ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 25, nº 2 (1989), p. 195-203

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http://www.numdam.org/

Ann. Inst. Henri Poincaré,

Vol. 25, n° 2, 1989, p. 195-203. *Probabilités et Statistiques*

On the asymptotic equidistribution of sums of independent identically distributed random variables

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ABSTRACT. — For a sum S_n of n I.I.D. random variables the idea of approximate equidistribution is made precise by introducing a notion of asymptotic translation invariance. The distribution of S_n is shown to be asymptotically translation invariant in this sense iff S_1 is nonlattice. Some ramifications of this result are given.

Key words : Sums of I.I.D. random variables, asymptotic equidistribution, asymptotic translation invariance.

RÉSUMÉ. — On introduit, pour une somme S_n de n variables aléatoires indépendantes équidistribuées, une notion d'invariance asymptotique par translation, qui permet de rendre précise l'idée d'équidistribution approximative. On montre que la loi de S_n est asymptotiquement invariante par translation en ce sens si, et seulement si, la loi de S_1 est non arithmétique. On donne quelques extensions de ce résultat.

1. INTRODUCTION

Let T_1, T_2, ... be a sequence of independent random variables with a common distribution $P^{T_n}=P^{T_1}$ and let $S_n=T_1+\ldots+T_n$. Intuitively, if

Classification A.M.S. : 60 G 50, 60 F 99.

$E(T_1)=0$ and $T_1 \not\equiv 0$, the mass of the probability measure P^{S_n} is expected to be approximately "equidistributed", as n becomes large. If $E(T_1)>0$, one is inclined to think of something like an "approach to uniformity at infinity". An old result of this kind is due to Robbins (1953). If T_1 is not concentrated on a lattice,

$$n^{-1} \sum_{i=1}^{n} h(S_i) \to \lim_{T \to \infty} (2T)^{-1} \int_{-T}^{T} h(x)\,dx, \qquad \text{as} \quad n \to \infty \tag{1.1}$$

for all almost periodic functions h (*i. e.* if h is the uniform limit of trigonometric polynomials). For a partial sharpening of this result *see* Theorem 3 of Stadje (1985). In the case when T_1 is not concentrated on a lattice, $E(T_1)=0$ and $0<\sigma^2:=\operatorname{Var}(T_1)<\infty$, the expectation of asymptotic equidistribution can also be justified by the limiting relation

$$\sigma(2\pi n)^{1/2} P(S_n \in I) \to \lambda(I), \qquad \text{as} \quad n \to \infty \tag{1.2}$$

which is valid for all bounded intervals $I \subset \mathbb{R}$, where λ denotes the Lebesgue measure (Shepp (1964), Stone (1965, 1967), Breiman (1968), chapt. 10).

One might try to interpret approximate uniformity of P^{S_n} by stating that $P(S_n \in I)$ asymptotically only depends on the length of I. Since $\lim_{n \to \infty} P(S_n \in I)=0$ for every bounded interval I, this idea should be made reasonable by examining the speed of convergence of $P(S_n \in I)$. This is done in (1.2) stating $a_n P^{S_n}$ approaches the Lebesgue measure, where $a_n=\sigma(2\pi n)^{1/2}$.

In this paper another approach to the idea of equidistribution of P^{S_n} is developed. The essential property of an "equidistribution" is the invariance under translations. To measure the degree of translation invariance of a probability measure Q on $\mathbb{R}$, we introduce, for $a \in \mathbb{R}$ and $t>s>0$, the quantities

$$d(a, t, Q):=\sum_{i=-\infty}^{\infty} \left| Q((a+it, a+(i+1)t]) - Q((a+(i-1)t, a+it]) \right| \tag{1.3}$$

$$D(t, Q):=\sup_{a \in \mathbb{R}} d(a, t, Q) \tag{1.4}$$

$$\tilde{D}(s, t, Q):=\sup_{s \leqq u \leqq t} D(u, Q). \tag{1.5}$$

We call a sequence $(Q_n)_{n \geqq 1}$ of probability measures *asymptotically translation invariant* (ATI), if $\lim_{n \to \infty} \tilde{D}(s, t, Q_n)=0$ for all $t>s>0$. The main theorem of this paper states that $(P^{S_n})_{n \geqq 1}$ is ATI if, and only if, P^{T_1} is not concentrated on a lattice. No moment conditions are needed for this equivalence. Let $D_n(t):=\tilde{D}(t^{-1}, t, P^{S_n})$, $t>1$. Regarding the speed of convergence of $D_n(t)$ we remark that

$$\liminf_{n \to \infty} n^{1/2} D_n(t)>0 \quad \text{for all } t>1, \qquad \text{if} \quad E(T_1^2)<\infty. \tag{1.6}$$

