ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 24, nº 1 (1988), p. 45-72
<http://www.numdam.org/item?id=AIHPB_1988__24_1_45_0>

Ann. Inst. Henri Poincaré,

Vol. 24, n° 1, 1988, p. 45-72. *Probabilités et Statistiques*

The martingale problem with sticky reflection conditions, and a system of particles interacting at the boundary

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ABSTRACT. — We give a martingale problem with Wentzell boundary conditions under its most natural and general form, without any assumption on the generator. We show the usual boundary sojourn-time condition is a consequence of the martingale problem as soon as the generator is sufficiently non-degenerate, and in the general case give a weaker condition which behaves well under limiting procedures. We develop the time-change theory, and the relationship with some generalized stochastic differential equations. We then give results on existence and uniqueness, some of the former by a limiting procedure, and some examples. We eventually construct a system of interacting particles, with an interaction in the sojourn term, as a limit in law.

Key words : Martingale problems, stochastic differential equations, sticky reflecting boundary conditions, local times, systems of interacting particles.

RÉSUMÉ. — Nous posons un problème de martingales avec conditions frontière de Ventcel sous sa forme la plus naturelle et générale, sans aucune hypothèse sur le générateur. Nous montrons que la condition habituelle de temps de séjour à la frontière découle du problème de martingales dès

Classification A.M.S. : Primary 60 H, 60 H 10, 60 K 35. Secondary : 60 B 10, 60 G 44, 60 J 50, 60 J 55.

que le générateur est suffisamment non dégénéré, et dans le cas général donnons une condition plus faible qui se comporte bien lors de passages à la limite. Nous développons la théorie du changement de temps, et la relation avec des équations différentielles stochastiques généralisées. Nous donnons alors des résultats d'existence et d'unicité, certains de ces premiers par des passages à la limite, et des exemples. Enfin nous construisons un système de particules en interaction, avec interaction dans le terme de séjour, en tant que limite en loi.

Mots clés : Problèmes de martingales, équations différentielles stochastiques, conditions frontière avec réflexion collante, temps locaux, systèmes de particules en interaction.

INTRODUCTION

Our first aim was to study a system of interacting particles reflecting in a domain θ with sticky boundary, with an interaction in the "sojourn" term. This was to model for example what happens in a chromatography tube, where molecules in a gaseous state pushed by a flow of neutral gas are absorbed and released by a liquid state deposited on the tube; what is measured is the "mean" time taken by a molecule to get through the tube.

The idea would be to construct such a system, and to investigate its asymptotic behaviour when the number of particles goes to infinity (supposing a mean-field interaction) : propagation of chaos, Gaussian fluctuations, large deviations, etc. But the construction itself is a non-classical problem, taking place in an "angular" domain θ^N, with an unusual boundary sojourn condition. For this construction, we will use the results in [9] plus a limiting procedure.

The "classical" martingale problem with boundary conditions, or the "classical" sub-martingale problem ([3], [5], [8]), are ill-suited to the limiting procedures we use for the construction and for the asymptotics. This is because of the presence of discontinuous terms $\mathbf{1}_\theta$ and $\mathbf{1}_{\partial\theta}$, either in the body of the (sub-)martingale problem or in the sojourn-time condition at the boundary. Thanks to the classical hypothesis of non-degeneracy of the generator, which enables the authors to "control" the component of the diffusion normal to the boundary, results using limiting procedures can nevertheless be obtained : *see* Proposition 20 in [3], Theorem 3.1 in [8].

For our purposes, we chose to develop an approach to the martingale problem, in its most natural form, which avoids the use of $\mathbf{1}_\theta$ or $\mathbf{1}_{\partial\theta}$ either explicitly or implicitly through the sojourn-time condition at the boundary. We show that under the classical hypothesis on the generator, the usual sojourn-time condition is implied by the martingale problem. As we do not assume this hypothesis, we replace the usual sojourn-time condition by a weaker one that behaves well under limiting procedures. Our approach is in accordance with the trajectorial approaches in [1], [2], [6], [9]. It is to be noted that the usual sojourn-time condition is too restrictive to get results when the generator is too degenerate.

