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Non equilibrium fluctuations for a zero range process

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RÉSUMÉ. — Nous étudions le champ de fluctuation de la densité d'un processus « zero-range » à plus proche voisin symétrique unidimensionnel hors de l'équilibre. Nous en prouvons la convergence en loi vers un processus de Ornstein-Uhlenbeck généralisé. Comme l'équation hydrodynamique de notre modèle est non linéaire, il faut prouver une version hors d'équilibre du principe de Gibbs-Boltzman afin de démontrer le théorème principal. Ce principe a été introduit pour la première fois par Brox and Rost pour l'étude des fluctuations à l'équilibre d'une grande classe de processus « zero-range ». Notre résultat est obtenu en appliquant ensuite la théorie de Holley et Stroock.

ABSTRACT. — We study the non equilibrium density fluctuation field of a one dimensional symmetric nearest neighbors zero range process, proving that it converges in law to a generalized Ornstein Uhlenbeck process. Since the hydrodynamical equation is non linear, to accomplish our main theorem we need to prove, for our model, a non equilibrium version of

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the Gibbs-Boltzmann principle. This was first introduced by Brox and Rost to study the equilibrium fluctuations for a large class of zero range models. Our result is then obtained by applying Holley and Stroock's theory.

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0. INTRODUCTION

There is by now a well established theory concerning the hydrodynamical features of stochastic systems of infinitely many interacting particles (*cf.* [6], and the references quoted in). Nevertheless, one still misses non trivial examples where all "equilibrium" and "non equilibrium" hydrodynamical properties can be proven to hold. Here, "non trivial" refers to a non constant diffusion coefficient in the hydrodynamical equation. In particular, the non equilibrium density fluctuation field has been studied, as far as we know, only in models where the diffusion coefficient is constant [21].

The goal of this paper is to complete such description of the hydrodynamical properties for a particular zero range model (*cf.* [9]), studying the density fluctuation field in non equilibrium situations. This provides the kind of example we were just mentioning, since in this case one has a non constant diffusion coefficient, as we shall see.

The model is the so called symmetric, nearest neighbors, zero range process, with constant intensity; more precisely, it is a Markov process $\xi(t)$, $t \geqq 0$, taking values on $\mathbb{N}^{\mathbb{Z}}$ ($\xi(u,t)=\xi(t)(u)$, $u \in \mathbb{Z}$, denotes the number of particles at site u, at time t) whose generator L acts on bounded cylindrical functions f as:

$$\mathrm{L}f(\xi)=\frac{1}{2}\sum_{u \in \mathbb{Z}} \mathbf{1}(\xi(u)>0)\,[f(\xi^{u,\,u+1})+f(\xi^{u,\,u-1})-2f(\xi)] \qquad (0.1\,a)$$

where for $\xi(u)>0$

$$\begin{aligned} \xi^{u,\,v}(z) &= \xi(z)+1 \quad \text{if} \quad z=v \\ &= \xi(z)-1 \quad \text{if} \quad z=u \\ &= \xi(z) \quad \text{if} \quad z \neq u, v. \end{aligned} \qquad (0.1\,b)$$

Questions on the existence and the ergodic properties of such Markov processes were considered in [15] (*cf.* also [16]); in [1] it is shown that the extremal invariant measures are reversible, and given by μ_ρ, $0<\rho<+\infty$, where μ_ρ is the product measure on $\mathbb{N}^{\mathbb{Z}}$ with

$$\mu_\rho(\xi(u)=k)=\frac{\rho^k}{(1+\rho)^{k+1}},\qquad k\in\mathbb{N} \tag{0.2}$$

From the results on local equilibrium [9] we can say that

$$\frac{\partial\rho}{\partial\tau}=\frac{1}{2}\frac{\partial}{\partial r}\left((1+\rho)^{-2}\frac{\partial}{\partial r}\rho\right) \tag{0.3}$$

is the "hydrodynamical equation" for this process, and so

$$D(\rho)=(1+\rho)^{-2} \tag{0.4}$$

represents the "bulk diffusion coefficient".

In fact, as in [9], let μ^ε, $\varepsilon\in(0,1]$ be a suitable class of initial measures, with a profile $\rho(r)$, (*i.e.*, around $\varepsilon^{-1}r$ μ^ε is approximately $\mu_{\rho(r)}$ as $\varepsilon\to 0$) where ρ is a smooth function bounded from below and above in $(0,+\infty)$. Then at time $\varepsilon^{-2}\tau$ and around $\varepsilon^{-1}r$ we have approximately $\mu_{\rho(r,\tau)}$ (as $\varepsilon\to 0$), where $\rho(r,\tau)$ is the solution of (0.3) with $\rho(r,0)=\rho(r)$. Also, as seen in [9], such solution $\rho(r,\tau)$ is "explicitely" given by

$$\rho(r,\tau)=\frac{1-p(z(r,\tau),\tau)}{p(z(r,\tau),\tau)} \tag{0.5a}$$

with $p(r,\tau)$, $z(r,\tau)$ given by

$$\frac{\partial}{\partial\tau}p=\frac{1}{2}\frac{\partial^2}{\partial r^2}p,\qquad p(z(r,0),0)=(1+\rho(r))^{-1} \tag{0.5b}$$

$$r=\int_{z(0,\tau)}^{z(r,\tau)}p(r',\tau)\,dr' \tag{0.5c}$$

$z(\tau)=z(0,\tau)$ given by

$$\frac{dz(\tau)}{d\tau}=\frac{1}{p(z(\tau),\tau)}\,j(z(\tau),\tau) \tag{0.5d}$$

with

$$j(r,\tau)=-\frac{\partial}{\partial r}p(r,\tau).$$

Before getting to the specific results derived in this paper let us remark that the "change of variables" transforming eq. (0.5*b*) into (0.3) can be thought of as coming from the isomorphism (at microscopic level) between our zero range model and the nearest nesghbors (n. n.) symmetric simple exclusion process which at time $t=0$ has a marked particle at the origin. This correspondance is a key fact also for the results proven here and which will be stated below, after fixing some basic notation.

Notations and definitions

When working on the simple exclusion process, S.E.P., we shall use $x, y, \ldots$ to denote the sites in $\mathbb{Z}$ and $\eta(., t)$ to denote the configuration at time t. $S(x)$ denotes the spatial translation by x (i.e. $S(x)\eta(y)=\eta(x+y)$), and we let $S(x)f(\eta)=f(S_{(x)}\eta)$ for $f: \mathbb{N}^{\mathbb{Z}} \to \mathbb{R}$. Similarly, when working on the zero range process we shall use $u, v \ldots$ to denote the sites, and again $S(u)$ will denote the spatial shift on $\mathbb{N}^{\mathbb{Z}}$.

$\mathscr{S}(\mathbb{R})$ denotes the space of rapidly decreasing test functions, and $\mathscr{S}'(\mathbb{R})$ its dual, i.e., the space of Schwartz distributions.

An immediate consequence of the results proven in [9] is the following:

THEOREM 0.1. — *For $\varepsilon \in (0, 1]$ let μ^ε be a probability on $\mathbb{N}^{\mathbb{Z}}$ which verifies:*

(i)
$$\lim_{\varepsilon \to 0} \sup_u |\mu^\varepsilon(S(u)f) - \mu_{\mathscr{S}(\varepsilon u)}(f)| = 0 \tag{0.6}$$

for any bounded cylindrical function f, where $\rho(.)$ is a C^∞ function with $0 < \rho_- \leqq \rho(r) \leqq \rho_+ < +\infty$ for all r;

(ii) *Assumptions* 1.2 (B) *of Section* 1.

Let us define the density field by

$$X_\tau^\varepsilon(\varphi) = \varepsilon \sum_u \varphi(\varepsilon u)\, \xi(u, \varepsilon^{-2}\tau) \tag{0.7}$$

for $\varphi \in \mathscr{S}(\mathbb{R})$, $\tau \geqq 0$.

Then as $\varepsilon \to 0$ the processes (X_τ^ε) converge in law on $D([0, +\infty), \mathscr{S}'(\mathbb{R}))$ to the deterministic process given by $X_\tau(\varphi) = \int \varphi(r)\rho(r, \tau)\, dr$ where $\rho(., .)$ is the solution of (0.3) *with $\rho(., 0) = \rho(.)$.*

The above theorem can be seen as a law of large numbers for this interacting system and it is now natural to look at the corresponding central limit theorem, i.e., to investigate the fluctuations of $X_\tau^\varepsilon(.)$ around its average: an interesting question also if the initial measure were some μ_ρ $\rho > 0$ (i.e., in equilibrium). In [3] the equilibrium case is solved for a larger class of zero range models. There have been extensions to other systems like some exclusion with speed change processes [7], and interacting brownian particles [25].

The main purpose of this paper is to study such fluctuation processes also in non equilibrium situations. Before we state this precisely it may be convenient, for sake of completeness, to recall the equilibrium case, which is a particular case of the results in [3].

THEOREM 0.2 [3]. — *Let* $(Y_\tau^\varepsilon, \tau \geqq 0)$ *be the* $\mathscr{S}'(\mathbb{R})$*-valued process defined by*

$$Y_\tau^\varepsilon(\varphi) = \sqrt{\varepsilon} \sum_u \varphi(\varepsilon u)\,[\xi(u, \varepsilon^{-2}\tau) - \rho] \tag{0.8}$$

for $\varphi \in \mathscr{S}(\mathbb{R})$ *and* $\tau \geqq 0$. *Let* $\mathscr{P}^\varepsilon$ *be its law on* $D([0, +\infty), \mathscr{S}'(\mathbb{R}))$, *when* μ_ρ *is the law of* $\xi(., 0)$.

Then, as ε *tends to zero,* $\mathscr{P}^\varepsilon$ *converges weakly to a probability* $\mathscr{P}$ *supported by* $C([0, +\infty), \mathscr{S}'(\mathbb{R}))$ *which is the law of a stationary generalized Ornstein-Uhlenbeck process, namely the* $\mathscr{S}'(\mathbb{R})$*-valued Gaussian process with zero averages and covariances*

$$\mathscr{P}(Y_0(\varphi)\, Y_\tau(\psi)) = \iint dr\, dr'\, \varphi(r)\, \psi(r')\, \mathscr{X}(\rho) \times \frac{1}{\sqrt{2\pi D(\rho)\tau}} \exp(-(r-r')^2/2D(\rho)\tau) \tag{0.9}$$

where $D(\rho)$ *is given in equation* (0.4) *and*

$$\mathscr{X}(\rho) = \mu_\rho(\xi(0)(\xi(0) - \rho)) \tag{0.10}$$

$D(\rho)$ *is the bulk diffusion coefficient given by* (0.4), *and* $Y_\tau(.)$ *denotes the canonical process on* $C([0, +\infty), \mathscr{S}'(\mathbb{R}))$.

