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# Asymptotic expansions of the invariant density of a Markov process with a small parameter

by

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ABSTRACT. — We consider the asymptotic behavior of the invariant density of a Markov process on $\mathscr{R}^d$ which is a perturbation of a dynamical system.

*Key words :* Asymptotic expansion, invariant density, S. Watanabe's theory.

RÉSUMÉ. — Nous considérons le comportement asymptotique de la densité invariante d'un processus de Markov sur $\mathscr{R}^d$ qui est une perturbation d'un système dynamique.

## INTRODUCTION

Let $(X^\varepsilon(t), P_x)$, $0 \leqq t$, $x \in \mathscr{R}^d$ be Markov processes on $\mathscr{R}^d$ solving the following stochastic differential equation:

$$dX^\varepsilon(t) = b(X^\varepsilon(t))\,dt + \varepsilon\,dW(t)$$
$$X^\varepsilon(0) = x$$

where $W(t)$ is an $d$-dimensional Wiener process, $b(x)$ is a function of $\mathscr{R}^d$ to $\mathscr{R}^d$ and $\varepsilon > 0$ is a small parameter. Then the following result is known; if $b(.) : \mathscr{R}^d \to \mathscr{R}^d$ is Lipschitz continuous,

$$P_x(\lim_{\varepsilon \to 0} \sup_{0<t<T} |X^\varepsilon(t) - \mathfrak{x}(t)| = 0) = 1 \qquad (x \in \mathscr{R}^d)$$

where we denote by $\mathfrak{x}(t)$ the dynamical system solving the next differential equation:

$$\frac{d}{dt}\mathfrak{x}(t)=b(\mathfrak{x}(t))$$
$$\mathfrak{x}(0)=x.$$

In this paper we study, by way of S. Watanabe's theory [6], the asymptotic behavior of the invariant density $\mathrm{p}^\varepsilon(x)$ of $\mathrm{X}^\varepsilon(t)$ of the following type:

$$\varepsilon^d \exp(\mathrm{V}(x)/\varepsilon^2)\,\mathrm{p}^\varepsilon(x)=\mathrm{p}^0(x)+\varepsilon^2\,\mathrm{p}^1(x)+\varepsilon^4\,\mathrm{p}^2(x)+\dots \quad (\text{as } \varepsilon\to 0)$$

where $\mathrm{V}(x)$ is the Wentzell-Freidlin quasi-potential. We also note that the invariant density $\mathrm{p}^\varepsilon(x)$ is the solution of the following equation:

$$\frac{\varepsilon^2}{2}\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}\mathrm{p}^\varepsilon(x)\text{-div}(b(x)\,\mathrm{p}^\varepsilon(x))=0 \qquad (x\in\mathscr{R}^d)$$

$$\int_{\mathscr{R}^d}\mathrm{p}^\varepsilon(x)\,dx=1.$$

The study is important when we study the exit distribution of $\mathrm{X}^\varepsilon(t)$ from the bounded domain in $\mathscr{R}^d$ (*cf.* M. V. Day [1], [2]).

With respect to this problem M. V. Day [3] got results only in the case $n=0$ but under weaker conditions than we give in this paper. As the special case, if $b(x)=\nabla\mathrm{U}(x)$ for an infinitely differentiable function U of $\mathscr{R}^d$ to $\mathscr{R}$ such that $\int_{\mathscr{R}^d}\exp(2\,\mathrm{U}(x)/\varepsilon^2)\,dx<+\infty$, then

$$\mathrm{p}^\varepsilon(x)=\mathrm{C}(\varepsilon)\exp(2\,\mathrm{U}(x)/\varepsilon^2)$$

where we put $\mathrm{C}(\varepsilon)=\left(\int_{\mathscr{R}^d}\exp(2\,\mathrm{U}(x)/\varepsilon^2)\,dx\right)^{-1}$.

In section 1 we state our results. In section 2 we give lemmas necessary for the proofs of our results. In section 3 we prove our results.

At last we give some notations which we use in this paper. For $(d, d)$-matrix $\mathrm{A}=(a_{ij})_{i,j=1}^d$, we put $|\mathrm{A}|=$ determinant of A, $\|\mathrm{A}\|=\left(\sum_{i,j=1}^{d}(a_{ij})^2\right)^{1/2}$ and $\mathrm{A}^*=(a_{ji})_{i,j=1}^d$.

For $u=(u_i)_{i=1}^d$ of $\mathscr{R}^d$, we put $|u|=\left(\sum_{i=1}^{d}(u_i)^2\right)^{1/2}$.

## 1. MAIN RESULTS

In this section we state our results.

We introduce the following conditions.

(A.0) $b(x)$ is an $d$-dimensional $C^\infty$-vector field on $\mathscr{R}^d$ with bounded derivatives of all orders.

(A.0)′ $b(x)$ is a function of $\mathscr{R}^d$ to $\mathscr{R}^d$ with Hölder continuous first derivatives for some exponent $\gamma\,(0<\gamma\leqq 1)$ and is $C^\infty$-class in a small neighborhood of $0\in\mathscr{R}^d$.

(A.1) $$b(0)=0.$$

(A.2) $$\sup\left(\sup\left(\sum_{i,j=1}^{d}\frac{\partial b^i}{\partial x_j}(x)\,e^i e^j;\ \sum_{i=1}^{d}(e^i)^2=1\right);\ x\in\mathscr{R}^d\right)<0.$$

(A.2)′ $$\left\{\begin{array}{l}\sup\left(\displaystyle\sum_{i,j=1}^{d}\frac{\partial b^i}{\partial x_j}(0)\,e^i e^j;\ \sum_{i=1}^{d}(e^i)^2=1\right)<0,\\ \displaystyle\lim_{t\to\infty}\mathfrak{x}(t)=0\qquad(\mathfrak{x}(0)\in\mathscr{R}^d).\end{array}\right.$$

(A.3) There exist positive constants $\varepsilon_0$, R and a nonnegative $C^2$-function $w(x)$ which tends to $\infty$ as $|x|\to\infty$ so that

$$\frac{\varepsilon^2}{2}\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}w(x)+\sum_{i=1}^{d}b^i(x)\frac{\partial w}{\partial x_i}(x)\leqq -1\qquad(|x|\geqq R,\ 0<\varepsilon\leqq\varepsilon_0).$$

*Remark.* — The assumptions (A.0), (A.1) and (A.2) imply the assumptions (A.0)′, (A.1), (A.2)′ and (A.3). Moreover the assumption (A.0)′ imply the regularity assumptions of M. V. Day [3] and (A.1), (A.2)′ and (A.3) imply the stability assumptions of M. V. Day [3].

The following proposition shows how our assumptions are strong.

PROPOSITION 1.1. — *Under the conditions* (A.0), (A.1) *and* (A.2), *Markov processes* $(X^\varepsilon(t), P_x)$, $0\leqq t$, $x\in\mathscr{R}^d$ *are positively recurrent and*

(1.1) $$|\mathfrak{x}(t)|\leqq\exp(-\lambda t)\,|\mathfrak{x}(0)|\qquad(t>0)$$

*where we put the quantity in* (A.2) $-\lambda$.

Before we state other results we give some notations. Let $V_T(y)$ denote the minimum of $\dfrac{1}{2}\displaystyle\int_0^T|\dot\varphi(t)-b(\varphi(t))|^2\,dt$ over all absolutely continuous functions $\varphi$ of $[0,T]$ to $\mathscr{R}^d$ such that $\varphi(0)=0$ and $\varphi(T)=y$. Let us put $V(y)=\text{infimum}\,(V_T(y);\ T>0)$. For a minimal path $\varphi$ of $V_T(y)$, let $Y^\varepsilon(t)$, $0\leqq t\leqq T$, be the solution of the following S.D.E.

(1.2) $$\left\{\begin{array}{l}dY^\varepsilon(t)=(b(Y^\varepsilon(t))-b(\varphi(t))+\dot\varphi(t))\,dt+\varepsilon\,dW(t)\\ \qquad\qquad Y^\varepsilon(0)=0\end{array}\right.$$

and let $Y^i(t)$, $0\leqq t\leqq T$, $(i=0,1,2,\ldots)$ be determined by the following formal expansion

(1.3) $$Y^\varepsilon(t)=\sum_{k=0}^{\infty}\varepsilon^k Y^k(t)\qquad(\text{as }\varepsilon\to 0).$$

To avoid confusion, we sometimes write $Y^{\varepsilon}(t)=Y^{\varepsilon,T}(t)$ and $Y^{i}(t)=Y^{i,T}(t)$.