Our paper is organized as follows:

In part I, we define our martingale problem and give different equivalent definitions including a sub-martingale problem. We show that if the non-degeneracy hypothesis holds, the usual sojourn condition is implied by the martingale problem. In the general case, we weaken the sojourn condition and use this to develop a time-change theory. Then we show the relationship with stochastic differential equations and use it to get existence and uniqueness results, some of which involving limiting procedures. We then give examples showing the greater generality obtained, and finish by a discussion of time-inhomogeneous problems.

In part II, we describe our system of interacting particles and give examples showing it cannot be treated by usual means. Then we give an original construction as a limit in law.

I. MARTINGALE PROBLEMS WITH BOUNDARY CONDITIONS

1. The framework

Let us consider a diffusion reflecting under the most general conditions (Wentzell's boundary conditions) in an open subset θ of R^d.

We suppose θ is given by $\Phi \in C_b^2(\mathbb{R}^d)$ as follows:

$$\theta = \{x \in \mathbb{R}^d,\ \Phi(x) > 0\},$$
$$\partial\theta = \{x \in \mathbb{R}^d,\ \Phi(x) = 0\}$$

with for $x \in \partial\theta$, $|\nabla\Phi(x)| = 1$. For such x, $\nabla\Phi(x)$ is then the normal vector n, pointing inwards.

Set $\Omega^0 = C(\mathbb{R}+, \bar{\theta})$, X the canonical process, $\mathscr{F}_t^0 = \sigma(X_s, s \leqq t)$.

For $1 \leqq i, j \leqq d$, a_{ij} and b_i will denote real-valued bounded measurable functions on $\bar{\theta} \times \mathbb{R}+$, such that the matrix $a=(a_{ij})$ is symmetric and $\geqq 0$; for $x \in \bar{\theta}$, $t \in \mathbb{R}+$, $f \in C_b^2(\bar{\theta})$, define

$$L_t f(x) = \frac{1}{2} \sum_{i,j} a_{ij}(x, t) \partial_{ij}^2 f(x) + \sum_i b_i(x, t) \partial_i f(x) \qquad (I.1)$$

the summations being from 1 to d.

Let us give a Wentzell-type boundary condition : for $1 \leqq i, j \leqq d$ α_{ij}, γ_i, ρ real-valued bounded measurable functions on $\partial\theta \times \mathbb{R}+$, such that $\alpha=(\alpha_{ij})$ is symmetric and $\geqq 0$, $\alpha n=0$, $\rho \geqq 0$; for $x \in \partial\theta$, $t \in \mathbb{R}+$, $f \in C_b^2(\partial\theta)$, define

$$\Lambda_t f(x) = \frac{1}{2} \sum_{i,j} \alpha_{ij}(x, t) \partial_{ij}^2 f(x). \qquad (I.2)$$

We say that a probability measure P on Ω^0 solves the martingale problem (L, Γ, ρ) starting at $x \in \bar{\theta}$ if $P(X_0=x)=1$ and if there exists an increasing, adapted process K, with $E(K_t) < +\infty$, $K_t = \int_0^t \mathbf{1}_{\partial\theta}(X_s)\, dK_s$, such that for $f \in C_b^2(\bar{\theta})$

$$M_t^f = f(X_t) - f(X_0) - \int_0^t L_s f(X_s)\, ds$$
$$- \int_0^t (\langle \gamma_s, \nabla \rangle + \Lambda_s - \rho_s L_s) f(X_s)\, dK_s \qquad (I.3)$$

is a continuous $(P, \mathscr{F}_t^0)$-martingale. Here, ∇f denotes the gradient of f, $\langle\ ,\ \rangle$ the scalar product in $\mathbb{R}^d$, and $\langle \gamma, \nabla \rangle$ is the derivative along γ of f, sometimes denoted by $\frac{\partial f}{\partial \gamma}$. Γ denotes $\langle \gamma, \nabla \rangle + \Lambda$, and it is easy to see that if P solves the martingale problem (L, Γ, ρ), then it will solve it for $(L, g\Gamma, g\rho)$, where g is any measurable function on $\partial\theta \times \mathbb{R}+$ such that for a constant c, $g > c > 0$; it suffices to change K_t to $\int_0^t g(X_s, s)^{-1}\, dK_s$.

