\phi(s,x,y)dy\right| < C\varepsilon.$$

Hence, $\{(P_t^{(n)}g)(x)\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers and so has a limit $f(x)$, say, as $n \to \infty$. Since $x \in D$ was arbitrary, this proves that the pointwise limit $f(x)$ of $\{(P_t^{(n)}g)(x)\}_{n=1}^{\infty}$ exists for all $x \in D$. But we already know that $\{P_t^{(n)}g\}_{n=1}^{\infty}$ converges weakly in $L^2(D)$ to $P_t g$. Thus, $f = P_t g\, m$-a.e. on D, by the uniqueness of limits. Then, by bounded convergence, $\{P_t^{(n)}g\}_{n=1}^{\infty}$ converges to $P_t g$ in $L^2(D)$. Hence,

$$\int_D (P_t^{(n)}g)^2 dx \to \int_D (P_t g)^2 dx \quad \text{as} \quad n \to \infty,$$

as required. □

LEMMA 3.3 : *$\{P_t, 0 \le t \le 1\}$ is a strongly continuous symmetric Markovian semigroup on $L^2(D)$.*

Remark : Here the term Markovian is taken in the sense of Fukushima [7], *i.e.*, for any $g \in L^2(D)$ satisfying $0 \leq g \leq 1$, we have $0 \leq P_t g \leq 1$ for all t.

Proof : Symmetry of $\{P_t\}$ has already been verified. The Markovian property is evident from the definition of P_t via conditional expectation. For the semigroup property, since $L^\infty(D)$ is dense in $L^2(D)$, it suffices to show that

$$\int_D f(P_t P_s g - P_{t+s} g)dx = 0 \qquad \forall f, g \in L^\infty(D), \tag{3.9}$$

for all $0 \leq s, t \leq 1$ satisfying $s + t \leq 1$. Now we have the decomposition

$$\begin{aligned} &\int_D f(P_t P_s g - P_{t+s} g)dx \\ &= \int_D f(P_t P_s g - P_t^{(n)} P_s^{(n)} g)dx + \int_D f(P_t^{(n)} P_s^{(n)} g - P_{t+s}^{(n)} g)dx \\ &\qquad + \int_D f(P_{t+s}^{(n)} g - P_{t+s} g)dx. \end{aligned}$$

From Lemma 3.2., the second and last integrals in the above equality approach zero as $n \to \infty$. The third integral is zero by the semigroup property of $\{P_t^{(n)}\}$ and the equivalence of the measure $P^{(n)}(dx)$ to dx on D. Thus (3.9) follows.

For the proof that $\{P_t\}$ is strongly continuous, let $g \in C_c^\infty(D)$. By Lemma 3.1 we have

$$\begin{aligned} E[|g(X_t) - g(X_0)|^2] &= E[g^2(X_t) - g^2(X_0)] - 2E[g(X_0)(g(X_t) - g(X_0))] \\ &= E\left[\frac{1}{2}\int_0^t \Delta g^2(X_s)ds\right] - E\left[g(X_0)\int_0^t \Delta g(X_s)ds\right] \\ &\to 0 \quad \text{as} \quad t \to 0. \end{aligned}$$

It follows that $E[|E[g(X_t)|X_0] - g(X_0)|^2] \to 0$ as $t \to 0$. Hence,

$$\int_D |P_t g - g|^2 dx \to 0 \quad \text{as} \quad t \to 0. \tag{3.10}$$

Since $C_c^\infty(D)$ is dense in $L^2(D)$ and P_t has norm one on $L^2(D)$, it follows that the above also holds for any $g \in L^2(D)$. □

THEOREM 3.1 : *Any weak limit process X of the sequence $\{X^{(n)}\}$ is a continuous Markov process with stationary measure $\gamma 1_D(x)dx$ and associated semigroup $\{P_t, 0 \leq t \leq 1\}$ on $L^2(D)$.*

Proof: To prove that X is a Markov process with semigroup $\{P_t, 0 \leq t \leq 1\}$ on $L^2(D)$, it suffices to show that

$$\gamma \int_D g_0 P_{s(1)}\{g_1 P_{s(2)}\{\ldots P_{s(m)} g_m\} dx = E[g_0(X_0) g_1(X_{s(1)}) \ldots g_m(X_{s(1)+\ldots+s(m)})], \tag{3.11}$$

for all $g_0, g_1, \ldots, g_m$ that are bounded continuous functions on D, and $s(1), \ldots, s(m)$ that are positive numbers satisfying $s(1) + \ldots + s(m) \leq 1$. Indeed, for such g_i's and $s(i)$'s, by the dominated convergence theorem we have