The following theorems show how the meaning of the expansion (1.3) is strong.

THEOREM 1.2. — *Under the conditions* (A.0), (A.1) *and* (A.2), *for any natural number m and n*

$$(1.4)\quad \sup_{\substack{0<T\\ \varepsilon<1}}\left(\sup_{\substack{0<t<T\\ \varphi(T)=y\in\mathscr{R}^d}}\left(\varepsilon^{-(n+1)}\left\|Y^{\varepsilon,T}(t)-\sum_{i=0}^{n}\varepsilon^{i}Y^{i,T}(t)\right\|_m\right)\right)<+\infty$$

*where we denote by* $\|.\|_m$ $L^m$*-norm.*

THEOREM 1.3. — *Under the conditions* (A.0), (A.1) *and* (A.2), *for any natural number m, n and sufficiently large q*

$$(1.5)\quad \sup_{\substack{0<T\\ \varepsilon<1}}\left(\sup_{\substack{0<t<T\\ \varphi(T)=y\in\mathscr{R}^d}}\left(\varepsilon^{-(n+1)}\left\|\left|D^{q}\left(Y^{\varepsilon,T}(t)-\sum_{i=0}^{n}\varepsilon^{i}Y^{i,T}(t)\right)\right|_{HS}\right\|_m\right)\right)<+\infty$$

*where we denote by* $|.|_{HS}$ *Hilbert Schmidt norm.*

The following proposition shows the uniform integrability of exponential moments which usually appear in such arguments.

PROPOSITION 1.4. — *Under the conditions* (A.0), (A.1) *and* (A.2), *there exists a constant* $p>1$ *such that*

$$(1.6)\quad \overline{\lim_{T\to\infty}}\sup\left(E\left[\exp\left(p\left|\left\langle\dot{\varphi}(T)-b(\varphi(T)),\frac{Y^{\varepsilon}(T)-Y^{0}(T)-\varepsilon Y^{1}(T)}{\varepsilon^{2}}\right\rangle\right|\right)\right]\right)<+\infty$$

*where the supremum is over all* $\varepsilon(0<\varepsilon<1)$ *and all* $\varphi(T)=y\in\mathscr{R}^d$ *for which* $V(y)<4\lambda^3(d^6\|\partial^2 b\|_\infty^2)^{-1}$. *Here we denote by* $\langle .,.\rangle$ *the inner product in* $\mathscr{R}^d$ *and we put*

$$\|\partial^2 b\|_\infty=\sup\left(\left|\frac{\partial^2 b^i(x)}{\partial x_j\partial x_k}\right|;x\in\mathscr{R}^d,i,j,k=1,\ldots,d\right).$$

The following results are what we want.

THEOREM 1.5. — *Suppose the conditions* (A.0), (A.1) *and* (A.2). *Then there exist functions* $p^k(x)$ $(k\geq 0)$ *such that for all* $n\geq 0$

$$(1.7)\quad \overline{\lim_{\varepsilon\to 0}}\left|\varepsilon^{d}\exp(V(x)/\varepsilon^{2})\,p^{\varepsilon}(x)-\sum_{i=0}^{n}\varepsilon^{2i}p^{i}(x)\right|\varepsilon^{-2(n+1)}<+\infty$$

*uniformly for all x for which* $|x|$ *is sufficiently small.*

From theorem 3 in M. V. Day [3] and theorem 1.5, we have the following corollary.

COROLLARY 1.6. — *Under the conditions* (A.0)′, (A.1), (A.2)′ *and* (A.3), *there exist functions* $\mathfrak{p}^k(x)$ $(k \geqq 0)$ *such that for all* $n \geqq 0$

$$(1.8) \qquad \overline{\lim_{\varepsilon \to 0}} \left| \varepsilon^d \exp(V(x)/\varepsilon^2)\, \mathfrak{p}^\varepsilon(x) - \sum_{i=0}^{n} \varepsilon^{2i} \mathfrak{p}^i(x) \right| \varepsilon^{-2(n+1)} < +\infty$$

*uniformly for all x for which* $|x|$ *is sufficiently small.*

## 2. LEMMAS

In this section we state the lemmas necessary for the proofs of our results. The next lemma is technically essential.

LEMMA 1. — *Let* $f(t)$ *be a continuous function of* $[0, \infty)$ *to* $\mathscr{R}$. *If there exist a positive constant* $\alpha$ *and a measurable function g of* $[0, \infty)$ *to* $\mathscr{R}$ *such that for any s, t* $(0 \leqq s < t)$

$$(2.1) \qquad f(t) - f(s) \leqq -\alpha \int_s^t f(u)\, du + \int_s^t g(u)\, du,$$

*then we have*

$$(2.2) \qquad f(t) \leqq \exp(-\alpha t) \left( \int_0^t \exp(\alpha s)\, g(s)\, ds + f(0) \right).$$

*Proof.* — If $t=0$, then (2.2) holds. Suppose that there exists a positive constant $t_0$ such that (2.2) does not hold. Put

$$(2.3) \; s_0 = \max \left\{ u < t_0;\ f(u) \leqq \exp(-\alpha u) \left( \int_0^u \exp(\alpha v)\, g(v)\, dv + f(0) \right) \right\}.$$

Then for $u\,(s_0<u<t_0)$, (2.2) does not hold. Therefore we have

$$\begin{aligned}
\exp(-\alpha t_0)&\left(\int_0^{t_0}\exp(\alpha u)\,g(u)\,du+f(0)\right)\\
&\qquad-\exp(-\alpha s_0)\left(\int_0^{s_0}\exp(\alpha u)\,g(u)\,du+f(0)\right)\\
&\qquad\qquad<f(t_0)-f(s_0)\\
&<-\alpha\int_{s_0}^{t_0}\exp(-\alpha u)\left(\int_0^{u}\exp(\alpha v)\,g(v)\,dv+f(0)\right)du+\int_{s_0}^{t_0}g(u)\,du\\
&\qquad=\exp(-\alpha t_0)\left(\int_0^{t_0}\exp(\alpha u)\,g(u)\,du+f(0)\right)\\
&\qquad\qquad-\exp(-\alpha s_0)\left(\int_0^{s_0}\exp(\alpha u)\,g(u)\,du+f(0)\right)
\end{aligned}$$

which is a contradiction.

Q.E.D.

The following lemma can be proved by lemma 1 and is used to prove theorem 1.5.

LEMMA 2. — *Let $f^n(t)$ $(n=1,2,\ldots)$ be continuous functions of $[0,\infty)$ to $\mathscr{R}$ such that $f^n(0)=0$ $(n=1,2,\ldots)$. Suppose that there exist positive constants $c$ and $c_n\,(n=1,2,\ldots)$ such that for any $s$, $t$ $(0\leqq s<t)$ and natural number $n$*

$$f^n(t)-f^n(s)\leqq -cn\int_s^t f^n(u)\,du+c_n\int_s^t f^{n-1}(u)\,du \tag{2.4}$$

*where we put $f^0(t)\equiv f^0=\text{constant}$. Then we have*

$$f^n(t)\leqq f^0\,(c^n n!)^{-1}\,(1-\exp(-ct))^n\prod_{k=1}^{n}c_k \tag{2.5}$$

*for all $t\geqq 0$ and natural numbers $n$.*

*Proof.* — We prove by induction.

When $n=1$, by lemma 1 we have

$$f^1(t)\leqq\exp(-ct)\int_0^t\exp(cs)\,(c_1 f^0)\,ds=f^0\,c^{-1}\,(1-\exp(-ct))\,c_1.$$

Assume that (2.5) holds for $n=k$. Then we have, by lemma 1,

$$f^{k+1}(t) \leqq \exp(-c(k+1)t) \int_0^t \exp(c(k+1)s)\, c_{k+1} f^k(s)\, ds$$

$$\leqq \exp(-c(k+1)t) \int_0^t \exp(c(k+1)s)\, c_{k+1} f^0 \times (c^k k!)^{-1}(1-\exp(-cs))^k \prod_{j=1}^{k} c_j\, ds$$

$$= f^0 (c^{k+1}(k+1)!)^{-1}(1-\exp(-ct))^{k+1} \prod_{j=1}^{k+1} c_j.$$

Q.E.D.

We use the following lemma to prove theorem 1.3.

