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by

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ABSTRACT. − Let X_t be a β-self-similar, $\beta > 0$, transient Markov process on $(0, \infty)$. We show that if X_{T_t} (T_t is the right continuous inverse of a continuous additive functional A_t) is an α-self-similar Markov process, $\alpha > 0$, then

$$A_t = k \int_0^t X_h^{1/\alpha - 1/\beta} \, dh \quad \text{for some } k > 0.$$

A result concerning time-changes of a transient Lévy process is also given.

Key words : Self-similar, Markov process, time-change, Lévy process.

RÉSUMÉ. − Soit X_t un processus β-self-similaire transient et de Markov sur $(0, \infty)$, $\beta > 0$. Notons T_t l'inverse continu à droite du fonctionnelle additive A_t. Nous montrons que si X_{T_t} est un processus α-self-similaire et de Markov, $\alpha > 0$, alors

$$A_t = k \int_0^t X_h^{1/\alpha - 1/\beta} \, dh \quad \text{pour quelque } k > 0.$$

Un résultat concernant le changement de temps d'un processus de Lévy transient est également donné.

0. INTRODUCTION

α-self-similar Markov processes (a-ssmp) on $(0, \infty)$ were introduced by
J. Lamperti [5]. The process (X_t, P^x) with a state space $(0, \infty)$ is called
α-ssmp, $\alpha > 0$, if there exists a Borel semigroup $(P_t(,))_{t \geq 0}$ on
$(0, \infty) \times \mathscr{B}(0, \infty)$ satisfying
   (i) $P_0(,) = I$;
   (ii) $P_t(x, A) = P_{at}(a^\alpha x, a^\alpha A)$ for all $t > 0$, $a > 0$, $x \in (0, \infty)$, $A \in \mathscr{B}(0, \infty)$,
such that (X_t, P^x) is a time homogeneous Markov process with a transition
function $(P_t(,))_{t \geq 0}$ and with sample paths that are P^x-almost surely
right-continuous with left limits for all $x \in (0, \infty)$.

It was proved in [6] that every α-ssmp with "nice paths" (*see* Notation)
on $(0, \infty)$ has a weak dual with respect to the measure $x^{1/\alpha - 1} dx$. In this
note we apply this result and characterize, the theory developed in [3] as
a main tool, all the possible ways to time-change a transient self-similar
process (in fact, an α-ssmp is transient iff it is cotransient; *see* Proposition)
to another self-similar process. We also obtain a result concerning time-
changes of a Lévy process. For simplicity, we assume $\alpha > 0$, but the results
can easily be generalized to negative α.

1. – NOTATION. DEFINITIONS

Ω notes the space of all functions ω from $[0, \infty) \to (0, \infty) \cup \{\Delta\}$ (Δ
denotes the point used as a "graveyard"; we assume Δ is an isolated
point), which satisfy
   (*a*) $\omega(t) = \Delta$ for $t \geq \zeta(\omega) = \inf \{ t \geq 0; \omega(t) = \Delta \}$;
   (*b*) ω is right continuous and ω or $1/\omega$ has left limits on $[0, \infty)$ at every
$t \in (0, \zeta(\omega)]$.
   Such Ω is called the space of "nice paths".

DEFINITION. – Let $\alpha > 0$ be given. A stochastic process (X_t, P^x) with a
state space $(0, \infty) \cup \{\Delta\}$ is called α-ssmp on $(0, \infty)$ if the following is
satisfied: there exists a Borel semigroup $(P_t(,))_{t \geq 0}$ on $(0, \infty) \times \mathscr{B}(0, \infty)$
with the properties:
   (i) $P_0(,) = I$;
   (ii) $P_t(x, A) = P_{at}(a^\alpha x, a^\alpha A)$ for all $t \geq 0$, $a > 0$, $x \in (0, \infty)$, $A \in \mathscr{B}(0, \infty)$,
such that (X_t, P^x) is a time homogeneous Markov process with a transition
function $(P_t(,)_{t \geq 0}$ and $t \to X_t \in \Omega$ P^x-a. s. for $x \in (0, \infty)$.
   Remark 1. – It was proved in [4] that every α-ssmp on $(0, \infty)$ automati-
cally is strongly Markov.
   A Markov process $(X_t, P^x, x \in (0, \infty))$ is said to be *in weak duality* with
a Markov process $(\hat{X}_t, \hat{P}^x, x \in (0, \infty))$ with respect to a σ-finite measure