$$\limsup_n \left| \int_D g_0 P^{(n)}_{s(1)}\{g_1 P^{(n)}_{s(2)}\{\ldots P^{(n)}_{s(m)} g_m\}(\gamma_n \exp\{-f_n\} - \gamma) dx \right| = 0,$$

and so by repeated application of Lemma 3.2,

$$\begin{aligned}
&\lim_n \int_D g_0 P^{(n)}_{s(1)}\{g_1 P^{(n)}_{s(2)}\{\ldots P^{(n)}_{s(m)} g_m\} P^{(n)}(dx) \\
&= \lim_n \left\{ \gamma \int_D g_0 P^{(n)}_{s(1)}\{g_1 P^{(n)}_{s(2)}\{\ldots P^{(n)}_{s(m)} g_m\} dx \right\} \\
&= \gamma \int_D g_0 P_{s(1)}\{g_1 P_{s(2)}\{\ldots P_{s(m) g(m)}\} dx.
\end{aligned}$$

On the other hand, since $\{P^{(n)}_t\}$ is the semigroup of the Markov process $X^{(n)}$,

$$\begin{aligned}
&\lim_n \int_D g_0 P^{(n)}_{s(1)}\{g_1 P^{(n)}_{s(2)}\{\ldots P^{(n)}_{s(m)} g_m\} P^{(n)}(dx) \\
&= \lim_n E\left[g_0(X^{(n)}_0) g_1(X^{(n)}_{s(1)}) \ldots g_m(X^{(n)}_{s(1)+\ldots+s(m)}) \right] \\
&= E[g_0(X_0) g_1(X_{s(1)}) \ldots g_m(X_{s(1)+\ldots+s(m)})].
\end{aligned}$$

Combining the above, yields (3.11) and therefore the theorem. □

We now show that the symmetric Markov process X, with semigroup $\{P_t, 0 \leq t \leq 1\}$ on $L^2(D)$, is equivalent in law to the stationary reflecting Brownian motion constructed by Fukushima [6, 7] using the theory of Dirichlet forms.

By Lemma 3.3, $\{P_t\}$ is a strongly continuous Markovian semigroup. Let $\mathcal{E}(\cdot,\cdot)$ denote the associated Dirichlet form and $\mathcal{D}(\mathcal{E})$ its domain in $L^2(D)$ (see Fukushima [7, §1.3]). Denote also by A the infinitesimal generator of $\{P_t\}$ and by $\mathcal{D}(A)$ its (strong) domain.

LEMMA 3.4. *For any $g \in C_c^\infty(D)$, we have $g \in \mathcal{D}(A)$ and*

$$Ag = \frac{1}{2} \Delta g \quad \text{on} \quad D.$$

Proof: By Lemma 3.1, for any $g \in C_c^\infty(D)$ we have

$$t^{-1}E[g(X_t) - g(X_0) \mid X_0] = (2t)^{-1}E\left[\int_0^t \Delta g(X_s)ds \;\middle|\; X_0\right],$$

where the right member tends boundedly, and hence in $L^2(\Omega, \sigma(X_0), P)$, to $\frac{1}{2}\Delta g(X_0)$, as $t \to 0$. By the correspondence between $L^2(D)$ and $L^2(\Omega, \sigma(X_0), P)$, it follows that $t^{-1}(P_t g - g)$ converges in $L^2(D)$ to $\frac{1}{2}\Delta g$. The desired result then follows the definition of A and its domain [7, §1.3]. □

By Lemma 3.4, A on $\mathcal{D}(A)$ is an extension of the symmetric operator $\frac{1}{2}\Delta$ defined from $C_c^\infty(D)$ into $L^2(D)$. It follows from [7, Lemma 2.3.4] that $\mathcal{D}(\mathcal{E}) \subset H^1(D)$, and

$$\mathcal{E}(g,g) \geq \frac{\gamma}{2}\int_D |\nabla g|^2 dx \quad \text{for all} \quad g \in \mathcal{D}(\mathcal{E}), \tag{3.12}$$

where $H^1(D) = \{g \in L^2(D) : \nabla g \in L^2(D)\}$ is endowed with the norm given by (1.3).