LEMMA 3. — *Let* $\psi(u)$ *be a measurable function of* $[0, \infty)$ *to* $\mathscr{R}$ *and let* $\mathrm{Y}(t)$ $(0 \leqq t < +\infty)$ *be the solution of the following differential equation:*

$$(2.6) \qquad \frac{d}{dt}\mathrm{Y}(t) = \partial b(\psi(t))\,\mathrm{Y}(t), \qquad \mathrm{Y}(0) = \mathrm{I}$$

*where* I *denote an* $(d, d)$*-identity matrix. Then under the conditions* (A.0) *and* (A.2), *for any positive constants* $s$, $t$ $(s<t)$, *we have*

$$(2.7) \qquad \|\mathrm{Y}(t)\mathrm{Y}(s)^{-1}\| \leqq d\{\exp(-\lambda(t-s))\}$$

*Proof.* — We put $\mathrm{Y}(t)\mathrm{Y}(s)^{-1} = \mathrm{Z}^{s,t} = (\mathrm{Z}_{ij}^{s,t})_{i,j=1}^d$.

Then for any $t_0$, $(t > t_0 > s)$, we have

$$\|\mathrm{Z}^{s,t}\|^2 - \|\mathrm{Z}^{s,t_0}\|^2 \leqq -2\lambda \int_{t_0}^t \|\mathrm{Z}^{s,u}\|^2\, du$$

since

$$\sum_{i,j=1}^d (\mathrm{Z}_{ij}^{s,t})^2 - \sum_{i,j=1}^d (\mathrm{Z}_{ij}^{s,t_0})^2 = \int_{t_0}^t 2 \sum_{i,j=1}^d \mathrm{Z}_{ij}^{s,u} \frac{d}{du}\mathrm{Z}_{ij}^{s,u}\, du,$$

$$\sum_{i,j=1}^d \mathrm{Z}_{ij}^{s,u}\frac{d}{du}\mathrm{Z}_{ij}^{s,u} = \sum_{i,j=1}^d \mathrm{Z}_{ij}^{s,u}\left(\sum_{k=1}^d \frac{\partial b^i}{\partial x_k}(\psi(u))\,\mathrm{Z}_{kj}^{s,u}\right)$$

$$= \sum_{j=1}^d \langle \mathrm{Z}_j^{s,u}, \partial b(\psi(u))\,\mathrm{Z}_j^{s,u} \rangle$$

$$\leqq -\lambda \sum_{j=1}^d |\mathrm{Z}_j^{s,u}|^2 = -\lambda \|\mathrm{Z}^{s,u}\|^2 \quad [\text{from (A.2)}]$$

where we put $\mathrm{Z}_j^{s,u} = (\mathrm{Z}_{ij}^{s,u})_{i=1}^d \in \mathscr{R}^d$ $(j=1, \ldots, d)$.

Therefore by lemma 1, we have

$$\| Z^{s,t} \|^2 \leqq \exp(-2\lambda(t-s)) \| Z^{s,s} \|^2 = \exp(-2\lambda(t-s))\, d^2.$$

Q.E.D.

*Remark.* — It is easy to see that $Y(t)^{-1}$ exists.

We use the next lemma when we prove theorem 1.2.

LEMMA 4. — *Under the conditions* (A.0) *and* (A.2), *for any natural number m and n,*

$$\text{(2.8)} \quad \sup(\sup(\| Y^{n,T}(t) \|_m;\, 0 \leqq t \leqq T,\, \varphi(T) = y \in \mathscr{R}^d);\, 0 \leqq T) < +\infty$$

*Proof.* — We prove by induction. Put $Y^{n,T}(t) = Y^n(t)$.

(When $n=1$.)

$$E[|Y^1(t)|^{2n}] \leqq ((2\lambda)^n n!)^{-1} \prod_{k=1}^{n} (kd + 2k(k-1)),$$

since

$$dY^1(t) = \partial b(Y^0(t))\, Y^1(t)\, dt + dW(t),$$

and by Ito's formula,

$$\begin{aligned} E[|Y^1(t)|^{2n}] - E[|Y^1(s)|^{2n}] &= 2n \int_s^t E[|Y^1(u)|^{2(n-1)} \langle Y^1(u), \partial b(Y^0(u))\, Y^1(u) \rangle]\, du \\ &\quad + (nd + 2n(n-1)) \int_s^t E[|Y^1(u)|^{2(n-1)}]\, du \\ &\leqq -2\lambda n \int_s^t E[|Y^1(u)|^{2n}]\, du \\ &\quad + (nd + 2n(n-1)) \int_s^t E[|Y^1(u)|^{2(n-1)}]\, du \end{aligned}$$

[from (A.2)], therefore from lemma 2, we have

$$E[|Y^1(t)|^{2n}] \leqq ((2\lambda)^n n!)^{-1} (1 - \exp(-2\lambda t))^n \prod_{k=1}^{n} (k\,d + 2k(k-1)).$$

(When $n \geqq 2$.) Since

$$Y^n(t) = \int_0^t [\partial b(Y^0(s))\, Y^n(s) + R_n(s)]\, ds,$$

where we put

$$R_n(s) = \frac{1}{2!} \sum_{\substack{i+j=n \\ i,j \geqq 1}} \partial^2 b(Y^0(s))\, Y^i(s) \otimes Y^j(s) + \ldots + \frac{1}{n!} \partial^n b(Y^0(s)) \overset{n}{\otimes} Y^1(s)$$

and for $a_i=(a_i^j)_{j=1}^d \in \mathscr{R}^d$ $(i=1, \ldots, n)$, we put

$$\partial^n b(x)\, a_1 \otimes \ldots \otimes a_n = \left( \sum_{i_1, \ldots, i_n=1}^{d} \frac{\partial^n b^i(x)}{\partial x_{i_1} \ldots \partial x_{i_n}} a_1^{i_1} \ldots a_n^{i_n} \right)_{i=1}^{d} \in \mathscr{R}^d,$$

we have, for any $s$, $t$ $(s<t)$ and $\alpha>0$,

$$(|Y^n(t)|^2+\alpha)^{1/2}-(|Y^n(s)|^2+\alpha)^{1/2}$$
$$=\int_s^t (|Y^n(u)|^2+\alpha)^{-1/2} \langle Y^n(u), \partial b(Y^0(u))\, Y^n(u)+R_n(u) \rangle\, du$$
$$\leqq -\lambda \int_s^t (|Y^n(u)|^2+\alpha)^{-1/2} |Y^n(u)|^2\, du + \int_s^t |R_n(u)|\, du.$$

Let $\alpha$ tend to 0 then we have

$$|Y^n(t)|-|Y^n(s)| \leqq -\lambda \int_s^t |Y^n(u)|\, du + \int_s^t |R_n(u)|\, du.$$

Therefore from lemma 1,

$$|Y^n(t)| \leqq \int_0^t \exp(-\lambda(t-s))\, |R_n(s)|\, ds.$$

Hence, by Hölder's inequality,

$$|Y^n(t)|^m \leqq \left( \int_0^t \exp\left( -\frac{m\lambda(t-s)}{2(m-1)} \right) ds \right)^{m-1}$$
$$\times \left( \int_0^t \exp\left( -\frac{m\lambda(t-s)}{2} \right) |R_n(s)|^m\, ds \right),$$

where we consider $\lambda(t-s)=\dfrac{\lambda(t-s)}{2}+\dfrac{\lambda(t-s)}{2}$ and $m^{-1}+(m/m-1)^{-1}=1$, which completes the proof.

At last we give the next lemma.

