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The Hausdorff measure of the closed support of super-Brownian motion (*)

by

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ABSTRACT. — If $d \geqq 3$ it is shown that the measure-valued super-Brownian motion (in $\mathbf{R}^d$) of Dawson and Watanabe distributes its mass over its closed support in a uniform manner with respect to Hausdorff φ-measure, where $\varphi(x) = x^2 \log^+ \log^+ 1/x$, and does so for all times simultaneously. This together with a less precise result for $d=2$ shows that the closed supports are Lebesgue null sets for all positive times a. s. when $d \geqq 2$.

Key words : Measure-valued diffusion, Hausdorff measure, super-Brownian motion.

RÉSUMÉ. — Lorsque $d \geqq 3$, on montre que le « super-Brownian motion » à valeurs mesure (dans $\mathbf{R}^d$) (étudié par Dawson et Watanabe), répartit, uniformément par rapport à la φ-mesure de Hausdorff, sa masse sur son support fermé, avec $\varphi(x) = x^2 \log^+ \log^+ 1/x$. En utilisant de plus, un résultat moins précis, lorsque $d=2$, on montre que, *p. s.*, pour tout temps positif, la mesure de Lebesgue des supports fermés est nulle si $d \geqq 2$.

Mots clés : Measure-valued diffusion, Hausdorff measure, super-Brownian motion.

Classification A.M.S. : Primary 60 G 17, 60 G 57, Secondary 60 H 15.

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1. INTRODUCTION

Let $M_F = M_F(\mathbf{R}^d)$ be the space of finite measures on $\mathbf{R}^d$ equipped with the topology of weak convergence. The DW super-Brownian motion [we will suppress the "DW" although this is not consistent with the terminology introduced by Dynkin (1988)] is the $M_F(\mathbf{R}^d)$-valued continuous strong Markov process X_t which is more commonly known as the critical multiplicative measure-valued branching Markov process with Brownian diffusion. The state space may be enlarged to a large class of infinite measures but this extension has no effect on the problem being considered here. Write $m(f)$ for the integral of f with respect to m. Let A be the generator of d-dimensional Brownian motion on its domain D(A) in the Banach space $C(\overline{\mathbf{R}^d})$ of continuous functions on the one-point compactification of $\mathbf{R}^d$. Given an initial measure m in M_F, X is the unique (in law) continuous M_F-valued process such that for any f in D(A)

$$X_t(f) = m(f) + Z_t(f) + \int_0^t X_s(Af)\,ds. \tag{1.1}$$

$Z_t(f)$ is a continuous F_t^X-martingale such that

$$\langle Z(f)\rangle_t = \int_0^t X_s(f^2)\,ds.$$

[*see* e.g. Ethier-Kurtz (1986, p. 406)]. $\mathscr{F}_t^X$ is the σ-field generated by X enlarged to satisfy the "usual hypotheses". Let Q^m denote the law of X on the canonicial space of continuous M_F-valued paths, $C([0, \infty), M_F)$.

The above characterization extends to Feller generators A. Let us consider the nature of X_t when A is the generator of a d-dimensional symmetric stable process of index α. If $d < \alpha$, then $X_t(dx) = Y(t, x)\,dx$ for some jointly continuous density Y [Roelly-Coppoletta (1986), Reimers (1987), Konno-Shiga (1987)]. If $d \geqq \alpha$, X_t is singular with respect to Lebesgue measure for all $t > 0$ a.s. [Dawson-Hochberg (1979), Perkins (1988 *a*)]. To describe the precise degree of singularity of X_t, let $\psi - m(A)$ denote the Hausdorff ψ-measure of A and let $\varphi_\alpha(x) = x^\alpha \log^+ \log^+ 1/x$.

THEOREM A [PERKINS (1988 *a*, THEOREM A)]. — *If $d > \alpha$ there are $0 < c(\alpha, d) \leqq C(\alpha, d) < \infty$ such that for any $m \in M_F$ and Q^m-a.a.ω.*

(1.2) *For all $t > 0$ there is a Borel set $\Lambda_t(\omega)$ which supports X_t and satisfies $c(\alpha, d)\,\varphi_\alpha - m(A \cap \Lambda_t) \leqq X_t(A) \leqq C(\alpha, d)\,\varphi_\alpha - m(A \cap \Lambda_t)$ for all Borel sets A in $\mathbf{R}^d$. In fact*

$$\Lambda_t = \{x : c'(\alpha, d) \leqq \varlimsup_{r \downarrow 0} X_t(B(x, r))\,\varphi(r)^{-1} \leqq C'(\alpha, d)\} \quad \textit{for some } 0 < c' \leqq C' < \infty. \tag{1.3}$$

Slightly less precise results are stated for $d=\alpha$. Theorem A says that X_t spreads its mass over a *Borel* support in a uniform manner according to a deterministic Hausdorff measure. The notion of a Borel support is highly non-unique and it is natural to ask if Theorem A also holds for the canonical closed support of X_t, which we denote by S_t. For $\alpha < 2$ this extension was shown to be false in Perkins (1988 *b*). In fact $S_t = \mathbf{R}^d$ or $\varnothing$ a. s. for each $t > 0$ [Perkins (1988 *b*, Cor. 1.6)]. In this work we prove that the extension to closed supports is true for $\alpha = 2$, *i. e.* for super-Brownian motion. Write $\varphi(x)$ for $\varphi_2(x) = x^2 \log^+ \log^+ 1/x$.

THEOREM 1. — *If $d > 2$ and S_t is the closed support of the super-Brownian motion X_t, there are constants $0 < c(d) \leqq C(d) < \infty$ such that for any $m \in M_F$ and Q^m a.a.ω*

$$(1.4)\quad c(d)\,\varphi - m(A \cap S_t) \leqq X_t(A) \leqq C(d)\,\varphi - m(A \cap S_t)$$
$$\text{for all Borel sets A in } \mathbf{R}^d \quad \text{and all} \quad t > 0.$$

The upper bound on X_t is clearly equivalent to the upper bound on X_t in (1.2). Replace A in (1.2) with $A \cap S_t$ to derive the upper bound in (1.4), and replace A in (1.4) with $A \cap \Lambda_t$ to derive the upper bound in (1.2). On the other hand, the lower bound in (1.4) clearly implies that in (1.2) because $\Lambda_t \subset S_t$ [by (1.3)]. The converse is non-obvious because one must find a better covering of the points of "low density" in $S_t - \Lambda_t$. This was done for the Brownian path by Lévy (1953) and Taylor (1964). The argument is complicated here by the fact that we must find such an economical covering for all $t > 0$ simultaneously. For a fixed t the result was stated without proof in Dawson *et al.* (1988, Theorem 7.1).

We have a less precise result for $d=2$. Let $\varphi_0(x) = x^2 (\log^+ 1/x)^2$.

THEOREM 2. — *If $d=2$ and X_t, S_t are as in Theorem 1, then there are constants $0 < c(2) \leqq C(2) < \infty$ such that for any m in M_F and $Q^m - a.a.\omega$.*

$$(1.5)\quad c(2)\,\varphi - m(A \cap S_t) \leqq X_t(A) \leqq C(2)\,\varphi_0 - m(A \cap S_t)$$
$$\textit{for all Borel sets A in } \mathbf{R}^d \quad \textit{and all} \quad t > 0.$$

The upper bound is again immediate from Perkins (1988 a, Theorem B). Theorems 1 and 2 immediately imply

COROLLARY 1.3. — *If $d \geqq 2$, S_t is a Lebesgue null subset of $\mathbf{R}^d$ for all $t > 0$ Q^m — a. s., for each m in M_F.*

All of the above results continue to hold when m is an infinite measure such that $\int e^{-\varepsilon |x|^2} dm(x) < \infty$ for all $\varepsilon > 0$. One need only apply Theorems 1.7 and 1.8 of Dawson *et al.* (1988) to obtain these results first for bounded sets A and hence general A by inner regularity.

The proof of Theorems 1 and 2 will use several technical estimates from Perkins (1988 *a*). The new ingredient needed to handle closed supports, and which is false in the α-stable case, is a Lévy modulus of continuity for S_t from Dawson *et al.* (1988, Theorem 1.1). It states that if

$$(1.5) \qquad A^{\varepsilon}=\{x: d(x, A) \leqq \varepsilon\} \qquad \text{and} \qquad h(u)=(u\log^{+} 1/u)^{1/2}$$

then for every $c > 2$ there is a $\delta(\omega, c) > 0$ a. s. such that

$$(1.6) \qquad S_u \subset S_t^{ch(u-t)} \qquad \text{for} \quad 0 < u-t < \delta(\omega, c).$$

This 1-sided modulus of continuity will allow us to cover all of S_t and not just most of it as in Dawson-Hochberg (1979) and Perkins (1988*a*). In addition it will allow us to interpolate coverings of $S_{i\Delta}$, $i \in \mathbf{N}$ to obtain coverings of all the S_t's.

Our arguments will not use (1.1) or (1.6) directly. Instead they are based on S. Watanabe's construction of X as a weak limit of branching Brownian motions, and a uniform modulus of continuity for these approximating branching systems which is stronger than (1.6). This construction and uniform modulus are recalled in Section 2. The proofs of Theorems 1 and 2 are given in Section 3.

