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## Asymptotic expansions of the invariant density of a Markov process with a small parameter

by

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**ABSTRACT.** — We consider the asymptotic behavior of the invariant density of a Markov process on  $\mathcal{R}^d$  which is a perturbation of a dynamical system.

*Key words* : Asymptotic expansion, invariant density, S. Watanabe's theory.

**RÉSUMÉ.** — Nous considérons le comportement asymptotique de la densité invariante d'un processus de Markov sur  $\mathcal{R}^d$  qui est une perturbation d'un système dynamique.

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### INTRODUCTION

Let  $(X^\varepsilon(t), P_x)$ ,  $0 \leq t$ ,  $x \in \mathcal{R}^d$  be Markov processes on  $\mathcal{R}^d$  solving the following stochastic differential equation:

$$\begin{aligned}dX^\varepsilon(t) &= b(X^\varepsilon(t)) dt + \varepsilon dW(t) \\ X^\varepsilon(0) &= x\end{aligned}$$

where  $W(t)$  is an  $d$ -dimensional Wiener process,  $b(x)$  is a function of  $\mathcal{R}^d$  to  $\mathcal{R}^d$  and  $\varepsilon > 0$  is a small parameter. Then the following result is known; if  $b(\cdot): \mathcal{R}^d \rightarrow \mathcal{R}^d$  is Lipschitz continuous,

$$P_x(\lim_{\varepsilon \rightarrow 0} \sup_{0 < t < T} |X^\varepsilon(t) - x(t)| = 0) = 1 \quad (x \in \mathcal{R}^d)$$

where we denote by  $\mathfrak{x}(t)$  the dynamical system solving the next differential equation:

$$\frac{d}{dt} \mathfrak{x}(t) = b(\mathfrak{x}(t))$$

$$\mathfrak{x}(0) = x.$$

In this paper we study, by way of S. Watanabe's theory [6], the asymptotic behavior of the invariant density  $p^\varepsilon(x)$  of  $X^\varepsilon(t)$  of the following type:

$$\varepsilon^d \exp(V(x)/\varepsilon^2) p^\varepsilon(x) = p^0(x) + \varepsilon^2 p^1(x) + \varepsilon^4 p^2(x) + \dots \quad (\text{as } \varepsilon \rightarrow 0)$$

where  $V(x)$  is the Wentzell-Freidlin quasi-potential. We also note that the invariant density  $p^\varepsilon(x)$  is the solution of the following equation:

$$\frac{\varepsilon^2}{2} \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} p^\varepsilon(x) - \operatorname{div}(b(x) p^\varepsilon(x)) = 0 \quad (x \in \mathcal{R}^d)$$

$$\int_{\mathcal{R}^d} p^\varepsilon(x) dx = 1.$$

The study is important when we study the exit distribution of  $X^\varepsilon(t)$  from the bounded domain in  $\mathcal{R}^d$  (cf. M. V. Day [1], [2]).

With respect to this problem M. V. Day [3] got results only in the case  $n=0$  but under weaker conditions than we give in this paper. As the special case, if  $b(x) = \nabla U(x)$  for an infinitely differentiable function  $U$  of

$\mathcal{R}^d$  to  $\mathcal{R}$  such that  $\int_{\mathcal{R}^d} \exp(2U(x)/\varepsilon^2) dx < +\infty$ , then

$$p^\varepsilon(x) = C(\varepsilon) \exp(2U(x)/\varepsilon^2)$$

where we put  $C(\varepsilon) = \left( \int_{\mathcal{R}^d} \exp(2U(x)/\varepsilon^2) dx \right)^{-1}$ .

In section 1 we state our results. In section 2 we give lemmas necessary for the proofs of our results. In section 3 we prove our results.

At last we give some notations which we use in this paper. For  $(d, d)$ -matrix  $A = (a_{ij})_{i,j=1}^d$ , we put  $|A| = \text{determinant of } A$ ,

$$\|A\| = \left( \sum_{i,j=1}^d (a_{ij})^2 \right)^{1/2} \quad \text{and} \quad A^* = (a_{ji})_{i,j=1}^d.$$

$$\text{For } u = (u_i)_{i=1}^d \text{ of } \mathcal{R}^d, \text{ we put } |u| = \left( \sum_{i=1}^d (u_i)^2 \right)^{1/2}.$$

## 1. MAIN RESULTS

In this section we state our results.

We introduce the following conditions.

(A. 0)  $b(x)$  is an  $d$ -dimensional  $C^\infty$ -vector field on  $\mathcal{R}^d$  with bounded derivatives of all orders.

(A. 0)'  $b(x)$  is a function of  $\mathcal{R}^d$  to  $\mathcal{R}^d$  with Hölder continuous first derivatives for some exponent  $\gamma (0 < \gamma \leq 1)$  and is  $C^\infty$ -class in a small neighborhood of  $0 \in \mathcal{R}^d$ .

(A. 1)  $b(0) = 0$ .

(A. 2)  $\sup \left( \sup \left( \sum_{i,j=1}^d \frac{\partial b^i}{\partial x_j}(x) e^i e^j; \sum_{i=1}^d (e^i)^2 = 1 \right); x \in \mathcal{R}^d \right) < 0$ .

(A. 2)'  $\left\{ \begin{array}{l} \sup \left( \sum_{i,j=1}^d \frac{\partial b^i}{\partial x_j}(0) e^i e^j; \sum_{i=1}^d (e^i)^2 = 1 \right) < 0, \\ \lim_{t \rightarrow \infty} \bar{x}(t) = 0 \quad (\bar{x}(0) \in \mathcal{R}^d). \end{array} \right.$

(A. 3) There exist positive constants  $\varepsilon_0, R$  and a nonnegative  $C^2$ -function  $w(x)$  which tends to  $\infty$  as  $|x| \rightarrow \infty$  so that

$$\frac{\varepsilon^2}{2} \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} w(x) + \sum_{i=1}^d b^i(x) \frac{\partial w}{\partial x_i}(x) \leq -1 \quad (|x| \geq R, 0 < \varepsilon \leq \varepsilon_0).$$

*Remark.* — The assumptions (A. 0), (A. 1) and (A. 2) imply the assumptions (A. 0)', (A. 1), (A. 2)' and (A. 3). Moreover the assumption (A. 0)' imply the regularity assumptions of M. V. Day [3] and (A. 1), (A. 2)' and (A. 3) imply the stability assumptions of M. V. Day [3].

The following proposition shows how our assumptions are strong.

PROPOSITION 1.1. — Under the conditions (A. 0), (A. 1) and (A. 2), Markov processes  $(X^\varepsilon(t), P_x)$ ,  $0 \leq t$ ,  $x \in \mathcal{R}^d$  are positively recurrent and

$$(1.1) \quad |\bar{x}(t)| \leq \exp(-\lambda t) |\bar{x}(0)| \quad (t > 0)$$

where we put the quantity in (A. 2)  $-\lambda$ .

Before we state other results we give some notations. Let  $V_T(y)$  denote the minimum of  $\frac{1}{2} \int_0^T |\dot{\varphi}(t) - b(\varphi(t))|^2 dt$  over all absolutely continuous functions  $\varphi$  of  $[0, T]$  to  $\mathcal{R}^d$  such that  $\varphi(0) = 0$  and  $\varphi(T) = y$ . Let us put  $V(y) = \infimum (V_T(y); T > 0)$ . For a minimal path  $\varphi$  of  $V_T(y)$ , let  $Y^\varepsilon(t)$ ,  $0 \leq t \leq T$ , be the solution of the following S.D.E.

$$(1.2) \quad \left\{ \begin{array}{l} dY^\varepsilon(t) = (b(Y^\varepsilon(t)) - b(\varphi(t)) + \dot{\varphi}(t)) dt + \varepsilon dW(t) \\ Y^\varepsilon(0) = 0 \end{array} \right.$$

and let  $Y^i(t)$ ,  $0 \leq t \leq T$ , ( $i = 0, 1, 2, \dots$ ) be determined by the following formal expansion

$$(1.3) \quad Y^\varepsilon(t) = \sum_{k=0}^{\infty} \varepsilon^k Y^k(t) \quad (\text{as } \varepsilon \rightarrow 0).$$

To avoid confusion, we sometimes write  $Y^\varepsilon(t) = Y^{\varepsilon, T}(t)$  and  $Y^i(t) = Y^{i, T}(t)$ .