η, if for all bounded f, g in $\mathscr{B}(0, \infty)$

$$\int f(x) \mathrm{E}^x g(\mathrm{X}_t) \eta(dx) = \int \hat{\mathrm{E}}^x f(\hat{\mathrm{X}}_t) g(x) \eta(dx), \quad \text{for all } t > 0.$$

Let $(\mathrm{X}_t, \mathrm{P}^x)$ be in weak duality with $(\hat{\mathrm{X}}_t, \hat{\mathrm{P}}^x)$ with respect to a measure η. $(\mathrm{X}_t, \mathrm{P}^x)$ is said to be *transient*, if

$$\mathrm{U}f(x) = \mathrm{E}^x \left\{ \int_0^\infty f(\mathrm{X}_t) dt \right\} < \infty$$

for all x, all bounded, non-negative Borel functions f on $(0, \infty)$ with compact support (*see* alternative definitions for transience in [2]). If the dual process $(\hat{\mathrm{X}}_t, \hat{\mathrm{P}}^x)$ is transient, then $(\mathrm{X}_t, \mathrm{P}^x)$ is said to be *cotransient*.

Remark 2. — It was shown in [6] that an α-ssmp on $(0, \infty)$ has a weak dual, with respect to the measure $x^{1/\alpha - 1} dx$, and the dual process is also an α-ssmp.

2. — THEOREMS

We assume throughout this paper that $(\mathrm{X}_t, \mathrm{P}^x)$ is transient (as we shall *see* in Proposition, for self-similar processes the transience is equivalent to the cotransience). According to [6], $(\mathrm{X}_{\mathrm{T}_t}, \mathrm{P}^x)$ is an α-ssmp if $(\mathrm{X}_t, \mathrm{P}^x)$ is β-ssmp and T_t is the right continuous inverse of an additive functional $k \int_0^t \mathrm{X}_h^{1/\alpha - 1/\beta} dh$. We shall show that this is the only possible way to time-change $(\mathrm{X}_t, \mathrm{P}^x)$ to an α-ssmp.

PROPOSITION. — *Let* $(\mathrm{X}_t, \mathrm{P}^x)$ *be a* β-*ssmp on* $(0, \infty)$, $\beta > 0$. *Then it is transient iff it is cotrasient.*

Proof. — According to [4] (Th. 2.3) and [5] (Th. 4.1), there is one to one correspondence between a β-ssmp X_t on $(0, \infty)$ and a Lévy process Z_t on $(-\infty, +\infty)$ (that is, Z_t is a strong Markov process which have stationary independent increments and right continuous paths with left limits) defined by $\mathrm{Z}_t = \log \mathrm{X}_{\mathrm{T}_t}$, where T_t is the right continuous inverse of an additive functional

$$\int_0^t \mathrm{X}_h^{-1/\beta} dh.$$

It is also easily seen that X_t is transient iff Z_t is transient. Now for Z_t there exists a weak dual $\hat{\mathrm{Z}}_t$ with respect to the Lebesgue measure such that also $\hat{\mathrm{Z}}_t$ is a Lévy process. As shown in [6], starting from $\hat{\mathrm{Z}}_t$ one can construct a β-ssmp $\hat{\mathrm{X}}_t$, which is a weak dual to X_t with respect to the

measure $x^{1/B-1} dx$. Now $\hat{X}_t$ is transient iff $\hat{Z}_t$ is transient and so it suffices to show that $\hat{Z}_t$ is transient iff it is cotransient. If (Z_t, Q^z) is a Lévy process then Z_t under Q^z has the same distribution as $z + Z_t$ under Q^0 and thus easy calculations show that $\hat{Z}_t$ under $\hat{Q}^z$ has the same distribution as $z - Z_t$ under Q^0. This shows that Z_t is transient iff $\hat{Z}_t$ is transient and gives thus the assertion.