It is natural to ask if one can take $c=C$ in (1.4). For a fixed $t > 0$ this is true and will be proved in a forthcoming paper [Dawson-Iscoe-Perkins (1989)]. Whether or not this is the case for all $t > 0$ a. s. remains an open problem.

Positive constants introduced in Section i are written $c_{i.j}$. These constants may depend on the dimension d but any other dependencies will be made explicit. Positive constants arising in the course of a proof and whose exact value is irrelevant are written $c_1, c_2, \ldots$

2. BRANCHING BROWNIAN MOTIONS

To construct a system of branching Brownian motions we use the labelling scheme in Perkins (1988*a*) and Dawson *et al.* (1988). Let $I=\bigcup_{n=0}^{\infty} Z_+ \times \{0, 1\}^n$. If $\beta=(\beta_0, \ldots, \beta_j)$ is a multi-index in I, $|\beta|=j$ is the length of β and $\beta \mid i=(\beta_0, \ldots, \beta_i)$ for $i \leqq j$. Call β a descendant of γ and write $\gamma < \beta$ if $\gamma=\beta \mid i$ for some $i \leqq |\beta|$.

Let $\{B^{\beta}: \beta \in I\}$ be a set of independent d-dimensional Brownian motions which start at zero, and let $\{e^{\beta}: \beta \in I\}$ be a collection of i. i. d. random variables which take on the values 0 or 2 each with probability 1/2. Assume these collections are independent and are defined on $(\Omega^2, \mathscr{A}^2, P^2)$. If $\hat{\mathbf{R}}^d=\mathbf{R}^d \cup \{\Delta\}$ and Δ is added as a discrete point, let $(\Omega^1, \mathscr{A}^1)=((\hat{\mathbf{R}}^d)^{Z_+}, \mathscr{B}(\hat{\mathbf{R}}^d)^{Z_+})$ and $(\Omega, \mathscr{A})=(\Omega^1 \times \Omega^2, \mathscr{A}^1 \times \mathscr{A}^2)$. If $\omega=((x_j), \omega^2) \in \Omega$, we write $B^{\beta}(\omega)$ and $e^{\beta}(\omega)$ for $B^{\beta}(\omega^2)$ and $e^{\beta}(\omega^2)$.

Fix $\eta \in \mathbf{N}$ and suppress dependence on η if there is no ambiguity. Let $T = T^{(\eta)} = \{j 2^{-\eta} : j \in Z_+\}$, and let $\lambda = \lambda^\eta$ denote the measure on T which assigns mass $2^{-\eta}$ to each point in T. For $t \geqq 0$, let

$$\{t\} = \{t\}^\eta = \max\{r \in T : r \leqq t\}.$$

Given $\omega = ((x_j), \omega^2)$ we construct a branching particle system as follows: a particle starts at each $x_j \neq \Delta$; subsequent particles die or split into two particles with equal probability at the times in $T - \{0\}$; in between consecutive times in T the particles follow independent Brownian paths. A more precise description follows.

If $\beta \in I$ and $t \geqq 0$ let

$$\hat{N}_t^\beta = \hat{N}_t^{\beta, \eta}((x_j), \omega^2) = \begin{cases} x_{\beta_0} + \sum_{i=0}^{|\beta|} \int 1(i 2^{-\eta} \leqq s < t \wedge ((i+1) 2^{-\eta})) \, dB_s^{\beta | i} & \text{if } x_{\beta_0} \neq \Delta \\ \Delta \quad \text{if } x_{\beta_0} = \Delta. \end{cases}$$

Use $\{e^\beta\}$ to define death times $\tau^\beta = \tau^{\beta, \eta}((x_j), \omega^2)$ by

$$\tau^\beta = \begin{cases} 0 \quad \text{if } x_{\beta_0} = \Delta \\ \min\{(i+1) 2^{-\eta} : e^{\beta | i} = 0\} \\ \text{if this set is non-empty and } x_{\beta_0} \neq \Delta \\ (|\beta| + 1) 2^{-\eta} \\ \text{if the above set is empty and } x_{\beta_0} \neq \Delta. \end{cases}$$

The system of branching Brownian motions is given by

$$N_t^\beta = N_t^{\beta, \eta} = \begin{cases} \hat{N}_t^\beta & \text{if } 0 \leqq t < \tau^\beta \\ \Delta & \text{if } t \geqq \tau^\beta \end{cases} \qquad \beta \in I.$$

NOTATION. — If $\beta \in I$ and $t \geqq 0$, write $\beta \sim t$ (or $\beta \overset{\eta}{\sim} t$) iff $|\beta| 2^{-\eta} \leqq t < (|\beta| + 1) 2^{-\eta}$. If δ_x denotes unit point mass at x, let

$$M_F^\eta(\mathbf{R}^d) = \left\{ 2^{-\eta} \sum_{i=0}^{K} \delta_{x_i} : K = -1, 0, 1, \ldots, x_i \in \mathbf{R}^d \right\} \subset M_F(\mathbf{R}^d).$$

Define $N = N^{(\eta)} : [0, \infty] \times \Omega \to M(\mathbf{R}^d) = \{m : m \text{ a measure on } \mathbf{R}^d\}$ by

$$N_t(\omega) = 2^{-\eta} \sum_{\beta \sim t} \delta_{N_t^\beta(\omega)} 1(N_t^\beta(\omega) \neq \Delta).$$

If Π^1 is the projection of Ω onto Ω^1, define a filtration on $(\Omega, \mathscr{A})$ by

$$\mathscr{A}_t = \mathscr{A}_t^\eta = \sigma(\Pi^1, B^\beta, e^\beta : |\beta| 2^{-\eta} < \{t\}) \\ \left(\bigcap_{u > t} \sigma(B_s^\beta - B_{\{t\}}^\beta : |\beta| 2^{-\eta} = \{t\}, \{t\} \leqq s < u) \right).$$

It is easy to check that N_t is $(\mathscr{A}_t)$-adapted [Lemma 2.1 of Dawson *et al.* (1988)].

If $\omega^1 = (x_j) \in \Omega^1$, let $P^{\omega^1} = \delta_{\omega^1} \times P^2$

and

$$m = m(\omega^1) = 2^{-\eta} \sum_{j=0}^{\infty} \delta_{x_j} 1(x_j \neq \Delta).$$

Note that (m, η) uniquely determines P^{ω^1} on $\sigma(N_t: t \geqq 0)$. We usually suppress dependence on the underlying labelling of the (x_j) and write P^m or $P^{m,\eta}$ for P^{ω^1}.

B_t denotes the d-dimensional Brownian motion defined on the canonical filtered space of continuous $\mathbf{R}^d$-valued paths, $(\Omega_0, \mathscr{F}^0, \mathscr{F}_t^0)$ and starting at x under the probability P_0^x. It should be clear from the above construction [*see* Perkins (1988 *a*, Lemma 2.1 (*c*)] if it is not that for $\beta \in I$, $t < (|\beta|+1) 2^{-\eta}$, $(x_i) \in \Omega^1$, $x_{\beta_0} \neq \Delta$ and $A \in \mathscr{F}_t^0$

$$P^{(x_i)}(N_{\cdot}^{\beta} \in A \mid N_t^{\beta} \neq \Delta) = P_0^{x_{\beta_0}}(B_{\cdot} \in A). \tag{2.1}$$

The next result is essentially due to S. Watanabe (1968, Theorem 4.1). The tightness arguments required for the convergence on function space may be found in Ethier-Kurtz (1986, p. 406), where a slightly different result is proved.

THEOREM 2.1. — *If* $m \in M_F(\mathbf{R}^d)$ *and* $m_\eta \in M_F^\eta(\mathbf{R}^d)$ *converge to* m *as* $\eta \to \infty$, *then*

$$P^{m_\eta, \eta}(N^{(\eta)} \in .) \xrightarrow{w} Q^m \text{ on } D([0, \infty), M_F(\mathbf{R}^d)) \text{ as } \eta \to \infty,$$

where the limit is supported by $C([0, \infty), M_F(\mathbf{R}^d))$.

The fact that we have taken a limit along the geometric sequence $\{2^\eta\}$ is irrelevant for the above result but will be convenient for technical reasons in Section 3.

If $t \geqq \varepsilon > 0$, let

$$I(t, \varepsilon) = \{\gamma \sim t - \varepsilon : \exists \beta \sim t, \beta > \gamma \text{ with } N_t^\beta \neq \Delta\}.$$

If ε, $r \in T^{(\eta)}$ and $\gamma \sim r$.

$$P(\gamma \in I(r+\varepsilon, \varepsilon) \mid \mathscr{A}_r) = p^\eta(\varepsilon) 1(N_r^\gamma \neq \Delta) = p(\varepsilon) 1(N_r^\gamma \neq \Delta), \tag{2.2}$$

where $p^\eta(\varepsilon)$ is the probability that a critical Galton-Watson process, starting with one individual and with offspring distribution $\delta_0/2 + \delta_2/2$, is not extinct after $\varepsilon 2^\eta$ generations. From Harris (1963, p. 21) we have

$$\lim_{\eta \to \infty} 2^\eta p^\eta(\varepsilon) = 2\varepsilon^{-1} \quad \text{for any dyadic rational } \varepsilon, \tag{2.3}$$

$$p^\eta(\varepsilon) \leqq c_{2.1} (2^\eta \varepsilon)^{-1} \quad \text{for all } \varepsilon \text{ in } T^\eta \quad \text{and all } \eta \text{ in } \mathbf{N}. \tag{2.4}$$

The following uniform modulus of continuity for $\{N^\beta: \beta \in I\}$ plays a major role in the proof of Theorems 1 and 2. Recall that $h(u) = (u \log^+ 1/u)^{1/2}$.