Remark. — This result relates the covariance of the density fluctuation field in equilibrium to the bulk diffusion coefficient $D(\rho)$, a parameter determined by the non equilibrium evolution [*cf.* (0.3)]. In the mathematical physics literature such relationship is called "Fluctuation Dissipation Theorem" since it connects the non equilibrium dissipative features of the system to its equilibrium fluctuations. A heuristic argument to understand such connections is the following (based on the so called "linearized theory"): We have in (0.9)

$$\mathscr{P}(Y_0(\varphi)\,Y_\tau(\psi)) = \lim_{\varepsilon \downarrow 0} \mathbb{P}_\rho\Big(\sqrt{\varepsilon}\sum_u \varphi(\varepsilon u)(\xi(u,0)-\rho) \times \sqrt{\varepsilon}\sum_v \psi(\varepsilon v)(\xi(v,\varepsilon^{-2}\tau)-\rho)\Big)$$

$$= \lim_{\varepsilon \downarrow 0} \mathbb{P}_\rho\Big(\sum_u \varphi(\varepsilon u)(\xi(u,0)-\rho)\,.\,\varepsilon\sum_v \psi(\varepsilon v)\,\xi(v,\varepsilon^{-2}\tau)\Big)$$

$$= \lim_{\varepsilon \downarrow 0} \mathbb{P}_\rho\Big(\sum_u \varphi(\varepsilon u)(\xi(u,0)-\rho)\,.\,X_\tau^\varepsilon(\psi)\Big) \tag{0.11}$$

$\mathbb{P}_\rho$ denoting the probability on some suitable probability space $(\Omega, \mathscr{A}, \mathbb{P}_\rho)$ where we have constructed the zero range process such that $\xi(.,0)$ is distributed as μ_ρ. Thus the covariance can be tought as the expectation of the density field $X_\tau^\varepsilon(\psi)$ if the "initial measure" of the zero range process were the signed measure

$$\mu_\rho\,.\sum_u \varphi(\varepsilon u)(\xi(u)-\rho) \tag{0.12}$$

Now, if we consider the probability measure

$$C_\lambda\,\mu_\rho\,.\exp\Big[\lambda\sum_u \varphi(\varepsilon u)(\xi(u)-\rho)\Big] \tag{0.13}$$

where C_λ is a suitable normalizing constant, and $\lambda>0$, and we expand it around $\lambda=0$, we see that the first order term is given by (0.12). On the other side, the time zero density profile determined by such probability measure is

$$\rho(r,0)=\rho+\lambda\,\mathscr{X}(\rho)\,\varphi(r)+o(\lambda)\equiv\rho+\lambda\,\delta\rho(r,0)+o(\lambda) \tag{0.14}$$

Then we would expect the r. h. s. of (0.11) to be of the form $\int \psi(r)\,\rho(r,\tau)\,dr$ with $\rho(r,\tau)$ being the solution of (0.3), with initial condition given by (0.14). If $\delta\rho(r,\tau)$ denotes the first order term in the λ-expansion of such profile $\rho(r,\tau)$, it should satisfy the linearized diffusion equation

$$\begin{aligned} &\frac{\partial}{\partial\tau}\delta\rho=\frac{1}{2}D(\rho)\frac{\partial^2}{\partial r^2}\delta\rho \\ &\delta\rho(r,0)=\mathscr{X}(\rho)\,\varphi(r) \end{aligned} \tag{0.15}$$

and so

$$\delta\rho(r,\tau)=\frac{1}{\sqrt{2\pi D(\rho)\tau}}\int\varphi(r')\,e^{-(r-r')^2/2\,D(\rho)\,\tau}\,dr'$$

which will then lead to (0.9) after inserting this into (0.14) and (0.11).

We may now state the main theorem of this paper.

THEOREM 0.3. — *Let μ^ε be as in Theorem 0.1, and let us consider the zero range process $\xi(.,t)$, with $\xi(.,0)$ distributed as μ^ε, and let $Y^\varepsilon_\tau(.)$ be its fluctuation density field, given by*

$$Y^\varepsilon_\tau(\varphi)=\sqrt{\varepsilon}\sum_u\varphi(\varepsilon u)\,(\xi(u,\varepsilon^{-2}\tau)-<\xi(u,\varepsilon^{-2}\tau)>)\tag{0.16}$$

for $\varphi\in\mathscr{S}(\mathbb{R})$, $\tau\geqq 0$, and where $\langle . \rangle$ denotes the expected value. Let $\mathscr{P}^\varepsilon$ denote the law of $(Y^\varepsilon_\tau, \tau\geqq 0)$ on $D([0,+\infty),\mathscr{S}'(\mathbb{R}))$.

Then $\mathscr{P}^\varepsilon$ converges weakly to a probability measure $\mathscr{P}$ supported by $C([0,+\infty),\mathscr{S}'(\mathbb{R}))$, and uniquely determined by the condition that under $\mathscr{P}$ the canonical process $Y_\tau(\varphi)$ satisfies (i) *and* (ii) *below.*

(i) *For any $\Phi\in C_0^\infty(\mathbb{R})$ and $\varphi\in\mathscr{S}(\mathbb{R})$*

$$\Phi(Y_\tau(\varphi))-\int_0^\tau\Phi'(Y_{\tau'}(\varphi))\,Y_{\tau'}(A_{\tau'}\varphi)\,d\tau'-\frac{1}{2}\int_0^\tau\|B_{\tau'}\varphi\|^2\,\Phi''(Y_{\tau'}(\varphi))\,d\tau'\tag{0.17}$$

is a $\mathscr{P}$-martingale with respect to the canonical filtration $\mathscr{F}_\tau=\sigma$-field generated by $(Y_{\tau'}(\varphi):0\leqq\tau'\leqq\tau,\ \varphi\in\mathscr{S}(\mathbb{R}))$, for $\tau\geqq 0$, A_τ and B_τ being defined in equations (0.19) *and* (0.20) *below.*

(ii) *Under $\mathscr{P}$, $Y_0(.)$ is Gaussian with $\mathscr{P}(Y_0(\varphi))=0$, and*

$$\mathscr{P}(Y_0(\varphi)\,Y_0(\psi))=\int\varphi(r)\,\psi(r)\,\mathscr{X}(\rho(r))\,dr\tag{0.18}$$

for each φ, $\psi\in\mathscr{S}(\mathbb{R})$.

A_τ, B_τ, $\mathscr{X}(\rho)$ are defined as follows:

$$A_\tau\varphi(r)=\frac{1}{2}(1+\rho(r,\tau))^{-2}\,\varphi''(r)\tag{0.19}$$

$$\|B_\tau\varphi\|^2=\int\rho(r,\tau)\,(1+\rho(r,\tau))^{-1}\,|\varphi'(r)|^2\,dr,\tag{0.20}$$

and $\mathscr{X}(\rho)$ *is given in* (0.10).

Remark. — From (0.17) and (0.18) we can compute the covariances

$$C_{\tau,\sigma}=E(Y_\tau(\varphi)\,Y_\sigma(\psi));$$

like in the equilibrium case, the different times covariance satisfies the linearized hydrodynamical equation. However, in contrast to what happens in equilibrium, the limiting fluctuation field distribution has a non trivial equal time covariance $C_{\tau,\tau}$. Its evolution is described in Theorem 0.3, and more in general by the Holley and Stroock's theory [11], on which this theorem is based.

As just mentioned, the proof of Theorems 0.2 and 0.3 makes strong use of the Holley and Stroock's theory of generalized Ornstein-Uhlenbeck processes ([11], [12], [13]). We also refer to [14], [17], [18], [19] for general criteria on the convergence of $\mathscr{S}'(\mathbb{R})$-valued processes. In some models, when the diffusion coefficient is constant, the theory of Holley and Stroock may be directly applied to prove convergence of the fluctuation density fields to generalized Ornstein-Uhlenbeck processes (*cf.*, for instance, [11], [12], [21], and section 6 of [6]). But this is not the case for density dependent diffusion coefficients, as considered here. The way to overcome this point was first understood by Hermann Rost [23]. It is based on the very natural idea that the fluctuation fields of non-conserved quantities change on a much faster time scale than the "conserved" ones; since here density is the only conserved quantity it is reasonable to expect that on a time integral only "the component" (projection) along the density fluctuation field survives. A first application is given in [3], where such property, the "Gibbs-Boltzmann principle", is proven to hold for a class of zero range models in equilibrium. In [7] and [25] the results are extended to some exclusion with speed change systems and interacting brownian particles, respectively.

The validity of such "principle" in non equilibrium may even be questionable. Our next theorem is an indication in the positive, since we show it for the non equilibrium symmetric simple exclusion and the zero range process under study here. It is important to stress that this last system has a density dependent diffusion coefficient, and the verification of the Gibbs-Boltzmann principle allows us to apply the theory of Holley and Stroock, in order to prove Theorem 0.3 (1).

(1) More recently the Gibbs Boltzmann principle has been proven in a model where the macroscopic equation is non linear ([27], [28]).

THEOREM 0.4 (Gibbs-Boltzmann principle). — *Let* μ^ε *be as in Theorem* 0.1 *and let* $(\Omega, \mathscr{A}, \mathbb{P}_\varepsilon)$ *be a probability space where we have constructed the zero range process* $(\xi(.,t), t \geqq 0)$ *with* $\xi(.,0)$ *distributed as* μ^ε. *Let* f *be a cylindrical function on* $\mathbb{N}^{\mathbb{Z}}$, *satisfying moreover the technical condition of being in* $\mathscr{C}$, cf. *Definition* 1.3. *For* $\varphi \in \mathscr{S}(\mathbb{R})$ *we set:*

$$Y_\tau^\varepsilon(f;\varphi) = \sqrt{\varepsilon}\sum_u \varphi(\varepsilon u)\,[S(u)f(\xi(\varepsilon^{-2}\tau)) - \mathbb{P}_\varepsilon(S(u)f(\xi(\varepsilon^{-2}\tau)))] \quad (0.21)$$

Then, for every $0<\tau'<\tau$:

$$\lim_{T\uparrow+\infty}\lim_{\varepsilon\downarrow 0}\sup_{\tau'\leqq\varepsilon^2 t\leqq\tau} \mathbb{P}_\varepsilon\left(\left|\frac{1}{T}\int_t^{t+T}[Y^\varepsilon_{\varepsilon^2 t'}(f;\varphi) - Y^\varepsilon_{\varepsilon^2 t'}((a\varphi)_{\varepsilon^2 t'})]\,dt'\right|\right) = 0 \quad (0.22)$$

with

$$(a\varphi)_\tau(r) = \varphi(r)\,a(f;r,\tau) \quad (0.23)$$

and

$$a(f;r,\tau) = \frac{d}{d\rho}\mu_\rho(f)\Big|_{\rho=\rho(r,\tau)}, \quad (0.24)$$

where $\rho(r,\tau)$ *is the solution of equation* (0.3), *with* $\rho(r,0)=\rho(r)$.

THEOREM 0.4′ (Gibbs-Boltzmann principle). — *Let* ν^ε *be a family of probability measures on* $\{0,1\}^{\mathbb{Z}}$ *verifying Assumptions* 1.2 (A) *of Section* 1, *and let* $(\Omega, \mathscr{A}, \mathbb{P}_\varepsilon)$ *be a probability space where we have constructed* $\{\eta(x,t), t\geqq 0, x\in\mathbb{Z}\}$, *a symmetric exclusion process with* $\eta(.,0)$ *distributed as* ν^ε. *Let* $\tilde{f}$ *be a cylindrical function on* $\{0,1\}^{\mathbb{Z}}$ *and let us define, for* $\varphi\in\mathscr{S}(\mathbb{R})$:

$$\tilde{Y}_\tau^\varepsilon(\tilde{f};\varphi) = \sqrt{\varepsilon}\sum_x \varphi(\varepsilon x)\,[S(x)\tilde{f}(\eta(\varepsilon^{-2}\tau)) - \mathbb{P}_\varepsilon(S(x)\tilde{f}(\eta(\varepsilon^{-2}\tau)))] \quad (0.25)$$

Then, for every $0<\tau'<\tau$:

$$\lim_{T\uparrow+\infty}\lim_{\varepsilon\downarrow 0}\sup_{\tau'\leq\varepsilon^2 t\leq\tau}\mathbb{P}_\varepsilon\left|\frac{1}{T}\int_t^{t+T}[(\tilde{Y}^\varepsilon_{\varepsilon^2 t'}(\mathcal{J};\varphi)-\tilde{Y}^\varepsilon_{\varepsilon^2 t'}((\tilde{a}\varphi)_{\varepsilon^2 t'}))]\,dt'\right|^2=0 \quad (0.26)$$

where

$$(\tilde{a}\varphi)_\tau(r)=\varphi(r)\,\tilde{a}(\mathcal{J};r,\tau), \tag{0.27}$$

$$\tilde{Y}^\varepsilon_\tau(\varphi)=\tilde{Y}^\varepsilon_\tau(g;\varphi) \qquad \textit{for} \quad g(\eta)=\eta(0),$$

and

$$\tilde{a}(\mathcal{J};r,\tau)=\frac{d}{dp}\nu_p(\mathcal{J})\big|_{p=p(r,\tau)}, \tag{0.28}$$

with $p(r,\tau)$ *being the solution of equation* (0.5 *b*) *for* $p(.,0)=p(.)$, *and* ν_p *the product of Bernoulli measures with* $\nu_p(\eta(0))=p$.

