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by

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ABSTRACT. — The tail empirical process is defined to be for each integer \( n \geq 1 \), \( a(n)^{-1/2} \alpha_n(a(n)s), 0 \leq s \leq 1 \), where \( \alpha_n \) is the uniform empirical process based on \( n \) independent uniform \((0,1)\) random variables and \( a(n) \) is a sequence of positive constants such that \( a(n) \to 0 \) and \( na(n) \to \infty \) as \( n \to \infty \). The tail empirical process converges weakly to a standard Wiener process on \([0,1]\). A strong approximation to the tail empirical process by a sequence of Gaussian processes in obtained which, among other results, leads to a functional law of the iterated logarithm for the tail empirical process. These results have application to the study of the almost sure limiting behavior of sums of extreme values.

Key words: Uniform empirical process, strong approximation, functional law of the iterated logarithm.

RéSUMÉ. — Le processus empirique de queue est défini, pour tout \( n \geq 1 \), par \( a(n)^{-1/2} \alpha_n(a(n)s), 0 \leq s \leq 1 \), où \( \alpha_n \) est le processus empirique uniforme basé sur \( n \) variables aléatoires indépendantes et uniformes sur \((0,1)\), et \( a(n) \) est une suite de constantes positives telles que \( a(n) \to 0 \) et \( na(n) \to \infty \) quand \( n \to \infty \). Le processus empirique de queue converge faiblement sur \([0,1]\) vers un processus de Wiener standard. On obtient une approximation...
forte du processus empirique de queue par une suite de processus gaussiens. Ceci mène, entre autres résultats, à une loi fonctionnelle du logarithme itéré pour le processus empirique de queue. Ces résultats peuvent être appliqués à l'étude du comportement asymptotique presque sûr de sommes de valeurs extrêmes.

1. INTRODUCTION AND STATEMENTS OF THE RESULTS

Let $U_1, U_2, \ldots$, be a sequence of independent uniform $(0,1)$ random variables and for each integer $n \geq 1$ let

$$G_n(s) = n^{-1} \sum_{i=1}^{n} 1(U_i \leq s), \quad 0 \leq s \leq 1,$$

where $1(x \leq y)$ denotes the indicator function, be the empirical distribution function based on the first $n$ of these uniform $(0,1)$ random variables. The uniform empirical process will be written

$$\alpha_n(s) = n^{1/2} \left\{ G_n(s) - s \right\}, \quad 0 \leq s \leq 1.$$ 

Throughout this paper $a(n)$ will denote a sequence of positive constants such that for all integers $n \geq 1$, $0 < a(n) < 1$, and $a(n) \to 0$ and $k(n) := na(n) \to \infty$ as $n \to \infty$.

In this paper we shall be concerned with the tail empirical process defined in terms of a sequence $a(n)$ to be

$$w_n(s) = a(n)^{-1/2} \alpha_n(a(n)s), \quad 0 \leq s \leq 1.$$ 

An easy application of Theorem 1 of Mason and van Zwet (1987) shows that on a rich enough probability space there exist a sequence of independent uniform $(0,1)$ random variables and a sequence of Brownian bridges $B_1, B_2, \ldots$, such that for any sequence $a(n)$

$$(1.1) \quad \sup_{0 \leq s \leq 1} \left| w_n(s) - a(n)^{-1/2} B_n(a(n)s) \right| = O_p(k(n)^{-1/2} \log k(n)) = o_p(1).$$

Writing for each integer $n \geq 1$

$$(1.2) \quad B_n(s) = W_n(s) - s W_n(1), \quad 0 \leq s \leq 1,$$

where $W_n$ is a standard Wiener process on $[0,1]$, we see that (1.1) leads to

$$(1.3) \quad \sup_{0 \leq s \leq 1} \left| w_n(s) - a(n)^{-1/2} W_n(a(n)s) \right| = o_p(1).$$
Next, by observing that \( a(n)^{-1/2} W_n(a(n)s), 0 \leq s \leq 1 \), is a standard Wiener process, it is clear that (1.3) implies \( w_n \) converges weakly in \( D[0, 1] \) to a standard Wiener process. [This weak convergence fact is implicit in the work of Cooil (1985) and M. Csörgő and Mason (1985).]

What is known about the almost sure behavior of the supremum of the tail empirical process can be summarized as follows [cf. Einmahl and Mason (1988a) and Kiefer (1972).]

Assume that the sequence \( a(n) \) satisfies

(A) \( a(n) \downarrow 0 \) and \( k(n) = na(n) \uparrow \infty \) as \( n \to \infty \),

(here and elsewhere \( \uparrow \) denotes non-decreasing and \( \downarrow \) non-increasing), and set

\[
l(n) = 2 \log \log (\max(n,3)).
\]

(1.4.i) Whenever \( o(k(n)) = \log \log n \), then

\[
\lim_{n \to \infty} \sup_{0 \leq s \leq 1} l(n)^{-1/2} |w_n(s)| = 1 \quad a.s.
\]

(1.4.ii) Whenever \( k(n) \sim c \log \log n \) for some \( 0 < c < \infty \), then

\[
\lim_{n \to \infty} \sup_{0 \leq s \leq 1} l(n)^{-1/2} |w_n(s)| = (c/2)^{1/2} (\beta_c - 1) \quad a.s.,
\]

where \( \beta_c > 1 \) and \( \beta_c (\log \beta_c - 1) + 1 = c^{-1} \).