The following theorems show how the meaning of the expansion (1.3) is strong.

**THEOREM 1.2.** — *Under the conditions (A.0), (A.1) and (A.2), for any natural number  $m$  and  $n$*

$$(1.4) \quad \sup_{\substack{0 < T \\ \varepsilon < 1}} \left( \sup_{\substack{0 < t < T \\ \varphi(T) = y \in \mathcal{R}^d}} \left( \varepsilon^{-(n+1)} \left\| Y^{\varepsilon, T}(t) - \sum_{i=0}^n \varepsilon^i Y^{i, T}(t) \right\|_m \right) \right) < +\infty$$

where we denote by  $\|\cdot\|_m$   $L^m$ -norm.

**THEOREM 1.3.** — *Under the conditions (A.0), (A.1) and (A.2), for any natural number  $m, n$  and sufficiently large  $q$*

$$(1.5) \quad \sup_{\substack{0 < T \\ \varepsilon < 1}} \left( \sup_{\substack{0 < t < T \\ \varphi(T) = y \in \mathcal{R}^d}} \left( \varepsilon^{-(n+1)} \left\| D^q \left( Y^{\varepsilon, T}(t) - \sum_{i=0}^n \varepsilon^i Y^{i, T}(t) \right) \right\|_{\text{HS}} \right) \right) < +\infty$$

where we denote by  $\|\cdot\|_{\text{HS}}$  Hilbert Schmidt norm.

The following proposition shows the uniform integrability of exponential moments which usually appear in such arguments.

**PROPOSITION 1.4.** — *Under the conditions (A.0), (A.1) and (A.2), there exists a constant  $p > 1$  such that*

$$(1.6) \quad \overline{\lim}_{T \rightarrow \infty} \sup \left( \mathbb{E} \left[ \exp \left( p \left\langle \dot{\varphi}(T) - b(\varphi(T)), \frac{Y^\varepsilon(T) - Y^0(T) - \varepsilon Y^1(T)}{\varepsilon^2} \right\rangle \right) \right] \right) < +\infty$$

where the supremum is over all  $\varepsilon (0 < \varepsilon < 1)$  and all  $\varphi(T) = y \in \mathcal{R}^d$  for which  $V(y) < 4\lambda^3 (d^6 \|\partial^2 b\|_\infty^2)^{-1}$ . Here we denote by  $\langle \cdot, \cdot \rangle$  the inner product in  $\mathcal{R}^d$  and we put

$$\|\partial^2 b\|_\infty = \sup \left( \left| \frac{\partial^2 b^i(x)}{\partial x_j \partial x_k} \right|; x \in \mathcal{R}^d, i, j, k = 1, \dots, d \right).$$

The following results are what we want.

**THEOREM 1.5.** — *Suppose the conditions (A.0), (A.1) and (A.2). Then there exist functions  $p^k(x) (k \geq 0)$  such that for all  $n \geq 0$*

$$(1.7) \quad \overline{\lim}_{\varepsilon \rightarrow 0} \left| \varepsilon^d \exp(V(x)/\varepsilon^2) p^\varepsilon(x) - \sum_{i=0}^n \varepsilon^{2i} p^i(x) \right| \varepsilon^{-2(n+1)} < +\infty$$

uniformly for all  $x$  for which  $|x|$  is sufficiently small.

From theorem 3 in M. V. Day [3] and theorem 1.5, we have the following corollary.

COROLLARY 1.6. — *Under the conditions (A.0)', (A.1), (A.2)' and (A.3), there exist functions  $p^k(x)$  ( $k \geq 0$ ) such that for all  $n \geq 0$*

$$(1.8) \quad \overline{\lim}_{\varepsilon \rightarrow 0} \left| \varepsilon^d \exp(V(x)/\varepsilon^2) p^n(x) - \sum_{i=0}^n \varepsilon^{2i} p^i(x) \right| \varepsilon^{-2(n+1)} < +\infty$$

uniformly for all  $x$  for which  $|x|$  is sufficiently small.

## 2. LEMMAS

In this section we state the lemmas necessary for the proofs of our results. The next lemma is technically essential.

LEMMA 1. — *Let  $f(t)$  be a continuous function of  $[0, \infty)$  to  $\mathcal{R}$ . If there exist a positive constant  $\alpha$  and a measurable function  $g$  of  $[0, \infty)$  to  $\mathcal{R}$  such that for any  $s, t$  ( $0 \leq s < t$ )*

$$(2.1) \quad f(t) - f(s) \leq -\alpha \int_s^t f(u) du + \int_s^t g(u) du,$$

then we have

$$(2.2) \quad f(t) \leq \exp(-\alpha t) \left( \int_0^t \exp(\alpha s) g(s) ds + f(0) \right).$$

*Proof.* — If  $t=0$ , then (2.2) holds. Suppose that there exists a positive constant  $t_0$  such that (2.2) does not hold. Put

$$(2.3) \quad s_0 = \max \left\{ u < t_0; f(u) \leq \exp(-\alpha u) \left( \int_0^u \exp(\alpha v) g(v) dv + f(0) \right) \right\}.$$

Then for  $u(s_0 < u < t_0)$ , (2.2) does not hold. Therefore we have

$$\begin{aligned} & \exp(-\alpha t_0) \left( \int_0^{t_0} \exp(\alpha u) g(u) du + f(0) \right) \\ & \quad - \exp(-\alpha s_0) \left( \int_0^{s_0} \exp(\alpha u) g(u) du + f(0) \right) \\ & \quad < f(t_0) - f(s_0) \\ & < -\alpha \int_{s_0}^{t_0} \exp(-\alpha u) \left( \int_0^u \exp(\alpha v) g(v) dv + f(0) \right) du + \int_{s_0}^{t_0} g(u) du \\ & = \exp(-\alpha t_0) \left( \int_0^{t_0} \exp(\alpha u) g(u) du + f(0) \right) \\ & \quad - \exp(-\alpha s_0) \left( \int_0^{s_0} \exp(\alpha u) g(u) du + f(0) \right) \end{aligned}$$

which is a contradiction.

Q.E.D.

The following lemma can be proved by lemma 1 and is used to prove theorem 1.5.

LEMMA 2. — Let  $f^n(t)$  ( $n=1, 2, \dots$ ) be continuous functions of  $[0, \infty)$  to  $\mathcal{R}$  such that  $f^n(0)=0$  ( $n=1, 2, \dots$ ). Suppose that there exist positive constants  $c$  and  $c_n$  ( $n=1, 2, \dots$ ) such that for any  $s, t$  ( $0 \leq s < t$ ) and natural number  $n$

$$(2.4) \quad f^n(t) - f^n(s) \leq -cn \int_s^t f^n(u) du + c_n \int_s^t f^{n-1}(u) du$$

where we put  $f^0(t) \equiv f^0 = \text{constant}$ . Then we have

$$(2.5) \quad f^n(t) \leq f^0 (c^n n!)^{-1} (1 - \exp(-ct))^n \prod_{k=1}^n c_k$$

for all  $t \geq 0$  and natural numbers  $n$ .

*Proof.* — We prove by induction.

When  $n=1$ , by lemma 1 we have

$$f^1(t) \leq \exp(-ct) \int_0^t \exp(cs) (c_1 f^0) ds = f^0 c^{-1} (1 - \exp(-ct)) c_1.$$

Assume that (2.5) holds for  $n=k$ . Then we have, by lemma 1,

$$\begin{aligned} f^{k+1}(t) &\leq \exp(-c(k+1)t) \int_0^t \exp(c(k+1)s) c_{k+1} f^k(s) ds \\ &\leq \exp(-c(k+1)t) \int_0^t \exp(c(k+1)s) c_{k+1} f^0 \\ &\quad \times (c^k k!)^{-1} (1 - \exp(-cs))^k \prod_{j=1}^k c_j ds \\ &= f^0 (c^{k+1} (k+1)!)^{-1} (1 - \exp(-ct))^{k+1} \prod_{j=1}^{k+1} c_j \end{aligned}$$

Q.E.D.