We shall also need the Dirichlet forms for the approximating processes $X^{(n)}$. Let $\mathcal{E}^{(n)}(\cdot,\cdot)$ denote the Dirichlet form associated with the strongly continuous symmetric Markovian semigroup $\{P_t^{(n)}\}$ and let $\mathcal{D}(\mathcal{E}^{(n)})$ denote the associated domain in $L^2(P^{(n)}) = L^2(D, \mathcal{B}(D), P^{(n)})$. Let $A^{(n)}$ denote the infinitesimal generator of $\{P_t^{(n)}\}$ and $\mathcal{D}(A^{(n)})$ denote its (strong) domain. By applying Itô's formula to $X^{(n)}$ given by (3.1), we can deduce in a similar manner to that for Lemma 3.4 that $C_c^\infty(D) \subset \mathcal{D}(A^{(n)})$ and

$$A^{(n)}g = S^{(n)} \quad \text{for all} \quad g \in C_c^\infty(D), \tag{3.13}$$

where

$$S^{(n)} = \frac{1}{2}e^{f_n}\nabla \cdot (e^{-f_n}\nabla \cdot).$$

Hence $(A^{(n)}, \mathcal{D}(A^{(n)}))$ is an extension of the symmetric operator $S^{(n)}$ defined from $C_c^\infty(D)$ into $L^2(P^{(n)})$. Since the coefficients of $S^{(n)}$ are in $C^\infty(D)$, one can check that the proof of Lemma 3.4 of [7], in particular, the use of Weyl's lemma, can be applied to conclude that the self-adjoint, non positive definite operator associated with the Dirichlet form defined by

$$(f,g) \to \frac{1}{2}\int_D (\nabla f \cdot \nabla g)(x)P^{(n)}(dx),$$

on

$$H^1(P^{(n)} \equiv \{g \in L^2(P^{(n)}) : \nabla g \in L^2(P^{(n)})\}$$

is the maximal self-adjoint, non positive definite Markovian extension of $(S^{(n)}, C_c^\infty(D))$, and so particular, $\mathcal{D}(\mathcal{E}^{(n)}) \subset H^1(P^{(n)})$ and

$$\mathcal{E}^{(n)}(g,g) \geq \frac{1}{2}\int_D |\nabla g|^2 P^{(n)}(dx) \quad \text{for all} \quad g \in \mathcal{D}(\mathcal{E}^{(n)}). \tag{3.15}$$

In fact we have the following whose proof was indicated to us by Y. Le Jan.

LEMMA 3.5 : *We have* $\mathcal{D}(\mathcal{E}^{(n)}) = H^1(P^{(n)})$ *and*

$$\mathcal{E}^{(n)}(g,g) = \frac{1}{2}\int_D |\nabla g|^2 P^{(n)}(dx) \quad \textit{for all} \quad g \in H^1(P^{(n)}). \tag{3.16}$$

Proof : By the above remarks, it suffices to show that $H^1(P^{(n)}) \subset \mathcal{D}(\mathcal{E}^{(n)})$ and (3.16) holds. For this, let $(f,g)_{L^2(P^{(n)})}$ denote $\int_D (fg)(x)P^{(n)}(dx)$ for all $f,g \in L^2(P^{(n)})$. By (3.13), the Corollary on page 19 of [7], and integration by parts, we have for each $g \in C_c^\infty(D)$,

$$\begin{aligned} \mathcal{E}^{(n)}(g,g) &= -(g, S^{(n)}g)_{L^2(P^{(n)})} \\ &= \frac{1}{2}\int_D |\nabla g|^2 P^{(n)}(dx). \end{aligned} \tag{3.17}$$

Let $H_0^1(P^{(n)})$ denote the closure of $C_c^\infty(D)$ in $H^1(P^{(n)})$ with respect to the norm $\|g\|_{H^1(P^{(n)})}$ defined by

$$\|g\|^2_{H^1(P^{(n)})} = \int_D |g|^2 P^{(n)}(dx) + \frac{1}{2}\int_D |\nabla g|^2 P^{(n)}(dx) \text{ for all } g \in H^1(P^{(n)}).$$

Since $\mathcal{D}(\mathcal{E}^{(n)})$ is complete with respect to the norm $\|\cdot\|_{\mathcal{D}(\mathcal{E}^{(n)})}$ defined by

$$\|g\|^2_{\mathcal{D}(\mathcal{E}^{(n)})} = \int_D |g|^2 P^{(n)}(dx) + \mathcal{E}^{(n)}(g,g), \quad g \in \mathcal{D}(\mathcal{E}^{(n)}),$$

and (3.16) holds on $C_c^\infty(D)$, it follows that $H_0^1(P^{(n)}) \subset \mathcal{D}(\mathcal{E}^{(n)})$ and (3.16) holds for all $g \in H_0^1(P^{(n)})$. For if $g \in H_0^1(P^{(n)})$, there is a sequence $\{g_m\}$ of functions in $C_c^\infty(D)$ that converges to g in $H^1(P^{(n)})$. Since this sequence is Cauchy in $H^1(P^{(n)})$ and (3.16) holds on $C_c^\infty(D)$, the sequence is also Cauchy in $\mathcal{D}(\mathcal{E}^{(n)})$ with the norm $\|\cdot\|_{\mathcal{D}(\mathcal{E}^{(n)})}$, and so by the completeness of $\mathcal{D}(\mathcal{E}^{(n)})$, the sequence converges in $\mathcal{D}(\mathcal{E}^{(n)})$. Since $\{g_m\}$ converges in