THEOREM 2.2. [Dawson *et al.* (1988, THEOREM 4.5)]. — *If* $\eta \in \mathbf{N}$ *and* $c > 2$ *there are positive constants* $c_{2.2}(c)$, $c_{2.3}(c)$ *and* $c_{2.4}(c)$ *and a random variable* $\delta(\omega, c, \eta)$ *such that for any* $m \in M_F^\eta(\mathbf{R}^d)$,

(2.5) $\quad P^m(\delta(c, \eta) \leqq \rho) \leqq c_{2.2}\, m(\mathbf{R}^d)\, \rho^{c_{2.3}} \quad$ *for* $\quad 0 \leqq \rho \leqq c_{2.4}$

(2.6) $\quad$ *If* $0 < t-s \leqq \delta(c, \eta)$, $\quad s, t \geqq 0$, $\quad \beta \sim t \quad$ *and*

$$N_t^{\beta, \eta} \neq \Delta, \quad \textit{then } \left| N_t^{\beta, \eta} - N_s^{\beta, \eta} \right| \leqq c h(t-s).$$

If $\varnothing \neq S \subset I$, let

$$\mathscr{F}(S) = \sigma(\Pi^1, B^{\beta \mid k}, e^{\beta \mid k}: \beta \in S, 0 \leqq k \leqq |\beta|)$$
$$\mathscr{G}(S) = \sigma(B^\beta, e^\beta: \beta \in S).$$

If in addition $\gamma \in I$, let

$$\sigma(S; \gamma) = \begin{cases} |\gamma| - \inf\{j \leqq |\gamma| : \gamma \mid j \notin \{\beta \mid j : \beta \in S, |\beta| \geqq j\} \\ \qquad\qquad\qquad \text{if this set is non-empty} \\ -1 \quad \text{otherwise.} \end{cases}$$

Thus $\sigma(S; \gamma)$ is the number of generations since γ split off from the family tree generated by S. Finally, let $\mathscr{F}_1(\gamma) = \sigma(\{N^\gamma_{|\gamma| 2^{-\eta}} \neq \Delta\})$.

$B(x, r)$ denotes the closed ball in $\mathbf{R}^d$, centered at x and with radius r.

3. PROOFS OF THEOREMS 1 AND 2

Let $d \geqq 2$ and set $a_k = 2^{-k/2}$ and $\varepsilon_n = a^2_{2^{n+1}} = 2^{-2^{n+1}}$. We will cover S_t with cubes in

$\Lambda_k = \{C: C$ an open d-dimensional cube of side length a_k and centered

$$\text{at } (\vec{j} + \vec{e})\, a_k \text{ where } \vec{j} \in \mathbf{Z}^d, \vec{e} \in \{0, 1/2\}^d, C \subset (-k, k)^d\}.$$

ρ will denote a positive constant whose exact value will be fixed after Lemma 3.4 below.

DEFINITION. — $C \in \Lambda_{2^{n+1}}$ is bad for a finite measure ν iff $\nu(C) > 0$ and

(3.1) $\quad \nu(C^{13 a_k}) < \rho \varphi(a_k) \quad$ for all k in $\{2^n, 2^n+1, \ldots, 2^{n+1}-n-1\}$.

$C \in \Lambda_{2^{n+1}}$ is good for ν iff $\nu(C) > 0$ and (3.1) fails.

We work with the system of branching Brownian motions introduced in the previous section and continue to suppress dependence on the choice of η if there is no ambiguity. m denotes the initial measure in $M_F^\eta(\mathbf{R}^d)$ and we write P for $P^m = P^{(x_i)}$.

NOTATION. – If $\gamma \in \mathrm{I}$, $r=|\gamma| 2^{-\eta}$ and $k \in \mathbf{N}$, let $\mathrm{B}^{\gamma}(a_k)=\mathrm{B}(\mathrm{N}_r^{\gamma}, a_k)$ if $\mathrm{N}_r^{\gamma} \neq \Delta$ and set $\mathrm{B}^{\gamma}(a_k)=\varnothing$ if $\mathrm{N}_r^{\gamma}=\Delta$. If $t \geqq r$, let

$$\mathrm{Y}_t^{\gamma}(a_k)=2^{-\eta} \sum_{\beta \sim t} 1(\sigma(\beta;\gamma) 2^{-\eta} \in (a_k^2, 2a_k^2], \mathrm{N}_r^{\beta} \in \mathrm{B}^{\gamma}(a_k))\, \delta_{\mathrm{N}_t^{\beta}}$$

$$\mathrm{Z}_t^{\gamma}(a_k)=\mathrm{Y}_t^{\gamma}(a_k)(\mathbf{R}^d) \qquad \text{and} \qquad \mathrm{Z}^{\gamma}(a_k)=\mathrm{Z}_r^{\gamma}(a_k).$$

$\mathrm{Y}_t^{\gamma}(a_k)$ represents the contribution to N_t of those particles which branched off from γ in $[r-2a_k^2, r-a_k^2)$ and were within a_k of N_r^{γ} at time r.

Let $\tau_n=\{j\varepsilon_n : j=0, 1, \ldots, n\varepsilon_n^{-1}-1\}$.

Each point in the support of N_t and in $(-2^{n+1}, 2^{n+1})^d$ is covered either by a good cube for N_t in $\Lambda_{2^{n+1}}$ or a bad cube for N_t in $\Lambda_{2^{n+1}}$. As in Taylor (1964) the contribution to the Hausdorff $\varphi-m$ from the good cubes may be bounded by a multiple of the N_t measure. Our task is therefore to bound the contribution of the bad cubes. More specifically if

$$\psi(a)=a^2(\log^+ 1/a)^2 \tag{3.2}$$

(the power 2 on the logarithm has been chosen arbitrarily), then we would like to bound

$$b_n(t, \omega)=b_n^{\eta}(t, \omega)=\sum_{\mathrm{C} \in \Lambda_{2^{n+1}}} 1(\mathrm{C} \text{ is bad for } \mathrm{N}_t^{(\eta)})\, \psi(a_{2^{n+1}}) \tag{3.3}$$

uniformly in t and large η.

Let $\eta \geqq 2^{n+1}$ and

$$\mathrm{A}_1^{\eta}(n)=\mathrm{A}_1(n)=\{\omega : \delta(\omega, 3, \eta) \leqq 4\varepsilon_n\},$$

where $\delta(\omega, c, \eta)$ is as in Theorem 2.2. Assume $\omega \notin \mathrm{A}_1(n)$ and $\mathrm{C} \in \Lambda_{2^{n+1}}$ is bad for N_t where $t > \varepsilon_n$. Choose $j \in \mathbf{N}$ so that $t \in (j\varepsilon_n, (j+1)\varepsilon_n]$ and let $r=(j-1)\varepsilon_n \in \mathrm{T}^{(\eta)} \cap \tau_n$. Choose a $\beta \sim t$ such that $\mathrm{N}_t^{\beta} \in \mathrm{C}$ and let $\gamma=\beta \mid r 2^{\eta} \in \mathrm{I}(r+\varepsilon_n, \varepsilon_n)$. Since $\omega \notin \mathrm{A}_1(n)$ we have

$$|\mathrm{N}_t^{\beta}-\mathrm{N}_r^{\beta}| \leqq 3h(t-r) \leqq 3h(2\varepsilon_n) \leqq 6a_k \quad \text{for all } k \leqq 2^{n+1}-n-1. \tag{3.4}$$

This implies

$$\mathrm{N}_r^{\gamma}=\mathrm{N}_r^{\beta} \in \mathrm{C}^{3h(2\varepsilon_n)} \tag{3.5}$$

and

$$\mathrm{N}_t(\mathrm{B}(\mathrm{N}_r^{\gamma}, 7a_k)) \leqq \mathrm{N}_t(\mathrm{B}(\mathrm{N}_t^{\beta}, 13a_k)) \leqq \mathrm{N}_t(\mathrm{C}^{13a_k}) < \rho\varphi(a_k) \quad \text{for all } k=2^n, 2^n+1, \ldots, 2^{n+1}-n-1. \tag{3.6}$$

If $\alpha \sim t$, $\mathrm{N}_t^{\alpha} \neq \Delta$ and $\mathrm{N}_r^{\alpha} \in \mathrm{B}^{\gamma}(a_k)$ then for $2^n \leqq k \leqq 2^{n+1}-n-1$,

$$|\mathrm{N}_t^{\alpha}-\mathrm{N}_r^{\gamma}| \leqq |\mathrm{N}_t^{\alpha}-\mathrm{N}_r^{\alpha}|+a_k \leqq 3h(2\varepsilon_n)+a_k \leqq 7a_k.$$

This means that all descendents at time t of the points in $\mathrm{B}^{\gamma}(a_k)$ at time r are in $\mathrm{B}(\mathrm{N}_r^{\gamma}, 7a_k)$. Therefore (3.6) implies

$$\mathrm{Z}_t^{\gamma}(a_k) \leqq \mathrm{N}_t(\mathrm{B}(\mathrm{N}_t^{\gamma}, 7a_k)) < \rho\varphi(a_k) \quad \text{for} \quad 2^n \leqq k \leqq 2^{n+1}-n-1. \tag{3.7}$$

We have proved

LEMMA 3.1. — *Let* $\eta \geqq 2^{n+1}$, $C \in \Lambda_{2^{n+1}}$ *and* $\omega \notin A_1(n)$. *Let* $t \in (j\varepsilon_n, (j+1)\varepsilon_n]$ *for* $j \in \mathbf{N}$, *and* $r = (j-1)\varepsilon_n \in \tau_n$. *If* C *is bad for* N_t, *then there is a* γ *in* $I(r+\varepsilon_n, \varepsilon_n)$ *such that* (3.5) *and* (3.7) *hold.*

The next result will allow us to replace $Z_t^\gamma(a_k)$ with $Z_r^\gamma(a_k)$ in (3.7) at the cost of doubling ρ.