LEMMA 5. — *Under the assumptions* (A.0) *and* (A.2), *we have, for the Malliavin's covariance*

$$\left\langle \frac{D(Y^\varepsilon(T)-Y^0(T))}{\varepsilon}, \frac{D(Y^\varepsilon(T)-Y^0(T))}{\varepsilon} \right\rangle_{HS},$$

$$(2.9) \quad \sup\left( \left\| \left\langle \frac{D(Y^\varepsilon(T)-Y^0(T))}{\varepsilon}, \frac{D(Y^\varepsilon(T)-Y^0(T))}{\varepsilon} \right\rangle_{HS} \right\|; \; T>0, \varepsilon>0, \varphi(T)=y \in \mathscr{R}^d \right) < +\infty$$

$$(2.10) \quad \underline{\lim}_{T \to \infty} \inf(\lambda_y^{\varepsilon, T}; \varepsilon>0, y \in \mathscr{R}^d) > 0$$

*where we denote by* $\lambda_y^{\varepsilon, \mathrm{T}}$ *the minimal eigenvalue of*

$$\left\langle \frac{\mathrm{D}(\mathrm{Y}^\varepsilon(\mathrm{T})-\mathrm{Y}^0(\mathrm{T}))}{\varepsilon}, \frac{\mathrm{D}(\mathrm{Y}^\varepsilon(\mathrm{T})-\mathrm{Y}^0(\mathrm{T}))}{\varepsilon} \right\rangle_{\mathrm{HS}} \qquad \textit{for} \quad y=\varphi(\mathrm{T}) \in \mathscr{R}^d.$$

*Proof* [proof of (2.9)]. – Let $\mathrm{Y}^\varepsilon(t)$ be the solution of (2.6) for $\psi(u)=\mathrm{Y}^\varepsilon(u)$ and put $\mathrm{Z}^{s,t,\varepsilon}=(\mathrm{Z}_{ij}^{s,t,\varepsilon})_{i,j=1}^d=\mathrm{Y}^\varepsilon(t)\,\mathrm{Y}^\varepsilon(s)^{-1}$ $(s \leqq t \leqq \mathrm{T})$. Since

$$\left\langle \frac{\mathrm{D}(\mathrm{Y}^\varepsilon(\mathrm{T})-\mathrm{Y}^0(\mathrm{T}))}{\varepsilon}, \frac{\mathrm{D}(\mathrm{Y}^\varepsilon(\mathrm{T})-\mathrm{Y}^0(\mathrm{T}))}{\varepsilon} \right\rangle_{\mathrm{HS}} = \int_0^{\mathrm{T}} \mathrm{Z}^{s,\mathrm{T},\varepsilon}(\mathrm{Z}^{s,\mathrm{T},\varepsilon})^* \, ds,$$

$$\begin{aligned} \left| \sum_{k=1}^d \int_0^{\mathrm{T}} \mathrm{Z}_{ik}^{s,\mathrm{T},\varepsilon} \mathrm{Z}_{jk}^{s,\mathrm{T},\varepsilon} \, ds \right| &\leqq \int_0^{\mathrm{T}} \| \mathrm{Z}^{s,\mathrm{T},\varepsilon} \|^2 \, ds \\ &\leqq \int_0^{\mathrm{T}} d^2 \exp(-2\lambda(\mathrm{T}-s))\, ds \quad \text{(from lemma 3).} \end{aligned}$$

Hence we conclude

$$\sup\left( \left\| \left\langle \frac{\mathrm{D}(\mathrm{Y}^\varepsilon(\mathrm{T})-\mathrm{Y}^0(\mathrm{T}))}{\varepsilon}, \frac{\mathrm{D}(\mathrm{Y}^\varepsilon(\mathrm{T})-\mathrm{Y}^0(\mathrm{T}))}{\varepsilon} \right\rangle_{\mathrm{HS}} \right\| ; \; \mathrm{T}>0, \varepsilon>0, \varphi(\mathrm{T})=y \in \mathscr{R}^d \right) \leqq \frac{d^3}{2\lambda}.$$

[proof of (2.10)].

We put $\mathrm{B}_\varepsilon^{s,\mathrm{T}}=\mathrm{Z}^{s,\mathrm{T},\varepsilon}(\mathrm{Z}^{s,\mathrm{T},\varepsilon})^*$ and denote by $\lambda_{1,\varepsilon}^{s,\mathrm{T}}$ the minimal eigenvalue of $\mathrm{B}_\varepsilon^{s,\mathrm{T}}$ and $\|\partial b\|_\infty = \sup\left( \left| \frac{\partial b^i}{\partial x_j}(x) \right| ; i,j=1, \ldots, d, x \in \mathscr{R}^d \right)$.

Then $\lambda_{1,\varepsilon}^{s,\mathrm{T}} \geqq \exp(-2d\|\partial b\|_\infty(\mathrm{T}-s))$, since

$$\lambda_{1,\varepsilon}^{s,\mathrm{T}} = \inf\{ |\mathrm{Z}^{s,\mathrm{T},\varepsilon} e| ; e=(e_i)_{i=1}^d \in \mathscr{R}^d \text{ and } |e|=1 \}$$

and for $t_1$, $t_2$ $(s<t_1,\ t_2<\mathrm{T})$,

$$\begin{aligned} |\mathrm{Z}^{s,t_2,\varepsilon} e|^2 - |\mathrm{Z}^{s,t_1,\varepsilon} e|^2 &= \int_{t_1}^{t_2} \sum_{i=1}^d 2\left( \sum_{k=1}^d \mathrm{Z}_{ik}^{s,u,\varepsilon} e_k \right)\left( \sum_{j=1}^d \frac{d}{du} \mathrm{Z}_{ij}^{s,u,\varepsilon} e_j \right) du \\ &= 2\int_{t_1}^{t_2} \langle \mathrm{Z}^{s,u,\varepsilon} e, \partial b(\mathrm{Y}^\varepsilon(u))\, \mathrm{Z}^{s,u,\varepsilon} e \rangle \, du \end{aligned}$$

therefore $\frac{d}{dt}|\mathrm{Z}^{s,t,\varepsilon} e|^2 \geqq -2d\|\partial b\|_\infty |\mathrm{Z}^{s,t,\varepsilon} e|^2$ and

$$|\mathrm{Z}^{s,t,\varepsilon} e|^2 \geqq \exp(-2d\|\partial b\|_\infty(t-s))\,|e|^2.$$

Hence

$$\begin{aligned}&\varliminf_{T\to\infty}\ \inf(\lambda_y^{\varepsilon,T};\ \varepsilon>0,\ y\in\mathscr{R}^d)\\&\geqq \varliminf_{T\to\infty}\int_0^T \exp(-2d\,\|\partial b\|_\infty(T-s))\,ds\\&=(2d\,\|\partial b\|_\infty)^{-1}.\end{aligned}$$

Q.E.D.

## 3. PROOFS OF MAIN RESULTS

In this section we give the proofs of our results.

*Proof of proposition* 1.1. [proof of the positively recurrent property of $X^\varepsilon(t)$]. — Since

$$\begin{aligned}\langle b(x),x\rangle&=\langle b(x),x\rangle-\langle b(0),x\rangle\\&=\int_0^1\langle \partial b(ux)\,x,x\rangle\,du\leqq -\lambda\,|x|^2\quad\text{[from (A.2)]},\end{aligned}$$

$X^\varepsilon(t)$ is positively recurrent (*cf.* Has'minskii [5]), [proof of $|\mathfrak{x}(t)|\leqq\exp(-\lambda t)\,|\mathfrak{x}(0)|$]. Since

$$|\mathfrak{x}(t)|^2-|\mathfrak{x}(s)|^2=2\int_s^t\langle \mathfrak{x}(u),b(\mathfrak{x}(u))\rangle du\leqq -2\lambda\int_s^t|\mathfrak{x}(u)|^2\,du,$$

$$|\mathfrak{x}(t)|^2\leqq\exp(-2\lambda t)\,|\mathfrak{x}(0)|^2\quad\text{(from lemma 1).}$$

Q.E.D.

Next we prove theorem 1.2.

*Proof of theorem* 1.2. — We put $R_n^\varepsilon(t)=Y^\varepsilon(t)-\sum_{i=0}^n\varepsilon^i Y^i(t)$.