(1.4.iii) Whenever \( k(n) = o(\log \log n) \), then

\[
\lim_{n \to \infty} \sup_{0 \leq s \leq 1} l(n)^{-1/2} |w_n(s)| = \infty \quad a.s.
\]

It is proven in Komlós, Major and Tusnády [KMT] (1975), that on a rich enough probability space there exist a sequence of independent uniform \((0,1)\) random variables \( U_2, U_3, \ldots \), and a sequence of independent Brownian bridges \( B_2, B_3, \ldots \), such that

\[
\sup_{0 \leq s \leq 1} \left| n^{1/2} \alpha_n(s) - \sum_{m=1}^{n} B_m(s) \right| = O((\log n)^2) \quad a.s.
\]

Writing \( B_m \) as in (1.2) for each \( m \geq 1 \) and noting that by the law of the iterated logarithm

\[
\sum_{m=1}^{n} W_m(1) = O((n \log \log n)^{1/2}) \quad a.s.,
\]

we obtain after some algebra that on the probability space of (1.5), with probability one,

\[
\sup_{0 \leq s \leq 1} \left| w_n(s) - k(n)^{-1/2} \sum_{m=1}^{n} W_m(a(n)s) \right|
\]

\[= O(k(n)^{-1/2}(\log n)^2 + (a(n) \log \log n)^{1/2}),\]
which for sequences $a(n)$ satisfying
\begin{equation}
\label{eq:1.7}
o(k(n)) = (\log n)^a / \log \log n
\end{equation}
yields
\begin{equation}
\label{eq:1.8}
\sup_{0 \leq s \leq 1} \left| w_n(s) - k(n)^{-1/2} \sum_{m=1}^{n} W_m(a(n)s) \right| = o((\log \log n)^{1/2}) \quad \text{a.s.}
\end{equation}
Thus if we knew that for sequences $a(n)$ satisfying (A)
\begin{equation}
\label{eq:1.9}
\limsup_{n \to \infty} \sup_{0 \leq s \leq 1} \left( l(n)k(n)^{-1/2} \sum_{m=1}^{n} W_m(a(n)s) \right) = 1 \quad \text{a.s.,}
\end{equation}
then part, but not all, of (1.4.i) would follow from strong invariance, that is, for sequences $a(n)$ satisfying (A) and (1.7) the lim sup statement in (1.4.i) would be a consequence of (1.8) and (1.9).

The main purpose of this paper is to completely characterize those sequences $a(n)$ satisfying (A) for which the strong approximation statement (1.8) holds on a suitably constructed probability space or for which it cannot hold on any probability space. It turns out that the condition $o(k(n)) = \log \log n$ of (1.4.i) is in a certain sense necessary and sufficient for (1.8). Theorem 2, below, implies that (1.9) is true for all sequences $a(n)$ satisfying (A). Thus (1.4.i) follows from Theorem 2 in combination with our strong approximation Theorem 1. Since the KMT (1975) strong approximation (1.5) only yields (1.8) for sequences $a(n)$ satisfying (1.7), we see that our Theorem 1 constitutes an improvement to their strong approximation in the tails, i.e. when $o(k(n)) = \log \log n$ but (1.7) does not hold.

We now state our results.

**Theorem 1.** — Let $a(n)$ satisfy (A). Whenever $o(k(n)) = \log \log n$, there exist a sequence of independent uniform $(0, 1)$ random variables $U_1, U_2, \ldots$, and a sequence of independent standard Wiener processes $W_1, W_2, \ldots$, sitting on the same probability space such that with probability one
\begin{equation}
\label{eq:1.10}
\limsup_{n \to \infty} l(n)^{-1/2} \left| w_n(s) - k(n)^{-1/2} \sum_{m=1}^{n} W_m(a(n)s) \right| = 0.
\end{equation}

Our next theorem provides the functional law of the iterated logarithm version of (1.9).

Let $B[0, 1]$ denote the space of bounded real-valued functions defined on $[0, 1]$ with the usual supremum norm and $K[0, 1]$ denote the set of absolutely continuous functions $f \in B[0, 1]$ such that
\[ f(0) = 0 \quad \text{and} \quad \int_0^1 (f'(s))^2 \, ds \leq 1. \]
For any sequence $a(n)$ and sequence of independent standard Wiener processes $W_1, W_2, \ldots$, set for integers $n \geq 1$

$$X_n(s) = k(n)^{-1/2} \sum_{m=1}^{n} W_m(a(n)s), \quad 0 \leq s \leq 1.$$

**Theorem 2.** Let $a(n)$ satisfy (A), then with probability one the sequence $l(n)^{-1/2} X_n$ is relatively compact in $B[0, 1]$ with set of limit points equal to $K[0, 1]$.

An immediate consequence of Theorem 2, (1.4.ii) and (1.4.iii) is the following corollary, which shows that a strong approximation to the tail empirical process of the type given in Theorem 1 is not possible when $k(n)/\log\log n \to c \in [0, \infty)$.