To see (1.6), let without loss of generality $E(T_1)=0$ and $E(T_1^2)=1$. Then, by Chebyshev's inequality,

$$P(|S_n|<n^{1/2})\geqq 1-n^{-1}. \tag{1.7}$$

The interval $(-n^{1/2}, n^{1/2})$ can be covered by $[2n^{1/2}/t]+1$ half-open intervals of length t. One of these intervals, say I, obviously satisfies

$$P(S_n\in I)\geqq(1-n^{-1})/([2n^{1/2}/t]+1)\geqq\frac{1}{2}\,\frac{1}{2n^{1/2}t+1},\qquad \text{if}\quad n\geqq 2. \tag{1.8}$$

Choose $a\in[0, t]$ and $i_0\in\mathbb{Z}$ such that $I=(a+i_0 t, a+(i_0+1)t]$.
Then

$$\begin{aligned}d(a, t, P^{S_n})\geqq&\sum_{i=-\infty}^{i_0}[P(S_n\in(a+it, a+(i+1)t])\\&\qquad -P(S_n\in(a+(i-1)t, a+it])]\\&=P(S_n\in(a+i_0 t, a+(i_0+1)t])\geqq\frac{1}{4t+2}n^{-1/2},\qquad n\geqq 2\end{aligned} \tag{1.9}$$

(1.6) follows from (1.9).

In order to derive a converse result to (1.6), we need a further notion. A distribution Q on $\mathbb{R}$ is called strongly nonlattice, if its characteristic function φ satisfies

$$\limsup_{|\zeta|\to\infty}|\varphi(\zeta)|<1. \tag{1.10}$$

The second main result of this note is that if P^{T_1} is strongly nonlattice,

$$\limsup_{n\to\infty} n^{1/2}D_n(t)<\infty\quad \text{for all}\quad t>1. \tag{1.11}$$

2. THE MAIN THEOREM

We shall prove

THEOREM 1. — *The following two statements are equivalent.*
(*a*) P^{T_1} *is nonlattice.*
(*b*) $(P^{S_n})_{n\geq 1}$ *is ATI.*

Proof. — (*a*) $\Rightarrow$ (*b*). Assume first that $E|T_1|^3<\infty$ and $E(T_1)=0$. Let $\sigma^2=E(T_1^2)$, $\mu_3=E(T_1^3)$ and denote the distribution function of S_n by F_n. Then

$$\begin{aligned}d(a, t, P^{S_n})=\sum_{i=-\infty}^{\infty}&|F_n(a+(i+1)t)-2F_n(a+it)\\&+F_n(a+(i-1)t)|.\end{aligned} \tag{2.1}$$

Since P^{T_1} is nonlattice, a well-known expansion for distribution functions yields

$$F_n(n^{1/2}\sigma x)=\Phi(x)+\frac{\mu_3}{6\sigma^3 n^{1/2}}(1-x^2)\varphi(x)+\varepsilon_n(x)n^{-1/2} \tag{2.2}$$

for all $x\in\mathbb{R}$, where

$$\varepsilon_n:=\sup_{x\in\mathbb{R}}|\varepsilon_n(x)|\to 0, \qquad \text{as} \quad n\to\infty \tag{2.3}$$

and Φ and φ are the distribution function and density of N(0, 1) (*see* e. g. Feller (1971), p. 539). It is easy to check that for each $j\in\mathbb{N}$ and $a\in[0, t]$

$$\sum_{i>j}|F_n(a+(i+1)t)-2F_n(a+it)+F_n(a+(i-1)t)|$$
$$\leqq 1-F_n(a+(j+1)t)+1-F_n(a+jt)\leqq 2P(S_n>jt) \tag{2.4}$$

and

$$\sum_{i<-j}|F_n(a+(i+1)t)-2F_n(a+it)+F_n(a+(i-1)t)|$$
$$\leqq F_n(a-jt)+F_n(a-(j+1)t)\leqq 2P(S_n\leqq-(j-1)t). \tag{2.5}$$