We use the following lemma to prove theorem 1.3.

LEMMA 3. — Let  $\psi(u)$  be a measurable function of  $[0, \infty)$  to  $\mathcal{R}$  and let  $Y(t)$  ( $0 \leq t < +\infty$ ) be the solution of the following differential equation:

$$(2.6) \quad \begin{aligned} \frac{d}{dt} Y(t) &= \partial b(\psi(t)) Y(t) \\ Y(0) &= I \end{aligned}$$

where  $I$  denote an  $(d, d)$ -identity matrix. Then under the conditions (A.0) and (A.2), for any positive constants  $s, t$  ( $s < t$ ), we have

$$(2.7) \quad \|Y(t) Y(s)^{-1}\| \leq d \{ \exp(-\lambda(t-s)) \}$$

*Proof.* — We put  $Y(t) Y(s)^{-1} = Z^{s,t} = (Z_{ij}^{s,t})_{i,j=1}^d$ .

Then for any  $t_0$ , ( $t > t_0 > s$ ), we have

$$\|Z^{s,t}\|^2 - \|Z^{s,t_0}\|^2 \leq -2\lambda \int_{t_0}^t \|Z^{s,u}\|^2 du$$

since

$$\begin{aligned} \sum_{i,j=1}^d (Z_{ij}^{s,t})^2 - \sum_{i,j=1}^d (Z_{ij}^{s,t_0})^2 &= \int_{t_0}^t 2 \sum_{i,j=1}^d Z_{ij}^{s,u} \frac{d}{du} Z_{ij}^{s,u} du, \\ \sum_{i,j=1}^d Z_{ij}^{s,u} \frac{d}{du} Z_{ij}^{s,u} &= \sum_{i,j=1}^d Z_{ij}^{s,u} \left( \sum_{k=1}^d \frac{\partial b^i}{\partial x_k}(\psi(u)) Z_{kj}^{s,u} \right) \\ &= \sum_{j=1}^d \langle Z_j^{s,u}, \partial b(\psi(u)) Z_j^{s,u} \rangle \\ &\leq -\lambda \sum_{j=1}^d |Z_j^{s,u}|^2 = -\lambda \|Z^{s,u}\|^2 \quad [\text{from (A.2)}] \end{aligned}$$

where we put  $Z_j^{s,u} = (Z_{ij}^{s,u})_{i=1}^d \in \mathcal{R}^d$  ( $j=1, \dots, d$ ).



Therefore by lemma 1, we have

$$\|Z^{s,t}\|^2 \leq \exp(-2\lambda(t-s)) \|Z^{s,s}\|^2 = \exp(-2\lambda(t-s)) d^2.$$

Q.E.D.

*Remark.* — It is easy to see that  $Y(t)^{-1}$  exists.

We use the next lemma when we prove theorem 1. 2.

LEMMA 4. — Under the conditions (A.0) and (A.2), for any natural number  $m$  and  $n$ ,

$$(2.8) \quad \sup(\sup(\|Y^{n,T}(t)\|_m; 0 \leq t \leq T, \varphi(T) = y \in \mathcal{R}^d; 0 \leq T) < +\infty$$

*Proof.* — We prove by induction. Put  $Y^{n,T}(t) = Y^n(t)$ .  
(When  $n = 1$ .)

$$E[|Y^1(t)|^{2n}] \leq ((2\lambda)^n n!)^{-1} \prod_{k=1}^n (kd + 2k(k-1)),$$

since

$$dY^1(t) = \partial b(Y^0(t)) Y^1(t) dt + dW(t),$$

and by Ito's formula,

$$\begin{aligned} E[|Y^1(t)|^{2n}] - E[|Y^1(s)|^{2n}] &= 2n \int_s^t E[|Y^1(u)|^{2(n-1)} \langle Y^1(u), \partial b(Y^0(u)) Y^1(u) \rangle] du \\ &\quad + (nd + 2n(n-1)) \int_s^t E[|Y^1(u)|^{2(n-1)}] du \\ &\leq -2\lambda n \int_s^t E[|Y^1(u)|^{2n}] du \\ &\quad + (nd + 2n(n-1)) \int_s^t E[|Y^1(u)|^{2(n-1)}] du \end{aligned}$$

[from (A.2)], therefore from lemma 2, we have

$$E[|Y^1(t)|^{2n}] \leq ((2\lambda)^n n!)^{-1} (1 - \exp(-2\lambda t))^n \prod_{k=1}^n (kd + 2k(k-1)).$$

(When  $n \geq 2$ .) Since

$$Y^n(t) = \int_0^t [\partial b(Y^0(s)) Y^n(s) + R_n(s)] ds,$$

where we put

$$R_n(s) = \frac{1}{2!} \sum_{\substack{i+j=n \\ i, j \geq 1}} \partial^2 b(Y^0(s)) Y^i(s) \otimes Y^j(s) + \dots + \frac{1}{n!} \partial^n b(Y^0(s)) \otimes Y^1(s)$$

and for  $a_i = (a_i^j)_{j=1}^d \in \mathcal{R}^d (i = 1, \dots, n)$ , we put

$$\partial^n b(x) a_1 \otimes \dots \otimes a_n = \left( \sum_{i_1, \dots, i_n=1}^d \frac{\partial^n b^i(x)}{\partial x_{i_1} \dots \partial x_{i_n}} a_1^{i_1} \dots a_n^{i_n} \right)_{i=1}^d \in \mathcal{R}^d,$$

we have, for any  $s, t (s < t)$  and  $\alpha > 0$ ,

$$\begin{aligned} & (|Y^n(t)|^2 + \alpha)^{1/2} - (|Y^n(s)|^2 + \alpha)^{1/2} \\ &= \int_s^t (|Y^n(u)|^2 + \alpha)^{-1/2} \langle Y^n(u), \partial b(Y^0(u)) Y^n(u) + R_n(u) \rangle du \\ &\leq -\lambda \int_s^t (|Y^n(u)|^2 + \alpha)^{-1/2} |Y^n(u)|^2 du + \int_s^t |R_n(u)| du. \end{aligned}$$

Let  $\alpha$  tend to 0 then we have

$$|Y^n(t)| - |Y^n(s)| \leq -\lambda \int_s^t |Y^n(u)| du + \int_s^t |R_n(u)| du.$$

Therefore from lemma 1,

$$|Y^n(t)| \leq \int_0^t \exp(-\lambda(t-s)) |R_n(s)| ds.$$

Hence, by Hölder's inequality,

$$\begin{aligned} |Y^n(t)|^m &\leq \left( \int_0^t \exp\left(-\frac{m\lambda(t-s)}{2}\right) ds \right)^{m-1} \\ &\quad \times \left( \int_0^t \exp\left(-\frac{m\lambda(t-s)}{2}\right) |R_n(s)|^m ds \right), \end{aligned}$$

where we consider  $\lambda(t-s) = \frac{\lambda(t-s)}{2} + \frac{\lambda(t-s)}{2}$  and  $m^{-1} + (m/m-1)^{-1} = 1$ ,

which completes the proof.

At last we give the next lemma.