In the proof of the following theorem actually cotransience (and not transience) is used.

THEOREM 2.1. — *Let* (X_t, P^x) *be a transient* β-*ssmp on* $(0, \infty)$, $\beta > 0$, *and let* A_t *be a continuous additive functional of* X_t *with* T_t *as the right continuous inverse*, i. e.

$$T_t = \inf \{ s \geq 0;\ A_s > t \}.$$

If the process (X_{T_t}, P^x) *is* α-*ssmp, then there exists* $k > 0$ *such that*

$$A_t(\omega) = k \int_0^t X_h^{1/\alpha - 1/\beta}(\omega)\, dh, \quad \text{for all } t < \zeta(\omega).$$

Proof. — As mentioned in Remark 2, (X_t, P^x) has a weak dual $(\hat{X}_t, \hat{P}^x)$ with respect to the measure $x^{1/\beta - 1} dx$ such that also $(\hat{X}_t, \hat{P}^x)$ is β-ssmp. Let A_t be a continuous additive functional of X_t and let (X_{T_t}, P^x), T_t is the right continuous inverse of A_t, be an α-ssmp. Let further v_A be the Revuz measure corresponding to A_t. According to the result of Getoor and Sharpe [3]

$$\int v_A(dx) f(x)\, \hat{U}(x, dy) = E^y \left\{ \int_0^\infty f(X_t)\, dA_t \right\} y^{1/\beta - 1}\, dy \qquad (2.1)$$

for any bounded, non-negative Borel function f with compact support.

The right side of (2.1) is equal to $E^y \left\{ \int_0^\infty f(X_{T_t})\, dt \right\} y^{1/\beta - 1}\, dy$, which, because of the α-self-similarity of (X_{T_t}, P^x), is equal to

$$E^{a^{\alpha_y}} \left\{ \int_0^\infty f(a^{-\alpha} X_{T_{at}})\, dt \right\} y^{1/\beta - 1}\, dy$$

$$= a^{-1} E^{a^{\alpha_y}} \left\{ \int_0^\infty f(a^{-\alpha} X_{T_t})\, dt \right\} y^{1/\beta - 1}\, dy$$

$$= a^{-1} E^{a^{\alpha_y}} \left\{ \int_0^\infty f(a^{-\alpha} X_t)\, dA_t \right\} y^{1/\beta - 1}\, dy \quad (2.2)$$

From (2.1) and (2.2) we obtain, by substituting $z = a^\alpha y$,

$$\int \nu_A(dx) f(x) \hat{U}(x, d(a^{-\alpha} z))$$

$$= a^{-\alpha/\beta - 1} E^z \left\{ \int_0^\infty f(a^{-\alpha} X_t) dA_t \right\} z^{1/\beta - 1} dz$$

$$= a^{-\alpha/\beta - 1} \int \nu_A(d(a^\alpha x)) f(x) \hat{U}(a^\alpha x, dz) \quad (2.3)$$

The β-self-similarity of $(\hat{X}_t, \hat{P}^x)$ implies

$$\hat{U}(x, d(a^{-\alpha} z)) = \int_0^\infty \hat{P}^x \{ \hat{X}_t \in d(a^{-\alpha} z) \} dt$$

$$= \int_0^\infty \hat{P}^{a^\alpha x} \{ \hat{X}_{a^{\alpha/\beta} t} \in dz \} dt = a^{-\alpha/\beta} \hat{U}(a^\alpha x, dx)$$

This, together with (2.3), gives

$$\nu_A(dx) \hat{U}(a^\alpha x, dz) = a^{-1} \nu_A(d(a^\alpha x)) \hat{U}(a^\alpha x, dz) \qquad (2.4)$$

Because X_t is cotransient we have $\hat{U} f(x) < +\infty$, $\forall x$. Thus

$$\nu_A(d(ax)) = a^{1/\alpha} \nu_A(dx), \qquad \forall a > 0. \qquad (2.5)$$

Applying the well-known uniqueness result for a Haar measure we obtain

$$\nu_A(dx) = k x^{1/\alpha - 1} dx = k (x^{1/\alpha - 1/\beta}) x^{1/\beta - 1} dx, \quad \text{for some } k > 0,$$

which gives

$$A_t = k \int_0^t X_h^{1/\alpha - 1/\beta} dh \text{ for some } k > 0.$$

Remark. – The special case of Theorem 1 is $\alpha = \beta$, which gives $A_t = kt$. This means that the only possible way to time-change a transient β-ssmp to another β-ssmp is a linear time-change.