**Corollary 1.** Let $a(n)$ satisfy (A) and $k(n) = c_n \log \log n$ where $c_n \to c$ as $n \to \infty$ with $0 \leq c < \infty$, then for any sequence of independent uniform $(0, 1)$ random variables $U_1, U_2, \ldots$, and sequence of independent standard Wiener processes $W_1, W_2, \ldots$, sitting on the same probability space with probability one

$$\limsup_{n \to \infty} \sup_{0 \leq s \leq 1} l(n)^{-1/2} \left| w_n(s) - k(n)^{-1/2} \sum_{m=1}^{n} W_m(a(n)s) \right| = \gamma(c),$$

where $0 < \gamma(c) < \infty$ when $0 < c < \infty$ and $\gamma(0) = \infty$.

Our second corollary gives the functional law of the iterated logarithm version of (1.4.i). It follows readily from Theorem 1 and Theorem 2, and may be viewed as the Finkelstein (1971) theorem for the tail empirical process.

**Corollary 2.** Let $a(n)$ satisfy (A) and $o(k(n)) = \log\log n$, then with probability one the sequence $l(n)^{-1/2} W_n$ is relatively compact in $B[0, 1]$ with set of limit points equal to $K[0, 1]$.

**Remark 1.** From (1.4.ii) and (1.4.iii) it is easy to see that Corollary 2 no longer holds when $a(n)$ satisfies the conditions of Corollary 1 in the sense that $K[0, 1]$ is no longer the set of limit points of $l(n)^{-1/2} W_n$.

In order to motivate our final three corollaries, consider now the following weighted versions of the tail empirical process defined for integers $n \geq 1$ and $0 < \nu < 1/2$ to be

$$w_n^{(\nu)}(s) = w_n(s) s^{-1/2 + \nu}, \quad 0 \leq s \leq 1.$$

Using Theorem 2 of Mason and van Zwet (1987), along with (1.2), it is straightforward to show that on a rich enough probability space there exist a sequence of independent uniform $(0, 1)$ random variables $U_1, U_2, \ldots$, and a sequence of standard Wiener processes $W_1, W_2, \ldots$, such
that

$$\sup_{0 \leq s \leq 1} \left| w_n^{(e)}(s) - a(n)^{-1/2} W_n(a(n)s)s^{-1/2+v}\right| = o_p(1),$$

which of course implies that $w_n^{(e)}$ converges weakly in $D[0,1]$ to $W^{-1/2+v}$, where here and elsewhere in this paper $W$ denotes a standard Wiener process and $I$ the identity function on $[0,1]$. [Again this weak convergence result is implicit in Cooil (1985) and M. Csörgő and Mason (1985).]

For any sequence $a(n)$ satisfying (A) and $0 < v < 1/2$ set for integers $n \geq 3$

$$e(n, v) = \{ nk(n)^{2v/(1 - 2v)}(\log \log n)^{1/(1 - 2v)}\}^{-1}.$$

As a special case of Theorem 1 of Einmahl and Mason (1988a) one obtains

$$(1.13) \quad \limsup_{n \to \infty} \sup_{0 \leq s \leq 1} l(n)^{-1/2} \left| w_n^{(e)}(s) \right| = 1 \quad \text{or} \quad \infty \quad \text{a.s.}$$

according as the following series is finite or infinite

$$(1.14) \quad \sum_{n=3}^{\infty} e(n, v).$$

It is natural then to consider the strong approximation analogue of Theorem 1 for these weighted tail empirical processes. The following corollary provides this analogue.

**Corollary 3. — Let $a(n)$ satisfy (A) and $0 < v < 1/2$.**

I) Whenever the series in (1.14) is finite, then on the probability space of Theorem 1 with probability one

$$(1.15) \quad \limsup_{n \to \infty} \sup_{0 \leq s \leq 1} l(n)^{-1/2} \left| w_n^{(e)}(s) - k(n)^{-1/2} \sum_{m=1}^{n} W_m(a(n)s)s^{-1/2+v}\right| = 0.$$

II) Whenever the series in (1.14) is infinite, then for any sequence of independent uniform $(0,1)$ random variables $U_1, U_2, \ldots$, and sequence of independent Wiener processes $W_1, W_2, \ldots$, sitting on the same probability space with probability one

$$(1.16) \quad \limsup_{n \to \infty} \sup_{0 \leq s \leq 1} l(n)^{-1/2} \times \left| w_n^{(e)}(s) - k(n)^{-1/2} \sum_{m=1}^{n} W_m(a(n)s)s^{-1/2+v}\right| = \infty.$$

For any $0 < v < 1/2$, let

$$K^{(v)}[0,1] = \{ f I^{-1/2+v} : f \in K[0,1] \}$$
and for any sequence \( a(n) \) and \( 0 < \nu < 1/2 \) set for integers \( n \geq 1 \)

\[
(1.17) \quad X_n^{(\nu)}(s) = k(n)^{-1/2} \sum_{m=1}^{n} W_m(a(n)s) s^{-1/2 + \nu}, \quad 0 \leq s \leq 1.
\]

Our next corollary provides the functional law of the iterated logarithm version of Theorem 2 for the sequence of processes \( X_n^{(\nu)} \).

**COROLLARY 4.** — Let \( a(n) \) satisfy (A) and \( 0 < \nu < 1/2 \), then with probability one the sequence \( l(n)^{-1/2} X_n^{(\nu)} \) is relatively compact in \( B[0, 1] \) with set of limit points equal to \( K^{(\nu)}[0, 1] \).

As our final corollary we obtain the functional law of the iterated logarithm form of (1.13), whose proof is immediate from Corollaries 3 and 4. This corollary may be viewed as the James (1975) theorem for the weighted tail empirical process.