NOTATION. — $A_2(r, n) = \{\omega: Z^\gamma(a_k) \geqq 2\rho\varphi(a_k)$ and $Z_t^\gamma(a_k) \leqq \rho\varphi(a_k)$ for some $\gamma \in I(r+\varepsilon_n, \varepsilon_n)$, $t \in [r, r+2\varepsilon_n]$ and $2^n \leqq k \leqq 2^{n+1}-(n+1)\}$, $r \in \tau_n$,

$$A_2(n) = \bigcup_{r \in \tau_n} A_2(r, n)$$

$$J(\gamma, \varepsilon) = \{\omega: \gamma \in I(r+\varepsilon, \varepsilon)(\omega)\}, \qquad \gamma \sim r.$$

LEMMA 3.2. — *There is a* $c_{3.1}$ *and a sequence* $\{\eta_{3.1}(n, \rho)\} \subset \mathbf{N}$ *such that if* $\eta \geqq \eta_{3.1}(n, \rho)$ *then* $\varepsilon_n \in T^{(\eta)}$ *and*

$$P(A_2(n)) \leqq c_{3.1}\, m(\mathbf{R}^d) \exp(-2^{n-2}(\log 2)(\rho(n-1)-20)).$$

Proof. — Assume $n \in \mathbf{N}$ satisfies $\eta \geqq 2^{n+1}$, so that $\tau_n \subset T^{(\eta)}$. Assume also that $r \in \tau_n$ and $\gamma \sim r$. It is clear from the definition of $Y^\gamma(a_k)$ that if C is a Borel subset of $D([0, \infty), \mathbf{R}^d)$, then

$$P(Z_{r+\cdot}^\gamma(a_k) \in C \mid \mathscr{A}_r) = P^{Y_r^\gamma(a_k)}(N_\cdot(\mathbf{R}^d) \in C). \tag{3.8}$$

Note that

$$J(\gamma, \varepsilon_n) \in \mathscr{A}_r \vee \mathscr{G}(\{\beta: \beta > \gamma\}) \tag{3.9}$$

and

$$Z_{r+\cdot}^\gamma(a_k) \in \mathscr{A}_r \vee \mathscr{G}(\{\beta: \sigma(\beta; \gamma)2^{-\eta} \in (a_k^2, 2a_k^2], |\beta|2^{-\eta} \geqq r\}) \equiv \mathscr{A}_r \vee \mathscr{G}_{\gamma, k}. \tag{3.10}$$

The σ-fields $\mathscr{A}_r$, $\mathscr{G}(\{\beta: \beta > \gamma\})$ and $\mathscr{G}_{\gamma, k}$ are independent because the original $\{e^\beta, B^\beta\}$ are independent. (3.9) and (3.10) therefore show that $J(\gamma, \varepsilon_n)$ and $Z_\cdot^\gamma(a_k)$ are conditionally independent given $\mathscr{A}_r$. This and (3.8) imply

$$\begin{aligned} P(A_2(r, n) \mid \mathscr{A}_r) &\leqq \sum_{\gamma \sim r} \sum_{k=2^n}^{2^{n+1}-(n+1)} P(J(\gamma, \varepsilon_n) \mid \mathscr{A}_r) 1(Z^\gamma(a_k) \geqq 2\rho\varphi(a_k)) \\ &\qquad \times P^{Y_r^\gamma(a_k)}(\inf_{t \leqq 2\varepsilon_n} N_t(\mathbf{R}^d) \leqq \rho\varphi(a_k)). \end{aligned} \tag{3.11}$$

By Lemma 4.1 of Perkins (1988*a*) there is an $\eta_1(\rho, k, n, \eta)$ such that $\lim_{\eta \to \infty} \eta_1(\rho, k, n, \eta) = 0$ and

$$\begin{aligned} &1(Z^\gamma(a_k) \geqq 2\rho\varphi(a_k)) \\ &\quad \times P^{Y_r^\gamma(a_k)}(\inf_{t \leqq 2\varepsilon_n} N_t(\mathbf{R}^d) \leqq \rho\varphi(a_k)) \\ &\qquad \leqq 1(N_r^\gamma \neq \Delta)(\exp(-\rho\varphi(a_k)/8\varepsilon_n) + \eta_1(\rho, k, n, \eta)). \end{aligned} \tag{3.12}$$

Let $\eta_2(\rho, n, \eta)=\max\{\eta_1(\rho, k, n, \eta): 2^n \leqq k \leqq 2^{n+1}-n-1\}$. Use (3.12) and (2.2) to bound the right (and therefore left) side of (3.11) by

$$\sum_{\gamma \sim r} 2^n p^{\eta}(\varepsilon_n) 1(N_r^\gamma \neq \Delta)(\exp(-\rho\varphi(a_{2^{n+1}-n-1})/8\varepsilon_n)+\eta_2(\rho, n, \eta))$$
$$\leqq c_{2.1} 2^n \varepsilon_n^{-1} N_r(\mathbf{R}^d) \times (\exp(-\rho 2^{n-2}(n-1)\log 2)+\eta_2(\rho, n, \eta)) \text{ (by (2.4))}.$$

Sum the above estimate over $r \in \tau_n$ and take expected values to conclude

$$P(A_2(n)) \leqq c_{2.1} 2^n n \varepsilon_n^{-2} m(\mathbf{R}^d)(\exp(-\rho 2^{n-2}(n-1)\log 2)+\eta_2(\rho, n, \eta))$$
$$\leqq c_{2.1} m(\mathbf{R}^d)(\exp(-2^{n-2}(\log 2)(\rho(n-1)-20))+n 2^n \varepsilon_n^{-2}\eta_2(\rho, n, \eta)).$$

Finally choose $\eta_{3.1}(n, \rho) \geqq 2^{n+1}$ and sufficiently large so that

$$n 2^n \varepsilon_n^{-2}\eta_2(\rho, n, \eta) \leqq \exp(-2^{n-2}(\log 2)(\rho(n-1)-20)) \quad \text{for all } \eta \geqq \eta_{3.1}.$$

This gives the desired inequality with $c_{3.1}=2c_{2.1}$. ■

NOTATION. — If $\gamma \in I$, $r \in T^{(n)}$ and $k, n \in \mathbf{N}$, let

$$B(\gamma, k)=\{\omega : Z^\gamma(a_k) \leqq 2\rho\varphi(a_k)\}$$
$$A(\gamma, n)=\bigcap_{k=2^n}^{2^{n+1}-n-1} B(\gamma, k)$$

and

$$W_r(n)=\sum_{\gamma \sim r} 1(J(\gamma, \varepsilon_n) \cap A(\gamma, n)).$$

Let $\tilde{\psi}(a)=a^2(\log^+ 1/a)^{2+d/2}$.

LEMMA 3.3. — *If* $\eta \geqq 2^{n+1}$ *and* $\omega \notin A_1(n) \cup A_2(n)$ $(n \in \mathbf{N})$ *then*

$$\sup_{4\varepsilon_{n-1} \leqq t \leqq n} b_n(t) \leqq c_{3.2} \sup\{W_r(n): 2\varepsilon_{n-1} \leqq r < n, r \in \tau_n\} \tilde{\psi}(a_{2^{n+1}}), \tag{3.13}$$

where $c_{3.2}$ *depends only on d.*

Proof. — Assume $\eta \geqq 2^{n+1}$ and $\omega \notin A_1(n) \cup A_2(n)$. Lemma 3.1 and the fact that $\omega \notin A_2(n)$ imply

$$\sup_{4\varepsilon_{n-1} \leqq t \leqq n} b_n(t) \leqq \sup_{2\varepsilon_{n-1} \leqq r \leqq n,\, r \in \tau_n} \sum_{C \in \Lambda_{2^{n+1}}} \sum_{\gamma \sim r} 1(J(\gamma, \varepsilon_n) \cap A(\gamma, n)) \times 1(N_r^\gamma \in C^{3h(2\varepsilon_n)}) \psi(a_{2^{n+1}}). \tag{3.14}$$

An easy calculation shows that for any $\gamma \sim r$ fixed

$$\sum_{C \in \Lambda_{2^{n+1}}} 1(N_r^\gamma \in C^{3h(2\varepsilon_n)}) \leqq c_{3.2}(\log 1/a_{2^{n+1}})^{d/2}.$$

Substitute the above inequality into (3.14) to derive (3.13). ■

As $A_1(n) \cup A_2(n)$ is a small set the above result shows that to control the contribution of the bad cubes uniformly in t we must bound $W_r(n)$

uniformly in $r \in \tau_n$. To do this we introduce

$$\bar{W}_r(n) = \sum_{\gamma \sim r} P(J(\gamma, \varepsilon_n) \cap A(\gamma, n) \mid \mathscr{F}_1(\gamma))$$

and will bound $\bar{W}_r(n)$ and $|W_r(n) - \bar{W}_r(n)|$ separately.