For any $t\,(0\leqq t\leqq T)$ and $\alpha>0$, since

$$\begin{aligned}R_n^\varepsilon(t)=&\int_0^t\left[b(Y^\varepsilon(s))-b\left(\sum_{i=0}^n\varepsilon^i Y^i(s)\right)\right]ds\\&+\int_0^t\left[b\left(\sum_{i=0}^n\varepsilon^i Y^i(s)\right)-\sum_{k=0}^n\frac{1}{k!}\,\frac{d^k}{d\varepsilon^k}b\left(\sum_{i=0}^n\varepsilon^i Y^i(s)\right)\bigg|_{\varepsilon=0}\varepsilon^k\right]ds,\end{aligned}$$

we have for any $s$, $t\,(0 \leqq s < t \leqq T)$,

$$
\begin{aligned}
&(|R_n^\varepsilon(t)|^2+\alpha)^{1/2}-(|R_n^\varepsilon(s)|^2+\alpha)^{1/2}\\
&\quad=\int_s^t \left\langle R_n^\varepsilon(u), \frac{d}{du} R_n^\varepsilon(u) \right\rangle (|R_n^\varepsilon(u)|^2+\alpha)^{-1/2}\,du\\
&\quad=\int_s^t (|R_n^\varepsilon(u)|^2+\alpha)^{-1/2} \Bigg\langle R_n^\varepsilon(u), b(Y^\varepsilon(u))\\
&\qquad -b\left(\sum_{i=0}^n \varepsilon^i Y^i(u)\right) \Bigg\rangle\,du+\int_s^t (|R_n^\varepsilon(u)|^2+\alpha)^{-1/2}\\
&\qquad \times\Bigg\langle R_n^\varepsilon(u), b\left(\sum_{i=0}^n \varepsilon^i Y^i(u)\right)\\
&\qquad -\sum_{i=0}^n \frac{1}{k!}\,\frac{d^k}{d\varepsilon^k} b\left(\sum_{i=0}^n \varepsilon^i Y^i(u)\right)\Bigg|_{\varepsilon=0} \varepsilon^k \Bigg\rangle\,du\\
&\quad=\int_s^t (|R_n^\varepsilon(u)|^2+\alpha)^{-1/2} \Bigg\langle R_n^\varepsilon(u), \partial b\Bigg(\sum_{i=0}^n \varepsilon^i Y^i(u)\\
&\qquad +\theta(u)\, R_n^\varepsilon(u)\Bigg) R_n^\varepsilon(u)\Bigg\rangle\,du\\
&\qquad +\int_s^t (|R_n^\varepsilon(u)|^2+\alpha)^{-1/2} \Bigg\langle R_n^\varepsilon(u),\\
&\qquad \frac{1}{(n+1)!}\,\frac{d^{n+1}}{d\gamma^{n+1}} b\left(\sum_{i=0}^n \gamma^i Y^i(u)\right)\Bigg|_{\gamma=\varepsilon\tilde\theta(u)} \varepsilon^{n+1}\Bigg\rangle\,du\\
&\quad\leqq -\lambda \int_s^t (|R_n^\varepsilon(u)|^2+\alpha)^{-1/2}\, |R_n^\varepsilon(u)|^2\,du\\
&\qquad +\int_s^t (|R_n^\varepsilon(u)|^2+\alpha)^{-1/2}\, |R_n^\varepsilon(u)|\\
&\qquad \left|\frac{1}{(n+1)!}\,\frac{d^{n+1}}{d\gamma^{n+1}} b\left(\sum_{i=0}^n \gamma^i Y^i(u)\right)\right|_{\gamma=\varepsilon\tilde\theta(u)} \varepsilon^{n+1}\,\Big|\,du
\end{aligned}
$$

for some $\theta(u)$, $\tilde\theta(u)$ such that $0 \leqq \theta(u),\ \tilde\theta(u) \leqq 1$ $(0 \leqq u \leqq T)$.

Let $\alpha \to 0$, then we have, for any $s$, $t\,(0 \leqq s < t \leqq T)$

$$
\begin{aligned}
|R_n^\varepsilon(t)|-|R_n^\varepsilon(s)| &\leqq -\lambda \int_s^t |R_n^\varepsilon(u)|\,du\\
&\quad +\int_s^t \left|\frac{1}{(n+1)!}\,\frac{d^{n+1}}{d\gamma^{n+1}} b\left(\sum_{i=0}^n \gamma^i Y^i(u)\right)\right|_{\gamma=\varepsilon\tilde\theta(u)} |\,\varepsilon^{n+1}\,du.
\end{aligned}
$$

Hence from lemma 1,

$$|R_n^\varepsilon(t)| \leqq \exp(-\lambda t) \int_0^t \exp(\lambda s) \times \left| \frac{1}{(n+1)!} \frac{d^{n+1}}{d\gamma^{n+1}} b\left(\sum_{i=0}^n \gamma^i Y^i(s)\right) \right|_{\gamma=\varepsilon\tilde{\theta}(s)} |\varepsilon^{n+1} ds.$$

which completes the proof by lemma 4.

We prove theorem 1.3 by induction.

*Proof of theorem* 1.3. — It is easy to see that for any natural number $n$ there exists a natural number $q(n)$ such that if $q \geqq q(n)$, then

$$D^q Y^i(h_1, \ldots, h_q)(t) = 0$$
$$(0 \leqq i \leqq n, 0 \leqq t \leqq T, h_1, \ldots, h_q \in H)$$

where H denote Cameron Martin space. Hence we only have to prove

$$(3.1) \quad \sup_{0 \leqq T, \varepsilon < 1} (\sup(\varepsilon^{-(n+1)} \| | D^q Y^{\varepsilon, T}(t) |_{HS} \|_m; \; 0 \leqq t \leqq T, \varphi(T) = y \in \mathscr{R}^d)) < +\infty$$

for sufficiently large $q$. From the following proposition, (3.1) holds.

Q.E.D.

PROPOSITION 3.1. — *For each natural number $n$, $t (0 \leqq t \leqq T)$ and $h_1, \ldots, h_n \in H$,*

$$\varepsilon^{-n} D^n Y^{\varepsilon, T}(h_1, \ldots, h_n)(t) = \int_0^t ds_1 \ldots \int_0^t ds_n g^{n, T}(s_1, \ldots, s_n; t) \dot{h}_1(s_1) \otimes \ldots \otimes \dot{h}_n(s_n)$$

*for some*

$$g^{n, T}(s_1, \ldots, s_n; t) = (g^{n, i, T}_{i_1, \ldots, i_n}(s_1, \ldots, s_n; t))^d_{i, i_1, \ldots, i_n = 1}$$

*where we put*

$$g^{n, T}(s_1, \ldots, s_n; t) \dot{h}_1(s_1) \otimes \ldots \otimes \dot{h}_n(s_n) = \left( \sum_{i_1, \ldots, i_n = 1}^d g^{n, i, T}_{i_1, \ldots, i_n}(s_1, \ldots, s_n; t) \times \dot{h}_1^{i_1}(s_1) \ldots \dot{h}_n^{i_n}(s_n) \right)_{i=1}^d \in \mathscr{R}^d.$$

*Moreover there exist nonrandom constants $C_n$ $(n = 1, 2, \ldots)$ such that*

$$\sup_{\substack{0 \leqq T \\ \varepsilon < 1}} \left( \sup_{\substack{0 \leqq t \leqq T \\ \varphi(T) = y \in \mathscr{R}^d}} \left( \int_0^t ds_1 \ldots \int_0^t ds_n \times \sum_{i, i_1, \ldots, i_n = 1}^d | g^{n, i, T}_{i_1, \ldots, i_n}(s_1, \ldots, s_n; t) |^2 \right) \right) < C_n.$$

*Proof of proposition* 3.1. — It is easy to see that

$$(3.2)\qquad DY^{\varepsilon}(h)(t)=\varepsilon h(t)+\int_0^t \partial b(Y^{\varepsilon}(s))\,DY^{\varepsilon}(h)(s)\,ds \qquad (h\in H)$$

$$(3.3)\qquad D^2 Y^{\varepsilon}(h_1,h_2)(t)=\int_0^t [\partial b(Y^{\varepsilon}(s))\,D^2 Y^{\varepsilon}(h_1,h_2)(s) + \partial^2 b(Y^{\varepsilon}(s))\,DY^{\varepsilon}(h_1)(s)\otimes DY^{\varepsilon}(h_2)(s)]\,ds\,(h_1,h_2\in H).$$

Inductively, in the same way, we can show that for all $n\geqq 2$

$$(3.4)\qquad \varepsilon^{-n} D^n Y^{\varepsilon}(\mathbf{h})(t)=\int_0^t [\partial b(Y^{\varepsilon}(s))\,\varepsilon^{-n} D^n Y^{\varepsilon}(\mathbf{h})(s) + f^n(\varepsilon^{-1} DY^{\varepsilon}(s),\ldots,\varepsilon^{-(n-1)} D^{n-1} Y^{\varepsilon}(s))(\mathbf{h})]\,ds$$

where we put $\mathbf{h}=(h_1,\ldots,h_n)$ and $f^n$ is a polynomial of

$$\varepsilon^{-1} DY^{\varepsilon}(s),\ldots,\varepsilon^{-(n-1)} D^{n-1} Y^{\varepsilon}(s)$$

with coefficient $\partial b(Y^{\varepsilon}(s)),\ldots,\partial^n b(Y^{\varepsilon}(s))$ and

$$\partial^k b(x)=\left(\frac{\partial^k b^i(x)}{\partial x_{i_1}\ldots\partial x_{i_k}}\right)^d_{i,i_1,\ldots,i_k=1} \qquad (1\leqq k\leqq n).$$

(When $n=1$.) We can put $g^{1,T}(s,t)=Z^{s,t,\varepsilon}$. In fact, it is easy to see that $\varepsilon^{-1} DY^{\varepsilon}(h)(t)=\int_0^t g^{1,T}(s,t)\,\dot{h}(s)\,ds$ from (3.2) and we have

$$\sup_{\substack{0\leqq T\\ \varepsilon<1}}\left(\sup_{\substack{0\leqq t\leqq T\\ \varphi(T)=y\in\mathscr{R}^d}} \left(\int_0^t \sum_{i,j=1}^d |g_j^{1,i,T}(s,t)|^2\,ds\right)\right)\leqq d^2(2\lambda)^{-1},$$

since

$$\sum_{i,j=1}^d |g_j^{1,i,T}(s,t)|^2=\|Z^{s,t,\varepsilon}\|^2\leqq d^2\exp(-2\lambda(t-s))$$

from lemma 3.