**COROLLARY 5.** — Let \( a(n) \) satisfy (A) and \( 0 < \nu < 1/2 \), then whenever the series in (1.14) is finite with probability one the sequence \( l(n)^{-1/2} w_n^{(\nu)} \) is relatively compact in \( B[0, 1] \) with set of limit points equal to \( K^{(\nu)}[0, 1] \).

**Remark 2.** — Whenever the series in (1.14) is infinite, it is obvious from (1.13) that Corollary 5 does not hold.

**Remark 3.** — The statements of Theorem 1 and Corollary 2 remain true when in the definition of \( w_n \) the uniform empirical process \( \alpha_n \) is replaced by the uniform quantile process \( \beta_n \). The resulting process is called the tail quantile process. These results are contained in Einmahl and Mason (1988 b).

For applications of these results to the derivation of functional laws of the iterated logarithm for sums of extreme value processes the interested reader is referred to Mason (1988).

## 2. PROOFS

**Proof of Theorem 1.** — First we must establish a number of lemmas, some of which may be of separate interest.

From now on we write for integers \( n \geq 1 \), \( b(n) = (k(n) l(n))^{-1/2} \) and \([x]\) will denote the integer part of \( x \).

**Lemma 1.** — Let \( a(n) \) satisfy (A) and \( o(k(n)) = \log \log n \), then for each \( 1 < \lambda < \infty \) there exist a sequence of independent uniform \((0, 1)\) random variables \( U_1, U_2, \ldots, \) and a sequence of independent standard Wiener processes \( W_1, W_2, \ldots, \) sitting on the same probability space \((\Omega, \mathcal{A}, \mathbb{P})\) such
that

\[
(2.1) \quad \lim_{r \to \infty} b(n_r) \sup_{0 \leq s \leq 1} \left| n_r^{1/2} \alpha_{n_r}(a(n_r)s) - \sum_{m=1}^{n_r} W_m(a(n_r)s) \right| = 0 \quad a.s.,
\]

where \( n_r = [\lambda^r] \).

**Proof.** Let \( m_1 = n_1 \) and \( m_r = n_r - n_{r-1} \) for \( r = 2, 3, \ldots \). For each integer \( r \geq 1 \) construct using Theorem 1 of Mason and van Zwet (1987) independent uniform \((0, 1)\) random variables \( U_1^{(r)}, \ldots, U_{m_r}^{(r)} \) and a standard Wiener process \( W^{(r)} \) sitting on the same probability space \((\Omega_r, \mathcal{A}_r, P_r)\) such that for universal positive constants \( C, K \) and \( \lambda \) independent of \( r \)

\[
P\left( \sup_{0 \leq s \leq k/m_r} m_r^{1/2} \left| \alpha_{m_r}^{(r)}(s) - (W^{(r)}(s) - s W^{(r)}(1)) \right| > C \log k + x \right) \leq K e^{-\lambda x}
\]

for all \( 0 \leq x < \infty \) and \( 1 \leq k \leq m_r \), where \( \alpha_{m_r}^{(r)} \) is the empirical processes based on \( U_1^{(r)}, \ldots, U_{m_r}^{(r)} \) which by an elementary bound for the tail of the standard normal distribution [cf. Feller (1968)] gives

\[
(2.2) \quad P\left( \sup_{0 \leq s \leq k/m_r} m_r^{1/2} \left| \alpha_{m_r}^{(r)}(s) - W^{(r)}(s) \right| > C \log k + x \right) < K_1 e^{-\lambda_1 x} + K_2 e^{-\lambda_2 x^2 m_r/k^2}
\]

for universal positive constants \( K_1, K_2, \lambda_1 \) and \( \lambda_2 \). We can assume that

\[
W^{(r)} = m_r^{-1/2} \sum_{i=1}^{m_r} W_i^{(r)},
\]

where \( W_1^{(r)}, \ldots, W_{m_r}^{(r)} \) are independent standard Wiener processes.

Now set

\[
(\Omega, \mathcal{A}, P) = \left( \times_{r=1}^{\infty} \Omega_r, \times_{r=1}^{\infty} \mathcal{A}_r, \times_{r=1}^{\infty} P_r \right),
\]

\[
U_i = U_i^{(1)} \quad W_i = W_i^{(1)} \quad \text{for} \quad i = 1, \ldots, m_1, \quad \text{and} \quad U_{n_r+i} = U_i^{(r+1)} \quad \text{and} \quad W_{n_r+i} = W_i^{(r+1)} \quad \text{for} \quad i = 1, \ldots, m_{r+1} \quad \text{for} \quad r = 1, 2, \ldots
\]

**Claim 1:**

\[
(2.3) \quad \lim_{r \to \infty} b(m_r) \sup_{0 \leq s \leq 1} m_r^{1/2} \left| \alpha_{m_r}^{(r)}(a(m_r)s) - W^{(r)}(a(m_r)s) \right| = 0 \quad a.s.
\]

**Proof.** Choose any \( \varepsilon > 0 \). We have by (2.2)

\[
P\left( \sup_{0 \leq s \leq 1} m_r^{1/2} \left| \alpha_{m_r}^{(r)}(a(m_r)s) - W^{(r)}(a(m_r)s) \right| > C \log k (m_r) + \varepsilon b^{-1}(m_r) \right) < K_1 \exp(-\varepsilon \lambda_1 b^{-1}(m_r)) + K_2 \exp(-\varepsilon^2 \lambda_2 (2 \log \log m_r)(m_r/k(m_r))),
\]

which since as \( r \to \infty \)

\[
b^{-1}(m_r)/\log \log m_r \to \infty \quad \text{and} \quad m_r/k(m_r) \to \infty
\]

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is less than $r^{-2}$ for all large enough $r$. Hence by the Borel-Cantelli lemma in combination with the fact that as $r \to \infty$

$$b(m_r) \log k(m_r) = (\log k(m_r))/(2 k(m_r) \log \log m_r)^{1/2} \to 0$$

we have shown (2.3).