Inserting (2.2)-(2.5) into (2.1) we obtain, for $a\in[0, t]$,

$$\begin{aligned} d(a, t, P^{S_n}) &\leqq 2P(|S_n|\geqq(j-1)t)+(2j+1)\varepsilon_n n^{-1/2}\\ &+\sum_{i=-\infty}^{\infty}\left|\Phi\left(\frac{a+(i+1)t}{\sigma n^{1/2}}\right)-2\Phi\left(\frac{a+it}{\sigma n^{1/2}}\right)+\Phi\left(\frac{a+(i-1)t}{\sigma n^{1/2}}\right)\right|\\ &+\frac{|\mu_3|}{6\sigma^3 n^{1/2}}\sum_{i=-\infty}^{\infty}|(1-x_{i+1}^2)\varphi(x_{i+1})\\ &-2(1-x_i^2)\varphi(x_i)+(1-x_{i-1}^2)\varphi(x_{i-1})| \end{aligned} \tag{2.6}$$

where $x_i:=(a+it)/\sigma n^{1/2}$. By Chebyshev's inequality,

$$2P(|S_n|\geqq(j-1)t)+(2j+1)\varepsilon_n n^{-1/2}\leqq\frac{2\sigma^2 n}{t^2(j-1)^2}+(2j+1)\varepsilon_n n^{-1/2}. \tag{2.7}$$

The smallest order of magnitude of the righthand side of (2.7) is attained for $j=j_n$ being equal to the integer part of $n^{1/2}\varepsilon_n^{-1/3}$; in this case

$$2P(|S_n|\geqq(j_n-1)t)+(2j_n+1)\varepsilon_n n^{-1/2}$$
$$=(1+t^{-2})O(\varepsilon_n^{2/3}), \qquad \text{as} \quad n\to\infty. \tag{2.8}$$

Next we estimate the two series in (2.6). Let X be a standard normal random variable. Then the first sum at the righthand side of (2.6) is equal to

$$\sum_{i=-\infty}^{\infty} |P(\sigma n^{1/2} X \in (a+it,\ a+(i+1)t])$$

$$-P(\sigma n^{1/2} X - t \in (a+it,\ a+(i+1)t)]|$$

$$\leqq (\sigma n^{1/2})^{-1} \int_{-\infty}^{\infty} |\varphi(x/\sigma n^{1/2}) - \varphi((x+t)/\sigma n^{1/2})|\, dx$$

$$= 2(\sigma^{1/2})^{-1} \left[\int_{-\infty}^{-t/2} \varphi((x+t)/\sigma n^{1/2})\, dx - \int_{-\infty}^{-t/2} \varphi(x/\sigma n^{1/2})\, dx \right]$$

$$= 2 \int_{-t/2\sigma n^{1/2}}^{t/2\sigma n^{1/2}} \varphi(u)\, du = t\, O(n^{-1/2}), \qquad \text{as} \quad n \to \infty. \quad (2.9)$$

To estimate the last sum at the right side of (2.6), note that the function $(1-x^2)\exp(-x^2/2)$ has four points of inflexion. Regarding the sequence

$$a_i := (1-x_{i+1}^2)\,\varphi(x_{i+1}) - (1-x_i^2)\,\varphi(x_i), \qquad i \in \mathbb{Z}$$

this implies that $(a_i - a_{i-1})_{i \in \mathbb{Z}}$ changes signs at most four times. Using its telescoping form the sum in question can be bounded from above as follows:

$$\sum_{i=-\infty}^{\infty} |a_i - a_{i-1}| \leqq 8 \sup_{-\infty < i < \infty} |a_i|. \qquad (2.10)$$

Further, by the mean value theorem,

$$|a_i| \leqq |x_{i+1} - x_i| \sup_{x \in \mathbb{R}} \left| \frac{d}{dx}(1-x^2)\,\varphi(x) \right| = K\, t/\sigma n^{1/2} \qquad (2.11)$$

for some constant K. Inserting (2.8)-(2.11) into (2.6) we arrive at

$$d(a,\ t,\ P^{S_n}) = (1+t^{-2})\, O(\varepsilon_n^{2/3}) + t\, O(n^{-1/2}), \qquad \text{as} \quad n \to \infty, \quad (2.12)$$

so that

$$D(t,\ P^{S_n}) = (1+t^{-2})\, O(\varepsilon_n^{2/3}) + t\, O(n^{-1/2}), \qquad \text{as} \quad n \to \infty. \quad (2.13)$$