LEMMA 3.4. — *There are constants* $c_{3.3}$, $c_{3.4}$, $c_{3.5}$ *and a sequence* $\{\eta_{3.2}(n, \rho) : n \in \mathbf{N}\}$ *in* $\mathbf{N}$ *such that if* $n \in \mathbf{N}$, $\eta \geqq \eta_{3.2}(n, \rho)$, $r \in T^{(\eta)}$ *and* $r \geqq 2\varepsilon_{n-1}$, *then*

$$\bar{W}_r(n) \leqq c_{3.3} N_r(\mathbf{R}^d) \varepsilon_n^{-1} \exp\{-c_{3.4} 2^{n(1-\rho c_{3.5})}\}.$$

Proof. — Let η, $n \in \mathbf{N}$, $r \in T^{(\eta)}$ and $\gamma \sim r$ satisfy $r \geqq 2\varepsilon_{n-1}$ and

$$\eta \geqq 1 + 2^{n+1} \text{ so that } \varepsilon_n \geqq 2^{1-\eta}. \tag{3.15}$$

Note that

$$J(\gamma, \varepsilon_n) \in \mathscr{F}(\gamma) \vee \mathscr{G}\{\beta : \beta > \gamma, \beta \neq \gamma\}) \tag{3.16}$$

and

$$B(\gamma, k) \in \mathscr{F}(\gamma) \vee \mathscr{G}(\{\beta : \sigma(\beta; \gamma) 2^{-\eta} \in (a_k^2, 2a_k^2]\}), \qquad k \geqq 2^n. \tag{3.17}$$

Recall that $\mathscr{G}(S_1), \ldots, \mathscr{G}(S_m)$ are independent σ-fields if $S_1, \ldots, S_m$ are disjoint subsets of I. (3.16) and (3.17) therefore show that $B(\gamma, k)$, $k = 2^n, \ldots, 2^{n+1} - n - 1$ and $J(\gamma, \varepsilon_n)$ are mutually conditionally independent given $\mathscr{F}(\gamma)$. Therefore

$$\begin{aligned} &P(J(\gamma, \varepsilon_n) \cap A(\gamma, n) \mid \mathscr{F}(\gamma)) \\ &\qquad = P(J(\gamma, \varepsilon_n) \mid \mathscr{F}(\gamma)) \prod_{k=2^n}^{2^{n+1}-n-1} P(B(\gamma, k) \mid \mathscr{F}(\gamma)). \end{aligned} \tag{3.18}$$

Note that $F(\gamma) \subset \mathscr{A}_{r+2^{-\eta}}$ (recall e^γ, $B^\gamma \in \mathscr{F}(\gamma)$). We may therefore use (2.4) and (3.15) to conclude

$$\begin{aligned} P(J(\gamma, \varepsilon_n) \mid \mathscr{F}(\gamma)) &\leqq c_{2.1} (2^\eta \varepsilon_n - 1)^{-1} 1(N_r^\gamma \neq \Delta) \\ &\leqq 2 c_{2.1} \varepsilon_n^{-1} 2^{-\eta} 1(N_r^\gamma \neq \Delta). \end{aligned} \tag{3.19}$$

Since $r \geqq \varepsilon_{n-1} = 2a_{2^n}^2$ we may use Lemma 5.2 of Perkins (1988 *a*) to find c_1, $c_{3.5}$ and $\eta_1(n, \rho) \in \mathbf{N}$ such that if $\eta \geqq \eta_1(n, \rho)$, then

$$\begin{aligned} &P(B(\gamma, k) \mid \mathscr{F}(\gamma)) \\ &\leqq \left(1 - c_1 (\log 1/a_k)^{-c_{3.5}\rho} \int_{(a_k^2, 2a_k^2]} s^{-1} 1(|N_r^\gamma - N_{r-s}^\gamma| \leqq a_k/2)\, d\lambda(s)\right) \\ &\qquad\qquad \text{for } k = 2^n, \ldots, 2^{n+1} - n - 1 \end{aligned} \tag{3.20}$$

(the restriction $r > 2a_{2^n}^2$ in the statement of Lemma 5.2 may be trivially changed to $r \geqq 2a_{2^n}^2$). Assume $\eta \geqq \eta_1(n, \rho)$ in what follows. (3.18), (3.19) and (3.20) show

$$\mathrm{P}(\mathrm{J}(\gamma,\varepsilon_n)\cap \mathrm{A}(\gamma,n)\mid \mathscr{F}(\gamma))$$
$$\leqq 2c_{2.1}\varepsilon_n^{-1}2^{-\eta}1(\mathrm{N}_r^\gamma\neq\Delta)\exp\left\{-c_1(2^n\log 2)^{-c_{3.5}\rho}\right.$$
$$\left.\times\int_{(a^2_{2n+1-n-1},\,2a^2_{2n}]} s^{-1}1(|\mathrm{N}_r^\gamma-\mathrm{N}_{r-s}^\gamma|\leqq\sqrt{s}\,2^{-3/2})\,d\lambda(s)\right\}.$$

Condition the above with respect to $\mathscr{F}_1(\gamma)$ and use (2.1) to see

$$(3.21)\quad \mathrm{P}(\mathrm{J}(\gamma,\varepsilon_n)\cap \mathrm{A}(\gamma,n)\mid \mathscr{F}_1(\gamma))$$
$$\leqq 2c_{2.1}\varepsilon_n^{-1}2^{-\eta}1(\mathrm{N}_r^\gamma\neq\Delta)\mathrm{E}_0^0\left(\exp\left\{-c_1 2^{-nc_{3.5}\rho}\right.\right.$$
$$\left.\left.\times\int_{(a^2_{2n+1-n-1},\,2a^2_{2n}]} s^{-1}1(|\mathrm{B}_s|\leqq\sqrt{s}\,2^{-3/2})\,d\lambda(s)\right\}\right)$$
$$\leqq 2c_{2.1}\varepsilon_n^{-1}2^{-\eta}1(\mathrm{N}_r^\gamma\neq\Delta)\left[\mathrm{E}_0^0\left(\exp\left\{-c_1 2^{-nc_{3.5}\rho}\right.\right.\right.$$
$$\left.\left.\left.\times\int_{(a^2_{2n+1-n-1},\,2a^2_{2n}]} s^{-1}1(|\mathrm{B}_s|s^{-1/2}\leqq 2^{-3/2})\,ds\right\}\right)+\delta(n,\rho,\eta)\right],$$

where $\lim_{\eta\to\infty}\delta(n,\rho,\eta)=0$. $\mathrm{Y}(u)=\mathrm{B}(e^u)e^{-u/2}$ is a stationary Ornstein-Uhlenbeck process and using well-known estimates [*see* e. g. Lemma 5.5 of Dawson *et al.* (1988)] one can bound the right-hand side of (3.21) by

$$2c_{2.1}\varepsilon_n^{-1}2^{-\eta}1(\mathrm{N}_r^\gamma\neq\Delta)\left(\mathrm{E}_0^0\left(\exp\left\{-c_1 2^{-nc_{3.5}\rho}\right.\right.\right.$$
$$\left.\left.\left.\times\int_0^{(2^n-n)\log 2} 1(|\mathrm{Y}_u|\leqq 2^{-3/2})\,du\right\}\right)+\delta(n,\rho,\eta)\right)$$
$$\leqq 2c_{2.1}\varepsilon_n^{-1}2^{-\eta}1(\mathrm{N}_r^\gamma\neq\Delta)$$
$$\times(c_3\exp\{-c_{3.4}2^{n(1-c_{3.5}\rho)}\}+\delta(n,\rho,\eta)).$$

If $\eta\geqq\eta_2(n,\rho)$ we may bound $\delta(n,\rho,\eta)$ by $c_3\exp\{-c_{3.4}2^{n(1-c_{3.5}\rho)}\}$ and so for $\eta\geqq\eta_{3.2}(n,\rho)=\max(1+2^{n+1},\eta_1(n,\rho),\eta_2(n,\rho))$ we have

$$\mathrm{P}(\mathrm{J}(\gamma,\varepsilon_n)\cap \mathrm{A}(\gamma,n)\mid \mathscr{F}_1(\gamma))$$
$$\leqq 4c_{2.1}c_3\varepsilon_n^{-1}2^{-\eta}1(\mathrm{N}_r^\gamma\neq\Delta)\exp\{-c_{3.4}2^{n(1-c_{3.5}\rho)}\}.$$

Sum the above inequality over $\gamma\sim r$ to complete the proof. ■

We may now set

$$(3.22)\quad \rho=(2c_{3.5})^{-1}\quad\text{and}\quad \eta_n=\max(\eta_{3.1}(n,\rho),\eta_{3.2}(n,\rho))>2^{n+1}.$$

DEFINITION. — Let $r \in T^{(\eta)}$ and $\beta=(\beta_1, \ldots, \beta_p)$ be a p-tuple in I^p with $|\beta_i|=2^\eta r$ for all $i \leqq p$. Let $D(\beta)=\{\omega: N_r^{\beta_i} \neq \Delta \text{ for all } i \leqq p\}$. If $b>0$, β is a b-good p-tuple iff $\sigma(\{\beta_i: i \neq j\}; \beta_j) \leqq 2^\eta b$ for all $j \leqq p$. $\sum\limits^{b}_{\beta_1, \ldots, \beta_p \sim r}$ denotes summation over all b-good p-tuples of length $2^\eta r$.