**CLAIM 2.** — **For each integer $k \geq 1$,**

$$\lim_{r \to \infty} \sup_{0 \leq s \leq 1} b(n_r) \left| \sum_{m=1}^{n_r-k} W_m(a(n_r)s) \right| \leq \lambda^{-k/2} \text{ a.s.} \quad (2.4)$$

and

$$\lim_{r \to \infty} \sup_{0 \leq s \leq 1} b(n_r) \left| n_r^{-1} \alpha_{n_r-k}(a(n_r)s) \right| \leq \lambda^{-k/2} \text{ a.s.} \quad (2.5)$$

**Proof.** — First consider (2.4). Notice that

$$\sum_{m=1}^{n_r-k} W_m(a(n_r)s) = (n_r-k)^{1/2} W(s), \quad 0 \leq s \leq 1.$$ 

Therefore for any $\varepsilon > 0$

$$P\left( \sup_{0 \leq s \leq 1} b(n_r) \left| \sum_{m=1}^{n_r-k} W_m(a(n_r)s) \right| > ((1 + \varepsilon) \lambda^{-k})^{1/2} \right)$$

$$= P\left( \sup_{0 \leq s \leq 1} |W(s)| > b^{-1}(n_r)(n_r-k)^{-1/2} ((1 + \varepsilon) \lambda^{-k})^{1/2} \right),$$

which is less than or equal to

$$4 P(W(1) > b^{-1}(n_r)(n_r-k)^{-1/2} ((1 + \varepsilon) \lambda^{-k})^{1/2}).$$

This expression is for all large enough $r$ less than or equal to

$$4 P(W(1) > 2(1+\varepsilon/2) \log \log n_r)^{1/2} \exp(-(1+\varepsilon/2) \log \log n_r).$$

The Borel-Cantelli lemma completes the proof of (2.4).

Turning to (2.5), we have by a straightforward application of Inequality 2 on p. 444 of Shorack and Wellner (1986) that for any $\varepsilon > 0$

$$\sum_{r=1}^{\infty} P\left( \sup_{0 \leq s \leq 1} b(n_r) \left| n_r^{-1/2} \alpha_{n_r-k}(a(n_r)s) \right| > ((1 + \varepsilon) \lambda^{-k})^{1/2} \right) < \infty.$$ 

This completes the proof of Claim 2.
To finish the proof of Lemma 1, we note that for each integer \( k \geq 1 \)
\[
\lim_{r \to \infty} \sup_{0 \leq s \leq 1} b(n_r) \left| n_r^{1/2} \alpha_{n_r}(a(n_r) s) - \sum_{m=1}^{n_r} W_m(a(n_r) s) \right|
\leq \lim_{r \to \infty} \sup_{0 \leq s \leq 1} b(n_r) \left| n_r^{1/2} \alpha_{n_r-k}(a(n_r) s) \right|
+ \lim_{r \to \infty} \sup_{0 \leq s \leq 1} b(n_r) \left| \sum_{m=1}^{n_r-k} W_m(a(n_r) s) \right|
+ \sum_{i=1}^{k} \lim_{r \to \infty} \sup_{0 \leq s \leq 1} b(m_{r+1-i}) m_{r+1-i}^{1/2}
\times \left| \alpha_{r+1-i}^{(r+1-i)}(a(m_{r+1-i} s) - W^{(r+1-i)}(a(m_{r+1-i} s)), \right|
\]
which by Claims 1 and 2 is almost surely less than or equal to \( 2 \lambda^{-k/2} \).
Since \( k \) can be chosen arbitrarily large we have (2.1).

For any integer \( n \geq 1 \) set
\[
S_n(s) = \sum_{j=1}^{n} W_j(s), \quad 0 \leq s \leq 1,
\]
and
\[
T_n(s) = \sum_{j=1}^{n} \{ 1(U_j \leq s) - s \}, \quad 0 \leq s \leq 1.
\]

**Lemma 2.** — For any \( 0 < a < 1 \), integer \( n \geq 1 \) and \( \varepsilon > 0 \)
\[
P( \max_{1 \leq j \leq n} \sup_{0 \leq s \leq a} |S_j(s)| > \varepsilon) \leq 2P( \sup_{0 \leq s \leq a} |S_n(s)| > \varepsilon)
\]
and if, in addition, \( \varepsilon > 2(2na)^{1/2} \)
\[
P( \max_{1 \leq j \leq n} \sup_{0 \leq s \leq a} |T_j(s)| > \varepsilon) \leq 2P( \sup_{0 \leq s \leq a} |T_n(s)| > \varepsilon/2).
\]