1. PRELIMINARIES

As mentioned in the introduction we shall make strong use of the isomorphism between our zero range model (Z.R.P.) and the one dimensional symmetric simple exclusion process, with nearest neighbors jumps (S.E.P.) with a tagged particle. For this let us recall a few definitions ([9], [10]), and set some notations.

1.1. Notations and definitions

We refer to [16] for the definition of more general exclusion processes including our S.E.P. The realization of the S.E.P. which we now present will be particularly useful for us, and may also be taken as its definition. ($x, y, \ldots$ will denote a site referring to S.E.P. and $u, v, \ldots$ will denote the label of a particle in S.E.P.)

Stirring process. At time $t=0$ each site $x\in\mathbb{Z}$ is occupied by a stirring particle which will keep the label x. The stirring process is defined as in [9], and $Y(x,t)$ denotes the position at time t of the "particle x" ($Y(x,0)=x$ by definition).

Labelled S.E.P. For $t=0$ we are given an initial configuration of particles in $\mathbb{Z}$ with at most one particle at each site. For each path of the stirring process we define the following evolution: each particle moves like the stirring particle which is at the same site unless this would determine an exchange of positions between particles and in this case it does not move. We will only consider initial configurations having a particle at $x=0$, called the zero-particle. For $u>0$ ($u<0$) the u-particle will be the u-th particle at the right (left) of the origin. From the definition it is immediate that the evolution preserves the order, *i. e.*, $u<v$ implies $q(u, t)<q(v, t)$ for all t, with $q(u, t)$ denoting the position of the u-particle at time t, for $u \in \mathbb{Z}$. For this evolution we let $\eta(x, t)=1$ (0) if the site x is occupied (empty) at time t, and write $\eta(x)$ for $\eta(x, 0)$. The process $\{\eta(x, t)\}$, constructed on a suitable probability space $(\Omega, \mathscr{A}, \mathbb{P})$ is then called S.E.P.

Notation. When ν is a probability measure on the space of initial configurations we let $(\Omega, \mathscr{A}, \mathbb{P}_\nu)$ be a suitable probability space where we have constructed the random variables $\{\eta(x, t), q(x, t), Y(x, t); x, u, t\}$ with $\eta(., 0)$ distributed as ν. When ν carries a superscript ν^ε, $\varepsilon \in (0, 1]$ we shall write simply $\mathbb{P}_\varepsilon$ if no confusion is possible.

Zero range process (Z.R.P.). The zero range process whose pre-generator is given by (0.1 *a*), (0.1 *b*) can be realized within the labelled S.E.P. just defined by setting

$$\xi(u, t)=q(u+1, t)-q(u, t)-1 \tag{1.1}$$

for $u \in \mathbb{Z}$, $t \geqq 0$, as one can easily check. We will write $\xi(u)=\xi(u, 0)$ and let μ, μ^ε, etc. denote the law of $\{\xi(u), u \in \mathbb{Z}\}$ on $\mathbb{N}^{\mathbb{Z}}$.

The correspondence given by (1.1) transforms the equilibrium measure μ_ρ ($\rho>0$) for the Z.R.P. into the equilibrium measure ν_p for the S.E.P. conditioned to $[\eta(0)=1]$, where

$$p=(1+\rho)^{-1}. \tag{1.2}$$

(ν_p is the product measure on $\{0, 1\}^{\mathbb{Z}}$ with $\nu_p(\eta(x)=1)=p$.)

1.2. Assumptions

In all theorems stated in the introduction we have the following assumptions on the initial measures μ^ε (Z.R.P.) or ν^ε (S.E.P.).

A. ν^ε will be the conditional probability $\bar{\nu}^\varepsilon(.\,|\,\eta(0)=1)$ where $\bar{\nu}^\varepsilon$ is a probability measure on $\{0,1\}^{\mathbb{Z}}$ which satisfy:

(i) (Finite range). There exists $R>0$ so that for $n\geqq 2$ and $x_1,\ldots,x_n$ with $|x_i-x_j|>R$ for $i\neq j$ then $\eta(x_1),\ldots,\eta(x_n)$ are conditionally independent given $(\eta(y): y\notin\{x_1,\ldots,x_n\})$.

(ii) (Uniformly bounded interactions.) There exist p', p'' such that for all x and all $\{\eta(y), y\neq x\}$

$$0<p'\leqq\bar{\nu}^\varepsilon(\eta(x)\,|\,\eta(y),\, y\neq x)\leqq p''<1.$$

(iii) (Uniform decay of correlations.) There exist b_1, $b_2>0$ so that for all k_1, $k_2\geqq 1$ and $x_{k_1}<\ldots<x_1<y_1<\ldots<y_{k_2}$

$$\left|\bar{\nu}^\varepsilon\left(\prod_{i=1}^{k_1}\eta(x_i)\prod_{j=1}^{k_2}\eta(y_j)\right)-\bar{\nu}^\varepsilon\left(\prod_{i=1}^{k_1}\eta(x_i)\right)\bar{\nu}^\varepsilon\left(\prod_{j=1}^{k_2}\eta(y_j)\right)\right|\leqq b_1\,e^{-b_2(y_1-x_1)}$$

(iv) (Smooth initial profile.) There exist $p\in C_b^\infty$ such that

$$\sup_x\left|\bar{\nu}^\varepsilon(S(x)f)-\nu_{p(\varepsilon x)}(f)\right|\underset{\varepsilon\downarrow 0}{\rightarrow} 0 \qquad \text{for } f \text{ cylindrical,}$$

and

$$\lim_{\varepsilon\downarrow 0}\sup_x \varepsilon^{-1}\left|\bar{\nu}^\varepsilon(\eta(x))-p(\varepsilon x)\right|=0$$

(v) There exist A_1, A_2 positive constants so that

$$\left|\nu^\varepsilon(\eta(x))-\bar{\nu}^\varepsilon(\eta(x))\right|\leqq A_1\,e^{-A_2|x|},\qquad x\in\mathbb{Z}.$$

B. For the initial measures on the Z.R.P. we assume they can be obtained via (1.1) from a family (ν^ε) verifying the assumptions in (A).

1.3. Definition

We denote by $\mathscr{C}$ the class of cylindrical functions on $\mathbb{N}^{\mathbb{Z}}$ for which there exist a cylindrical function $\tilde{f}$ on $\{0,1\}^{\mathbb{Z}}$ with basis contained in $\{0,1,\ldots\}$ such that if $\xi(.)$ is related to $\eta(.)$ by (1.1) then

$$S(u)f(\xi)=S(q(u)+1)\tilde{f}(\eta) \tag{1.3}$$

This is the class for which we shall prove the Gibbs-Boltzmann principle. The main example needed for Theorem 0.3 is $f(\xi)=\mathbf{1}\,(\xi(0)>0)$.

2. PROOF OF THE RESULTS

In this section we shall prove Theorems 0.3 and 0.4. For Theorem 0.3 we shall also need the theory of Holley and Stroock for generalized Ornstein-Uhlenbeck processes via martingale problems [11]. In our case we use that (i) and (ii) in Theorem 0.3 determine exactly one probability measure on $C([0, +\infty), \mathscr{S}'(\mathbb{R}))$, which is an easy generalization of Theorem 1.4 of [11].

2.1. *Remark:* It may be convenient to notice that in this characterization of the measure $\mathscr{P}$ we may change (i) in Theorem 0.3 to the following:

(i) For every $\varphi \in \mathscr{S}(\mathbb{R})$

$$M_\tau(\varphi) \stackrel{\text{def}}{=} Y_\tau(\varphi) - \int_0^\tau Y_{\tau'}(A_{\tau'}\varphi)\, d\tau' \tag{2.1 a}$$

and

$$(M_\tau(\varphi))^2 - \int_0^\tau \| B_{\tau'}\varphi \|^2\, d\tau' \tag{2.1 b}$$

are $\mathscr{P}$-martingales with respect ot the canonical filtration $(\mathscr{F}_\tau)$, where A_τ and B_τ are defined by (0.19) and (0.20), respectively.

The above remark follows easily from stochastic calculus, after noticing that (2.1) implies that

$$Z_\varphi^{\tau_0}(\tau) \stackrel{\text{def}}{=} \exp\Bigg[i(Y_{\tau_0 \vee \tau}(\varphi) - Y_{\tau_0}(\varphi) - \int_{\tau_0}^{\tau_0 \vee \tau} d\tau'\, Y_{\tau'}(A_{\tau'}\varphi) - \frac{1}{2}\int_{\tau_0}^{\tau_0 \vee \tau} \| B_{\tau'}\varphi \|^2\, d\tau \Bigg]$$

is a $\mathscr{P}$-martingale, and from this one also gets (0.17). (*Cf.* [26].)

2.2. COROLLARY. — *Theorem* 0.3 *follows if the following conditions are verified:*

(*a*) *The family* $(\mathscr{P}^\varepsilon : 0 < \varepsilon \leq 1)$ *is tight on* $D([0, +\infty), \mathscr{S}'(\mathbb{R}))$ *and any weak limit point is supported by* $C([0, +\infty), \mathscr{S}'(\mathbb{R}))$.

(*b*) *Any weak limit point of* $\mathscr{P}^\varepsilon$ *solves the martingale problem described by* (i) *and* (ii) *in Theorem* 0.3.

Condition (*b*) of the above corollary will be a consequence of the Gibbs-Boltzmann principle. Thus we postpone its verification until Theorem 0.4 is proven. Now we concentrate on the thightness condition, for which we shall use the following criterium from [19], an improvement to those stated in [11], [12], [17], [18].