$L^2(P^{(n)})$ to g, it follows that this must also be the limit in $\mathcal{D}(\mathcal{E}^{(n)})$. Hence $g \in \mathcal{D}(\mathcal{E}^{(n)})$.

Note that $f_n \in H^1(P^{(n)})$ (c.f. (3.3)) and $\lim_{x \to \partial D} f_n(x) = +\infty$. Let $\phi : [0, \infty) \to \mathbb{R}$ be an infinitely differentiable, non-decreasing function on $[0, \infty)$ such that $\phi(x) = x$ for $x \in [0, \frac{1}{2}]$, $\phi(x) = 1$ for $x \geq 2$, and $\phi'(x) \leq 1$ for all $x \geq 0$. For each positive integer m, h_m defined by $h_m = \phi(m^{-1} f_n)$ is in $H^1(P^{(n)})$, $\|h_m\|_{H^1(P^{(n)})} \downarrow 0$ as $m \uparrow \infty$, and $1 - h_m \in H_0^1(P^{(n)})$. Then, for $g \in H^1(P^{(n)}) \cap L^\infty(D)$, $g_m \equiv g(1 - h_m)$ is in $H_0^1(P^{(n)})$ and

$$\begin{aligned} \|g - g_m\|^2_{H^1(P^{(n)})} &= \|g h_m\|^2_{H^1(P^{(n)})} \\ &\leq \|g\|^2_\infty \|h_m\|^2_{H^1(P^{(n)})} + \int_D |\nabla g|^2 |h_m|^2 P^{(n)}(dx). \end{aligned}$$

As $m \to \infty$, the first term in the last line above tends zero by the previously stated properties of h_m, and the second term also tends to zero by bounded convergence, since for each $x \in D$, $1 \geq h_m(x) \downarrow 0$ as $m \uparrow \infty$. It then follows by the completeness of $\mathcal{D}(\mathcal{E}^{(n)})$ with respect to $\|\cdot\|_{\mathcal{D}(\mathcal{E}^{(n)})}$ and the fact that (3.16) holds on $H_0^1(P^{(n)})$ that $g \in \mathcal{D}(\mathcal{E}^{(n)})$ and (3.16) holds for g.

Finally, consider $g \in H^1(P^{(n)})$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable, non-decreasing function such that $\psi(x) = x$ for $|x| \leq \frac{1}{2}$, $\psi(x) = 1$ for $x \geq 2$, $\psi(x) = -1$ for $x \leq -2$ and $\psi'(x) \leq 1$ for all $x \in \mathbb{R}$. Then for each positive integer m, $g_m \equiv m\psi(m^{-1} g)$ is in $H^1(P^{(n)}) \cap L^\infty(D)$, and $g_m \to g$ in $H^1(P^{(n)})$ as $m \to \infty$. It then follows from the completeness of $\mathcal{D}(\mathcal{E}^{(n)})$ with respect to $\|\cdot\|_{\mathcal{D}(\mathcal{E}^{(n)})}$ and the last paragraph above that $g \in \mathcal{D}(\mathcal{E}^{(n)})$ and (3.16) holds for g. □

Remark : Note that the proof of the above lemma did not use anything particular about the Dirichlet form $(\mathcal{E}^{(n)}, \mathcal{D}(\mathcal{E}^{(n)}))$ other than the fact that (3.16) holds on $C_c^\infty(D) \subset \mathcal{D}(\mathcal{E}^{(n)})$. It follows that any Dirichlet form corresponding to a selfadjoint, non positive definite Markovian extension of the symmetric operator $S^{(n)}$ defined from $C_c^\infty(D)$ into $L^2(P^{(n)})$ has domain equal to $H^1(P^{(n)})$ and (3.16) holds there. It follows that there is only one such extension, *i.e.*, the minimal extension equals the maximal extension equals the extension corresponding to the Dirichlet form given by the right member of (3.16) on the space $H^1(P^{(n)})$.