LEMMA 5. — Under the assumptions (A.0) and (A.2), we have, for the Malliavin's covariance

$$\begin{aligned} & \left\langle \frac{D(Y^\varepsilon(T) - Y^0(T))}{\varepsilon}, \frac{D(Y^\varepsilon(T) - Y^0(T))}{\varepsilon} \right\rangle_{\text{HS}}, \\ (2.9) \quad & \sup \left( \left\| \left\langle \frac{D(Y^\varepsilon(T) - Y^0(T))}{\varepsilon}, \frac{D(Y^\varepsilon(T) - Y^0(T))}{\varepsilon} \right\rangle_{\text{HS}} \right\|; \right. \\ & \left. T > 0, \varepsilon > 0, \varphi(T) = y \in \mathcal{R}^d \right) < +\infty \end{aligned}$$

$$(2.10) \quad \liminf_{T \rightarrow \infty} (\lambda_y^{\varepsilon, T}; \varepsilon > 0, y \in \mathcal{R}^d) > 0$$

where we denote by  $\lambda_{y,T}^{\varepsilon}$  the minimal eigenvalue of

$$\left\langle \frac{D(Y^\varepsilon(T) - Y^0(T))}{\varepsilon}, \frac{D(Y^\varepsilon(T) - Y^0(T))}{\varepsilon} \right\rangle_{\text{HS}} \quad \text{for } y = \varphi(T) \in \mathcal{R}^d.$$

*Proof* [proof of (2.9)]. — Let  $Y^\varepsilon(t)$  be the solution of (2.6) for  $\psi(u) = Y^\varepsilon(u)$  and put  $Z^{s,t,\varepsilon} = (Z_{ij}^{s,t,\varepsilon})_{i,j=1}^d = Y^\varepsilon(t) Y^\varepsilon(s)^{-1}$  ( $s \leq t \leq T$ ). Since

$$\begin{aligned} \left\langle \frac{D(Y^\varepsilon(T) - Y^0(T))}{\varepsilon}, \frac{D(Y^\varepsilon(T) - Y^0(T))}{\varepsilon} \right\rangle_{\text{HS}} &= \int_0^T Z^{s,T,\varepsilon} (Z^{s,T,\varepsilon})^* ds, \\ \left| \sum_{k=1}^d \int_0^T Z_{ik}^{s,T,\varepsilon} Z_{jk}^{s,T,\varepsilon} ds \right| &\leq \int_0^T \|Z^{s,T,\varepsilon}\|^2 ds \\ &\leq \int_0^T d^2 \exp(-2\lambda(T-s)) ds \quad (\text{from lemma 3}). \end{aligned}$$

Hence we conclude

$$\begin{aligned} \sup \left( \left\| \left\langle \frac{D(Y^\varepsilon(T) - Y^0(T))}{\varepsilon}, \frac{D(Y^\varepsilon(T) - Y^0(T))}{\varepsilon} \right\rangle_{\text{HS}} \right\|; \right. \\ \left. T > 0, \varepsilon > 0, \varphi(T) = y \in \mathcal{R}^d \right) \leq \frac{d^3}{2\lambda}. \end{aligned}$$

[proof of (2.10)].

We put  $B_\varepsilon^{s,T} = Z^{s,T,\varepsilon} (Z^{s,T,\varepsilon})^*$  and denote by  $\lambda_{1,\varepsilon}^{s,T}$  the minimal eigenvalue of  $B_\varepsilon^{s,T}$  and  $\|\partial b\|_\infty = \sup \left( \left| \frac{\partial b^i}{\partial x_j}(x) \right|; i, j = 1, \dots, d, x \in \mathcal{R}^d \right)$ .

Then  $\lambda_{1,\varepsilon}^{s,T} \geq \exp(-2d\|\partial b\|_\infty(T-s))$ , since

$$\lambda_{1,\varepsilon}^{s,T} = \inf \{ |Z^{s,T,\varepsilon} e|; e = (e_i)_{i=1}^d \in \mathcal{R}^d \text{ and } |e| = 1 \}$$

and for  $t_1, t_2$  ( $s < t_1, t_2 < T$ ),

$$\begin{aligned} |Z^{s,t_2,\varepsilon} e|^2 - |Z^{s,t_1,\varepsilon} e|^2 &= \int_{t_1}^{t_2} \sum_{i=1}^d 2 \left( \sum_{k=1}^d Z_{ik}^{s,u,\varepsilon} e_k \right) \left( \sum_{j=1}^d \frac{d}{du} Z_{ij}^{s,u,\varepsilon} e_j \right) du \\ &= 2 \int_{t_1}^{t_2} \langle Z^{s,u,\varepsilon} e, \partial b(Y^\varepsilon(u)) Z^{s,u,\varepsilon} e \rangle du \end{aligned}$$

therefore  $\frac{d}{dt} |Z^{s,t,\varepsilon} e|^2 \geq -2d\|\partial b\|_\infty |Z^{s,t,\varepsilon} e|^2$  and

$$|Z^{s,t,\varepsilon} e|^2 \geq \exp(-2d\|\partial b\|_\infty(t-s)) |e|^2.$$

Hence

$$\begin{aligned} & \liminf_{T \rightarrow \infty} (\lambda_y^{\varepsilon, T}; \varepsilon > 0, y \in \mathcal{R}^d) \\ & \geq \liminf_{T \rightarrow \infty} \int_0^T \exp(-2d \|\partial b\|_{\infty} (T-s)) ds \\ & = (2d \|\partial b\|_{\infty})^{-1}. \end{aligned}$$

Q.E.D.

### 3. PROOFS OF MAIN RESULTS

In this section we give the proofs of our results.

*Proof of proposition 1. 1.* [proof of the positively recurrent property of  $X^{\varepsilon}(t)$ ]. — Since

$$\begin{aligned} \langle b(x), x \rangle &= \langle b(x), x \rangle - \langle b(0), x \rangle \\ &= \int_0^1 \langle \partial b(ux) x, x \rangle du \leq -\lambda |x|^2 \quad [\text{from (A. 2)}], \end{aligned}$$

$X^{\varepsilon}(t)$  is positively recurrent (cf. Has'minskii [5]), [proof of  $|\mathbf{x}(t)| \leq \exp(-\lambda t) |\mathbf{x}(0)|$ ]. Since

$$\begin{aligned} |\mathbf{x}(t)|^2 - |\mathbf{x}(s)|^2 &= 2 \int_s^t \langle \mathbf{x}(u), b(\mathbf{x}(u)) \rangle du \leq -2\lambda \int_s^t |\mathbf{x}(u)|^2 du, \\ |\mathbf{x}(t)|^2 &\leq \exp(-2\lambda t) |\mathbf{x}(0)|^2 \quad (\text{from lemma 1}). \end{aligned}$$

Q.E.D.

Next we prove theorem 1. 2.

*Proof of theorem 1. 2.* — We put  $R_n^{\varepsilon}(t) = Y^{\varepsilon}(t) - \sum_{i=0}^n \varepsilon^i Y^i(t)$ .

For any  $t (0 \leq t \leq T)$  and  $\alpha > 0$ , since

$$\begin{aligned} R_n^{\varepsilon}(t) &= \int_0^t \left[ b(Y^{\varepsilon}(s)) - b\left(\sum_{i=0}^n \varepsilon^i Y^i(s)\right) \right] ds \\ &+ \int_0^t \left[ b\left(\sum_{i=0}^n \varepsilon^i Y^i(s)\right) - \sum_{k=0}^n \frac{1}{k!} \frac{d^k}{d\varepsilon^k} b\left(\sum_{i=0}^n \varepsilon^i Y^i(s)\right) \Big|_{\varepsilon=0} \varepsilon^k \right] ds, \end{aligned}$$