To establish the assertion without moment conditions we first remark that $D(t, Q)$ is translation invariant in the sense that

$$D(t,\ Q) = D(t,\ Q * \varepsilon_x) \quad \text{for all } x \in \mathbb{R}, \quad t > 0, \qquad (2.14)$$

where $*$ denotes convolution and ε_x is the point mass at x. Thus, (2.13) holds, if $E|T_1|^3 < \infty$ (without the assumption $E(T_1) = 0$). Further, for probability measures Q and R we have

$$D(t,\ Q * R) \leqq D(t,\ Q). \qquad (2.15)$$

(2.15) is proved as follows:

$$\begin{aligned}
D(t, Q*R) &= \sup_a \sum_{i=-\infty}^{\infty} \left| (Q*R)((a+it, a+(i+1)t]) \right. \\
&\qquad \left. -(Q*R)((a+(i-1)t, a+it]) \right| \\
&\leqq \sup_a \sum_{i=-\infty}^{\infty} \int_{-\infty}^{\infty} \left| Q((a+it-x, a+(i+1)t-x]) \right. \\
&\qquad \left. - Q((a+(i-1)t-x, a+it-x)]) \right| dR(x) \\
&\leqq \int_{-\infty}^{\infty} \sup_a \sum_{i=-\infty}^{\infty} \left| Q((a+it-x, a+(i+1)t-x]) \right. \\
&\qquad \left. - Q((a+(i-1)t-x, a+it-x]) \right| dR(x) \\
&= \int_{-\infty}^{\infty} D(t, Q)\, dR(x) = D(t, Q). \qquad (2.16)
\end{aligned}$$

Next suppose that $P^{T_1} = \alpha Q + (1-\alpha) R$ for some $\alpha \in (0, 1]$ and probability measures Q and R such that Q satisfies, for some constants K_1, K_2,

$$D(t, Q^{*n}) \leqq (1+t^{-2}) K_1 \varepsilon_n^{2/3} + t K_2 n^{-1/2}, \qquad \text{as} \quad n \to \infty \qquad (2.17)$$

(Q^{*n} is the n-fold convolution of Q with itself). Then

$$\begin{aligned}
D(t, P^{S_n}) &= \sup_a \sum_{i=-\infty}^{\infty} \left| \sum_{l=0}^{n} \binom{n}{l} \alpha^l (l-\alpha)^{n-l} \{ Q^{*l} * R^{*(n-l)} ((a+it, \right. \\
&\qquad \left. a+(i+1)t]) - Q^{*l} * R^{*(n-l)} ((a+(i-1)t, a+it]) \} \right| \\
&\leqq \sum_{l=0}^{n} \binom{n}{l} \alpha^l (1-\alpha)^{n-l} D(t, Q^{*l} * R^{*(n-l)}) \\
&\leqq \sum_{l=0}^{n} \binom{n}{l} \alpha^l (1-\alpha)^{n-l} D(t, Q^{*l}). \qquad (2.18)
\end{aligned}$$

Let $\delta_n := \sup_{l>n} \varepsilon_l$. Then $\delta_n \downarrow 0$ and, by Chebyshev's inequality for the binomal distribution and (2.17), it follows that, for arbitrary $\varepsilon \in (0, \alpha)$,

$$\begin{aligned}
\sum_{l=0}^{n} \binom{n}{l} \alpha^l (1-\alpha)^{n-l} D(t, Q^{*l}) &\leqq 2 \sum_{l \leqq (\alpha-\varepsilon) n} \binom{n}{l} \alpha^l (1-\alpha)^{n-l} \\
&\qquad + (1+t^{-2}) K_1 \sup_{l > (\alpha-\varepsilon) n} \varepsilon_l + t K_2 ((\alpha-\varepsilon) n)^{-1/2} \\
&\leqq 2\alpha(1-\alpha)\varepsilon^{-2} n^{-1} + K[(1+t^{-2}) \delta_{(\alpha-\varepsilon) n} + t n^{-1/2}] \qquad (2.19)
\end{aligned}$$

where $K = \max(K_1, K_2)$. (2.19) implies that $D_n(t) \to 0$, as $n \to \infty$, for all $t > 1$.