COROLLARY 3.1 : *We have*

$$H^1(D) \subset \mathcal{D}(\mathcal{E}^{(n)}). \tag{3.18}$$

Proof: Since the density of $P^{(n)}$ is $\gamma_n e^{-f_n}$, which is bounded uniformly as n varies, it follows that $H^1(D) \subset H^1(P^{(n)})$ and then (3.18) follows from Lemma 3.5. □

THEOREM 3.2 : *We have* $\mathcal{D}(\mathcal{E}) = H^1(D)$ *and*

$$\mathcal{E}(g,g) = \frac{\gamma}{2}\int_D |\nabla g|^2 dx \quad \text{for all} \quad g \in H^1(D). \tag{3.19}$$

It follows that $\{P_t\}$ is the strongly continuous symmetric Markovian semigroup associated with the Dirichlet form (3.19) on $H^1(D)$. In particular, it is unique and does not depend on the particular subsequence of $\{X^{(n)}\}$ chosen to converge to X.

Proof : Since there is a one-to-one correspondence [7; Theorem 1.3.1, Lemma 1.3.2] between Dirichlet forms and strongly continuous symmetric Markovian semigroups, it suffices to verify the first statement of the theorem.

Since we have (3.12), it suffices to prove that $H^1(D) \subset \mathcal{D}(\mathcal{E})$ and that for all $g \in H^1(D)$,

$$\mathcal{E}(g,g) \leq \frac{\gamma}{2}\int_D |\nabla g|^2 dx. \tag{3.20}$$

Moreover, since $\mathcal{D}(\mathcal{E})$ is complete with respect to the norm $\|\cdot\|_{\mathcal{D}(\mathcal{E})}$ defined by

$$\|g\|^2_{\mathcal{D}(\mathcal{E})} = \gamma\int_D |g|^2 dx + \mathcal{E}(g,g), \quad g \in \mathcal{D}(\mathcal{E}),$$

and $L^\infty(D) \cap H^1(D)$ is dense in $H^1(D)$, it suffices to prove that for any $g \in L^\infty(D) \cap H^1(D)$, $g \in \mathcal{D}(\mathcal{E})$ and (3.20) holds. By Corollary 3.1 and Lemma 3.5, any $g \in H^1(D)$ is in $\mathcal{D}(\mathcal{E}^{(n)})$ and (3.16) holds for g. Then, by Lemma 1.3.4 of [7], for each $t > 0$,

$$t^{-1}\int_D g(g - P_t^{(n)} g)P^{(n)}(dx) \leq \mathcal{E}^{(n)}(g,g). \tag{3.21}$$

Since the density $\gamma_n \exp(-f_n)$ of $P^{(n)}$ converges boundedly to γ, and by Lemma 3.2, $P_t^{(n)}$ converges in $L^2(D)$ to $P_t g$, it follows that for $g \in L^\infty(D) \cap H^1(D)$, the left member above converges to $t^{-1}\gamma \int_D g(g - P_t g)dx$ as $n \to \infty$. The right member, given by (3.16), is easily seen to converge to $\frac{1}{2}\int_D |\nabla g|^2 dx$ as $n \to \infty$. Hence,

$$t^{-1}\gamma\int_D g(g - P_t g)dx \leq \frac{\gamma}{2}\int_D |\nabla g|^2 dx < \infty \quad \text{for all} \quad g \in L^\infty(D) \cap H^1(D), t > 0.$$

Invoking Lemma 1.3.4 of [7] again, we conclude that $g \in \mathcal{D}(\mathcal{E})$ and (3.20) holds for all $g \in L^\infty(D) \cap H^1(D)$. □

Thus, we have given an alternative method of constructing stationary reflecting Brownian motions in domains that have finite Lebesgue measure. Our construction is more sample path oriented than the Dirichlet form approach of [6, 7] and it has the additional advantage that it immediately gives a process with paths in the Euclidean closure $\overline{D}$ of D.

IV. SKOROKHOD REPRESENTATION

In Meyer-Zheng [12], it is proved that a sequence of quasi-martingales which have uniformly bounded conditional variations is tight in law under the topology of pseudo-paths on $C([0,1],\mathbb{R}^d)$ and any limit process is a quasi-martingale. We will apply this result to prove that there is a Skorokhod-like semimartingale representation for our reflecting Brownian motion under condition (C.1). The following lemma is key to this.