**Proof.** — First, (2.8) follows from the Banach space version of Lévy’s inequality for sums of independent symmetric random variables [cf. Theorem 2.6 of Araujo and Gine (1980)]. Next we turn to (2.9). Choose any \( 0 < a < 1 \) and let \( \{ r_k : k \geq 0 \} \) with \( r_0 = 0 \) be a denumeration of the rationals in \([0, a]\). For integers \( 1 \leq j \leq n \) and \( k \geq 1 \) set
\[
A_{j, k} = \{ \sup_{i \leq j, 1 < k} |T_i(r_i)| \leq \varepsilon, |T_j(r_k)| > \varepsilon \}
\]
and
\[
B_{j, k} = \{ |T_n(r_k) - T_j(r_k)| \leq \varepsilon/2 \}.
\]
Notice that
\[
P\left( \bigcup_{j=1}^{n} \bigcup_{k=1}^{\infty} (A_{j,k} \cap B_{j,k}) \right) \leq P\left( \sup_{0 \leq s \leq a} \left| T_{n}(s) \right| > \varepsilon/2 \right)
\]
and by Chebyshev's inequality and the choice of \( \varepsilon \)
\[
P(B_{j,k}) \geq 1 - 4(n-j)/a \varepsilon^{2} \geq 1/2.
\]
Proceeding now as in the proof of Lemma 2.3 of James (1975) we obtain (2.9).

For any \( 1 < \lambda < \infty \) write for integers \( r \geq 1 \)
\[
\Delta_{r} = \max_{n_{r} \leq n \leq n_{r}+1} \sup_{0 \leq s \leq 1} \left| \sum_{j=1}^{n-n_{r}} W_{j+n_{r}}(a(n_{r})s) \right|
\]
and
\[
\Delta_{r}^{*} = \max_{n_{r} \leq n \leq n_{r}+1} \sup_{0 \leq s \leq 1} \left| \sum_{j=1}^{n-n_{r}} \left\{ 1(U_{j+n_{r}} \leq a(n_{r})s) - a(n_{r})s \right\} \right|
\]
where \( n_{r} = [\lambda r] \).

**Lemma 3.** — Let \( a(n) \) satisfy (A), then for each \( 1 < \lambda < \infty \)
\[
(2.10) \quad \lim_{r \to \infty} \sup b(n_{r}) \Delta_{r} \leq (\lambda - 1)^{1/2} \quad a.s.
\]
and if, in addition, \( o(k(n)) = \log \log n \), then
\[
(2.11) \quad \lim_{r \to \infty} \sup b(n_{r}) \Delta_{r}^{*} \leq 2(\lambda - 1)^{1/2} \quad a.s.
\]

**Proof.** — First consider (2.10). Choose any \( \varepsilon > 0 \). By Lemma 2
\[
P(b(n_{r}) \Delta_{r} > ((1+\varepsilon)(\lambda - 1))^{1/2})
\]
\[
\leq 2P\left( \sup_{0 \leq s \leq 1} \left| S_{n_{r}+1}(a(n_{r})s) \right| > ((1+\varepsilon)(\lambda - 1))^{1/2}/b(n_{r}) \right)
\]
\[
= 2P\left( \sup_{0 \leq s \leq 1} \left| W(s) \right| > (2(1+\varepsilon)(\lambda - 1)/n_{r}/m_{r+1})^{1/2}(\log \log n_{r})^{1/2} \right),
\]
which for all large enough \( r \) is less than
\[
2P\left( \sup_{0 \leq s \leq 1} \left| W(s) \right| > (2(1+\varepsilon/2)(\log \log n_{r})^{1/2} \right),
\]
and noting that for all \( x > 0 \),
\[
P\left( \sup_{0 \leq s \leq 1} \left| W(s) \right| > x \right) \leq 2P\left( \sup_{0 \leq s \leq 1} W(s) > x \right) = 4P(W(1) > x),
\]
we see that the previous expression is for all large \( r \)
\[
\leq 8P(W(1) \geq (2(1+\varepsilon/2)(\log \log n_{r})^{1/2} < \exp(-(1+\varepsilon/2)(\log \log n_{r})
\]
An application of the Borel-Cantelli lemma finishes the proof of (2.10).
Assertion (2.11) is proven similarly, applying at the appropriate step Inequality 2 on p. 444 of Shorack and Wellner (1986).

**Lemma 4.** Let \( a(n) \) satisfy (A) and \( o(k(n)) = \log \log n \), then for every \( \varepsilon > 0 \) there exist a sequence of independent uniform \((0, 1)\) random variables \( U_1, U_2, \ldots \), and a sequence of independent standard Wiener processes \( W_1, W_2, \ldots \), sitting on the same probability space such that

\[
(2.12) \quad \lim_{n \to \infty} \sup_{0 \leq s \leq 1} b(n) \left| n^{1/2} \alpha_n(a(n)s) - \sum_{m=1}^{n} W_m(a(n)s) \right| \leq \varepsilon \quad a.s.
\]

**Proof.** Choose any \( \varepsilon > 0 \) and \( 1 < \lambda < \infty \) such that \( 3(\lambda - 1)^{1/2} < \varepsilon \). Let \( U_1, U_2, \ldots \), and \( W_1, W_2, \ldots \), be constructed as in Lemma 1. Observe that since \( a(n) \downarrow 0 \) and \( b(n) \downarrow 0 \) for any integers \( r \geq 1 \) and \( n_r < n \leq n_{r+1} \)

\[
b(n) \sup_{0 \leq s \leq 1} \left| n^{1/2} \alpha_n(a(n)s) - \sum_{m=1}^{n_r} W_m(a(n)s) \right| \leq b(n_r) \sup_{0 \leq s \leq 1} \left| n_r^{1/2} \alpha_{n_r}(a(n_r)s) - \sum_{m=1}^{n_r} W_m(a(n_r)s) \right| + b(n_r) \Delta_r + b(n_r) \Delta_r^*.
\]

Obviously now, (2.12) follows from (2.1), (2.10) and (2.11) combined with the above inequality.