2.3. THEOREM (cf. [19]). — *Let* $(\Omega, \mathscr{A})$ *be a measurable space with some right continuous filtration (usual conditions)* $(\mathscr{A}_\tau^\varepsilon)_{\tau \geqq 0}$ *and probability measures* $\mathbb{P}_\varepsilon$, $0<\varepsilon\leqq 1$. *Let* $(Y_\tau^\varepsilon, \tau \geqq 0)$ *be an* $(\mathscr{A}_\tau^\varepsilon)$*-adapted process with paths in* $D([0, +\infty), \mathscr{S}'(\mathbb{R}))$ *and let us suppose there exists, for each* $\varphi \in \mathscr{S}(\mathbb{R})$, $\mathscr{A}_\tau^\varepsilon$*-predictable processes* $\gamma_1^\varepsilon(.,\varphi)$, $\gamma_2^\varepsilon(.,\varphi)$ *so that*

$$M_\tau^\varepsilon(\varphi) \overset{\text{def}}{=} Y_\tau^\varepsilon(\varphi) - \int_0^\tau \gamma_1^\varepsilon(\tau', \varphi)\, d\tau' \tag{2.2a}$$

and

$$(M_\tau^\varepsilon(\varphi))^2 - \int_0^\tau \gamma_2^\varepsilon(\tau', \varphi)\, d\tau' \tag{2.2b}$$

are $(\mathbb{P}_\varepsilon, \mathscr{A}_\cdot^\varepsilon)$*-martingales. Assume further that:*

(*c*.1) *For every* $\tau_0 \geqq 0$ *and* $\varphi \in \mathscr{S}(\mathbb{R})$

$$\sup_{0<\varepsilon\leqq 1} \sup_{0\leqq\tau\leqq\tau_0} \mathbb{P}_\varepsilon[(Y_\tau^\varepsilon(\varphi))^2] < +\infty \tag{2.3}$$

$$\sup_{0<\varepsilon\leqq 1} \sup_{0\leqq\tau\leqq\tau_0} \mathbb{P}_\varepsilon[(\gamma_i^\varepsilon(\tau, \varphi))^2] < +\infty, \qquad \textit{for} \quad i=1,2. \tag{2.4}$$

(*c*.2) *For every* $\varphi \in \mathscr{S}(\mathbb{R})$ *there exists* $\delta(\tau, \varphi, \varepsilon)$ *with* $\lim_{\varepsilon \downarrow 0} \delta(\tau, \varphi, \varepsilon) = 0$, *and*

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}_\varepsilon\Big(\sup_{0\leqq\tau'\leqq\tau} \big| Y_{\tau'}^\varepsilon(\varphi) - Y_{\tau'-}^\varepsilon(\varphi) \big| \geqq \delta(\tau, \varphi, \varepsilon)\Big) = 0. \tag{2.5}$$

Then, if $\mathscr{P}^\varepsilon$ *is the law induced on* $D([0, +\infty), \mathscr{S}'(\mathbb{R}))$ *by* (Y_τ^ε) *under* $\mathbb{P}_\varepsilon$, *we can say that the family* $(\mathscr{P}^\varepsilon : 0<\varepsilon\leqq 1)$ *is tight and that any weak limit point is supported by* $C([0, +\infty), \mathscr{S}'(\mathbb{R}))$.

2.4. *Proof of tightness in Theorem* 0.3: We must find γ_1^ε, γ_2^ε verifying (2.2) and also check condition (*c*.1), since (*c*.2) follows immediately from the definition. With the notation of Section 1 we know that if $F : \mathbb{N}^{\mathbb{Z}} \times [0, +\infty) \to \mathbb{R}$ is a bounded function such that $F(\xi, .)$ is a C^1 function and $F(., t)$ is in the domain of the closure of L [def. by eq. (0.1)], then

$$F(\xi(.,\varepsilon^{-2}\tau),\varepsilon^{-2}\tau)-\int_0^\tau \varepsilon^{-2}\Big[LF(.,\varepsilon^{-2}\tau').\xi(\varepsilon^{-2}\tau') \\ +\frac{\partial}{\partial\sigma}F(\xi(\varepsilon^{-2}\tau'),\sigma)\big|_{\sigma=\varepsilon^{-2}\tau'}\Big]d\tau' \quad (2.5\,bis)$$

is a martingale on the basic space $(\Omega, \mathscr{A}, \mathbb{P}_\varepsilon)$ of Section 1 for the filtration $\mathscr{A}_\tau^\varepsilon=\sigma$-field generated by $(\xi(.,t):0\leqq t\leqq\varepsilon^{-2}\tau)$. We would like to apply this to

$$F(\xi,t)=\sqrt{\varepsilon}\sum_u \varphi(\varepsilon u)(\xi(u)-\mathbb{P}_\varepsilon(\xi(u,t))) \quad (2.6)$$

which is not bounded in ξ. *A priori* we would then have only local martingales in (2.5). But then the argument given below can be applied if we stop the process at suitable stopping times $T_\eta^\varepsilon\to+\infty$ a. s., and as seen below, this allows to conclude that

$$\sup_{0\leqq\tau\leqq\tau_0}\ \sup_{0<\varepsilon\leqq 1}\mathbb{P}_\varepsilon(F^2(\xi(\varepsilon^{-2}\tau),\varepsilon^{-2}\tau))<+\infty$$

so that (2.5) will hold for $F(.)$ given by (2.6). (*Cf.* [26].)

Now, using equation (0.1) we get:

$$\gamma_1^\varepsilon(\tau,\varphi)=\frac{\sqrt{\varepsilon}}{2}\sum_u \Delta_\varepsilon\varphi(\varepsilon u)(\mathbf{1}(\xi(u,\varepsilon^{-2}\tau)>0)-\mathbb{P}_\varepsilon(\xi(u,\varepsilon^{-2}\tau)>0)) \quad (2.7)$$

where

$$\Delta_\varepsilon\varphi(x)=\varepsilon^{-2}(\varphi(x+\varepsilon)+\varphi(x-\varepsilon)-2\varphi(x)). \quad (2.8)$$

Hence $\gamma_1^\varepsilon(\tau,\varphi)=Y_\tau^\varepsilon(g;\Delta_\varepsilon\varphi)$, with $g(\xi)=\mathbf{1}(\xi(0)>0)$, according to the definition introduced in the statement of Theorem 0.4. Similarly we get

$$\gamma_2^\varepsilon(\tau,\varphi)=\varepsilon^{-1}\sum_u \mathbf{1}(\xi(u,\varepsilon^{-2}\tau)>0)\,[(\varphi(\varepsilon u+\varepsilon)-\varphi(\varepsilon u))^2 \\ +(\varphi(\varepsilon u-\varepsilon)-\varphi(\varepsilon u))^2] \quad (2.9)$$

and from this we see that equation (2.4) for $i=2$ holds, since

$$0\leqq\gamma_2^\varepsilon(\tau,\varphi)\leqq c\sup_r(1+r^2)\,|\varphi'(r)|^2$$

for a suitable constant $c\in(0,+\infty)$.

We now want to check equation (2.4) for $i=1$. From the above expression for $\gamma_1^\varepsilon(.,.)$ it is easy to see that it suffices to get a bound as (2.3) for $Y_\tau^\varepsilon(g;\varphi)$ with $\varphi\in\mathscr{S}(\mathbb{R})$ and $g(\xi)=\mathbf{1}(\xi(0)>0)$. Following the general strategy in the article we shall realize such fields in the S.E.P., and get the suitable bound after reduction to simpler expressions throug a Taylor expansion. For this let us introduce the new field

$$\hat{Y}_\tau^\varepsilon(g;\varphi)=\sqrt{\varepsilon}\sum_u \varphi(\varepsilon u)(\mathbf{1}(\xi(u,t)>0)-\beta(u,\varepsilon,t)) \tag{2.10}$$

where $t=\varepsilon^{-2}\tau$ (here and in the rest of this proof), and the $\beta(u,\varepsilon,t)$ are numbers which will be specified later. In any case $\mathbb{P}_\varepsilon(Y_\tau^\varepsilon(g;\varphi)^2)\leqq\mathbb{P}_\varepsilon(\hat{Y}_\tau^\varepsilon(g;\varphi)^2)$ and we want to prove (2.3) for $\hat{Y}_\tau^\varepsilon(g;\varphi)$. In the S.E.P. $\hat{Y}_\tau^\varepsilon(g;\varphi)$ becomes

$$\hat{Y}_\tau^\varepsilon(g;\varphi)=\sqrt{\varepsilon}\sum_x \varphi(\varepsilon u(x,t))\,\eta(x,t)(1-\eta(x+1,t)-\beta(u(x,t),\varepsilon,t)) \tag{2.11}$$

where $u(x,t)$ is defined as

$$u(x,t)=u \qquad \text{iff} \quad q(u,t)\leqq x<q(u+1,t) \tag{2.12}$$

for $x,\ u\in\mathbb{Z},\ t\geqq 0$.

The function $u(.,t)$ is non decreasing Setting

$$\bar{u}(x,\varepsilon,t)=\mathbb{P}_\varepsilon(u(x,t)) \tag{2.13}$$

we have

$$\bar{u}(x+1,\varepsilon,t)-\bar{u}(x,\varepsilon,t)=\mathbb{P}_\varepsilon(\eta(x+1,t)).$$

We easily see that $\bar{u}(x,\varepsilon,t)$ extends to a smooth function $\bar{u}(.,\varepsilon,t)$ on $\mathbb{R}$, which is strictly increasing (by Assumption 1.2) and for which there exist $0<m<M<+\infty$ so that

$$m\leqq\frac{\partial}{\partial r}\bar{u}(r,\varepsilon,t)\leqq M \tag{2.14}$$

for $t=\varepsilon^{-2}\tau$ and $0\leqq\tau\leqq\tau_0$. Thus we may define the inverse function $R(.,\varepsilon,t)$ so that

$$R(\bar{u}(r,\varepsilon,t),\varepsilon,t)=r \qquad \text{for all} \quad r\in\mathbb{R} \tag{2.15}$$

and we have a bound as (2.14) for $(\partial/\partial u)R(.,\varepsilon,t)$, when $t=\varepsilon^{-2}\tau$ and $\tau\leqq\tau_0$. We then set

$$\beta(u,\varepsilon,t)=1-p(\varepsilon R(u,\varepsilon,t),\tau) \tag{2.16}$$

and notice that $\beta(\bar{u}(x,\varepsilon,t),\varepsilon,t)=1-p(\varepsilon x,\tau)$.

From the Taylor-Lagrange expansion to first order we can write

$$\hat{Y}_\tau^\varepsilon(g;\varphi)=\mathrm{I}+\mathrm{II}+\mathrm{III} \tag{2.17}$$

where

$$\mathrm{I}=-\sqrt{\varepsilon}\sum_x \varphi(\varepsilon\bar{u}(x,\varepsilon,t))\,\eta(x,t)(\eta(x+1,t)-p(\varepsilon x,\tau))$$

$$\mathrm{II}=\sqrt{\varepsilon}\sum_x \varphi'(\varepsilon\tilde{u}(x,\varepsilon,t)\,\eta(x,t)(1-\eta(x+1,t)$$
$$-\beta(\tilde{u}(x,\varepsilon,t),\varepsilon,t))\,\varepsilon(u(x,t)-\bar{u}(x,\varepsilon,t))$$

$$\mathrm{III}=-\sqrt{\varepsilon}\sum_x \varphi(\varepsilon\tilde{u}(x,\varepsilon,t))\,\eta(x,t)\,\mathrm{C}(\tilde{u}(x,\varepsilon,t),\varepsilon,t)\,\varepsilon(u(x,t)-\bar{u}(x,\varepsilon,t))$$

where $\tilde{u}(x,\varepsilon,t)$ is some (random) point in the interval with endpoints $u(x,t)$ and $\bar{u}(x,\varepsilon,t)$, and

$$\mathrm{C}(u,\varepsilon,t)=\frac{\partial}{\partial u}\mathrm{R}(u,\varepsilon,t).\ \frac{\partial}{\partial r}p(r,\tau)\big|_{r=\varepsilon\mathrm{R}(u,\varepsilon,t)} \tag{2.18}$$

In the sequel we shall estimate each term in (2.17); we start with I:

$$\mathbb{P}_\varepsilon(\mathrm{I}^2)=\varepsilon\sum_{x,y}\varphi(\varepsilon\bar{u}(x,\varepsilon,t))\,\varphi(\varepsilon\bar{u}(y,\varepsilon,t))$$
$$\times\mathbb{P}_\varepsilon[\eta(x,t)\,\eta(y,t)(\eta(x+1,t)$$
$$-p(\varepsilon x,\tau))(\eta(y+1,t)-p(\varepsilon y,\tau))] \tag{2.19}$$

But from Lemma A.1 we know that

$$\varepsilon^{-1}\,\mathbb{P}_\varepsilon(\eta(x,t)\,\eta(y,t)(\eta(x+1,t)-p(\varepsilon x,\tau))(\eta(y+1,t)-p(\varepsilon y,\tau))$$

is uniformly bounded, for $|x-y|>1$ and $0\leqq\tau\leqq\tau_0$. Thus we get

$$\mathbb{P}_\varepsilon(\mathrm{I}^2)\leqq \mathrm{C}\sup_r|(1+r^2)\,\varphi(\varepsilon\bar{u}(r,\varepsilon,t))|^2\{\varepsilon\sum_x(1+(\varepsilon x)^2)^{-1}\}^2 \tag{2.20}$$

As easily seen in Lemma A.2 there exist a, $b>0$ so that [*see* also (2.14)]

$$|\bar{u}(x,\varepsilon,t)-bx|\leqq a \tag{2.21}$$

for $\tau\leqq\tau_0$, $\varepsilon>0$. Using (2.20) and (2.21) we get the required estimate for $\mathbb{P}_\varepsilon(\mathrm{I}^2)$.