we have for any  $s, t$  ( $0 \leq s < t \leq T$ ),

$$\begin{aligned}
& (|\mathbf{R}_n^\varepsilon(t)|^2 + \alpha)^{1/2} - (|\mathbf{R}_n^\varepsilon(s)|^2 + \alpha)^{1/2} \\
&= \int_s^t \left\langle \mathbf{R}_n^\varepsilon(u), \frac{d}{du} \mathbf{R}_n^\varepsilon(u) \right\rangle (|\mathbf{R}_n^\varepsilon(u)|^2 + \alpha)^{-1/2} du \\
&= \int_s^t (|\mathbf{R}_n^\varepsilon(u)|^2 + \alpha)^{-1/2} \left\langle \mathbf{R}_n^\varepsilon(u), b(\mathbf{Y}^\varepsilon(u)) \right. \\
&\quad \left. - b\left(\sum_{i=0}^n \varepsilon^i \mathbf{Y}^i(u)\right) \right\rangle du + \int_s^t (|\mathbf{R}_n^\varepsilon(u)|^2 + \alpha)^{-1/2} \\
&\quad \times \left\langle \mathbf{R}_n^\varepsilon(u), b\left(\sum_{i=0}^n \varepsilon^i \mathbf{Y}^i(u)\right) \right. \\
&\quad \left. - \sum_{i=0}^n \frac{1}{k!} \frac{d^k}{d\varepsilon^k} b\left(\sum_{i=0}^n \varepsilon^i \mathbf{Y}^i(u)\right) \Big|_{\varepsilon=0} \varepsilon^k \right\rangle du \\
&= \int_s^t (|\mathbf{R}_n^\varepsilon(u)|^2 + \alpha)^{-1/2} \left\langle \mathbf{R}_n^\varepsilon(u), \partial b\left(\sum_{i=0}^n \varepsilon^i \mathbf{Y}^i(u)\right) \right. \\
&\quad \left. + \theta(u) \mathbf{R}_n^\varepsilon(u) \right\rangle \mathbf{R}_n^\varepsilon(u) du \\
&\quad + \int_s^t (|\mathbf{R}_n^\varepsilon(u)|^2 + \alpha)^{-1/2} \left\langle \mathbf{R}_n^\varepsilon(u), \right. \\
&\quad \left. \frac{1}{(n+1)!} \frac{d^{n+1}}{d\gamma^{n+1}} b\left(\sum_{i=0}^n \gamma^i \mathbf{Y}^i(u)\right) \Big|_{\gamma=\varepsilon\tilde{\theta}(u)} \varepsilon^{n+1} \right\rangle du \\
&\leq -\lambda \int_s^t (|\mathbf{R}_n^\varepsilon(u)|^2 + \alpha)^{-1/2} |\mathbf{R}_n^\varepsilon(u)|^2 du \\
&\quad + \int_s^t (|\mathbf{R}_n^\varepsilon(u)|^2 + \alpha)^{-1/2} |\mathbf{R}_n^\varepsilon(u)| \\
&\quad \left| \frac{1}{(n+1)!} \frac{d^{n+1}}{d\gamma^{n+1}} b\left(\sum_{i=0}^n \gamma^i \mathbf{Y}^i(u)\right) \Big|_{\gamma=\varepsilon\tilde{\theta}(u)} \varepsilon^{n+1} \right| du
\end{aligned}$$

for some  $\theta(u), \tilde{\theta}(u)$  such that  $0 \leq \theta(u), \tilde{\theta}(u) \leq 1$  ( $0 \leq u \leq T$ ).

Let  $\alpha \rightarrow 0$ , then we have, for any  $s, t$  ( $0 \leq s < t \leq T$ )

$$\begin{aligned}
|\mathbf{R}_n^\varepsilon(t)| - |\mathbf{R}_n^\varepsilon(s)| &\leq -\lambda \int_s^t |\mathbf{R}_n^\varepsilon(u)| du \\
&\quad + \int_s^t \left| \frac{1}{(n+1)!} \frac{d^{n+1}}{d\gamma^{n+1}} b\left(\sum_{i=0}^n \gamma^i \mathbf{Y}^i(u)\right) \Big|_{\gamma=\varepsilon\tilde{\theta}(u)} \varepsilon^{n+1} \right| du.
\end{aligned}$$

Hence from lemma 1,

$$|R_n^\varepsilon(t)| \leq \exp(-\lambda t) \int_0^t \exp(\lambda s) \times \left| \frac{1}{(n+1)!} \frac{d^{n+1}}{d\gamma^{n+1}} b \left( \sum_{i=0}^n \gamma^i Y^i(s) \right) \right|_{\gamma=\varepsilon \tilde{\theta}(s)} |\varepsilon^{n+1} ds.$$

which completes the proof by lemma 4.

We prove theorem 1.3 by induction.

*Proof of theorem 1.3.* — It is easy to see that for any natural number  $n$  there exists a natural number  $q(n)$  such that if  $q \geq q(n)$ , then

$$D^q Y^i(h_1, \dots, h_q)(t) = 0 \\ (0 \leq i \leq n, 0 \leq t \leq T, h_1, \dots, h_q \in H)$$

where  $H$  denote Cameron Martin space. Hence we only have to prove

$$(3.1) \quad \sup_{0 \leq T, \varepsilon < 1} (\sup_{0 \leq t \leq T} (\varepsilon^{-(n+1)} \| |D^q Y^{\varepsilon, T}(t)|_{HS} \|_m)) < +\infty$$

for sufficiently large  $q$ . From the following proposition, (3.1) holds.

Q.E.D.

**PROPOSITION 3.1.** — *For each natural number  $n$ ,  $t$  ( $0 \leq t \leq T$ ) and  $h_1, \dots, h_n \in H$ ,*

$$\varepsilon^{-n} D^n Y^{\varepsilon, T}(h_1, \dots, h_n)(t) \\ = \int_0^t ds_1 \dots \int_0^t ds_n g^{n, T}(s_1, \dots, s_n; t) \dot{h}_1(s_1) \otimes \dots \otimes \dot{h}_n(s_n)$$

for some

$$g^{n, T}(s_1, \dots, s_n; t) = (g_{i_1, \dots, i_n}^{n, i, T}(s_1, \dots, s_n; t))_{i_1, \dots, i_n=1}^d$$

where we put

$$g^{n, T}(s_1, \dots, s_n; t) \dot{h}_1(s_1) \otimes \dots \otimes \dot{h}_n(s_n) \\ = \left( \sum_{i_1, \dots, i_n=1}^d g_{i_1, \dots, i_n}^{n, i, T}(s_1, \dots, s_n; t) \right. \\ \left. \times \dot{h}_1^{i_1}(s_1) \dots \dot{h}_n^{i_n}(s_n) \right)_{i=1}^d \in \mathcal{R}^d.$$

Moreover there exist nonrandom constants  $C_n$  ( $n=1, 2, \dots$ ) such that

$$\sup_{\substack{0 \leq T \\ \varepsilon < 1}} \left( \sup_{0 \leq t \leq T} \left( \int_0^t ds_1 \dots \int_0^t ds_n \right. \right. \\ \left. \left. \times \sum_{i_1, \dots, i_n=1}^d |g_{i_1, \dots, i_n}^{n, i, T}(s_1, \dots, s_n; t)|^2 \right) \right) < C_n.$$

*Proof of proposition 3.1.* — It is easy to see that

$$(3.2) \quad DY^\varepsilon(h)(t) = \varepsilon h(t) + \int_0^t \partial b(Y^\varepsilon(s)) DY^\varepsilon(h)(s) ds \quad (h \in H)$$

$$(3.3) \quad D^2 Y^\varepsilon(h_1, h_2)(t) = \int_0^t [\partial b(Y^\varepsilon(s)) D^2 Y^\varepsilon(h_1, h_2)(s) + \partial^2 b(Y^\varepsilon(s)) DY^\varepsilon(h_1)(s) \otimes DY^\varepsilon(h_2)(s)] ds \quad (h_1, h_2 \in H).$$

Inductively, in the same way, we can show that for all  $n \geq 2$

$$(3.4) \quad \varepsilon^{-n} D^n Y^\varepsilon(\mathbf{h})(t) = \int_0^t [\partial b(Y^\varepsilon(s)) \varepsilon^{-n} D^n Y^\varepsilon(\mathbf{h})(s) + f^n(\varepsilon^{-1} DY^\varepsilon(s), \dots, \varepsilon^{-(n-1)} D^{n-1} Y^\varepsilon(s))(\mathbf{h})] ds$$

where we put  $\mathbf{h} = (h_1, \dots, h_n)$  and  $f^n$  is a polynomial of  $\varepsilon^{-1} DY^\varepsilon(s), \dots, \varepsilon^{-(n-1)} D^{n-1} Y^\varepsilon(s)$

with coefficient  $\partial b(Y^\varepsilon(s)), \dots, \partial^n b(Y^\varepsilon(s))$  and

$$\partial^k b(x) = \left( \frac{\partial^k b^i(x)}{\partial x_{i_1} \dots \partial x_{i_k}} \right)_{i, i_1, \dots, i_k=1}^d \quad (1 \leq k \leq n).$$

(When  $n=1$ .) We can put  $g^{1,T}(s, t) = Z^{s,t,\varepsilon}$ . In fact, it is easy to see that

$$\varepsilon^{-1} DY^\varepsilon(h)(t) = \int_0^t g^{1,T}(s, t) \dot{h}(s) ds \text{ from (3.2) and we have}$$

$$\sup_{\substack{0 \leq T \\ \varepsilon < 1}} \left( \sup_{\substack{0 \leq t \leq T \\ \varphi(T) = y \in \mathcal{A}^d}} \int_0^t \sum_{i,j=1}^d |g_j^{1,i,T}(s, t)|^2 ds \right) \leq d^2 (2\lambda)^{-1},$$

since

$$\sum_{i,j=1}^d |g_j^{1,i,T}(s, t)|^2 = \|Z^{s,t,\varepsilon}\|^2 \leq d^2 \exp(-2\lambda(t-s))$$

from lemma 3.