The next result is proved for even p in Lemma 3.3 of Perkins (1988 *a*). The proof given there easily extends to general p. This extension does change the way the constants depend on p from that given in Perkins (1988 *a*). The result we need is

LEMMA 3.5. — *If $r \in T^{(\eta)}$, $2^{1-\eta} \leqq b < r$ and $p \in N^{>2}$, then*

$$2^{-\eta p} \sum^{b}_{\beta_1, \ldots, \beta_p \sim r} P(D(\beta)) \leqq 4^p (p-1) b^{p-[p/2]} E(N_r(\mathbf{R}^d)^{[p/2]}).$$

LEMMA 3.6. — *There are constants $\{c_{3.6}(p): p=2, 3, \ldots\}$ such that for $n \in N$, $\eta \in N$, and $r \in T^{(\eta)}$ satisfying $r \geqq 2\varepsilon_{n-1} \geqq 2^{1-\eta}$,*

$$E((W_r(n)-\bar{W}_r(n))^{2p}) \leqq c_{3.6}(p)\, e^{2m(\mathbf{R}^d)}(1+r)^p \varepsilon_n^{-2p}(\varepsilon_n^2+\varepsilon_{n-1}^p).$$

Proof. — Fix $p \in N^{\geqq 2}$ and assume $r \geqq 2\varepsilon_{n-1} \geqq 2^{1-\eta}$, $r \in T$. Then

$$E((W_r(n)-\bar{W}_r(n))^{2p}) = \sum_{\gamma_1, \ldots, \gamma_{2p} \sim r} E(\Pi(\gamma_1, \ldots, \gamma_{2p}))$$

where

$$\Pi(\gamma_1, \ldots, \gamma_{2p}) = \prod_{i=1}^{2p} (1(J(\gamma_i, \varepsilon_n) \cap A(\gamma_i, n)) - P(J(\gamma_i, \varepsilon_n) \cap A(\gamma_i, n) \mid \mathscr{F}_1(\gamma_i))).$$

Fix $\gamma_1, \ldots, \gamma_{2p}$ such that $\gamma_i \sim r$ for $i \leqq 2p$ and

$$\sigma(\{\gamma_i: i<2p\}; \gamma_{2p})2^{-\eta} > 2\varepsilon_{n-1}.$$

Define

$$\begin{aligned}\mathscr{H} &= \sigma(B^\beta: \sigma(\beta; \gamma_{2p})2^{-\eta} \leqq 2\varepsilon_{n-1}) \\ &\quad \vee \sigma(e^\beta: \sigma(\beta; \gamma_{2p})2^{-\eta} \leqq 2\varepsilon_{n-1} \text{ and } \beta \neq \gamma_{2p} \mid i \text{ for all } i < |\gamma_{2p}|)\end{aligned}$$

and

$$\mathscr{G} = \mathscr{G}(\{\beta: \sigma(\beta; \gamma_{2p})2^{-\eta} > 2\varepsilon_{n-1}\}).$$

Then for $i < 2p$, $\mathscr{F}_1(\gamma_i) \subset \mathscr{G}$ and $J(\gamma_i, \varepsilon_n) \cap A(\gamma_i, n) \in \mathscr{G}$. Therefore

$$(3.23)\quad E(\Pi(\gamma_1, \ldots, \gamma_{2p})) = E\Bigg(\prod_{i=1}^{2p-1} (1(J(\gamma_i, \varepsilon_n) \cap A(\gamma_i, n)) - P(J(\gamma_i, \varepsilon_n) \cap A(\gamma_i, n) \mid \mathscr{F}_1(\gamma_1))) \times E((1(J(\gamma_{2p}, \varepsilon_n) \cap A(\gamma_{2p}, n)) - P(J(\gamma_{2p}, \varepsilon_n) \cap A(\gamma_{2p}, n) \mid \mathscr{F}_1(\gamma_{2p}))) \mid \mathscr{F}_1(\gamma_{2p}) \vee \mathscr{G})\Bigg).$$

There is a set D in $\mathscr{H}$ such that

$$A(\gamma_{2p}, n) \cap J(\gamma_{2p}, \varepsilon_n) = \{N_r^{\gamma_{2p}} \neq \Delta\} \cap D.$$

The independence of $\{e^\beta, B^\beta : \beta \in I\}$ shows $\mathscr{H}$ is independent of $\mathscr{F}_1(\gamma_{2p}) \vee \mathscr{G}$ and therefore

$$\begin{aligned}P(J(\gamma_{2p}, \varepsilon_n) \cap A(\gamma_{2p}, n) \mid \mathscr{F}_1(\gamma_{2p}) \vee \mathscr{G}) &= 1(N_r^{\gamma_{2p}} \neq \Delta)\, P(D \mid \mathscr{F}_1(\gamma_{2p}) \vee \mathscr{G}) = 1(N_r^{\gamma_{2p}} \neq \Delta)\, P(D)\\ &= P(\{N_r^{\gamma_{2p}} \neq \Delta\} \cap D \mid \mathscr{F}_1(\gamma_{2p})) = P(J(\gamma_{2p}, \varepsilon_n) \cap A(\gamma_{2p}, n) \mid \mathscr{F}_1(\gamma_{2p})).\end{aligned}$$

Substitute this into (3.23) to see that $E(\Pi(\gamma_1, \ldots, \gamma_{2p})) = 0$ whenever $\sigma(\{\gamma_i : i < 2p\}; \gamma_{2p}) > 2\varepsilon_{n-1}$, or more generally whenever

$$\sigma(\{\gamma_i : i \neq i_0\}; \gamma_{i_0}) > 2\varepsilon_{n-1}$$

for some $i_0 \leqq 2p$. We have therefore proved that

$$\begin{aligned}(3.24)\quad E((W_r(n) - \bar{W}_r(n))^{2p}) &= \sum_{\gamma_1, \ldots, \gamma_{2p} \sim r}^{2\varepsilon_{n-1}} E(\Pi(\gamma_1, \ldots, \gamma_{2p}))\\ &\leqq \sum_{\gamma_1, \ldots, \gamma_{2p} \sim r}^{2\varepsilon_{n-1}} E\left(\prod_{i=1}^{2p} (1(J(\gamma_i, \varepsilon_n)) + P(J(\gamma_i, \varepsilon_n) \mid \mathscr{F}_1(\gamma_i)))\right).\end{aligned}$$

If $\gamma = (\gamma_1, \ldots, \gamma_{2p})$ let $k(\gamma) = \operatorname{card}\{\gamma_1, \ldots, \gamma_{2p}\}$. If $k(\gamma) = k$ and $\{\gamma_1, \ldots, \gamma_{2p}\} = \{\hat{\gamma}_1, \ldots, \hat{\gamma}_k\}$, then

$$\begin{aligned}&E\left(\prod_{i=1}^{2p} (1(J(\gamma_i, \varepsilon_n)) + P(J(\gamma_i, \varepsilon_n) \mid \mathscr{F}_1(\gamma_i)))\right)\\ &\quad = 2^{2p} E\left(E\left(\prod_{i=1}^{2p} ((1(J(\gamma_i, \varepsilon_n)) + p(\varepsilon_n))/2) \mid \mathscr{A}_r\right) 1(D(\gamma))\right)\\ &\quad \leqq 2^{2p} E\left(E\left(\prod_{j=1}^{k} ((1(J(\hat{\gamma}_j, \varepsilon_n)) + p(\varepsilon_n))/2) \mid \mathscr{A}_r\right) 1(D(\gamma))\right)\\ &\quad = 2^{2p} E\left(\prod_{j=1}^{k} ((P(J(\hat{\gamma}_j, \varepsilon_n) \mid \mathscr{A}_r) + p(\varepsilon_n))/2)\, 1(D(\gamma))\right).\end{aligned}$$

Here we have used the fact that $\{J(\hat{\gamma}_j, \varepsilon_n) : j \leqq k\}$ are conditionally independent given $\mathscr{A}_r$. On $D(\gamma)$, $P(J(\hat{\gamma}_j, \varepsilon_n) \mid \mathscr{A}_r) = p(\varepsilon_n)$ and so the above is equal to $2^2 p(\varepsilon_n)^k P(D(\gamma))$. Substitute this into (3.24) and use (2.4) to see that if

$$L_r(n, k) = E\left(\sum_{\gamma_1, \ldots, \gamma_{2p} \sim r}^{2\varepsilon_{n-1}} 1(k(\gamma) = k)\, 2^{-\eta k}\, 1(D(\gamma))\right),$$

then

$$(3.25) \qquad E((W_r(n) - \bar{W}_r(n))^{2p}) \leqq c_1(p) \sum_{k=1}^{2p} \varepsilon_n^{-k} L_r(n, k).$$

Lemma 3.5 implies

$$(3.26) \qquad L_r(n, 2p) \leqq 4^{2p}(2p-1)!\,(2\varepsilon_{n-1})^p E(N_r(\mathbf{R}^d)^p).$$