(When $n=2$.) We put, for $i$, $i_1$, $i_2$ $(=1,\ldots,d)$,

$$g^{2,i,T}_{i_1,i_2}(u_1,u_2;t)=\int_{u_1\vee u_2}^t \langle (g^{1,T}(s,t))^i,\partial^2 b(Y^{\varepsilon}(s)) \times (g^{1,T}(u_1,s))_{i_1}\otimes(g^{1,T}(u_2,s))_{i_2}\rangle\,ds$$

where we put, for $i$, $j$ $(=1,\ldots,d)$,

$$(g^{1,T}(u,s))^i=(g_k^{1,i,T}(u,s))^d_{k=1}$$

and

$$(g^{1,T}(u,s))_j=(g_j^{1,k,T}(u,s))^d_{k=1}.$$

Then from (3.3), it is easy to see that

$$\varepsilon^{-2} D^2 Y^\varepsilon(h_1, h_2)(t) = \int_0^t du_1 \int_0^t du_2\, g^{2,T}(u_1, u_2; t)\, \dot{h}_1(u_1) \otimes \dot{h}_2(u_2),$$

$$\sup_{\substack{0 \leqq T \\ \varepsilon < 1}} \Bigg( \sup_{\substack{0 \leqq t \leqq T \\ \varphi(T) = y \in \mathscr{R}^d}} \Bigg( \int_0^t du_1 \int_0^t du_2 \sum_{i, i_1, i_2 = 1}^{d} | g^{2, i, T}_{i_1, i_2}(u_1, u_2; t) |^2 \Bigg)\Bigg) < C_2$$

for some nonrandom constant $C_2$. In fact

$$\sum_{i, i_1, i_2 = 1}^{d} | g^{2, i, T}_{i_1, i_2}(u_1, u_2; t) |^2$$

$$= \sum \Bigg| \int_{u_1 \vee u_2}^{t} \langle (g^{1,T}(s,t))^i, \partial^2 b(Y^\varepsilon(s))$$
$$\times (g^{1,T}(u_1, s))_{i_1} \otimes (g^{1,T}(u_2, s))_{i_2} \rangle ds \Bigg|^2$$
$$= \sum \int_{u_1 \vee u_2}^{t} \langle (g^{1,T}(s_1,t))^i, \partial^2 b(Y^\varepsilon(s_1))$$
$$\times (g^{1,T}(u_1, s_1))_{i_1} \otimes (g^{1,T}(u_2, s_1))_{i_2} \rangle ds_1$$
$$\times \int_{u_1 \vee u_2}^{t} \langle (g^{1,T}(s_2,t))^i, \partial^2 b(Y^\varepsilon(s_2))$$
$$\times (g^{1,T}(u_1, s_2))_{i_1} \otimes (g^{1,T}(u_2, s_2))_{i_2} \rangle ds_2$$
$$\leqq \Bigg( \int_{u_1 \vee u_2}^{t} \Bigg( \sum \langle (g^{1,T}(s,t))^i, \partial^2 b(Y^\varepsilon(s))$$
$$\times (g^{1,T}(u_1, s))_{i_1} \otimes (g^{1,T}(u_2, s))_{i_2} \rangle^2 \Bigg)^{1/2} ds \Bigg)^2$$

where $\Sigma$ is over all $i$, $i_1$, $i_2 (=1, \ldots, d)$ and

$$| \langle (g^{1,T}(s,t))^i, \partial^2 b(Y^\varepsilon(s))$$
$$\times (g^{1,T}(u_1, s))_{i_1} \otimes (g^{1,T}(u_2, s))_{i_2} \rangle |$$
$$\leqq | (g^{1,T}(s,t))^i | . | \partial^2 b(Y^\varepsilon(s))$$
$$\times (g^{1,T}(u_1, s))_{i_1} \otimes (g^{1,T}(u_2, s))_{i_2} |$$
$$\leqq | (g^{1,T}(s,t))^i | . \| \partial^2 b(Y^\varepsilon(s)) \| . |$$
$$\times (g^{1,T}(u_1, s))_{i_1} | . | (g^{1,T}(u_2, s))_{i_2} |$$

where we put

$$\| \partial^2 b(y) \| = \Bigg( \sum_{i, j, k = 1}^{d} \Bigg| \frac{\partial^2 b^i(y)}{\partial x_j \partial x_k} \Bigg|^2 \Bigg)$$

and hence

$$\int_0^t du_1 \int_0^t du_2 \sum_{i, i_1, i_2 = 1}^{d} |g_{i_1, i_2}^{2, i, T}(u_1, u_2; t)|^2$$

$$\leqq \int_0^t du_1 \int_0^t du_2 \|\partial^2 b\|_\infty d^3 \left( \int_{u_1 \vee u_2}^t \|g^{1, T}(s, t)\| \times \|g^{1, T}(u_1, s)\| . \|g^{1, T}(u_2, s)\| ds \right)^2$$

$$\leqq \|\partial^2 b\|_\infty d^3 \sup_{\substack{0 \leqq T \\ \varepsilon < 1}} \left( \left[ \sup_{\substack{0 \leqq t \leqq T \\ \varphi(T) \in \mathscr{R}^d}} \left( \int_0^t \|g^{1, T}(s, t)\| ds \right)^2 \right] \times \left[ \sup_{\substack{0 \leqq t \leqq T \\ \varphi(T) \in \mathscr{R}^d}} \left( \int_0^t \|g^{1, T}(s, t)\|^2 ds \right)^2 \right] \right)$$

$$\leqq \|\partial^2 b\|_\infty d^9 (4 \lambda^4)^{-1}$$

since $\|g^{1, T}(s, t)\| \leqq d \exp(-\lambda(t - s))$.

(Assume that proposition 3.1 holds when $n = k$.) From (3.4)

$$\varepsilon^{-(k+1)} D^{k+1} Y^\varepsilon (h_1, \ldots, h_{k+1})(t) = \int_0^t Z^{s, t, \varepsilon} f^{k+1} (\varepsilon^{-1} D Y^\varepsilon (s), \ldots, \varepsilon^{-k} D^k Y^\varepsilon (s))(h_1, \ldots, h_{k+1}) ds$$

and $f^{k+1}$ can be written as the following;

$$\int_0^s ds_1 \ldots \int_0^s ds_{k+1} \tilde{g}^{k, T}(s_1, \ldots, s_{k+1}; s) \dot{h}_1(s_1) \otimes \ldots \otimes \dot{h}_{k+1}(s_{k+1})$$

for some

$$\tilde{g}^{k, T}(s_1, \ldots, s_{k+1}; s) = (\tilde{g}_{i_1, \ldots, i_{k+1}}^{k, i, T}(s_1, \ldots, s_{k+1}; s))_{i, i_1, \ldots, i_{k+1} = 1}^d$$

such that

$$\sup_{\substack{0 \leqq T \\ \varepsilon < 1}} \left( \sup_{\substack{0 \leqq t \leqq T \\ \varphi(T) = y \in \mathscr{R}^d}} \left( \int_0^t ds_1 \ldots \int_0^t ds_{k+1} \times \sum |\tilde{g}_{i_1, \ldots, i_{k+1}}^{k, i, T}(s_1, \ldots, s_{k+1}; t)|^2 \right) \right) < \tilde{C}_{k+1}$$