(When  $n=2$ .) We put, for  $i, i_1, i_2 (= 1, \dots, d)$ ,

$$g_{i_1, i_2}^{2, i, T}(u_1, u_2; t) = \int_{u_1, u_2}^t \langle (g^{1,T}(s, t))^{i_1}, \partial^2 b(Y^\varepsilon(s)) \times (g^{1,T}(u_1, s))_{i_1} \otimes (g^{1,T}(u_2, s))_{i_2} \rangle ds$$

where we put, for  $i, j (= 1, \dots, d)$ ,

$$(g^{1,T}(u, s))^{i_1} = (g_k^{1, i_1, T}(u, s))_{k=1}^d$$

and

$$(g^{1,T}(u, s))_{i_2} = (g_j^{1, i_2, T}(u, s))_{j=1}^d.$$

Then from (3.3), it is easy to see that

$$\varepsilon^{-2} D^2 Y^\varepsilon(h_1, h_2)(t) = \int_0^t du_1 \int_0^t du_2 g^{2, T}(u_1, u_2; t) \dot{h}_1(u_1) \otimes \dot{h}_2(u_2),$$

$$\sup_{\substack{0 \leq t \leq T \\ \varepsilon < 1}} \left( \sup_{\substack{0 \leq t \leq T \\ \varphi(T) = y \in \mathcal{R}^d}} \left( \int_0^t du_1 \int_0^t du_2 \sum_{i, i_1, i_2=1}^d |g_{i_1, i_2}^{2, i, T}(u_1, u_2; t)|^2 \right) \right) < C_2$$

for some nonrandom constant  $C_2$ . In fact

$$\begin{aligned} & \sum_{i, i_1, i_2=1}^d |g_{i_1, i_2}^{2, i, T}(u_1, u_2; t)|^2 \\ &= \sum \left| \int_{u_1 \vee u_2}^t \langle (g^{1, T}(s, t))^i, \partial^2 b(Y^\varepsilon(s)) \right. \\ & \quad \left. \times (g^{1, T}(u_1, s))_{i_1} \otimes (g^{1, T}(u_2, s))_{i_2} \rangle ds \right|^2 \\ &= \sum \int_{u_1 \vee u_2}^t \langle (g^{1, T}(s_1, t))^i, \partial^2 b(Y^\varepsilon(s_1)) \rangle \\ & \quad \times (g^{1, T}(u_1, s_1))_{i_1} \otimes (g^{1, T}(u_2, s_1))_{i_2} \rangle ds_1 \\ & \quad \times \int_{u_1 \vee u_2}^t \langle (g^{1, T}(s_2, t))^i, \partial^2 b(Y^\varepsilon(s_2)) \rangle \\ & \quad \times (g^{1, T}(u_1, s_2))_{i_1} \otimes (g^{1, T}(u_2, s_2))_{i_2} \rangle ds_2 \\ &\leq \left( \int_{u_1 \vee u_2}^t \left( \sum \langle (g^{1, T}(s, t))^i, \partial^2 b(Y^\varepsilon(s)) \right. \right. \\ & \quad \left. \left. \times (g^{1, T}(u_1, s))_{i_1} \otimes (g^{1, T}(u_2, s))_{i_2} \rangle^2 \right)^{1/2} ds \right)^2 \end{aligned}$$

where  $\Sigma$  is over all  $i, i_1, i_2 (= 1, \dots, d)$  and

$$\begin{aligned} & |\langle (g^{1, T}(s, t))^i, \partial^2 b(Y^\varepsilon(s)) \\ & \quad \times (g^{1, T}(u_1, s))_{i_1} \otimes (g^{1, T}(u_2, s))_{i_2} \rangle| \\ & \leq |(g^{1, T}(s, t))^i| \cdot |\partial^2 b(Y^\varepsilon(s)) \\ & \quad \times (g^{1, T}(u_1, s))_{i_1} \otimes (g^{1, T}(u_2, s))_{i_2}| \\ & \leq |(g^{1, T}(s, t))^i| \cdot \|\partial^2 b(Y^\varepsilon(s))\| \cdot \\ & \quad \times (g^{1, T}(u_1, s))_{i_1} \cdot |(g^{1, T}(u_2, s))_{i_2}| \end{aligned}$$

where we put

$$\|\partial^2 b(y)\| = \left( \sum_{i, j, k=1}^d \left| \frac{\partial^2 b^i(y)}{\partial x_j \partial x_k} \right|^2 \right)$$



and hence

$$\begin{aligned} & \int_0^t du_1 \int_0^t du_2 \sum_{i, i_1, i_2=1}^d |g_{i_1, i_2}^{2, i, T}(u_1, u_2; t)|^2 \\ & \leq \int_0^t du_1 \int_0^t du_2 \|\partial^2 b\|_\infty d^3 \left( \int_{u_1 \vee u_2}^t \|g^{1, T}(s, t)\| \right. \\ & \quad \left. \times \|g^{1, T}(u_1, s)\| \cdot \|g^{1, T}(u_2, s)\| ds \right)^2 \\ & \leq \|\partial^2 b\|_\infty d^3 \sup_{\substack{0 \leq t \leq T \\ \varepsilon < 1}} \left[ \sup_{\substack{0 \leq t \leq T \\ \varphi(T) \in \mathcal{A}^d}} \left( \int_0^t \|g^{1, T}(s, t)\| ds \right)^2 \right] \\ & \quad \times \left[ \sup_{\substack{0 \leq t \leq T \\ \varphi(T) \in \mathcal{A}^d}} \left( \int_0^t \|g^{1, T}(s, t)\|^2 ds \right)^2 \right] \\ & \leq \|\partial^2 b\|_\infty d^9 (4\lambda^4)^{-1} \end{aligned}$$

since  $\|g^{1, T}(s, t)\| \leq d \exp(-\lambda(t-s))$ .

(Assume that proposition 3.1 holds when  $n=k$ .) From (3.4)

$$\begin{aligned} & \varepsilon^{-(k+1)} \mathbf{D}^{k+1} \mathbf{Y}^\varepsilon(h_1, \dots, h_{k+1})(t) \\ & = \int_0^t \mathbf{Z}^{s, t, \varepsilon} f^{k+1}(\varepsilon^{-1} \mathbf{D} \mathbf{Y}^\varepsilon(s), \dots, \\ & \quad \varepsilon^{-k} \mathbf{D}^k \mathbf{Y}^\varepsilon(s))(h_1, \dots, h_{k+1}) ds \end{aligned}$$

and  $f^{k+1}$  can be written as the following;

$$\int_0^s ds_1 \dots \int_0^s ds_{k+1} \tilde{g}^{k, T}(s_1, \dots, s_{k+1}; s) \dot{h}_1(s_1) \otimes \dots \otimes \dot{h}_{k+1}(s_{k+1})$$

for some

$$\tilde{g}^{k, T}(s_1, \dots, s_{k+1}; s) = (\tilde{g}_{i_1, \dots, i_{k+1}}^{k, i, T}(s_1, \dots, s_{k+1}; s))_{i, i_1, \dots, i_{k+1}=1}^d$$

such that

$$\begin{aligned} & \sup_{\substack{0 \leq t \leq T \\ \varepsilon < 1}} \left( \sup_{\substack{0 \leq t \leq T \\ \varphi(T) = y \in \mathcal{A}^d}} \left( \int_0^t ds_1 \dots \int_0^t ds_{k+1} \right. \right. \\ & \quad \left. \left. \times \sum \left| \tilde{g}_{i_1, \dots, i_{k+1}}^{k, i, T}(s_1, \dots, s_{k+1}; t) \right|^2 \right) \right) < \tilde{C}_{k+1} \end{aligned}$$

for some nonrandom constant  $\check{C}_{k+1}$ , where  $\sum$  is over all  $i, i_1, \dots, i_{k+1} (=1, \dots, d)$ . Hence we have to prove that the following quantity is bounded by some nonrandom constant  $C_n$