If $\hat{\gamma}_1, \ldots, \hat{\gamma}_k$ are distinct and $k < 2p$, there are at most k^{2p} $2p$-tuples $(\gamma_1, \ldots, \gamma_{2p})$ such that $\{\gamma_1, \ldots, \gamma_{2p}\} = \{\hat{\gamma}_1, \ldots, \hat{\gamma}_k\}$. Whence

$$(3.27) \quad \sum_{k=1}^{2p-2} \varepsilon_n^{-k} L_r(n, k) \leqq \sum_{k=1}^{2p-2} \varepsilon_n^{-k} k^{2p} 2^{-\eta k}$$

$$\times E\Big(\sum_{\gamma_1, \ldots, \gamma_k \sim r} 1(D(\gamma_1, \gamma_2, \ldots, \gamma_k))\Big) \leqq \sum_{k=1}^{2p-2} \varepsilon_n^{-k} k^{2p} E(N_r(\mathbf{R}^d)^k)$$

$$\leqq (2p-2)^{2p} \varepsilon_n^{2-2p} E\left(\sum_{k=1}^{2p-2} N_r(\mathbf{R}^d)^k\right).$$

Finally we must bound $L_r(n, 2p-1)$. Let I_1 denote the set of $2\varepsilon_{n-1}$-good $2p$-tuples $(\gamma_1, \ldots, \gamma_{2p})$ such that each $\gamma_i \sim r$, $\gamma_1 = \gamma_2$, $k(\gamma) = 2p-1$ and $\sigma(\{\gamma_i : i > 2\}; \gamma_1) \leqq 2\varepsilon_{n-1} 2^{\eta}$. Let I_2 be the set of $2p$-tuples satisfying the same conditions but with $>$ in place of $\leqq$ in the last inequality. If $(\gamma_1, \ldots, \gamma_{2p}) \in I_1$, then $(\gamma_2, \ldots, \gamma_{2p})$ is a $2\varepsilon_{n-1}$-good $(2p-1)$-tuple and if $(\gamma_1, \ldots, \gamma_{2p}) \in I_2$, then $(\gamma_3, \ldots, \gamma_{2p})$ is a $2\varepsilon_{n-1}$-good $(2p-2)$-tuple. Therefore by symmetry we have

$$(3.28) \quad L_r(n, 2p-1) \leqq \binom{2p}{2} 2^{\eta(1-2p)} \Big(E\Big(\sum_{\gamma_1, \ldots, \gamma_{2p} \sim r} 1(\gamma \in I_1, D(\gamma))\Big)$$

$$+ E\Big(\sum_{\gamma_1, \ldots, \gamma_{2p} \sim r} 1(\gamma \in I_2, D(\gamma))\Big)\Big)$$

$$\leqq \binom{2p}{2} \Bigg[E\left(\sum_{\gamma_2, \ldots, \gamma_{2p} \sim r}^{2\varepsilon_{n-1}} 2^{-\eta(2p-1)} 1(D(\gamma_2, \ldots, \gamma_{2p}))\right)$$

$$+ E\left(\sum_{\gamma_3, \ldots, \gamma_{2p} \sim r}^{2\varepsilon_{n-1}} 2^{-\eta 2(p-1)} 1(D(\gamma_3, \ldots, \gamma_{2p}))\right.$$

$$\times E\Big(2^{-\eta} \sum_{\gamma_1 \sim r} 1(\sigma(\{\gamma_i : i \geqq 3\}; \gamma_1) < 2^{\eta} r)\, 1(N_r^{\gamma_1} \neq \Delta)$$

$$\left.\left. + 2^{-\eta} \sum_{\gamma_1 \sim r} 1(\sigma(\{\gamma_i : i \geqq 3; \gamma_1) = 2^{\eta} r)\, 1(N_r^{\gamma_1} \neq \Delta) \;\Big|\; \mathcal{F}(\gamma_3, \ldots, \gamma_{2p})\Big)\right)\right]$$

$$\leq \binom{2p}{2}[4^{2p-1}(2p-2)(2\varepsilon_{n-1})^p \mathrm{E}(\mathrm{N}_r(\mathbf{R}^d)^{p-1}) + 4^{2(p-1)}(2p-3)!\,(2\varepsilon_{n-1})^{p-1}\mathrm{E}(\mathrm{N}_r(\mathbf{R}^d)^{p-1}) \times (2(p-1)(r+2^{-\eta})+\mathrm{E}(\mathrm{N}_r(\mathbf{R}^d)))].$$

In the last line we have used Lemma 3.5 and Lemma 2.4 (c) of Perkins (1988 a). Combine (3.25), (3.26), (3.27) and (3.28) and conclude

$$\text{(3.29)}\quad \mathrm{E}((\mathrm{W}_r(n)-\bar{\mathrm{W}}_r(n))^{2p}) \leq c_2(p)\Big[\varepsilon_n^{2-2p}\mathrm{E}\Big(\sum_{k=1}^{2p-2}\mathrm{N}_r(\mathbf{R}^d)^k\Big) + \varepsilon_n^{1-2p}\varepsilon_{n-1}^{p-1}\mathrm{E}(\mathrm{N}_r(\mathbf{R}^d)^{p-1})(r+1+m(\mathbf{R}^d)) + \varepsilon_n^{-2p}\varepsilon_{n-1}^{p}\mathrm{E}(\mathrm{N}_r(\mathbf{R}^d)^p)\Big].$$

It is easy to bound the moments of $\mathrm{N}_t(\mathbf{R}^d)$ in terms of $m(\mathbf{R}^d)$, for example, by using the bound on $\mathrm{E}(e^{\theta \mathrm{N}_t(\mathbf{R}^d)})$ in Proposition 2.6 (c) of Perkins (1988 a) to get

$$\mathrm{E}(\mathrm{N}_r(\mathbf{R}^d)^p) \leq c_3(p)(1+r)^p \exp(m(\mathbf{R}^d)).$$

Use this in (3.29) and obtain

$$\mathrm{E}(\mathrm{W}_r(n)-\bar{\mathrm{W}}_r(n))^{2p}) \leq c_{3.6}(p)\,e^{m(\mathbf{R}^d)}(1+r)^p \times[\varepsilon_n^{2-2p}+\varepsilon_n^{-2p}\varepsilon_{n-1}^{p}(1+m(\mathbf{R}^d))] \leq c_{3.6}(p)\,e^{2m(\mathbf{R}^d)}(1+r)^p\varepsilon_n^{-2p}(\varepsilon_n^2+\varepsilon_{n-1}^p). \quad \blacksquare$$

LEMMA 3.7. — *There is an* $n_{3.1}\in\mathbf{N}$ *such that if* $n \geq n_{3.1}$, $\eta \geq \eta_n$ (η_n *as in* (3.22)), *then*

$$\mathrm{P}\Big(\sup_{4\varepsilon_{n-1}\leq t\leq n} b_n^\eta(t) \geq 2^{-\eta}\Big) \leq e^{2m(\mathbf{R}^d)}2^{-n}.$$

Proof. — Lemma 3.3 shows that for $\eta \geq \eta_n$,

$$\text{(3.30)}\quad \mathrm{P}\Big(\sup_{4\varepsilon_{n-1}\leq t\leq n} b_n(t)\geq 2^{-\eta}\Big) \leq \mathrm{P}(\mathrm{A}_1(n))+\mathrm{P}(\mathrm{A}_2(n)) + \mathrm{P}(\sup(\mathrm{W}_r(n): r\in\tau_n,\ 2\varepsilon_{n-1}\leq r<n)\geq c_{3.2}^{-1}2^{-n}\tilde{\psi}(a_{2^{n+1}})^{-1})$$
$$\leq c_{2.2}(3)\,m(\mathbf{R}^d)(4\varepsilon_n)^{c_{2.3}(3)}+c_{3.1}\,m(\mathbf{R}^d)\exp\{-2^{n-2}\log 2(\rho(n-1)-20)\} + \mathrm{P}\Big(\sup_{t\leq n}\mathrm{N}_t(\mathbf{R}^d)\geq c_{3.2}^{-1}2^{-1-n}\tilde{\psi}(a_{2^{n+1}})^{-1}c_{3.3}\,\varepsilon_n\exp(c_{3.4}2^{n/2})\Big) + \mathrm{P}(\sup(|\mathrm{W}_r(n)-\bar{\mathrm{W}}_r(n)|: r\in\tau_n,\ 2\varepsilon_{n-1}\leq r<n) \geq c_{3.2}^{-1}2^{-1-n}\tilde{\psi}(a_{2^{n+1}})^{-1}),$$

where we have used Theorem 2.2, Lemma 3.2, Lemma 3.4 and the choice of ρ to bound $\mathrm{P}(\mathrm{A}_1(n))$, $\mathrm{P}(\mathrm{A}_2(n))$ and $\bar{\mathrm{W}}_r(n)$, respectively. Use Doob's maximal inequality for martingales to bound the third term on the extreme right-hand side of (3.30) by

$$(3.31)\quad P(\sup_{t \leq n} N_t(\mathbf{R}^d) \geq (c_{3.2}c_{3.3})^{-1} \times (\log 2)^{-2-d/2} 2^{-(3+d/2)n-1} \exp(c_{3.4} 2^{n/2})) \leq c_{3.2}c_{3.3}(\log 2)^{2+d/2} 2^{(3+d/2)n+1} \times \exp(-c_{3.4} 2^{n/2})\, m(\mathbf{R}^d)\, (\ll 2^{-n}).$$