LEMMA 4.1 : *Suppose there is a family of open sets $\{G_\lambda, \lambda \in \Lambda\}$ such that*

(i) $\overline{D} \subset \cup_\lambda G_\lambda$, *and*

(ii) *the following inequality holds for each $\lambda \in \Lambda$,*

$$\sup_n \int_{G_\lambda \cap D} |\nabla \exp\{-f_n(x)\}| dx < \infty. \tag{4.1}$$

Then any weak limit X of $\{X^{(n)}\}$ under the pseudo-path topology (or the topology of uniform convergence) on $C([0,1],\mathbb{R}^d)$ is a semimartingale, with decomposition relative to the filtration generated by X of the form

$$X_t = X_0 + W_t + V_t, \quad t \in [0,1],$$

where W is a Brownian motion martingale and V is a continuous adapted process of bounded variation.

Remark : The pseudo-path topology on $C([0,1],\mathbb{R}^d)$ is weaker than the topology of uniform convergence.

Proof : If we can prove X is a continuous semimartingale in each G_λ, then X is also a semimartingale on their union (see Zheng [18]). To prove the former, it suffices to show that $u(X.)$ is a continuous semimartingale for any twice continuously differentiable function u with compact support in G_λ. Consider $u \in C_c^2(G_\lambda)$. Since the probability density of $X_t^{(n)}$ is $\gamma_n \exp\{-f_n\}$ on D and $\gamma_n \to \gamma e \in (0,\infty)$ as $n \to \infty$, it follows that (4.1) is equivalent to

$$\sup_n \int_0^1 E\Big[1_{G_\lambda}(X_t^{(n)}) |\nabla f_n(X_t^{(n)})|\Big] dt < \infty, \tag{4.2}$$

which implies

$$(4.3)\quad \sup_n \int_0^1 \left\{ E\Big[|\nabla u(X_t^{(n)}) \cdot \nabla f_n(X_t^{(n)})|\Big] + E\Big[|\Delta u(X_t^{(n)})|\Big] \right\} dt < \infty.$$

From the above inequality and Itô's formula, we deduce that the conditional variations of the $\{u(X^{(n)}.)\}_{n=1}^{\infty}$ are uniformly bounded. Applying the above-mentioned result from Meyer and Zheng [12, Theorem 4], we deduce that the laws of the $\{u(X^{(n)})\}_{n=1}^{\infty}$ are tight under the topology of pseudo-paths on $C([0,1],\mathbb{R}^d)$ and any of their limit processes is a quasi-martingale with conditional variation bounded by the left member in (4.3). But from the discussion in Section 3, we know that the laws of the $\{u(X^{(n)})\}_{n=1}^{\infty}$ are also tight under the topology of uniform convergence on $C([0,1],\mathbb{R}^d)$ and any of their limit processes is continuous. Thus, by taking subsequences twice, we may conclude that for any weak limit process X of $X^{(n)}$ under the topology on $C([0,1],\mathbb{R}^d)$ of pseudo-paths (or of uniform convergence) the process $u(X)$ is a continuous semimartingale.

To verify the form of the decomposition of X, we recall that

$$X^{(n)} = X_0^{(n)} + W^{(n)} + U^{(n)},$$

where $W^{(n)}$ is a Brownian motion and $U_t^{(n)} = -\frac{1}{2}\int_0^t \nabla f_n(X_s^{(n)})ds$ is a continuous process that is of bounded variation (*i.e.*, almost surely the paths are of bounded variation on [0,1]). By passing to a subsequence and applying the tightness criterion of Billingsley [3, p. 55] component by component, we may assume that $(X^{(n)}, W^{(n)})$ converges weakly to (X, W), where W is a Brownian motion. Then, $U^{(n)}$ converges weakly to $U \equiv X - W$. Now, for each positive integer m, let $F_m = \{x \in \overline{D} : |x| \le m\}$. Since F_m is compact, there is a finite collection $\{G_{\lambda_i}, i = 1, \ldots, k_m\}$ such that $F_m \subset \cup_{i=1}^{k_m} G_{\lambda_i}$. Then, by (4.2),

$$\begin{aligned} C_m &\equiv \sup_n E\left[\int_0^1 1_{F_m}\Big(X_t^{(n)}\Big)|dU_t^{(n)}|\right] \\ &\le \sum_{i=1}^{k_m} \sup_n E\left[\int_0^1 1_{G_{\lambda_i}}\Big(X_t^{(n)}\Big)\Big|\nabla f_n\Big(X_t^{(n)}\Big)\Big|dt\right] \\ &< \infty. \end{aligned}$$