$$\sup_{\substack{0 \leq T \\ \varepsilon < 1}} \left( \sup_{\substack{0 \leq t \leq T \\ \varphi(T) = y \in \mathcal{R}^d}} \left( \int_1 \sum \left| \int_2 \sum_{j=1}^d g_j^{1, i, T}(u, t) \tilde{g}_{i_1, \dots, i_{k+1}}^{k, j, T}(s_1, \dots, s_{k+1}; u) \right|^2 \right) \right)$$

where  $\int_1$  is over all  $s_1, \dots, s_{k+1}$  ( $0 \leq s_1, \dots, s_{k+1} \leq t$ ) and  $\int_2$  is over all  $u (s_1 \vee \dots \vee s_{k+1} \leq u \leq t)$  and  $\sum$  is over all  $i, i_1, \dots, i_{k+1} (=1, \dots, d)$ . In fact

$$\begin{aligned} & \int_1 \sum \left| \int_2 \sum_{j=1}^d g_j^{1, i, T}(u, t) \tilde{g}_{i_1, \dots, i_{k+1}}^{k, j, T}(s_1, \dots, s_{k+1}; u) \right|^2 \\ & \leq d^{k+2} \int_1 \left( \int_2 \|g^{1, T}(u, t)\| \cdot \|\tilde{g}^{k, T}(s_1, \dots, s_{k+1}; u)\| \right)^2 \\ & \leq d^{k+2} \sup \left( \left[ \sup \left\{ \left( \int_0^t \|g^{1, T}(s, t)\| ds \right)^2; 0 \leq t \leq T, \varphi(T) \in \mathcal{R}^d \right\} \right] \right. \\ & \quad \times \left[ \sup \left\{ \int_0^t ds_1, \dots, \int_0^t ds_{k+1} \|\tilde{g}^{k, T}(s_1, \dots, s_{k+1}; t)\|^2; \right. \right. \\ & \quad \left. \left. 0 \leq t \leq T, \varphi(T) \in \mathcal{R}^d \right\} \right]; 0 \leq T, \varepsilon < 1 \Big) \leq d^{k+4} \lambda^{-2} \check{C}_{k+1}. \end{aligned}$$

Q.E.D.

Next we prove proposition 1.4.

*Proof of proposition 1.4.* — Since

$$\begin{aligned} & \langle \dot{\varphi}(T) - b(\varphi(T)), Y^\varepsilon(T) - Y^0(T) - \varepsilon Y^1(T) \rangle \\ & = \int_0^T \langle \dot{\varphi}(s) - b(\varphi(s)), b(Y^\varepsilon(s)) - b(Y^0(s)) \\ & \quad - \partial b(Y^0(s))(Y^\varepsilon(s) - Y^0(s)) \rangle ds, \\ & |\langle \dot{\varphi}(T) - b(\varphi(T)), Y^\varepsilon(T) - Y^0(T) - \varepsilon Y^1(T) \rangle| \leq \varepsilon^{-2} \\ & \leq \int_0^T |\dot{\varphi}(s) - b(\varphi(s))|^2 \|\partial^2 b\|_\infty 2^{-1} (|Y^\varepsilon(s) - Y^0(s)| \varepsilon^{-1})^2 ds. \end{aligned}$$

From this we have

$$\begin{aligned}
 & E[\exp(p \langle \dot{\phi}(T) - b(\phi(T)), Y^\varepsilon(T) - Y^0(T) - \varepsilon Y^1(T) \rangle | \varepsilon^{-2})] \\
 & \leq \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{2} p d^2 \|\partial^2 b\|_{\infty} \right)^n E \left[ \left( \int_0^T |\dot{\phi}(s) - b(\phi(s))| \right. \right. \\
 & \quad \left. \left. \times |(Y^\varepsilon(s) - Y^0(s)) \varepsilon^{-1}|^2 ds \right)^n \right] \\
 & \leq \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{2} p d^2 \|\partial^2 b\|_{\infty} \right)^n \left( \int_0^T |\dot{\phi}(s) - b(\phi(s))| \right. \\
 & \quad \left. \times \|(Y^\varepsilon(s) - Y^0(s)) \varepsilon^{-1}\|_{2,n}^2 ds \right)^n \\
 & \leq \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{2} p d^2 \|\partial^2 b\|_{\infty} \right)^n \left( \prod_{k=1}^n \frac{2k+d-2}{2\lambda} \right) \left( \frac{4}{\lambda} V_T(y) \right)^{n/2} \\
 & \leq \sum_{n=0}^{\infty} \left( \frac{1}{2} p d^2 \|\partial^2 b\|_{\infty} \frac{d}{2\lambda} \left( \frac{4}{\lambda} V_T(y) \right)^{1/2} \right)^n < +\infty
 \end{aligned}$$

if  $V_T(y) < 4\lambda^3 (p^2 d^6 \|\partial^2 b\|_{\infty}^2)^{-1}$ , since

$$(3.5) \quad E[|(Y^\varepsilon(t) - Y^0(t)) \varepsilon^{-1}|^{2n}] \leq (2\lambda)^{-n} \prod_{k=1}^n (2k+d-2)$$

$$(3.6) \quad \int_0^T |\dot{\phi}(s) - b(\phi(s))| ds \leq 2(\lambda^{-1} V_T(y))^{1/2}.$$

Under the conditions (A. 0), (A. 1) and (A. 2), the limit of  $V_T(y)$  as  $T \rightarrow \infty$  exists and equals to  $V(y)$  for each  $y$  (cf. M. I. Freidlin and A. D. Wentzell [4]). Therefore we only have to prove (3. 5) and (3. 6).

[Proof of (3. 5).] We put  $Z^\varepsilon(t) = Y^\varepsilon(t) - Y^0(t)$ . Since

$$Y^\varepsilon(t) - Y^0(t) = \varepsilon W(t) + \int_0^T [b(Y^\varepsilon(s)) - b(Y^0(s))] ds,$$

by Ito's formula, we have

$$\begin{aligned}
 & |Z^\varepsilon(t)|^{2n} - |Z^\varepsilon(s)|^{2n} \\
 &= 2n \int_s^t |Z^\varepsilon(u)|^{2(n-1)} \langle Z^\varepsilon(u), \varepsilon dW(u) + \{b(Y^\varepsilon(u)) - b(Y^0(u))\} \rangle du \\
 &\quad + \varepsilon^2 (nd + 2n(n-1)) \int_s^t |Z^\varepsilon(u)|^{2(n-1)} du \\
 &\leq 2n \int_s^t |Z^\varepsilon(u)|^{2(n-1)} \langle Z^\varepsilon(u), \varepsilon dW(u) \rangle - 2n\lambda \int_s^t |Z^\varepsilon(u)|^{2n} du \\
 &\quad + \varepsilon^2 (nd + 2n(n-1)) \int_s^t |Z^\varepsilon(u)|^{2(n-1)} du \quad [\text{from (A. 2)}].
 \end{aligned}$$

Hence

$$E[|(Y^\varepsilon(t) - Y^0(t))\varepsilon^{-1}|^{2n}] \leq (2\lambda)^{-n} \prod_{k=1}^n (2k + d - 2),$$

since for any  $s, t$  ( $0 \leq s < t \leq T$ ),

$$\begin{aligned}
 & E[|Z^\varepsilon(t)|^{2n}] - E[|Z^\varepsilon(s)|^{2n}] \\
 &\leq -2n\lambda \int_s^t E[|Z^\varepsilon(u)|^{2n}] du + \varepsilon^2 (nd + 2n(n-1)) \\
 &\quad \times \int_s^t E[|Z^\varepsilon(u)|^{2(n-1)}] du
 \end{aligned}$$

and from lemma 2 we have

$$\begin{aligned}
 & E[|(Y^\varepsilon(t) - Y^0(t))\varepsilon^{-1}|^{2n}] \leq (2\lambda)^{-n} (n!)^{-1} \\
 &\quad \times (1 - \exp(-2\lambda t))^n \prod_{k=1}^n (2k(k-1) + kd) \\
 &\leq (2\lambda)^{-n} \prod_{k=1}^n (2k + d - 2).
 \end{aligned}$$