Now we choose a function f on $\mathbb{R}$ such that $0<f(x)<1$ for all $x\in\mathbb{R}$ and

$$\int_{-\infty}^{\infty} |x|^3 f(x)\, d\mathrm{P}^{\mathrm{T}_1}(x)<\infty.$$

Let $\alpha := \int_{-\infty}^{\infty} f(x)\, d\mathrm{P}^{\mathrm{T}_1}(x)$ and define the probability measures Q and R by $d\mathrm{Q} := \alpha^{-1} f\, d\mathrm{P}^{\mathrm{T}_1}$, $d\mathrm{R} = (1-\alpha)^{-1}(1-f)\, d\mathrm{P}^{\mathrm{T}_1}$. Then the third moment of Q is finite and Q is nonlattice so that Q satisfies (2.17). Since $\mathrm{P}^{\mathrm{T}_1} = \alpha\,\mathrm{Q} + (1-\alpha)\,\mathrm{R}$, it follows that $\mathrm{D}_n(t)\to 0$.

$(b)\Rightarrow(a)$. Let $\mathrm{P}^{\mathrm{T}_1}$ have span $\lambda>0$. If $t\in(0, \lambda/2)$, at most one of the successive intervals $(a+(i-1)t,\ a+it]$ and $(a+it,\ a+(i+1)t]$ contains a multiple of λ. Therefore it is obvious that $d(a, t, \mathrm{P}^{\mathrm{S}_n})=2$ for all $a\in\mathbb{R}$, $t\in(0, \lambda/2)$ and $n\in\mathbb{N}$.

3. THE STRONGLY NONLATTICE CASE

Concerning the speed of convergence of $\mathrm{D}_n(t)$ we shall now prove

THEOREM 2. — *If* $\mathrm{P}^{\mathrm{T}_1}$ *is strongly nonlattice,*

$$\limsup_{n\to\infty} n^{1/2}\, \mathrm{D}_n(t)<\infty \quad \text{for all } t>1. \tag{3.1}$$

Proof. — Let $\eta := \limsup_{|\zeta|\to\infty} |\varphi(\zeta)|<1$. We can decompose $\mathrm{P}^{\mathrm{T}_1} = \alpha\,\mathrm{Q} + (1-\alpha)\,\mathrm{R}$, where $\alpha\in(0, 1]$ and Q and R are probability measures such that Q is strongly nonlattice and concentrated on a bounded interval. If $\mathrm{P}^{\mathrm{T}_1}$ itself is concentrated on a bounded interval, this is trivial. Otherwise let $\alpha_{\mathrm{N}} := \mathrm{P}(\mathrm{T}_1\in[-\mathrm{N}, \mathrm{N}])$, where N is large enough to ensure $0<\alpha_{\mathrm{N}}<1$. Define, for Borel sets B,

$$\mathrm{Q}_{\mathrm{N}}(\mathrm{B}) := \alpha_{\mathrm{N}}^{-1}\, \mathrm{P}(\mathrm{T}_1\in \mathrm{B}\cap[-\mathrm{N}, \mathrm{N}])$$
$$\mathrm{R}_{\mathrm{N}}(\mathrm{B}) := (1-\alpha_{\mathrm{N}})^{-1}\, \mathrm{P}(\mathrm{T}_1\in \mathrm{B}\setminus[-\mathrm{N}, \mathrm{N}]).$$

Then the characteristic functions $\tilde{\varphi}_{\mathrm{N}}$ and $\tilde{\tilde{\varphi}}_{\mathrm{N}}$ of Q_{N} and R_{N} satisfy $\tilde{\varphi}_{\mathrm{N}} = \alpha_{\mathrm{N}}^{-1}(\varphi-(1-\alpha_{\mathrm{N}})\,\tilde{\tilde{\varphi}}_{\mathrm{N}})$ so that

$$\limsup_{|\zeta|\to\infty} |\tilde{\varphi}_{\mathrm{N}}(\zeta)| \leqq \alpha_{\mathrm{N}}^{-1}(\eta+1-\alpha_{\mathrm{N}}), \tag{3.2}$$

and the righthand side of (3.2) is smaller than 1 for sufficiently large N, because $\alpha_{\mathrm{N}}\uparrow 1$.