By the lower semicontinuity [12, Lemma 8] of $w \to \int_0^1 |dw_t|$ and the continuity of $w \to \max_{0\le t\le 1} |w_t|$ on $C([0,1],\mathbb{R}^d)$, we have for each $r > 0$ and

positive integer m,

$$\begin{aligned}
&P\left(\int_0^1 |dU_t| > r, \max_{0\le t\le 1} |X_t| < m\right)\\
&\le \liminf_{n\to\infty} P\left(\int_0^1 |dU_t^{(n)}| > r, \max_{0\le t\le 1} |X_t^{(n)}| < m\right)\\
&\le \liminf_{n\to\infty} P\left(\int_0^1 1_{F_m}\left(X_t^{(n)}\right)\left|dU_t^{(n)}\right| > r\right)\\
&\le \frac{1}{r}\sup_n E\left[\int_0^1 1_{F_m}\left(X_t^{(n)}\right)\left|dU_t^{(n)}\right|\right]\\
&\le r^{-1}C_m.
\end{aligned}$$

Letting $r \to \infty$ and then $m \to \infty$, it follows that $\int_0^1 |dU_t| < \infty$ almost surely. Thus, U is a process of bounded variation. Hence,

$$X_t = X_0 + W_t + U_t,$$

where W is a Brownian motion and U is a continuous process of bounded variation. Note that do not know that W and U are adapted to X. However, since the mutual variation does not depend on the choice of filtration, we may deduce that $\langle X^i, X^j\rangle_t = \langle W^i, W^j\rangle_t = \delta_{ij}t$, where δ_{ij} is the Kronecker delta, equal to one when $i = j$ and equal to zero otherwise. Finally, let $X = X_0 + M + V$ be the semimartingale decomposition of X *relative to the filtration generated by* X, where M is a continuous local martingale and V is a continuous process of bounded variation, and $M_0 = V_0 = 0$. Then $\langle M, M\rangle = \langle X, X\rangle$ and it follows that M is a Brownian motion. Thus, we have desired decomposition of X. □

THEOREM 4.1 : *Suppose condition (C.1) holds. Then any weak limit process X of $\{X^{(n)}\}$ under topology of pseudo-paths (or of uniform convergence) on $C([0,1],\mathbb{R}^d)$ is a continuous semimartingale, with decomposition relative to the filtration generated by X of the form*

$$X_t = X_0 + W_t + V_t, \quad t \in [0,1], \tag{4.4}$$

where W is a Brownian motion martingale and V is a continuous adapted process of bounded variation. Moreover, for all $v \in C_c^2(\mathbb{R}^d,\mathbb{R}^d)$,

$$E\left[\int_0^1 v(X_t)\cdot dV_t\right] = -\frac{\gamma}{2}\int_D \operatorname{div}(v)dx, \tag{4.5}$$

where $C_c^2(\mathbb{R}^d, \mathbb{R}^d)$ denotes the space of twice continuously differentiable $\mathbb{R}^d$-valued functions defined on $\mathbb{R}^d$ that have compact support.

Proof : (i) We first verify the theorem, except for the last sentence. According to Lemma 4.1, we only need to show (4.1) holds. Taking $q(y) = \exp(1 - \exp\{(ny)^{-1}\})$ in Lemma 2.2, we see that the hypotheses of Lemma 4.1 are satisfied with $G_\lambda = B_n$.

(ii) Now we are going to prove (4.5). We first note that the left member of (4.5) is finite. Indeed, the conditional variation of V on the compact support of v is less than or equal to the conditional variation of $u(X)$ for any $u \in C_c^2(\mathbb{R}^d)$ satisfying $u(x) = x$ on the support of v. Since the compact support of u can be covered by finitely many of the sets B_n, it follows in a similar manner to that in the first paragraph of the proof of Lemma 4.1 that the conditional variation of $u(X)$ is finite.

To verify (4.5), note that from (4.4) and the relation between Itô's integral and Stratonovich's integral, we have

$$\begin{aligned} E\left[\int_0^1 v(X_t)\cdot dV_t\right] &= E\left[\int_0^1 v(X_t)\cdot dX_t\right] \\ &= E\left[\int_0^1 v(X_t) * dX_t\right] - \frac{1}{2}E\left[\int_0^1 \operatorname{div}(v)(X_t)dt\right], \end{aligned} \tag{4.6}$$

where $*$ denotes integration in the Stratonovich sense. From the symmetry of X, we know the Stratonovich integral $\int_0^1 v(X_t) * dX_t$ is a difference of a forward martingale and a backward martingale (see Lyons and Zheng [11,(4.5)]), both having initial value zero; and so the expectation of this integral vanishes. By stationarity of X, the remaining term in (4.6) is just

$$-\frac{\gamma}{2}\int_D \operatorname{div}(v)dx. \qquad \square$$

Since a bounded Lipschitz domain D satisfies the condition $m(D) < \infty$, for such a D our process X exists and by Theorem 3.2 it is the stationary symmetric Markov process associated with the Dirichlet form (1.1) on $H^1(D)$. In this case, by the results of Bass and Hsu [2], X has a semimartingale decomposition of the Skorokhod form (1.2), where W is a martingale relative to the filtration generated by X. By uniqueness of such a decomposition, it follows that for D bounded and Lipschitz, V in our decomposition (4.4) has the specific form of the last term in (1.2). One can view (4.5) as a weaker version of this form that applies to more general domains than those that are bounded and Lipschitz.