For $\varepsilon>0$ and $\tau\leqq\tau_0$

$$\mathbb{P}_\varepsilon(\mathrm{II}^2)\leqq\varepsilon^2\sum_{x,y}\mathbb{P}_\varepsilon(\mathrm{G}_\varepsilon(x)\,\mathrm{G}_\varepsilon(y))\leqq\{\varepsilon\sum_x[\mathbb{P}_\varepsilon(\mathrm{G}_\varepsilon^2(x))]^{1/2}\}^2 \quad (2.22\,a)$$

where

$$\mathrm{G}_\varepsilon(x)=|\varphi'(\varepsilon\tilde{u}(x,\varepsilon,t))\sqrt{\varepsilon}(u(x,t)-\bar{u}(x,\varepsilon,t))|. \quad (2.22\,b)$$

We shall now prove that there exists a constant c_1 such that

$$[\mathbb{P}_\varepsilon(\mathrm{G}_\varepsilon^2(x))]^{1/2}\leqq c_1\sup_r|\varphi'(r)(1+r^2)|(1+(\varepsilon x)^2)^{-1} \quad (2.23)$$

and this will give the desired estimate for $\mathbb{P}_\varepsilon(\mathrm{II}^2)$. Since $\mathbb{P}_\varepsilon(\mathrm{III}^2)$ is completely analogous we will have proven (2.4).

To prove equation (2.23) let $\mathscr{X}_\varepsilon=\mathbf{1}(|u(x,t)-\bar{u}(x,\varepsilon,t)|\leqq\varepsilon^{-\alpha})$, for some $\alpha\in(3/4,1)$. Then

$$[\mathbb{P}_\varepsilon(\mathrm{G}_\varepsilon^2(x))]^{1/2}\leqq\tilde{\varphi}(\varepsilon\bar{u}(x,\varepsilon,t))[\varepsilon\,\mathbb{P}_\varepsilon(u(x,t)-\bar{u}(x,\varepsilon,t))^2]^{1/2} +[\mathbb{P}_\varepsilon(\mathrm{G}_\varepsilon^2(x)(1-\mathscr{X}_\varepsilon))]^{1/2} \quad (2.24)$$

where

$$\tilde{\varphi}(y)=\sup(|\varphi'(y+x)|:|x|\leqq\varepsilon^{1-\alpha}).$$

Since one can find a constant $c<+\infty$ so that

$$\sup_x\mathbb{P}_\varepsilon(\varepsilon(u(x,t)-\bar{u}(x,\varepsilon,t))^2)\leqq c \quad (2.25)$$

for $\varepsilon\in(0,1]$ and $\tau\leqq\tau_0$, as proven in Lemma A.2, just as before we see that the first term in (2.24) gives rise to something of the required form. For the second term we write

$$[\mathbb{P}_\varepsilon(\mathrm{G}_\varepsilon^2(x)(1-\mathscr{X}_\varepsilon)]^{1/2}\leqq[\mathbb{P}_\varepsilon(\mathrm{G}_\varepsilon^4(x))]^{1/4}.[\mathbb{P}_\varepsilon(1-\mathscr{X}_\varepsilon)]^{1/4}.$$

Now the important estimates come from [10] and allows us to say that (as in Lemma A.2) for each $n\geqq 1$ there exists $d_n<+\infty$ so that (2)

$$\mathbb{P}_\varepsilon([u(x,t)-\bar{u}(x,\varepsilon,t)]^{2n})\leqq d_n\,\varepsilon^{-2n(1-\delta)} \quad (2.26)$$

(2) Joseph Fritz has shown us how to derive an exponential estimate for $u-\bar{u}$. He can actually prove, using Prop. 1.7, ch. 8, of [16], that for any $\alpha>1/2$ there exist β, a, and b positive so that $\mathrm{P}(|u-\bar{u}|>\varepsilon^\alpha)\leqq a.\exp(-b\,\varepsilon^{-\beta})$.

with $0<\delta<1/4$. In particular

$$[\mathbb{P}_\varepsilon(1-\mathscr{X}_\varepsilon)]^{1/4} \leqq d_n \varepsilon^{n(\delta+\alpha-1)/2} \tag{2.27}$$

and we can choose δ so that $\delta+\alpha>1$.

We now compute $\mathbb{P}_\varepsilon(G_\varepsilon^4(x))$ by decomposing the integral according to the sign of $|u(x,t)-\bar{u}(x,\varepsilon,t)|-(1/2)b|x|$, and using once more the Cauchy Schwartz inequality.

This leads to

$$\begin{aligned}
[\mathbb{P}_\varepsilon(G_\varepsilon^4(x))]^{1/4} &\leqq \sqrt{\varepsilon} \sup_r (|(1+r^2)\varphi'(r)| \vee 1) \\
&\quad \times \Bigg[\Bigg\{\left(1+\varepsilon^2\left(\frac{b}{2}|x|-a\right)^2\right)^{-1} \mathbf{1}(|x|>2a|b) \\
&\quad + \mathbf{1}(|x| \leqq 2a/b)\Bigg\} [\mathbb{P}_\varepsilon(u(x,t)-\bar{u}(x,\varepsilon,t))^4]^{1/4} \\
&\quad + \left[\mathbb{P}_\varepsilon\left(|u(x,t)-\bar{u}(x,\varepsilon,t)|>\frac{b}{2}|x|\right)\right]^{1/8} [\mathbb{P}_\varepsilon(|u(x,t)-\bar{u}(x,\varepsilon,t)|^8)]^{1/8}\Bigg]
\end{aligned}$$

and from (2.26):

$$\mathbb{P}_\varepsilon\left(|u(x,t)-\bar{u}(x,\varepsilon,t)| \geqq \frac{b}{2}|x|\right) \leqq d_8 \left(\frac{2}{b|x|}\right)^{16} \varepsilon^{-16(1-\delta)}$$

which gives the desired dependence on x; by taking n large enough in (2.27) we get positive powers of ε and (2.23) follows. [In fact, we have seen that the contribution to (2.23) coming from $G_\varepsilon^2(1-\mathscr{X}_\varepsilon)$ vanishes as ε tends to zero.]

To complete the proof to tightness it remains to check equation (2.3). For this we write

$$\hat{Y}_\tau^\varepsilon(\varphi) = \sqrt{\varepsilon} \sum_u \varphi(\varepsilon u)(\xi(u,t)-v(u,\varepsilon,t)) \tag{2.28 a}$$

where $t=\varepsilon^{-2}\tau$ and

$$v(u,\varepsilon,t) = (1-p(\varepsilon R(u,\varepsilon,t),\tau))/p(\varepsilon R(u,\varepsilon,t),\tau) \tag{2.28 b}$$

It is enough to check equation (2.3) for $\hat{Y}_\tau^\varepsilon(\varphi)$. In the S.E.P. we can write $\hat{Y}_\tau^\varepsilon(\varphi)$ as

$$\hat{Y}_\tau^\varepsilon(\varphi) = \sqrt{\varepsilon} \sum_x \varphi(\varepsilon u(x,t))\, \eta(x,t)[N_t(x)-v(u(x,t),\varepsilon,t)]$$

where $N_t(x)$ denotes the number of empty sites in between x and the next (to the right) occupied site, at time t. From this we easily see that

$$\begin{aligned}\hat{Y}^\varepsilon_\tau(\varphi) &= \sqrt{\varepsilon}\sum_x \varphi(\varepsilon u(x,t))[1-\eta(x,t)-\eta(x,t)v(u(x,t),\varepsilon,t)]\\ &= \tilde{I}+II+III \end{aligned} \tag{2.29}$$

with

$$\tilde{I} = -\sqrt{\varepsilon}\sum_x \varphi(\varepsilon\bar{u}(x,\varepsilon,t))(p(\varepsilon x,\tau))^{-1}(\eta(x,t)-p(\varepsilon x,\tau))$$

$$\begin{aligned}II = \sqrt{\varepsilon}\sum_x \varphi'(\varepsilon\tilde{u}(x,\varepsilon,t))[1-\eta(x,t)\\ -\eta(x,t)v(\tilde{u}(x,\varepsilon,t),\varepsilon,t)\varepsilon(u(x,t)-\bar{u}(x,\varepsilon,t))\end{aligned}$$

$$III = -\sqrt{\varepsilon}\sum \varphi(\varepsilon\tilde{u}(x,\varepsilon,t))\eta(x,t)D(\tilde{u}(x,\varepsilon,t),\varepsilon,t)\varepsilon(u(x,t)-\bar{u}(x,\varepsilon,t))$$

for some suitable (random) $\tilde{u}(x,\varepsilon,t)$ in the interval with endpoints $u(x,t)$ and $\bar{u}(x,\varepsilon,t)$, and where

$$D(u,\varepsilon,t) = -R'(u)(p(r,\tau))^{-2}\frac{\partial}{\partial r}p(r,\tau)\big|_{r=\varepsilon R(u,\varepsilon,t)}. \tag{2.30}$$

Each of these terms is treated analogously to the corresponding ones in $\hat{Y}^\varepsilon_\tau(g;\varphi)$. This concludes the proof of tightness.

2.5. *Proof of Theorem* 0.4': It is enough to consider $\mathcal{J}(\eta)=\prod_{i=1}^k \eta(x_i)$, where $x_1<\ldots<x_k$ are integers. Here $\tilde{a}(\mathcal{J};x,\tau)=kp(x,\tau)^{k-1}$, and we shall write $\mathcal{J}_t$ for $\mathcal{J}(\eta(.,t))$. Also we will simply denote the $\mathbb{P}_\varepsilon$-expectations by $\langle . \rangle$. Writting

$$\begin{aligned}S(x)\mathcal{J}_t = \prod_{i=1}^k \eta(x+x_i,t) = \sum_{\Delta 7\{1,\ldots,k\}}\prod_{i\in\Delta}[\eta(x+x_i,t)\\ -\langle\eta(x+x_i,t)\rangle]\prod_{i\notin\Delta}\langle\eta(x+x_i,t)\rangle\end{aligned} \tag{2.31 a}$$

and

$$\begin{aligned}\langle S(x)\mathcal{J}_t\rangle = \sum_{\Delta 7\{1,\ldots,k\}}\langle\prod_{i\in\Delta}(\eta(x+x_i,t)\\ -\langle\eta(x+x_i,t)\rangle)\rangle\prod_{j\notin\Delta}\langle\eta(x+x_j,t)\rangle\end{aligned} \tag{2.31 b}$$