for some nonrandom constant $\tilde{C}_{k+1}$, where $\sum$ is over all $i$, $i_1, \ldots, i_{k+1}\,(=1, \ldots, d)$. Hence we have to prove that the following quantity is bounded by some nonrandom constant $C_n$

$$\sup_{\substack{0\leqq T\\ \varepsilon<1}}\left(\sup_{\substack{0\leqq t\leqq T\\ \varphi(T)=y\in\mathscr{R}^d}}\left(\int_1\sum\left|\int_2\sum_{j=1}^{d} g_j^{1,i,T}(u,t)\tilde{g}_{i_1,\ldots,i_{k+1}}^{k,j,T}(s_1,\ldots,s_{k+1};u)\right|^2\right)\right)$$

where $\int_1$ is over all $s_1, \ldots, s_{k+1}$ $(0\leqq s_1, \ldots, s_{k+1}\leqq t)$ and $\int_2$ is over all $u\,(s_1\vee\ldots\vee s_{k+1}\leqq u\leqq t)$ and $\sum$ is over all $i, i_1, \ldots, i_{k+1}$ $(=1, \ldots, d)$. In fact

$$\int_1\sum\left|\int_2\sum_{j=1}^{d} g_j^{1,i,T}(u,t)\tilde{g}_{i_1,\ldots,i_{k+1}}^{k,j,T}(s_1,\ldots,s_{k+1};u)\right|^2$$

$$\leqq d^{k+2}\int_1\left(\int_2\|g^{1,T}(u,t)\|.\|\tilde{g}^{k,T}(s_1,\ldots,s_{k+1};u)\|\right)^2$$

$$\leqq d^{k+2}\sup\left(\left[\sup\left\{\left(\int_0^t\|g^{1,T}(s,t)\|\,ds\right)^2;0\leqq t\leqq T,\varphi(T)\in\mathscr{R}^d\right\}\right]\right.$$

$$\times\left[\sup\left\{\int_0^t ds_1,\ldots\int_0^t ds_{k+1}\|\tilde{g}^{k,T}(s_1,\ldots,s_{k+1};t)\|^2;\right.\right.$$

$$\left.\left.\left.0\leqq t\leqq T,\varphi(T)\in\mathscr{R}^d\right\}\right];0\leqq T,\varepsilon<1\right)\leqq d^{k+4}\lambda^{-2}\tilde{C}_{k+1}.$$

Q.E.D.

Next we prove proposition 1.4.

*Proof of proposition* 1.4. – Since

$$\langle\dot{\varphi}(T)-b(\varphi(T)),Y^\varepsilon(T)-Y^0(T)-\varepsilon Y^1(T)\rangle$$

$$=\int_0^T\langle\dot{\varphi}(s)-b(\varphi(s)),b(Y^\varepsilon(s))-b(Y^0(s))$$

$$-\partial b(Y^0(s))(Y^\varepsilon(s)-Y^0(s))\rangle\,ds,$$

$$|\langle\dot{\varphi}(T)-b(\varphi(T)),Y^\varepsilon(T)-Y^0(T)-\varepsilon Y^1(T)\rangle|\,\varepsilon^{-2}$$

$$\leqq\int_0^T|\dot{\varphi}(s)-b(\varphi(s))|\,d^2\,\|\partial^2 b\|_\infty 2^{-1}(|Y^\varepsilon(s)-Y^0(s)|\,\varepsilon^{-1})^2\,ds.$$

From this we have

$$\mathrm{E}\,[\exp(p\,|\langle\dot\varphi(\mathrm{T})-b(\varphi(\mathrm{T})),\mathrm{Y}^{\varepsilon}(\mathrm{T})-\mathrm{Y}^{0}(\mathrm{T})-\varepsilon\,\mathrm{Y}^{1}(\mathrm{T})\rangle\,|\,\varepsilon^{-2})]$$

$$\leqq\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}pd^{2}\,\|\partial^{2}b\|_{\infty}\right)^{n}\mathrm{E}\left[\left(\int_{0}^{\mathrm{T}}|\dot\varphi(s)-b(\varphi(s))|\right.\right.$$

$$\left.\left.\times|(\mathrm{Y}^{\varepsilon}(s)-\mathrm{Y}^{0}(s))\,\varepsilon^{-1}|^{2}\,ds\right)^{n}\right]$$

$$\leqq\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}pd^{2}\,\|\partial^{2}b\|_{\infty}\right)^{n}\left(\int_{0}^{\mathrm{T}}|\dot\varphi(s)-b(\varphi(s))|\right.$$

$$\left.\times\|(\mathrm{Y}^{\varepsilon}(s)-\mathrm{Y}^{0}(s))\,\varepsilon^{-1}\|_{2n}^{2}\,ds\right)^{n}$$

$$\leqq\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}pd^{2}\,\|\partial^{2}b\|_{\infty}\right)^{n}\left(\prod_{k=1}^{n}\frac{2k+d-2}{2\lambda}\right)\left(\frac{4}{\lambda}\mathrm{V}_{\mathrm{T}}(y)\right)^{n/2}$$

$$\leqq\sum_{n=0}^{\infty}\left(\frac{1}{2}pd^{2}\,\|\partial^{2}b\|_{\infty}\frac{d}{2\lambda}\left(\frac{4}{\lambda}\mathrm{V}_{\mathrm{T}}(y)\right)^{1/2}\right)^{n}<+\infty$$

if $\mathrm{V}_{\mathrm{T}}(y)<4\lambda^{3}(p^{2}d^{6}\,\|\partial^{2}b\|_{\infty}^{2})^{-1}$, since

$$\mathrm{E}[|(\mathrm{Y}^{\varepsilon}(t)-\mathrm{Y}^{0}(t))\,\varepsilon^{-1}|^{2n}]\leqq(2\lambda)^{-n}\prod_{k=1}^{n}(2k+d-2)\tag{3.5}$$

$$\int_{0}^{\mathrm{T}}|\dot\varphi(s)-b(\varphi(s))|\,ds\leqq 2(\lambda^{-1}\mathrm{V}_{\mathrm{T}}(y))^{1/2}.\tag{3.6}$$

Under the conditions (A.0), (A.1) and (A.2), the limit of $\mathrm{V}_{\mathrm{T}}(y)$ as $\mathrm{T}\to\infty$ exists and equals to $\mathrm{V}(y)$ for each $y$ (*cf.* M. I. Freidlin and A. D. Wentzell [4]). Therefore we only have to prove (3.5) and (3.6).

[Proof of (3.5).] We put $\mathrm{Z}^{\varepsilon}(t)=\mathrm{Y}^{\varepsilon}(t)-\mathrm{Y}^{0}(t)$. Since

$$\mathrm{Y}^{\varepsilon}(t)-\mathrm{Y}^{0}(t)=\varepsilon\,\mathrm{W}(t)+\int_{0}^{\mathrm{T}}[b(y^{\varepsilon}(s))-b(\mathrm{Y}^{0}(s))]\,ds,$$

by Ito's formula, we have

$$|Z^\varepsilon(t)|^{2n}-|Z^\varepsilon(s)|^{2n}$$
$$=2n\int_s^t |z^\varepsilon(u)|^{2(n-1)}\langle Z^\varepsilon(u), \varepsilon\, dW(u)+\{b(Y^\varepsilon(u))-b(Y^0(u))\}\rangle du$$
$$+\varepsilon^2(nd+2n(n-1))\int_s^t |Z^\varepsilon(u)|^{2(n-1)}du$$
$$\leqq 2n\int_s^t |z^\varepsilon(u)|^{2(n-1)}\langle Z^\varepsilon(u), \varepsilon dW(u)\rangle - 2n\lambda\int_s^t |Z^\varepsilon(u)|^{2n}du$$
$$+\varepsilon^2(nd+2n(n-1))\int_s^t |Z^\varepsilon(u)|^{2(n-1)}du \quad \text{[from (A.2)].}$$