APPENDIX

Here we review several main steps in the construction of the regularized distance function of Lemma 2.1. See Stein [14, p. 168-171] for more details.

(i) Consider the lattice of points in $\mathbb{R}^d$ whose coordinates take integer values. This lattice determines a mesh Ψ_0, which is a collection of cubes : namely, all cubes with sides of unit length whose vertices are points of the above lattice. The mesh Ψ_0 yields a doubly infinite chain of meshes $\{\Psi_\alpha\}_{\alpha=-\infty}^{\infty}$, where $\Psi_\alpha = 2^{-\alpha}\Psi_0$ and α runs over all of the integers. Thus each cube in the mesh Ψ_α gives rise to 2^d cubes in the mesh $\Psi_{\alpha+1}$ by bisecting the sides. The diameter of a cube in Ψ_α is $2^{-\alpha}\sqrt{d}$.

(ii) Define strips Θ_α in D by

$$\Theta_\alpha = \{x \in D : c2^{-\alpha} < \theta(x) \leq c2^{-\alpha+1}\},$$

where $c = 2\sqrt{d}$.

(iii) We now make an initial choice of cubes, and denote the resulting collection by Γ_0. Our choice is made as follows. For each α, we consider the cubes of the mesh Ψ_α and include a cube of this mesh in Γ_0 if it intersects Θ_α. Then we have

$$\bigcup_{Q\in\Gamma_0} Q = D$$

By the choice of c in (ii) above, we have

$$\text{diam}(Q) \leq \text{dist}(Q, \partial D) \leq 4 \text{ diam}(Q), \qquad \forall\, Q \in \Gamma_0.$$

(iv) Start now with any cube $Q \in \Gamma_0$, and consider the the maximal cube in Γ_0 which contains it. Let $\Gamma = \{Q_1, Q_2, \ldots\}$ denote the collection of maximal cubes of Γ_0. Then the following four properties can be readily verified.

(a) $\cup_{Q\in\Gamma} Q = D$.

(b) The cubes of Γ are disjoint.

(d) $\text{diam}(Q) \leq \text{dist}(Q, \partial D) \leq 4 \text{ diam}(Q)$, $\forall\, Q \in \Gamma$.

(d) If $Q, Q' \in \Gamma$ and these cubes touch, then

$$(1/4)\text{diam}(Q') \leq \text{diam}(Q) \leq 4 \text{ diam}(Q').$$

(v) Fix any $\varepsilon : 0 < \varepsilon < 1/4$. Denote by Q'_j the cube which has the same center as Q_j but is expanded by the factor $1+\varepsilon$. Then we can prove that each point in D is contained in at most $(12)^d$ of the cubes Q'_j.

(vi) Fix a C^∞ real-valued function ψ defined on $\mathbb{R}$ that is symmetric about $x = 0$ and satisfies $0 \leq \psi \leq 1 : \psi(x) = 1$, for $-\frac{1}{2} \leq x \leq \frac{1}{2}$;

$\psi(x) = 0$, for $|x| > \frac{1}{2}(1+\varepsilon)$, and ψ is strictly increasing on $(-\frac{1}{2}(1+\varepsilon), 0)$ and strictly decreasing on $(0, \frac{1}{2}(1+\varepsilon))$. Define

$$\phi(x) = \prod_{i=1}^{d} \psi(x_i) \quad \text{for all} \quad x \in \mathbb{R}^d.$$

For each $Q_j \in \Gamma$, let x^j denote the center of Q_j and let l_j be the common lenght of its sides. Define

$$\phi_j(x) = \phi\Big(\frac{x - x^j}{l_j}\Big) \quad \text{for} \quad x \in \mathbb{R}^d.$$

(vii) Finally define

$$\delta(x) = \sum_j \operatorname{diam}(Q_j)\phi_j(x).$$

Then δ has the properties of the regularized distance function described in Lemma 2.1. □

The reader should note that in the above we have used a more specific form of the function ϕ than in Stein [14]. The reason for this is that this form is needed for the proof of Lemma 2.2.

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