The term corresponding to $\Delta=\varnothing$ cancels when taking $S(x)\tilde{f}_t - \langle S(x)\tilde{f}_t \rangle$. Also, it is easily seen that the terms corresponding to $|\Delta|=1$ in $Y^{\varepsilon}_{\varepsilon^2 t}(\tilde{f}; \varphi)$ "cancel" with $Y^{\varepsilon}_{\varepsilon^2 t}((\tilde{a}\varphi)_{\varepsilon^2 t})$ in equation (0.26). Thus, it remains to prove that for $k \geqq 2$, and for $0<\tau'<\tau<+\infty$

$$\lim_{T\uparrow+\infty} \lim_{\varepsilon\downarrow 0} \sup_{\tau'\leqq\varepsilon^2 t\leqq\tau} \mathbb{P}_\varepsilon\left[\left\{\frac{1}{T}\int_t^{t+T} ds\sqrt{\varepsilon}\sum_x \varphi(\varepsilon x) g_s(x)\right\}^2\right]=0 \quad (2.32)$$

where

$$g_s(x)=\prod_{i=1}^{k}(\eta(x+x_i, s)-p_i(x, s)) \quad (2.33)$$

for $x_1, \ldots, x_k$ distinct integers, $k \geqq 2$, and $p_i(x, s)=\mathbb{P}_\varepsilon(\eta(x+x_i, s))$. But the expectation in (2.32) can be written as

$$\frac{2}{T}\int_t^{t+T} ds\, \varepsilon \sum_x \varphi(\varepsilon x)\frac{1}{T}\int_0^{T+t-s} A^\varepsilon(x, s, s')\, ds' \quad (2.34)$$

with

$$A^\varepsilon(x, s, s')=\sum_y \varphi(\varepsilon y)\,\mathbb{P}_\varepsilon(g_s(x) g_{s+s'}(y)). \quad (2.35)$$

For $R>0$ we decompose $A^\varepsilon(.)$ as

$$A^\varepsilon(x, s, s')=C^\varepsilon_R(x, s, s')+D^\varepsilon_R(x, s, s'),$$

where:

$$C^\varepsilon_R(x, s, s')=\sum_{y:|y-z|\leqq R}\varphi(\varepsilon y)\,\mathbb{P}_\varepsilon(g_s(x) g_{s+s'}(y))$$

$$D^\varepsilon_R(x, s, s')=\sum_{y:|y-x|>R}(\varepsilon y)\,\mathbb{P}_\varepsilon(g_s(x) g_{s+s'}(y)).$$

For C^ε_R we use local equilibrium (*cf.* [9]) to get:

$$\left|C^\varepsilon_R(x, s, s')-\sum_{|y|\leqq R}\varphi(\varepsilon(y+x))\,\nu_{p(\varepsilon x, \varepsilon^2 s)}(g(0)\,e^{s'\tilde{L}}g(y))\right|$$

$$\leqq \Psi(\varepsilon, R, s').\sup_r|\varphi(r)| \quad (2.36\,a)$$

where $\tilde{L}$ is the generator of S.E.P., $g(y)=\prod_{i=1}^{k}(\eta(x_i+y)-p(\varepsilon x, \varepsilon^2 s))$, and

$$\lim \Psi(\varepsilon, R, s')=0 \quad (2.36\,b)$$

$\{\Psi(\varepsilon, R, .)\}_\varepsilon$ is uniformly bounded on each $[0, T]$.

On the other side a simple computation gives, for all p:

$$\sum_{|y|\leqq R}\left|\nu_p\left(\prod_{i=1}^{k}(\eta(x_i)-p)e^{t\tilde{L}}\prod_{j=1}^{k}(\eta(x_j+y)-p)\right|\leqq\bar{\Psi}(t)\quad(2.37\,a)$$

where

$$\lim_{t\to+\infty}\bar{\Psi}(t)=0.\quad(2.37\,b)$$

From these two facts we get

$$\frac{1}{T}\int_0^T C_R^\varepsilon(x, s, s')\,ds'\leqq\frac{1}{T}\int_0^T[\bar{\Psi}(s')+\Psi(\varepsilon, R, s')]\,ds'.\sup_r(|\varphi(r)|)\quad(2.38)$$

Now, let us estimate $D_R^\varepsilon(.)$. Using the initial construction of stirring particles on some $(\Omega, \mathscr{A}, \mathbb{P})$, we denote by $Y_s(x, t)$, $0\leqq t\leqq s$ the path of that stirring particle such that $Y_s(x, s)=x$. The duality gives:

$$D_R^\varepsilon(x, s, s')=\sum_{|y|>R}\varphi(\varepsilon(y+x))\,\mathbb{P}(G^\varepsilon(x, y, s, s'))\quad(2.39)$$

where

$$G^\varepsilon(x, y, s, s')=\prod_{i=1}^{k}(\eta(Y_s(\tilde{x}_i, 0)) -\tilde{p}(i, s))\prod_{j=1}^{k}(\eta(Y_{s+s'}(\tilde{x}_j+y, 0))-\tilde{p}(j, s+s'))$$

with

$$\tilde{x}_i=x_i+y$$
$$\tilde{p}(i, s)=\mathbb{P}_\varepsilon(\eta(\tilde{x}_i, s))$$
$$\tilde{p}(j, s+s')=\mathbb{P}_\varepsilon(\eta(\tilde{x}_j, s+s')).$$

We decompose the integral in (2.39) according to the set

$$B(y)=\{\omega:\exists\, i, j\in\{1, \ldots, k\}\text{ s. t. }Y_{s+s'}(\tilde{x}_i, s)=\tilde{x}_j\}.$$

We have

$$\sum_{|y|>R} \left| \varphi(\varepsilon(y+x)) \mathbb{P}(\mathbf{1}_{\Omega \setminus B(y)} G^{\varepsilon}(x, y, s, s') \right|$$

$$\leqq \sum_{y} |\varphi(\varepsilon(y+x))| \sum_{\substack{\tilde{y}_1, \ldots, \tilde{y}_k \\ \tilde{y}_i \neq \tilde{x}_j, \forall i, j}} \mathbb{P}(Y_{s+s'}(\tilde{x}_i, s) = \tilde{y}_i, i = 1, \ldots, k).$$

$$\times \left| \mathbb{P}_{\varepsilon}\left(\prod_{i=1}^{k} (\eta(\tilde{x}_i, s) - \tilde{p}(i, s))(\eta(\tilde{y}_i, s) - \tilde{p}(i, s+s')) \right| \quad (2.40\,a)$$

Now, it is not difficult to see that $\tilde{p}(i, s+s') - \mathbb{P}_{\varepsilon}(\eta(\tilde{y}_i, s))$ is of the order ε (for T fixed) and we shall get (*cf.* [10]):

$$\text{r.h.s. of } (2.40\,a) \leqq \varepsilon \sum_{y} |\varphi(\varepsilon y)| . \gamma(T, k, \varepsilon) \qquad (2.40\,b)$$

where for each $T < +\infty, k \geqq 2$

$$\lim_{\varepsilon \downarrow 0} \gamma(T, k, \varepsilon) = 0. \qquad (2.40\,c)$$

We still need to consider the contribution of $B(y)$ to (2.39). But for R sufficiently large

$$\mathbb{P}(B(y)) \leqq CP(T_{y/2} < T)$$

where T_x = hitting time of the origin for a symmetric simple random walk which starts at x. Thus

$$\sum_{|y|>R} |\varphi(\varepsilon(y+x))| \mathbb{P}(\mathbf{1}_{B(y)} G^{\varepsilon}(x, y, s, s'))$$

$$\leqq \tilde{c} \sum_{|y|>R} P(T_{y/2} < T) \leqq \Delta(R^2/T) \quad (2.41\,a)$$

where $\lim_{x \to \infty} \Delta(x) = 0$.

Taking, for instance, $R = T$, (2.37)-(2.41) prove (2.32), and so the theorem.

2.6. *Proof of Theorem* 0.4: Let $f \in \mathscr{B}$ (*cf.* Definition 1.3) and write $f(u, t) = S(u) f(\xi(., t))$. From Definition 1.3 there exists $\tilde{f}$ bounded cylindrical function on the S.E.P. so that $f(u, t) = S(q(u, t) + 1) \tilde{f}_t$, where $\tilde{f}_t = \tilde{f}(\eta(., t))$.

Thus, for $t=\varepsilon^{-2}\tau$ we may write:

$$Y_\tau^\varepsilon(f;\varphi)=\sqrt{\varepsilon}\sum_x \varphi(\varepsilon u(x,t))\,\eta(x,t)\,[S(x+1)\tilde{f}_t-h(u(x,t),\varepsilon,t)] \quad (2.42)$$

with

$$h(u,\varepsilon,t)=\mathbb{P}_\varepsilon(f(u,t)). \quad (2.43)$$

As before we let $\hat{Y}_\tau^\varepsilon(f;\varphi)$ be defined by changing h to $\hat{h}$ in (2.42), where

$$\hat{h}(u,\varepsilon,t)=\nu_p(\tilde{f}) \qquad \text{with} \quad p=p(\varepsilon R(u,\varepsilon,t),\tau), \quad (2.44)$$

Notice that, from the relations (0.5):

$$a(f;r,\tau)=\frac{d}{d\rho}\mu_\rho(f)\big|_{\rho=\rho(r,\tau)}=-p^2\frac{d}{dp}\nu_p(\tilde{f})\big|_{p=p(z(r,\tau),\tau)} \quad (2.45\,a)$$

We set

$$\hat{a}(f;r,\tau)=-p^2\frac{d}{dp}\nu_p(\tilde{f})\big|_{p=p(\varepsilon R(\varepsilon^{-1}r,\varepsilon,t),\tau)} \quad (2.45\,b)$$

and let $(\hat{a}\varphi)_\tau(r)=\hat{a}(f;r,\tau)\,\varphi(r)$. Thus, letting $\hat{Y}_\tau^\varepsilon(.)$ be as in (2.28 a) we can write

$$\hat{Y}_\tau^\varepsilon((\hat{a}\varphi)_\tau)=\sqrt{\varepsilon}\sum_x(\hat{a}\varphi)_\tau(\varepsilon u(x,t))$$

$$\times(1-\eta(x,t)-\eta(x,t)\,v(u(x,t),\varepsilon,t)) \quad (2.46)$$

As before, we expand such fields around $\bar{u}(x,\varepsilon,t)$:

$$\hat{Y}_\tau^\varepsilon(f;\varphi)=I+II+III+E \quad (2.47\,a)$$

$$\hat{Y}_\tau^\varepsilon((\hat{a}\varphi)_\tau)=I'+II'+III'+E' \quad (2.47\,b)$$

where (writing u and $\bar{u}$ for $u(x, t)$ and $\bar{u}(x, \varepsilon, t)$, respectively):

$$\mathrm{I}=\sqrt{\varepsilon}\sum_{x}\varphi(\varepsilon\bar{u})\,\eta(x,t)\,[\mathrm{S}(x+1)\tilde{f}_t-\nu_{p(\varepsilon x,\tau)}(\tilde{f})] \tag{2.48 a}$$

$$\mathrm{II}=-\sqrt{\varepsilon}\sum_{x}\varphi(\varepsilon\bar{u})\,\eta(x,t)\,\mathrm{H}(\bar{u})\,\varepsilon(u-\bar{u}) \tag{2.48 b}$$

$$\mathrm{III}=\sqrt{\varepsilon}\sum_{x}\varphi'(\varepsilon\bar{u})\,\eta(x,t)\,[\mathrm{S}(x+1)\tilde{f}_t-\nu_{p(\varepsilon x,\tau)}(\tilde{f})]\,\varepsilon(u-\bar{u}) \tag{2.48 c}$$

$$\mathrm{I}'=-\sqrt{\varepsilon}\sum_{x}\varphi(\varepsilon\bar{u})\,\hat{a}(f;\varepsilon\bar{u})\,p(\varepsilon x,\tau)^{-1}\,[\eta(x,t)-p(\varepsilon,x,\tau)] \tag{2.48 d}$$

$$\mathrm{II}'=-\sqrt{\varepsilon}\sum_{x}\varphi(\varepsilon\bar{u})\,\hat{a}(f;\varepsilon\bar{u})\,\eta(x,t)\,\mathrm{D}(\bar{u})\,\varepsilon(u-\bar{u}) \tag{2.48 e}$$

$$\mathrm{III}'=\sqrt{\varepsilon}\sum_{x}(\hat{a}(f;\,.\,,\tau)\,\varphi(\,.\,))'(\varepsilon\bar{u})\,p(\varepsilon x,\tau)^{-1}$$

$$\times[\eta(x,t)-p(\varepsilon x,\tau)]\,\varepsilon(u-\bar{u}) \tag{2.48 f}$$

where $\mathrm{D}(u)=\mathrm{D}(u, \varepsilon, t)$ was defined in (2.30),

$$\mathrm{H}(u)=\frac{d}{dp}\nu_p(\tilde{f})\big|_{p=p(\varepsilon\mathrm{R}(u,\varepsilon,t),\tau)}\mathrm{R}'(u).\,\frac{\partial}{\partial r}p(r,\tau)\big|_{r=\varepsilon\mathrm{R}(u,\varepsilon,t)} \tag{2.49}$$

and E, E′ are the second order terms in the Taylor-Lagrange expansions. First notice that $\mathrm{H}(\bar{u})=\hat{a}(f;\varepsilon x,\tau)\,\mathrm{D}(\bar{u})$ and so $\mathrm{II}=\mathrm{II}'$. We now prove