Lemma 3.6 shows that the last term on the extreme right side of (3.30) is less than

$$\sum_{r \in \tau_n,\, 2\varepsilon_{n-1} < r < n} c_{3.2}^{2p} 2^{(1+n)2p} 2^{(2+d/2)n2p} (\log 2)^{(2+d/2)2p} \varepsilon_n^{2p} \times E(|W_r(n) - \bar{W}_r(n)|^{2p}) \leq c_1(p)\, n\, \varepsilon_n^{-1} 2^{(3+d/2)2np} e^{2m(\mathbf{R}^d)} (1+n)^p (\varepsilon_n^2 + \varepsilon_{n-1}^p).$$

If $p=4$, this is less than

$$c_2 e^{2m(\mathbf{R}^d)} n^5 2^{(3+d/2)8n} \varepsilon_n (\ll 2^{-\eta}).$$

Substitute this bound and (3.31) into the right side of (3.30) to see that for $n \geq n_{3.1}$ ($n_{3.1}$ depending only on d),

$$P(\sup_{4\varepsilon_{n-1} \leq t \leq n} b_n(t) \geq 2^{-n}) \leq e^{2m(\mathbf{R}^d)} 2^{-n}. \quad \blacksquare$$

Standard covering arguments now complete the proofs of Theorems 1 and 2.

LEMMA 3.8. — *If* $E = \bigcup_{i=1}^{m} C_i$ *and each* C_i *is a cube in* $\Lambda_{2k(i)}$ *for some* $k(i)$, *then there is a subset* $\{i_n\} \subset \{1, \ldots, m\}$ *such that* $E = \bigcup C_{i_n}$ *and no point of* E *is contained in more than* 2^d *of the cubes* $\{C_{i_n}\}$.

This is a slight perturbation of Lemma 1 of Taylor (1964). We are dealing with open cubes of side 2^{-k} rather than closed squares of side 2^{-2k}. The proof remains unchanged provided one replaces the factor 4 in Taylor's result in $\mathbf{R}^2$ with 2^d.

To complete the proof of Theorems 1 and 2 consider $d \geq 2$ and for a given initial measure m in $M_F(\mathbf{R}^d)$ choose $m_\eta \in M_F^\eta(\mathbf{R}^d)$ such that $m_\eta \xrightarrow{w} m$ as $\eta \to \infty$. Hence $N^{(\eta)} \xrightarrow{w} X$ on $D([0, \infty), M_F(\mathbf{R}^d))$ by Theorem 2.1, where X has law Q^m. By a theorem of Skorokhod (Skorokhod (1956, Thm. 3.1.1, p. 281)) we may work on a probability space $(\Omega, \mathscr{F}, P)$ on which $N^{(\eta)} \to X$ a. s. in $D([0, \infty), M_F)$ as $\eta \to \infty$. Since $X_.$ is a. s. continuous this means that $N_t^\eta \to X_t$ for all $t \geq 0$ a. s. By redefining $\{N^{(\eta)}\}$ and X off a null set we may (and shall) assume $N_t^{(\eta)} \to X_t$ for all $t \geq 0$ and all $\omega \in \Omega$.

Let

$$\begin{aligned}
G_n(t) &= \{C \in \Lambda_{2^{n+1}} : C \text{ is good for } X_t\} \\
B_n(t) &= \{C \in \Lambda_{2^{n+1}} : C \text{ is bad for } X_t\} \\
b_n^\eta(\omega) &= \sup\{b_n^\eta(t, \omega) : 4\varepsilon_{n-1} \leqq t \leqq n\}, \ b_n^\eta(t) \text{ as in (3.3)} \\
b_n(\omega) &= \sup\{\sum_{C \in \Lambda_{2^{n+1}}} 1(C \in B_n(t))\psi(a_{2^{n+1}}) : 4\varepsilon_{n-1} \leqq t \leqq n\}.
\end{aligned}$$

Assume C is bad for X_t. Then $X_t(C) > 0$ and $X_t(C^{13 a_k}) < \rho\varphi(a_k)$ for all $k = 2^n, \ldots, 2^{n+1}-n-1$. Since C is open and $C^{13 a_k}$ is closed this implies

$$\underline{\lim}_{\eta \to \infty} N_t^{(\eta)}(C) > 0 \text{ and } \overline{\lim}_{\eta \to \infty} N_t^{(\eta)}(C^{13 a_k}) < \rho\varphi(a_k) \quad \text{for} \quad k = 2^n, \ldots, 2^{n+1}-n-1.$$

This means that C is bad for $N_t^{(\eta)}$ for large enough η. Therefore

$$\sum_{C \in \Lambda_{2^{n+1}}} 1(C \text{ is bad for } X_t) \leqq \underline{\lim}_{\eta \to \infty} \sum_{C \in \Lambda_{2^{n+1}}} 1(C \text{ is bad for } N_t^{(\eta)})$$

and it follows easily that

$$(3.32) \qquad b_n(\omega) \leqq \underline{\lim}_{\eta \to \infty} b_n^\eta(\omega) \equiv \bar{b}_n(\omega).$$

Lemma 3.7 shows that if $n \geqq n_{3.1}$ and $\eta \geqq \eta_n$,

$$P(b_n^\eta \geqq 2^{-n}) \leqq \exp(2 m_\eta(\mathbf{R}^d)) 2^{-n}.$$

Let $\eta \to \infty$ in the above to conclude that

$$P(\bar{b}_n \geqq 2^{-n}) \leqq 2^{2m(\mathbf{R}^d)} 2^{-n} \quad \text{for} \quad n \geqq n_{3.1},$$

and hence by Borel-Cantelli and (3.32),

$$(3.33) \qquad \lim_{n \to \infty} b_n(\omega) = 0 \quad \text{a. s.}$$

Fix ω such that (3.33) holds. Let $t > 0$. If $n \geqq 4$ and $C \in G_n(t)$, there is a $k \in [2^n, 2^{n+1}-n-1] \cap \mathbf{N}$ such that $X_t(C^{13 a_k}) \geqq \rho\varphi(a_k)$. The definition of Λ_k shows that if $k' = k-12$ or $k-13$, whichever one is even, then there is a $C' \in \Lambda_{k'}$ such that $C^{13 a_k} \subset C'$ and hence

$$(3.34) \qquad X_t(C') \geqq \rho\varphi(a_k) \geqq \rho 2^{-13} \varphi(a_{k'}).$$

Choose one such C′ for every C in $G_n(t)$ and let $G_n'(t)$ denote the resulting connection of C′'s. By Lemma 3.8 there is a subset $G_n''(t)$ of $G_n'(t)$ such that

(3.35) $E(t) = \cup\{C : C \in G_n''(t)\} = \cup\{C : C \in G_n'(t)\} \supset \cup\{C : C \in G_n(t)\}$

and

(3.36) No point in $E(t)$ is covered by more than 2^d sets in $G_n''(t)$.

Let A be a compact set in $\mathbf{R}^d$. If n is large enough so that $A \subset (-n, n)^d$ and $x \in A \cap S_t$ then $x \in C$ for some C in $\Lambda_{2^{n+1}}$. Since $X_t(C) > 0$, C must be in $B_n(t)$ or $G_n(t)$. By (3.35) we have

$$(3.37) \quad A \cap S_t \subset \cup \{C : C \in G_n''(t),\ C \cap A \neq \varnothing\} \cup (\cup \{C : C \in B_n(t)\}).$$

Every cube C in $G_n''(t)$ has diameter dC at most $c_1(d)\, a_{2^n-13} = r_n$. Therefore if n is also large enough so that $t \in [4\varepsilon_{n-1}, n]$ then

$$\begin{aligned}
\sum_{C \in G_n''(t)} 1(C \cap A \neq \varnothing)\, \varphi(dC) + \sum_{C \in B_n(t)} \varphi(dC) \\
\leqq c_2(d)\, \rho^{-1} 2^{13} \sum_{C \in G_n''(t)} 1(C \cap A \neq \varnothing)\, X_t(C) \\
+ c_2(d)\, \varphi(a_{2^n+1})\, \psi(a_{2^n+1})^{-1}\, b_n \quad (\text{by } (3.34)) \\
\leqq c_2(d)\, \rho^{-1} 2^{13+d} X_t(A^{r_n}) \\
+ c_2(d)\, \varphi(a_{2^n+1})\, \psi(a_{2^n+1})^{-1}\, b_n \quad (\text{by } (3.36)).
\end{aligned}$$

Let $n \to \infty$ and use (3.33) and (3.37) to conclude

$$(3.38) \qquad \varphi - m(A \cap S_t) \leqq c_2(d)\, \rho^{-1} 2^{13+d} X_t(A)$$

(recall A is closed). The inner regularity of $\varphi - m$ with respect to compact sets [Rogers (1970, Theorems 47, 48 and the ensuing Corollaries)] shows that (3.38) holds for any analytic set A. This proves the required lower bound on X. As was pointed out in the Introduction, the necessary upper bound for X for $d > 2$ and $d = 2$ are immediate from Theorems A and B of Perkins (1988 *a*), respectively. ■

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