In [5] J. Lamperti introduced a continuous additive functional

$$A_t(\omega) = \int_0^t X_h^{-1/\beta}(\omega) dh,$$

where (X_t, P^x) is a β-ssmp on $(0, \infty)$. He showed that if T_t is the right continuous inverse of A_t, then the time-changed process (X_{T_t}, P^x) is a strong Markov process on $(0, \infty)$ such that it is *multiplicatively invariant*, i.e.

$$Q_t(x, A) = Q_t(ax, aA), \quad \text{for all } t > 0, a > 0, X \in (0, \infty), A \in \mathscr{B}(0, \infty),$$

where $Q_t(\ ,\ )$ is a transition function for (X_{T_t}, P^x).

The following theorem says that this, possibly multiplied by a constant, is the only way to time-change (X_t, P^x) to a multiplicatively invariant

process:

THEOREM 2.2. − *Let* (X_t, P^x) *be a transient* β-*ssmp on* $(0, \infty)$ *and let* A_t *be a continuous additive functional of* (X_t, P^x) *and* T_t *the right continuous inverse of* A_t. *If* (X_{T_t}, P^x) *is multiplicatively invariant, then there exists* $k > 0$ *such that*

$$A_t = k \int_0^t X_t^{-1/\beta} \, dh \quad \text{for all } t < \xi.$$

The proof is similar to that of Theorem 2.1 and will therefore be omitted.

Finally, we shall present a result concerning time-changes of a Lévy process. It was already remarked by Lamperti [5] that (Y_t) is a multiplicatively invariant strong Markov process with "nice paths" (*see* Notation) on $(0, \infty)$ iff $(Z_t) = (\log Y_t)$ is a Lévy process on $(-\infty, -\infty)$. We will show

THEOREM 2.3. − *The only possible way to time-change a transient Lévy process on* $(-\infty, +\infty)$ *to another Lévy process is a linear time-change* $t \to kt$, $k > 0$.

We need the following Lemma:

LEMMA. − *Let* (Y_t, P^x) *be a transient, multiplicatively invariant strong Markov process with "nice paths"* (see *Notation*) *on* $(0, \infty)$. *Then the only possible way to time-change* (Y_t, P^x) *to another process of the same type is a linear time-change* $t \to kt$, $k > 0$.

Proof. − What we have to show is, that if A_t is a continuous additive functional of (Y_t) with T_t as the right continuous inverse and (Y_{T_t}, P^x) is multiplicatively invariant, then there exists $k > 0$ such that $A_t = kt$ for all $t < \xi$. According to [6], (Y_t, P^x) has a weak dual with respect to the measure $x^{-1} \, dx$. We can now show, by the same way as in Theorem 2.1, that A_t has a Revuz measure

$$v_A(dx) = kx^{-1} \, dx, \quad \text{for some } k > 0,$$

which gives the assertion.

Proof of Theorem 2.3. − Let (Z_t, Q^z) be a transient Lévy process on $(-\infty, +\infty)$. Then, as remarked in the proof of Proposition, the weak dual $(\hat{Z}_t, \hat{Q}^z)$, which also is a Lévy process, is transient. Now $(\exp Z_t, Q^{\log x})$, $x > 0$, is multiplicatively invariant and has, as shown in [6], a weak dual $(\exp \hat{Z}_t, \hat{Q}^{\log x})$ with respect to the measure $x^{-1} \, dx$. It is easily seen that (Z_t) is transient iff $(\exp Z_t)$ is transient and so, according to Lemma, $(\exp Z_t)$ cannot be time-changed to another multiplicatively invariant process with "nice paths" otherwise than by the linear time change $t \to kt$, $k > 0$. Thus one to one correspondence between the class of Lévy

processes and the class of multiplicatively invariant processes with "nice paths" gives the assertion.

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