2.7. LEMMA. – *With the same notations as in* (2.47), (2.48) *we have that for each t*

$$\lim_{\varepsilon\downarrow 0}\mathbb{P}_\varepsilon(|\mathrm{III}|+|\mathrm{III}'|+|\mathrm{E}|+|\mathrm{E}'|)=0.$$

Proof: Let us first look at $\mathbb{P}_\varepsilon(|\mathrm{E}|)$. Notice that E can be written as a sum of three similar expressions, of the form

$$\sqrt{\varepsilon}\sum_{x}\psi(\varepsilon\tilde{u}(x,\varepsilon,t),\varepsilon,t)\,g(x,\eta_t)\,\varepsilon^2(u(x,t)-\bar{u}(x,\varepsilon,t))^2$$

where $g(.)$ is bounded, $\tilde{u}(x, \varepsilon, t)$ is a suitable point in the interval with extremes $u(x, t)$ and $\bar{u}(x, \varepsilon, t)$, and $\psi(u, \varepsilon, t)$ is a linear combination of terms like $\mathrm{D}(u)$, $(a\varphi)_\tau(u)$ or their first and second derivatives. Thus we want to see that

$$\lim_{\varepsilon\downarrow 0}\mathbb{P}_\varepsilon\{\sqrt{\varepsilon}\sum_{x}|\psi(\varepsilon\tilde{u}(x,\varepsilon,t))|\,\varepsilon^2(u(x,t)-\bar{u}(x,\varepsilon,t))^2\}=0.$$

The procedure is very similar to that used when estimating (2.22), and it is therefore omitted. The same holds for E′.

We now prove that $\mathbb{P}_\varepsilon(|\mathrm{III}|) \to 0$. Let $\mathcal{S}^\varepsilon$ be the partition of $\mathbb{R}$ in intervals $(k\varepsilon^{-\beta}, (k+1)\varepsilon^{-\beta}]$, $k \in \mathbb{Z}$ where $\beta \in (0, 1/2)$. We denote such intervals by $\mathcal{J}$ and let $\mathcal{J}(x)$ be that interval which contains x (of course we omit writing explicitely the ε-dependence); $y < \mathcal{J}(x)$ ($y > \mathcal{J}(x)$) will mean that y is to the left (right) of $\mathcal{J}(x)$, for $y \in \mathbb{Z}$. For $x > 0$, we set

$$b(x, 0) = \sum_{0 < y < \mathcal{J}(x)} \eta(y), \qquad \bar{b}(x, \varepsilon, 0) = \mathbb{P}_\varepsilon(b(x, 0)) \tag{2.50a}$$

$$d(x, t) = \sum_{y < \mathcal{J}(x)} \eta(y) \mathbf{1}(\mathrm{Y}(x, t) > \mathcal{J}(x)) - \sum_{y > \mathcal{J}(x)} \eta(y) \mathbf{1}(\mathrm{Y}(x, t) < \mathcal{J}(x)) \tag{2.50b}$$

$$\bar{d}(x, \varepsilon, t) = \mathbb{P}_\varepsilon(d(x, t)) \tag{2.50c}$$

and make the analogous definitions for $x < 0$. Also, we get

$$\alpha(x, \varepsilon, t) = b(x, 0) + d(x, t) - \bar{b}(x, \varepsilon, 0) - \bar{d}(x, \varepsilon, t). \tag{2.50d}$$

The difference between $b(x, 0) + d(x, t)$ and $u(x, t)$ is due to particles in $\mathcal{J}(x)$; it is therefore not surprising that:

$$|u(x, t) - \bar{u}(x, \varepsilon, t) - \alpha(x, \varepsilon, t)| \leq 10\,\varepsilon^{-\beta} \tag{2.51}$$

Similar arguments are given in the proof of Lemma A.2 to which we refer for more details.

So, if we call $\widehat{\mathrm{III}}$ the value of III when $u(x, t) - \bar{u}(x, \varepsilon, t)$ is substituted by $\alpha(x, \varepsilon, t)$, we can find constants c, $c' < +\infty$ so that, with probability one:

$$|\widehat{\mathrm{III}} - \mathrm{III}| \leq c\,\varepsilon^{3/2-\beta} \sum_x |\varphi'(\varepsilon \bar{u}(x, \varepsilon, t))| \leq c'\,\varepsilon^{1/2-\beta} \sup_y |\varphi'(y)(1 + y^2)|. \tag{2.52}$$

On the other side:

$$\mathbb{P}_\varepsilon(|\widehat{\mathrm{III}}|) \leq \sum_{k \in \mathbb{Z}} \varepsilon^{3/2} \{\mathbb{P}_\varepsilon(\alpha^2(k\varepsilon^{-\beta}, \varepsilon, t))\}^{1/2} \times \{\mathbb{P}_\varepsilon(\sum_{\substack{k\varepsilon^{-\beta} \leq x \leq (k+1)\varepsilon^{-\beta} \\ x \in \mathbb{Z}}} \varphi'(\varepsilon \bar{u}(x, \varepsilon, t))\, \eta(x, t) \times [\mathrm{S}(x+1)\bar{f}_t - \nu_{p(\varepsilon x, \tau)}(\bar{f})])^2]\}^{1/2} \tag{2.53}$$

From (2.26) and (2.51) we have

$$\sup_x \varepsilon\, \mathbb{P}_\varepsilon(\alpha^2(x, \varepsilon, t)) < +\infty. \tag{2.54}$$

Moreover, from [10] and Theorem 7.1 of [6] we know:

$$\mathbb{P}_\varepsilon(\eta(x,t)\,\eta(y,t)\,[S(x+1)\tilde{f}_t-\nu_{p(\varepsilon x,\tau)}(\tilde{f})]$$
$$\times[S(y+1)\tilde{f}_t-\nu_{p(\varepsilon y,\tau)}(\tilde{f})])\leqq c_1\,\varepsilon+c_2\,\tilde{\delta}_{x,y} \quad (2.55)$$

for suitable constants c_1, c_2 and where

$$\tilde{\delta}_{x,y}=1 \quad \text{if} \quad x+1+z\notin \mathrm{B} \text{ and } y+1+z\notin \mathrm{B} \text{ for some } z\notin \mathrm{B}$$
$$=0 \quad \text{otherwise.}$$

B being the basis of the cylindrical function $\tilde{f}$. From (2.52)-(2.55) we easily conclude that $\mathbb{P}_\varepsilon(|\mathrm{III}|)\to 0$.

The term III′ can be treated in the same way, and the details are omitted. The lemma is therefore proven.

So far we have proven the following: if we take the limit in equation (0.22) with $\hat{Y}^\varepsilon_\tau(f;\varphi)$ and $\hat{Y}^\varepsilon_\tau((\hat{a}\,\varphi)_\tau)$ replacing $Y^\varepsilon_\tau(f;\varphi)$ and $Y^\varepsilon_\tau((a\,\varphi)_\tau)$ then only I and I′ can survive. Now, $\tilde{f}$ has basis on $\{0,1,\ldots\}$ and ν_p is a product measure, so that:

$$\eta(x,t)\,[S(x+1)\tilde{f}_t-\nu_{p(\varepsilon x,\tau)}(\tilde{f})]=S(x+1)(\eta(-1,t)\tilde{f}_t)$$
$$-\nu_{p(\varepsilon x,\tau)}(\eta(-1)\tilde{f})-(\eta(x,t)-p(\varepsilon x,\tau))\,\nu_{p(\varepsilon x,\tau)}(\tilde{f}).$$

From (2.45), and the expressions for I and I′ we are very close to the condition for applying (0.26); the only problem is that the test function, *i.e.*, $(\hat{a}\,\varphi)_\tau$ is time dependent. Since the time dependence is smooth and the variations on $(\hat{a}\,\varphi)_\tau$ on the integral in (0.26) (for each fixed T) are of order ε^2, they can be neglected.

We have therefore proven for $\mathrm{I}-\mathrm{I}'$ even a L_2-estimate.

It remains to prove that $\hat{Y}^\varepsilon_\tau(f;\varphi)-Y^\varepsilon_\tau(f;\varphi)$ and $\hat{Y}^\varepsilon_\tau((\hat{a}\,\varphi)_\tau)-Y^\varepsilon_\tau((a\,\varphi)_\tau)$ tend to zero in L_1-norm, as ε tends to zero.

The first difference is non random, and since $\mathbb{P}_\varepsilon(Y^\varepsilon_\tau(f;\varphi))=0$, it remains to see that $\mathbb{P}_\varepsilon(\hat{Y}^\varepsilon_\tau(f;\varphi))$ tends to zero, as $\varepsilon\to 0$. From Lemma 2.7 it is enough to see that $\lim_{\varepsilon\downarrow 0}\mathbb{P}_\varepsilon(\mathrm{I}+\mathrm{II})=0$. For this we write $\eta(x,t)$ in (2.40 *b*) as $(\eta(x,t)-p(\varepsilon x,\tau))+p(\varepsilon x,\tau)$. Obviously, the only contribution comes from the terms with $\eta(x,t)-p(\varepsilon x,\tau)$ and for these we may use the same argument as in Lemma 2.7 to see that their sum will tend to zero in L_1-norm.

For $\hat{Y}^\varepsilon_\tau((\hat{a}\,\varphi)_\tau)-Y^\varepsilon_\tau((a\,\varphi)_\tau)$ we can write it as a deterministic part, which is treated as $\hat{Y}^\varepsilon_\tau(f;\varphi)-Y^\varepsilon_\tau(f;\varphi)$, and a random part, which is

$$\sqrt{\varepsilon}\sum_u(\hat{a}(f;\varepsilon u,\tau)-a(f;\varepsilon u,\tau))\,\varphi(\varepsilon u))\,(\xi(u,t)-h(u,\varepsilon,t)).$$

After translating this into the S.E.P. we expand, as before, around $\bar{u}(x, \varepsilon, t)$ and write it as $\tilde{\mathrm{I}}+\mathrm{II}+\mathrm{III}+\tilde{\mathrm{E}}$ where $\tilde{\mathrm{I}}, \ldots, \tilde{\mathrm{E}}$ are completely analogous to $\mathrm{I}', \ldots, \mathrm{E}'$ defined in (2.48) but with $(\hat{a}(f; ., \tau)-a(f; ., \tau))\,\varphi(.)$ instead of $\varphi(.)$. From the smooth dependence on r of $a(f; r, \tau)$ and the definition of $\hat{a}(f; ., \tau)$ it is then easy to see that each of these tends to zero in L_1-norm. Since the τ-dependence is smooth (0.22) follows.