Hence

$$E[|(Y^\varepsilon(t)-Y^0(t))\varepsilon^{-1}|^{2n}]\leqq(2\lambda)^{-n}\prod_{k=1}^{n}(2k+d-2),$$

since for any $s$, $t$ $(0\leqq s<t\leqq T)$,

$$E[|Z^\varepsilon(t)|^{2n}]-E[|Z^\varepsilon(s)|^{2n}]$$
$$\leqq -2n\lambda\int_s^t E[|Z^\varepsilon(u)|^{2n}]du+\varepsilon^2(nd+2n(n-1))$$
$$\times\int_s^t E[|Z^\varepsilon(u)|^{2(n-1)}]du$$

and from lemma 2 we have

$$E[|(Y^\varepsilon(t)-Y^0(t))\varepsilon^{-1}|^{2n}]\leqq(2\lambda)^{-n}(n!)^{-1}$$
$$\times(1-\exp(-2\lambda t))^n\prod_{k=1}^{n}(2k(k-1)+kd)$$
$$\leqq(2\lambda)^{-n}\prod_{k=1}^{n}(2k+d-2).$$

[Proof of (3.6).] Since we have

$$|\dot\varphi(t)-b(\varphi(t))|^2-|\dot\varphi(0)|^2$$
$$=\int_0^t 2\left\langle \dot\varphi(s)-b(\varphi(s)), \frac{d}{ds}(\dot\varphi(s)-b(\varphi(s)))\right\rangle ds$$
$$=-2\int_0^t \langle \dot\varphi(s)-b(\varphi(s)), \partial b(\varphi(s))^*(\dot\varphi(s)-b(\varphi(s))\rangle ds$$
$$\geqq 2\lambda\int_0^t |\dot\varphi(s)-b(\varphi(s))|^2 ds \quad \text{[from (A.2)]}$$

where we use Euler's equation for $\varphi(t)$ and from this for $\rho>0$,

$$\int_0^T |\dot\varphi(s)-b(\varphi(s))|^2 (|\dot\varphi(s)-b(\varphi(s))|^2+\rho)^{-1/2}\,ds$$

$$\leq \int_0^T |\dot\varphi(s)-b(\varphi(s))|^2 \left(2\lambda\int_0^s |\dot\varphi(u)-b(\varphi(u))|^2\,du+\rho\right)^{-1/2} ds$$

$$=\left[\lambda^{-1}\left(2\lambda\int_0^s |\dot\varphi(u)-b(\varphi(u))|^2\,du+\rho\right)^{1/2}\right]_{s=0}^{T}$$

$$=\lambda^{-1}[(4\lambda V_T(y)+\rho)^{1/2}-\rho^{1/2}]\to\left(\frac{4}{\lambda}V_T(y)\right)^{1/2} \qquad (\rho\to 0),$$

$$\int_0^T |\dot\varphi(s)-b(\varphi(s))|\,ds\leq \underline{\lim}_{\rho\to 0}\int_0^T |\dot\varphi(s)-b(\varphi(s))|^2$$

$$\times(|\dot\varphi(s)-b(\varphi(s))|^2+\rho)^{-1/2}\,ds\leq 2(\lambda^{-1}V_T(y))^{1/2}.$$

Q.E.D.

Now let us prove theorem 1.5.

*Proof of theorem* 1.5. – From lemma 5, theorems 1.2, 1.3 and proposition 1.4, by S. Watanabe's theory, for the transition probability density $p^\varepsilon(t,x,y)$ of $X^\varepsilon(t)$, there exist functions $p^i(T,0,y)$ $(i\geq 0)$ and constants $C^i_{T,y}$ $(i\geq 0)$ such that for all $n\,(=0,1,\ldots)$

$$(3.7)\quad \varepsilon^{-2(n+1)}\left|\varepsilon^d\exp(V_T(y)/\varepsilon^2)\,p^\varepsilon(T,0,y)-\sum_{i=0}^{n}\varepsilon^{2i}p^i(T,0,y)\right|<C^i_{T,y}$$

if $V_T(y)<4\lambda^3(d^6\|\partial^2 b\|_\infty^2)^{-1}$. Moreover for any $\alpha$ $(0<\alpha<1)$,

$$(3.8)\quad \begin{cases}\overline{\lim}_{T\to\infty}\sup(C^i_{T,y};V_T(y)<\alpha[4\lambda^3(d^6\|\partial^2 b\|_\infty^2)^{-1}])<+\infty\\ \qquad (i=0,1,\ldots).\end{cases}$$

Since $X^\varepsilon(t)$ is positively recurrent from proposition 1.1, the limits of $p^\varepsilon(T,0,y)$ as $T\to\infty$ exist for each $\varepsilon>0$ and $y\in\mathscr{R}^d$ (*cf.* Has'minskii [5]) and from this the limits of $p^i(T,0,y)$ as $T\to\infty$ exist for all $i\,(=0,1,\ldots)$ and $y$ for which $V(y)<4\lambda^3(d^6\|\partial^2 b\|_\infty^2)^{-1}$. In fact, for $n=0$, since

$$\varepsilon^{-2}\left|\varepsilon^d\exp(V_T(y)/\varepsilon^2)\,p^\varepsilon(T,0,y)-p^0(T,0,y)\right|<C^0_{T,y},$$

$$-\varepsilon^2\,\overline{\lim}_{T\to\infty}C^0_{T,y}+\varepsilon^d\exp(V(y)/\varepsilon^2)\,p^\varepsilon(y)\leq\underline{\lim}_{T\to\infty}p^0(T,0,y)$$

$$\leq\overline{\lim}_{T\to\infty}p^0(T,0,y)\leq\varepsilon^2\,\overline{\lim}_{T\to\infty}C^0_{T,y}+\varepsilon^d\exp(V(y)/\varepsilon^2)\,p^\varepsilon(y).$$

Therefore

$$(3.9) \quad \lim_{\varepsilon \to 0} \varepsilon^d \exp(V(y)/\varepsilon^2)\, p^\varepsilon(y) = \underline{\lim}_{T \to \infty} p^0(T, 0, y) = \overline{\lim}_{T \to \infty} p^0(T, 0, y).$$

In the same way we can prove inductively that the limits of $p^i(T, 0, y)$ as $T \to \infty$ exist for all $i (=0, 1, \ldots)$.

Q.E.D.

At last we prove corollary 1.6.

*Proof of corollary* 1.6. — Let $X_1^\varepsilon(t)$, $0 \leqq t$ be the solution of the following stochastic differential equation:

$$(3.10) \quad \begin{cases} dX_1^\varepsilon(t) = b_1(X_1^\varepsilon(t))\, dt + \varepsilon\, dW(t) \\ X_1^\varepsilon(0) = x_1 \qquad (x_1 \in \mathscr{R}^d) \end{cases}$$

where $b_1(x)$ satisfies the conditions (A.0), (A.1) and (A.2) and $b_1(x) = b(x)$ if $|x| < r$ for some $r > 0$. Then from theorem 3 in M. V. Day [3], we have the following:

$$(3.11) \quad \overline{\lim}_{\varepsilon \to 0} \sup_{y \in \mathscr{K}} (\varepsilon \log | p^\varepsilon(x) - p_1^\varepsilon(x) |) \leqq -\min(V(x); |x| = r)$$

for any compact subset $\mathscr{K}$ of the set $(x; |x| < r)$ where we denote by $p_1^\varepsilon(x)$ the invariant density of $X_1^\varepsilon(t)$ and from theorem 1.5, if $r$ is sufficiently small, (1.7) holds for $p^\varepsilon(x) = p_1^\varepsilon(x)$ uniformly for $|x| < r$. Therefore for any $\alpha < \min(V(x); |x| = r)$, (1.7) also holds for $p^\varepsilon(x)$ uniformly for $x$ of the set $\{y; V(y) \leqq \alpha\}$, since

$$\begin{aligned} \varepsilon^{-2(n+1)} \left| \varepsilon^d \exp(V(y)/\varepsilon^2)\, p^\varepsilon(y) - \sum_{i=0}^{n} \varepsilon^{2i} p^i(y) \right| & \\ \leqq \varepsilon^{-2(n+1)} | \varepsilon^d \exp(V(y)/\varepsilon^2)(p^\varepsilon(y) - p_1^\varepsilon(y)) | & \\ + \varepsilon^{-2(n+1)} \left| \varepsilon^d \exp(V(y)/\varepsilon^2)\, p_1^\varepsilon(y) - \sum_{i=0}^{n} \varepsilon^{2i} p^i(y) \right| & \end{aligned}$$

and the first term is bounded by

$$\varepsilon^{d-2(n+1)} \exp\left( \frac{\alpha - \min(V(x); |x| = r)}{2\varepsilon^2} \right)$$

for sufficiently small $\varepsilon$, uniformly for $x \in \{y; V(y) \leqq \alpha\}$.

Q.E.D.

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