[Proof of (3.6).] Since we have

$$\begin{aligned}
 & |\dot{\varphi}(t) - b(\varphi(t))|^2 - |\dot{\varphi}(0)|^2 \\
 &= \int_0^t 2 \left\langle \dot{\varphi}(s) - b(\varphi(s)), \frac{d}{ds} (\dot{\varphi}(s) - b(\varphi(s))) \right\rangle ds \\
 &= -2 \int_0^t \langle \dot{\varphi}(s) - b(\varphi(s)), \partial b(\varphi(s))^* (\dot{\varphi}(s) - b(\varphi(s))) \rangle ds \\
 &\geq 2\lambda \int_0^t |\dot{\varphi}(s) - b(\varphi(s))|^2 ds \quad [\text{from (A. 2)}]
 \end{aligned}$$

where we use Euler's equation for  $\varphi(t)$  and from this for  $\rho > 0$ ,

$$\begin{aligned} & \int_0^T |\dot{\varphi}(s) - b(\varphi(s))|^2 (|\dot{\varphi}(s) - b(\varphi(s))|^2 + \rho)^{-1/2} ds \\ & \leq \int_0^T |\dot{\varphi}(s) - b(\varphi(s))|^2 \left( 2\lambda \int_0^s |\dot{\varphi}(u) - b(\varphi(u))|^2 du + \rho \right)^{-1/2} ds \\ & = \left[ \lambda^{-1} \left( 2\lambda \int_0^s |\dot{\varphi}(u) - b(\varphi(u))|^2 du + \rho \right)^{1/2} \right]_{s=0}^T \\ & = \lambda^{-1} [(4\lambda V_T(y) + \rho)^{1/2} - \rho^{1/2}] \rightarrow \left( \frac{4}{\lambda} V_T(y) \right)^{1/2} \quad (\rho \rightarrow 0), \end{aligned}$$

$$\begin{aligned} \int_0^T |\dot{\varphi}(s) - b(\varphi(s))| ds & \leq \lim_{\rho \rightarrow 0} \int_0^T |\dot{\varphi}(s) - b(\varphi(s))|^2 \\ & \quad \times (|\dot{\varphi}(s) - b(\varphi(s))|^2 + \rho)^{-1/2} ds \leq 2(\lambda^{-1} V_T(y))^{1/2}. \end{aligned}$$

Q.E.D.

Now let us prove theorem 1. 5.

*Proof of theorem 1. 5.* — From lemma 5, theorems 1. 2, 1. 3 and proposition 1. 4, by S. Watanabe's theory, for the transition probability density  $p^\varepsilon(t, x, y)$  of  $X^\varepsilon(t)$ , there exist functions  $p^i(T, 0, y)$  ( $i \geq 0$ ) and constants  $C_{T,y}^i$  ( $i \geq 0$ ) such that for all  $n (= 0, 1, \dots)$

$$(3. 7) \quad \varepsilon^{-2(n+1)} \left| \varepsilon^d \exp(V_T(y)/\varepsilon^2) p^\varepsilon(T, 0, y) - \sum_{i=0}^n \varepsilon^{2i} p^i(T, 0, y) \right| < C_{T,y}^i$$

if  $V_T(y) < 4\lambda^3 (d^6 \|\partial^2 b\|_\infty^2)^{-1}$ . Moreover for any  $\alpha$  ( $0 < \alpha < 1$ ),

$$(3. 8) \quad \left\{ \overline{\lim}_{T \rightarrow \infty} \sup (C_{T,y}^i; V_T(y) < \alpha [4\lambda^3 (d^6 \|\partial^2 b\|_\infty^2)^{-1}]) \right\} < +\infty$$

( $i = 0, 1, \dots$ ).

Since  $X^\varepsilon(t)$  is positively recurrent from proposition 1. 1, the limits of  $p^\varepsilon(T, 0, y)$  as  $T \rightarrow \infty$  exist for each  $\varepsilon > 0$  and  $y \in \mathcal{R}^d$  (cf. Has'minskii [5]) and from this the limits of  $p^i(T, 0, y)$  as  $T \rightarrow \infty$  exist for all  $i (= 0, 1, \dots)$  and  $y$  for which  $V(y) < 4\lambda^3 (d^6 \|\partial^2 b\|_\infty^2)^{-1}$ . In fact, for  $n = 0$ , since

$$\begin{aligned} & \varepsilon^{-2} \left| \varepsilon^d \exp(V_T(y)/\varepsilon^2) p^\varepsilon(T, 0, y) - p^0(T, 0, y) \right| < C_{T,y}^0, \\ & -\varepsilon^2 \overline{\lim}_{T \rightarrow \infty} C_{T,y}^0 + \varepsilon^d \exp(V(y)/\varepsilon^2) p^\varepsilon(y) \leq \lim_{T \rightarrow \infty} p^0(T, 0, y) \\ & \leq \overline{\lim}_{T \rightarrow \infty} p^0(T, 0, y) \leq \varepsilon^2 \overline{\lim}_{T \rightarrow \infty} C_{T,y}^0 + \varepsilon^d \exp(V(y)/\varepsilon^2) p^\varepsilon(y). \end{aligned}$$

Therefore

$$(3.9) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^d \exp(V(y)/\varepsilon^2) p^\varepsilon(y) = \lim_{T \rightarrow \infty} p^0(T, 0, y) = \overline{\lim}_{T \rightarrow \infty} p^0(T, 0, y).$$

In the same way we can prove inductively that the limits of  $p^i(T, 0, y)$  as  $T \rightarrow \infty$  exist for all  $i (= 0, 1, \dots)$ .

Q.E.D.

At last we prove corollary 1.6.

*Proof of corollary 1.6.* — Let  $X_1^\varepsilon(t)$ ,  $0 \leq t$  be the solution of the following stochastic differential equation:

$$(3.10) \quad \begin{cases} dX_1^\varepsilon(t) = b_1(X_1^\varepsilon(t)) dt + \varepsilon dW(t) \\ X_1^\varepsilon(0) = x_1 \quad (x_1 \in \mathcal{R}^d) \end{cases}$$

where  $b_1(x)$  satisfies the conditions (A.0), (A.1) and (A.2) and  $b_1(x) = b(x)$  if  $|x| < r$  for some  $r > 0$ . Then from theorem 3 in M. V. Day [3], we have the following:

$$(3.11) \quad \overline{\lim}_{\varepsilon \rightarrow 0} \sup_{y \in \mathcal{X}} (\varepsilon \log |p^\varepsilon(x) - p_1^\varepsilon(x)|) \leq -\min(V(x); |x| = r)$$

for any compact subset  $\mathcal{X}$  of the set  $\{x; |x| < r\}$  where we denote by  $p_1^\varepsilon(x)$  the invariant density of  $X_1^\varepsilon(t)$  and from theorem 1.5, if  $r$  is sufficiently small, (1.7) holds for  $p^\varepsilon(x) = p_1^\varepsilon(x)$  uniformly for  $|x| < r$ . Therefore for any  $\alpha < \min(V(x); |x| = r)$ , (1.7) also holds for  $p^\varepsilon(x)$  uniformly for  $x$  of the set  $\{y; V(y) \leq \alpha\}$ , since

$$\begin{aligned} \varepsilon^{-2(n+1)} \left| \varepsilon^d \exp(V(y)/\varepsilon^2) p^\varepsilon(y) - \sum_{i=0}^n \varepsilon^{2i} p^i(y) \right| \\ \leq \varepsilon^{-2(n+1)} \left| \varepsilon^d \exp(V(y)/\varepsilon^2) (p^\varepsilon(y) - p_1^\varepsilon(y)) \right| \\ + \varepsilon^{-2(n+1)} \left| \varepsilon^d \exp(V(y)/\varepsilon^2) p_1^\varepsilon(y) - \sum_{i=0}^n \varepsilon^{2i} p^i(y) \right| \end{aligned}$$

and the first term is bounded by

$$\varepsilon^{d-2(n+1)} \exp\left(\frac{\alpha - \min(V(x); |x| = r)}{2\varepsilon^2}\right)$$

for sufficiently small  $\varepsilon$ , uniformly for  $x \in \{y; V(y) \leq \alpha\}$ .

Q.E.D.

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