We proceed by proving the assertion for Q instead of $\mathrm{P}^{\mathrm{T}_1}$. Obviously we may assume that $\int x\, d\mathrm{Q}(x)=0$. Let F_n be the distribution function of

Q^{*n} and $\sigma^2 := \int x^2\, dQ(x)$. Since Q is strongly nonlattice and possesses moments of all orders, a well-known expansion yields, for every $r \geqq 3$,

$$F_n(n^{1/2}\sigma x) - \Phi(x) - \varphi(x) \sum_{k=3}^{r} n^{-(k/2)+1} R_k(x) = o(n^{-(r/2)+1}) \quad (3.3)$$

uniformly in x, where R_k is a polynomial depending only on the first r moments of Q (*see*, e. g., Feller (1971), p. 541). Letting $r = 5$ and proceeding as in (2.4)-(2.6) we obtain, for arbitrary j,

$$\begin{aligned} d(a, t, Q^{*n}) \leqq {} & 2\, Q^{*n}(\mathbb{R} \setminus [-(j-1)t, (j-1)t]) \\ & + (2j+1)\, o(n^{-3/2}) + \sum_{i=-j}^{j} \left| \Phi\left(\frac{a+(i+1)t}{\sigma n^{1/2}}\right) \right. \\ & \left. - 2\Phi\left(\frac{a+it}{\sigma n^{1/2}}\right) + \Phi\left(\frac{a+(i-1)t}{\sigma n^{1/2}}\right) \right| \\ & + \sum_{k=3}^{5} n^{-(k/2)+1} \sum_{i=-j}^{j} |\varphi(x_{i+1}) R_k(x_{i+1}) - 2\varphi(x_i) R_k(x_i) \\ & + \varphi(x_{i-1}) R_k(x_{i-1})|. \end{aligned} \quad (3.4)$$

Here again $x_i = (a+it)/\sigma n^{1/2}$. Since each function $\varphi(x) R_k(x)$ has a bounded derivative and only a finity number of points of inflexion, the same reasoning as in the proof of Theorem 1 (for $R(x) = 1 - x^2$) shows that, for $k = 3, 4, 5$,

$$\begin{aligned} \sum_{i=-\infty}^{\infty} & |\varphi(x_{i+1}) R_k(x_{i+1}) - \varphi(x_i) R_k(x_i) \\ & - [\varphi(x_i) R_k(x_i) - \varphi(x_{i-1}) R_k(x_{i-1})]| \\ & \leqq L \sup_{-\infty < i < \infty} |\varphi(x_{i+1}) R_k(x_{i+1}) - \varphi(x_i) R_k(x_i)| \\ & \leqq \tilde{L} \sup_{-\infty < i < \infty} |x_{i+1} - x_i| = \tilde{L}\, t/\sigma n^{1/2}, \end{aligned} \quad (3.5)$$

where L and $\tilde{L}$ are appropriate constants. Thus the last term at the right side of (3.4) is $t\, O(n^{-1/2})$. Further using (2.9) for the remaining sum in (3.4) and Chebyshev's inequality we arrive at

$$\begin{aligned} d(a, t, Q^{*n}) \leqq {} & 2\, Q^{*n}(\mathbb{R} \setminus [-(j-1)t, (j-1)t]) \\ & + (2j+1)\, o(n^{-3/2}) + t\, O(n^{-1/2}) \\ = {} & t^{-2}\, O(n/j^2) + (2j+1)\, o(n^{-3/2}) + t\, O(n^{-1/2}). \end{aligned} \quad (3.6)$$

Choosing $j = j_n = n^{5/6}$, (3.6) implies that

$$d(a, t, Q^{*n}) = t^{-2}\, O(n^{-2/3}) + t\, O(n^{-1/2}). \quad (3.7)$$

Now arguing similarly as in (2.18) and (2.19),

$$D(t, P^{S_n}) \leqq \sum_{l=0}^{n} \binom{n}{l} a^l (1-\alpha)^{n-l} D(t, Q^{*l})$$
$$\leqq 2\alpha(1-\alpha)\varepsilon^{-2} n^{-1} + K[t^{-2} O(n^{-2/3}) + t\, O(n^{-1/2})], \quad (3.8)$$

where $\varepsilon > 0$ and K are constants. It follows that $D_n(t) = O(n^{-1/2})$ for each $t > 1$, as claimed.

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(*Manuscrit reçu le* 9 *juin* 1988.)