Lemma 4 allows us to apply a trick of Major (1976) [cf. also p. 397-398 of Gaenssler and Stute (1977) and p. 261-262 of Philipp and Stout (1986)] to construct a sequence of independent uniform \((0, 1)\) random variables \( U_1, U_2, \ldots \), and a sequence of independent standard Wiener processes \( W_1, W_2, \ldots \), sitting on the same probability space so that (1.10) holds. For the sake of brevity these details are omitted. This completes the proof of Theorem 1.

**Proof of Theorem 2.** The proof of Theorem 2 is based on Theorem 1 of Lai (1974). For any \( 1 < \lambda < \infty \), set for integers \( r \geq 1 \)

\[
Y_r(s) = X_{n_r}(s), \quad 0 \leq s \leq 1,
\]

where \( n_r = [\lambda^r] \) and \( X_{n_r} \) is as in (1.11).

**Lemma 5.** Let \( a(n) \) satisfy (A), then for any \( 1 < \lambda < \infty \) with probability one the sequence \( l(n_r)^{-1/2} Y_r \) is relatively compact in \( B[0,1] \) with set of limit points equal to \( K[0,1] \).

**Proof.** First note that each \( Y_r \) is a standard Wiener process on \([0,1]\). For each integer \( r \geq 1 \) let \( F_r \) denote the \( \sigma \)-field generated by \( \{ Y_m(s): 0 \leq s \leq 1, 1 \leq m \leq r \} \). Obviously we have for each \( r, k = 1, 2, \ldots \), and \( 0 \leq s \leq 1 \)

\[
E \{ E(Y_{r+k}(s) | F_r) \}^2 = s n_{r} / n_{r+k}.
\]
Since
\[ \lim_{r,k \to \infty} \frac{n_r}{n_{r+k}} = 0, \]
the assertion of the lemma follows from Theorem 1 of Lai (1974).
For any \(1 < \lambda < \infty\) set for integers \(r \geq 1\)
\[ D_r = \max_{n_r < n \leq n_r + 1} \sup_{0 \leq s \leq 1} \left| l(n_{r})^{-1/2} X_{n_r}(s) - l(n)^{-1/2} X_n(s) \right| \]
where \(n_r = [\lambda r]\).

**Lemma 6.** Let \(a(n)\) satisfy (A), then for any \(1 < \lambda < \infty\)
\[ (2.13) \quad \lim_{r \to \infty} \sup D_r \leq \psi(\lambda) \quad a.s., \]
where \(\psi(\lambda) = (\lambda - 1)^{1/2} + 1 - \lambda^{-1/2} + (1 - \lambda^{-2})^{1/2}.\)

**Proof.** Recall the definitions of \(X_n\) in (1.11) and \(S_n\) in (2.6). Notice that
\[ D_r \leq \max_{n_r < n \leq n_r + 1} \sup_{0 \leq s \leq 1} \left\{ \sum_{j=1}^{m-n_r} W_{j+n_r} b(n_r)(n_r) s \right\} \]
\[ \quad + \sup_{0 \leq s \leq 1} l(n_{r})^{-1/2} X_{n_r}(s) \{ 1 - b(n_{r+1})/b(n_r) \} \]
\[ \quad + \max_{n_r < n \leq n_r + 1} \sup_{0 \leq s \leq 1} b(n_r) |S_{n_r}(a(n_r)s) - S_{n_r}(a(n)s)| \]
\[ : = b(n_r) \Delta_r + D_r^{(1)} + D_r^{(2)}. \]

First, by Lemma 3 we have
\[ (2.14) \quad \lim_{r \to \infty} \sup b(n_r) \Delta_r \leq (\lambda - 1)^{1/2} \quad a.s. \]

Observe that
\[ b(n_{r+1})/b(n_r) = (k(n_r) \log \log n_r / (k(n_{r+1}) \log \log n_{r+1}))^{1/2}, \]
and since \(a(n_{r+1}) \leq a(n_r), k(n_r)/k(n_{r+1}) \geq n_r/n_{r+1}.\) Therefore
\[ 1 - b(n_{r+1})/b(n_r) \leq 1 - (n_r \log \log n_r / (n_{r+1} \log \log n_{r+1}))^{1/2}, \]
and hence
\[ (2.15) \quad \lim_{r \to \infty} \sup \{ 1 - b(n_{r+1})/b(n_r) \} \leq 1 - \lambda^{-1/2}. \]

Statement (2.15) in combination with Lemma 5 gives
\[ (2.16) \quad \lim_{r \to \infty} \sup D_r^{(1)} \leq 1 - \lambda^{-1/2} \quad a.s. \]