2.8. *Conclusion of the proof of Theorem* 0.3: We must check that any weak limit point $\mathscr{P}$ of the family $\mathscr{P}^{\varepsilon}(\varepsilon \to 0)$ verifies (*a*) and (*b*) of Remark 2.1.

Condition (*a*) is a direct consequence of Assumptions 1.2. Using (0.22) with $f(\xi)=\mathbf{1}(\xi(0)>0)$ it is easy to derive the first martingale relation in (*b*), after recalling (2.2*a*) and (2.7). The second relation in (*b*) follows easily from (2.26), (2.9) and Theorem 0.1.

5. APPENDIX

LEMMA A.1: *Let* ν^{ε} *satisfy Assumption* 1.2 *A and let* $\tilde{\nu}^{\varepsilon}$ *be the product probability measure on* $\{0, 1\}^{\mathbb{Z}}$ *with* $\tilde{\nu}^{\varepsilon}(\eta(x))=p(\varepsilon x)$. *Let* $\mathbb{P}_{\varepsilon}$ *and* $\tilde{\mathbb{P}}_{\varepsilon}$ *denote the laws of S.E.P. starting at time* 0 *with* ν^{ε} *and* $\tilde{\nu}^{\varepsilon}$, *respectively. Then for each* $\tau>0$ *fixed and* $t=\varepsilon^{-2}\tau$

$$\limsup_{\varepsilon \downarrow 0} \sup_{x_1 \neq x_2} \varepsilon^{-1}\left|\mathbb{P}_{\varepsilon}\left[\prod_{i=1}^{2}(\eta(x_i, t)-\mathbb{P}_{\varepsilon}(\eta(x_i, t)))\right]\right.$$
$$\left.-\tilde{\mathbb{P}}_{\varepsilon}\left[\prod_{i=1}^{2}(\eta(x_i, t)-\tilde{\mathbb{P}}_{\varepsilon}(\eta(x_i, t)))\right]\right|=0 \quad (a.1)$$

Proof: Let $\bar{\nu}^{\varepsilon}$ be as in Assumption 1.2 (A), and let $\bar{\mathbb{P}}_{\varepsilon}$ be the law of S.E.P. with initial measure $\bar{\nu}^{\varepsilon}$. From Theorem 5.1 of [5] it is enough to prove (*a*.1) with $\bar{\mathbb{P}}_{\varepsilon}$ instead of $\tilde{\mathbb{P}}_{\varepsilon}$. But this follows once we prove that

$$\lim_{\varepsilon \downarrow 0} \sup_{x_1 \neq x_2} \Gamma_{\varepsilon}(x_1, x_2)=0 \quad (a.2)$$

where

$$\Gamma_{\varepsilon}(x_1, x_2)=\varepsilon^{-1}\left|\mathbb{P}_{\varepsilon}\left(\prod_{i=1}^{2}(\eta(x_i, t)-p_i)\right)-\bar{\mathbb{P}}_{\varepsilon}\left(\prod_{i=1}^{2}(\eta(x_i, t)-p_i)\right)\right| \quad (a.3)$$

and

$$p_i=\sum_{x} \mathrm{P}(\mathrm{Y}(x_i, t)=z)\, p(\varepsilon z). \quad (a.4)$$

Also, from duality:

$$\Gamma_\varepsilon(x_1, x_2) = \varepsilon^{-1} \sum_{z_1, z_2} (\mathrm{Y}(x_1, t = z_1, \mathrm{Y}(x_2, t) = z_2)$$

$$\times \left[\nu^\varepsilon\left(\prod_{i=1}^{2} (\eta(z_i) - p_i) \right) - \bar{\nu}^\varepsilon\left(\prod_{i=1}^{2} (\eta(z_i) - p_i) \right) \right]. \quad (a.5)$$

Taking $\delta > 0$ small, splitting this sum according to: (a) $|z_i| > \varepsilon^{-\delta}$, $i = 1, 2$; (b) $|z_1| < \varepsilon^{-\delta}$, $|z_2| > 2\varepsilon^{-\delta}$ or $|z_2| < \varepsilon^{-\delta}$, $|z_1| > 2\varepsilon^{-\delta}$; (c) otherwise, and using the exponential decay of correlations for $\bar{\nu}^\varepsilon$ we get:

$$\Gamma_\varepsilon(x_1, x_2) \leqq 2\varepsilon^{-1} b_1' e^{-b_2 \varepsilon^{-\delta}}$$

$$+ 2\varepsilon^{-1} \mathrm{P}(|\mathrm{Y}(x_1, t)| \leqq 2\varepsilon^{-\delta}, |\mathrm{Y}(x_2, t)| \leqq 2\varepsilon^{-\delta})$$

$$+ 2\varepsilon^{-1} \sum_{z_1, z_2} \mathbf{1}(|z_1| < \varepsilon^{-\delta}) \mathbf{1}(|z_2| > 2\varepsilon^{-\delta}) \mathrm{P}(\mathrm{Y}(x_i, t) = z_i, i = 1, 2)$$

$$\times \left| \nu^\varepsilon\left(\prod_{i=1}^{2} (\eta(z_i) - p_i) \right) - \bar{\nu}^\varepsilon\left(\prod_{i=1}^{2} (\eta(z_i) - p_i) \right) \right|. \quad (a.6)$$

We must estimate the second and third terms on the r. h. s. of $(a.6)$. For the second term recall the coupling Q introduced in Section 4 of [5] between $(\mathrm{Y}(x_i, t),\ i = 1, 2)$ and $(\mathrm{Y}^0(x_i, t),\ i = 1, 2)$ independent random walks, so that for

$$\mathrm{Q}(|\mathrm{Y}(x_i, t) - \mathrm{Y}^0(x_i, t)| < \varepsilon^{-\beta'}\ i = 1, 2) > 1 - d_1 e^{-d_2 \varepsilon^{-(\beta'/2) + (1/4)}}$$

for some positive constants d_1, d_2. From this we easily get that

$$\mathrm{P}(|\mathrm{Y}(x_i, t)| \leqq 2\varepsilon^{-\delta}, i = 1, 2) \leqq \mathrm{C}\varepsilon^{2-\delta-\beta'} + d_1 e^{-d_2 \varepsilon^{-(\beta'/2)+(1/4)}}$$

and thus, taking $\beta' > 1/2$ and $\delta > 0$ small enough so that $\delta + \beta' < 1$ we get the desired estimate for the second term in $(a.6)$. Let us now look at the third term on the r. h. s. of $(a.6)$. From Assumption 1.2(a) this can be bounded above by

$$|\varepsilon^{-1} \Sigma \mathbf{1}(|z_1| < \varepsilon^{-\delta}) \mathbf{1}(|z_1| > 2\varepsilon^{-\delta}) \mathrm{P}(\mathrm{Y}(x_i, t) = z_i, i = 1, 2)$$

$$\times \nu^\varepsilon(\eta(z_1) - p_1) \bar{\nu}^\varepsilon(\eta(z_2) - p_2))|$$

$$+ \varepsilon^{-1} \Sigma \mathbf{1}(|z_1| < \varepsilon^{-\delta}) \mathbf{1}(|z_2| > 2\varepsilon^{-\delta}) \mathrm{P}(\mathrm{Y}(x_i, t) = z_i, i = 1, 2) \bar{\nu}^\varepsilon(\eta(z_1) - p_1)$$

$$\bar{\nu}^\varepsilon(\eta(z_2) - p_2) + 2\varepsilon^{-1} b_1'' e^{-b_2 \varepsilon^{-\delta}}| \quad (a.7)$$

Now these two terms are treated similarly. We give the argument for the first (the other is analogous, but simpler).

Dropping the condition $|z_2| > 2\varepsilon^{-\delta}$ does not make difference for $(a.2)$, since the error in doing this is of the order $\varepsilon^{-1}\ \varepsilon^{2-\delta-\beta'} = \varepsilon^{1-\delta-\beta'}$. Now, by Assumption 1.2 [condition (v)], and using the coupling Q we have

$$\Big|\varepsilon^{-1}\sum_{z_1,z_2} P(Y(x_i,t)=z_i, i=1,2)\,\mathbf{1}(|z_1|<\varepsilon^{-\delta})\,v^\varepsilon(\eta(z_1)-p_1)\,\bar{v}^\varepsilon(\eta(z_2)-p_2)\Big|$$

$$\begin{aligned}\leqq \Big|\varepsilon^{-1}\sum_{z_2,z_1,z_2^0} & P(Y(x_i,t)=z_i, Y^0(x_2,t)=z_2)\\ & \times \mathbf{1}(|z_2-z_2^0|<\varepsilon^{-\beta'}).\mathbf{1}(|z_1|<\varepsilon^{-\delta})\\ & \times v^\varepsilon(\eta(z_1)-p_1)(p(\varepsilon z_2)-p(\varepsilon z_2^0)+\varepsilon\theta_\varepsilon(z_2))\\ & + d_1\varepsilon^{-1}\exp(-d_2\varepsilon^{-\beta'/2+1/4}).\end{aligned} \quad (a.8)$$

where $\limsup_{\varepsilon\downarrow 0}\sup_z \theta_\varepsilon(z)=0$. Since p has a bounded derivative we easily get that the expression in $(a.8)$ is bounded by

$$d_1\varepsilon^{-1}e^{-d_2\varepsilon^{-(\beta'/2)+(1/4)}}+c\varepsilon^{-\beta'}P(|Y(x_1,t)|<\varepsilon^{-\delta})$$
$$\leqq d_1\varepsilon^{-1}e^{-d_2\varepsilon^{-(\beta'/2)+(1/4)}}+c\varepsilon^{1-\beta'-\delta}$$

and the lemma follows.

LEMMA A.2. — *Equations* (2.21), (2.25) *and* (2.26) *hold.*

Proof: To simplify let us just consider $x>0$. Then

$$u(x,0)=\sum_{0<y<x}\eta(y) \quad (a.9)$$

$$u(x,t)-u(x,0)=\sum_{y>x}\eta(y)\mathbf{1}(Y(y,t)\leqq x)-\sum_{y\leqq x}\eta(y)\mathbf{1}(Y(y,t)>x) \quad (a.10)$$

Taking averages in equations $(a.9)$ and $(a.10)$ we easily prove (2.21). For equation (2.25) we take squares in equations $(a.9)$ and $(a.10)$. A typical term will be $(\langle . \rangle=\mathbb{P}_\varepsilon(.))$

$$\varepsilon\sum_{\substack{y,y'\geqq x'\\ y\neq y'}}(\eta(y)\mathbf{1}(Y(y,t)\leqq x)-\langle . \rangle)(\eta(y')\mathbf{1}(Y(y',t)\leqq x)-\langle . \rangle)$$

where we used Liggett's inequality for $\mathbf{1}(Y(y,t)<x)$ and $\mathbf{1}(Y(y,t)<x)$ *cf.* [2] and [19]. By Assumption 1.2, the above sum is bounded by some suitable constant c. The other terms are treated similarly and equation (2.25) is therefore proven. We now prove equation (2.26). The term

$$\sum_{y\leqq x}(\eta(y)\mathbf{1}(Y(y,t)>x)-\langle . \rangle)$$

can be interpreted as the fluctuation on the number of particles which at time t are at the right of x if the initial distribution has particles only at the left of x and these are distributed according to (restricted to

$\mathbb{N} \cap (-\infty, x]$). The estimate follows from Theorem in [10], *cf.* also the footnote before equation (2.26).

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