**Claim 3:**
\[ (2.17) \quad \lim_{r \to \infty} \sup D_r^{(2)} \leq (1 - \lambda^{-2})^{1/2} \quad a.s. \]
Proof. — We see that

\[ D_r^{(2)} \leq \sup_{sa (n_r + 1)/\alpha (n_r) \leq t \leq s \leq 1} b (n_r) \left| S_{n_r} (a (n_r) s) - S_{n_r} (a (n_r) t) \right| = D_r^{(3)} \]

and

\[ D_r^{(3)} = l (n_r)^{-1/2} \sup_{sa (n_r + 1)/\alpha (n_r) \leq t \leq s \leq 1} \left| W (s) - W (t) \right| = D_r^{(4)}. \]

Since for all large enough \( r \)

\[ \lambda^{-2} \leq k (n_r + 1) n_r / (k (n_r) n_r + 1) = a (n_r + 1) / a (n_r) \leq 1, \]

we have for any \( \varepsilon > 0 \) and all large enough \( r \)

\[ P(D_r^{(4)} > ((1 + \varepsilon) (1 - \lambda^{-2}))^{1/2}) \]

\[ \leq P(\sup_{s \lambda^{-2} \leq t \leq s \leq 1} \left| W (s) - W (t) \right| > ((1 + \varepsilon) (1 - \lambda^{-2}) 2 \log \log n_r)^{1/2}) \]

\[ \leq P(\sup_{0 \leq v \leq 1} \sup_{0 \leq u \leq 1 - \lambda^{-2}} \left| W (u + v) - W (v) \right| \]

\[ \geq ((1 + \varepsilon) (1 - \lambda^{-2}) 2 \log \log n_r^{1/2}), \]

which by Lemma 1.2.1 on p. 29 of M. Csörgö and Révész (1981) is

\[ \leq c (\varepsilon, \lambda) \exp (- (1 + \varepsilon/2) \log \log n_r), \]

for some constant \( 0 < c (\varepsilon, \lambda) < \infty \) depending on only \( \varepsilon > 0 \) and \( \lambda > 1 \). The Borel-Cantelli lemma completes the proof of (2.17).

From statements (2.14), (2.15) and (2.17) we conclude (2.13), finishing the proof of Lemma 6.

Since \( \psi (\lambda) \to 0 \) as \( \lambda \downarrow 1 \), the assertion of Theorem 2 follows from an easy argument based on Lemmas 5 and 6. [See, for instance, Section 3 of Lai (1974).]

Proofs of Corollaries 3 and 4. — We require the following two lemmas.

Lemma 7. — Let \( a (n) \) satisfy (A) and \( 0 < v < 1/2 \).

\[ \lim_{n \to \infty} \sup_{0 \leq s \leq \delta \downarrow 0} l (n)^{-1/2} \left| X_n^{(v)} (s) \right| = 0 \; a.s. \quad (2.18) \]

Proof. — From Orey and Pruitt (1973) or Wichura (1973) [cf. also Theorem 1.12.2 on p. 61 of M. Csörgö and Révész (1981)] it is easy to conclude using a change of time scale with \( S_n \) as in (26) that

\[ \lim_{n \to \infty} \sup_{0 \leq s \leq \delta \downarrow 0} \left| S_n (a (n) s) \right| \left( \frac{4 sk (n) \log \log \left( \frac{n^2}{sk (n)} \right)}{4 sk (n) \log \log \left( \frac{n^2}{sk (n)} \right)} \right)^{1/2} = 1 \; a.s. \quad (2.19) \]

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\[ (2.20) \quad \sup_{0 \leq s \leq \delta} l(n)^{-1/2} |X_n^{(s)}(s)| \leq \sup_{0 \leq s \leq \delta} \left| S_n(a(n)s) \right|/\left(4sk(n)\log\log \left(\frac{n^2}{sk(n)}\right)\right)^{1/2} g_n(\delta), \]
where
\[ g_n(\delta) = 2^{1/2} \sup_{0 \leq s \leq \delta} s^v ((\log \{ \log (n^2/k(n)) + \log (1/s) \})/\log\log n)^{1/2}. \]
Since \( g_n(\delta) \to 0 \) as \( n \to \infty \) and \( \delta \downarrow 0 \), (2.18) follows from (2.19) and inequality (2.20).

**Lemma 8.** — Let \( a(n) \) satisfy (A) and \( 0 < v < 1/2 \). Whenever the series in (1.14) is finite.
\[ (2.21) \lim_{\delta \downarrow 0} \lim_{n \to \infty} \sup_{0 \leq s \leq \delta} l(n)^{-1/2} |w_n^{(s)}(s)| = 0 \quad a.s. \]

**Proof.** — It is clear that the series in (1.14) being finite implies by the (1.13) statement in this case that
\[ \lim_{n \to \infty} \sup_{0 \leq s \leq \delta} l(n)^{-1/2} \delta^{-v} |w_n^{(s)}(s \delta)| = 1 \quad a.s., \]
which gives (2.21).

The first part of Corollary 3 obviously follows from Lemmas 7 and 8 and Theorem 1, i.e. here we necessarily have \( o(k(n)) = \log \log n. \) Corollary 4 is a consequence of Lemma 7 and Theorem 2. Finally, the second part of Corollary 3 is implied by Corollary 4 and the fact that the lim sup in (1.13) is infinite almost surely when the series in (1.14) is infinite.

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**References**


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