Note that we don't ask for any non-degeneracy assumption on L, and that $\langle n, an \rangle$ may well vanish. In [3], [5], the authors ask for $\langle n, an \rangle > c > 0$, and in [8] a is uniformely elliptic.

Remark 1. — We could consider a martingale problem on $C(\mathbb{R}+, \bar{\theta}) \times C(\mathbb{R}+, \mathbb{R})$, the canonical process being then (X, K). This is sometimes more convenient, especially when taking limits. If we check that K is an adapted process of X, then the law of X will solve our martingale problem. All results to our martingale problem that don't use the adaptation of K to X extend to the martingale problem on $C(\mathbb{R}+, \bar{\theta}) \times C(\mathbb{R}+, \mathbb{R})$, and some can be used to prove this adaptation.

Remark 2. — We could as well take the coefficients to be predictable processes.

Remark 3. — In order to simplify our notations, we take θ to be $\{x \in \mathbb{R}^d, x^1 > 0\}$ and $\Phi(x) = 1 - \exp(-x^1)$ or even x^1 if we don't need the boundedness. Then for example $\alpha n = 0$ means the first row (and the first column) of α is equal to zero. All proofs could be carried out with a general Φ, and to interpret the results we only need to remplace $1 - \exp(-x^1)$ by the general Φ, a_{11} by $n^* an = \langle n, an \rangle$, b_1 by $\langle n, b \rangle$, γ_1 by $\langle n, \gamma \rangle$, etc. Remarks will help this interpretation. Γ may then be written $\gamma_1 \partial_1 + \Lambda^1$, where Λ^1 is a second order operator on $\{x^1 = 0\}$.

For $v \in \mathbb{R}^d$, $Y \in \mathbb{R}^d$, A $d \times d$ matrix, set

$$Y^v = \langle v, Y \rangle, \qquad A^{vv} = v^* A v = \langle v, A v \rangle.$$

For a continuous martingale M, $\langle M \rangle$ is its quadratic variational process as in [5], sometimes referred to as the increasing process of the martingale.

THEOREM I.1. — *The following propositions are all equivalent :*

(*a*)
$$M_t^f = f(X_t) - f(X_0) - \int_0^t L_s f(X_s)\, ds - \int_0^t (\langle \gamma_s, \nabla \rangle + \Lambda_s - \rho_s L_s) f(X_s)\, dK_s$$

is for all $f \in C_0^\infty(\bar{\theta})$ *a continuous* $(P, \mathscr{F}_t^0)$*-martingale.*

(*b*) M_t^f *is for all* $f \in C_b^2(\bar{\theta})$ *a continuous* $(P, \mathscr{F}_t^0)$*-martingale.*

(*c*) M_t^f *is for all* $f \in C^2(\bar{\theta})$ *a continuous* $(P, \mathscr{F}_t^0)$*-local martingale.*

(*d*)
$$M_t^v = X_t^v - X_0^v - \int_0^t b^v(X_s, s)(ds - \rho_s(X_s)\, dK_s) - \int_0^t \gamma^v(X_s, s)\, dK_s$$

is for all $v \in \mathbb{R}^d$ *a continuous* $(P, \mathscr{F}_t^0)$*-local martingale, and*

$$\langle M^v \rangle_t = \int_0^t a^{vv}(X_s, s)(ds - \rho_s(X_s)\, dK_s) + \int_0^t \alpha^{vv}(X_s, s)\, dK_s.$$

Moreover, for any $f \in C^2$, *we have*

$$\langle M^f \rangle_t = \int_0^t \langle \nabla f, a \nabla f \rangle (X_s, s) (ds - \rho_s (X_s) d K_s) + \int_0^t \langle \nabla f, \alpha \nabla f \rangle (X_s, s) d K_s$$

and using the boundedness of the coefficients and $E(K_t) < +\infty$, *we see that the* M^v *for* $v \in \mathbb{R}^d$ *and the* M^f *for* f *with a bounded first derivative are actually* L^2-*martingales.*

Proof. — The proof is the same as in [3], [8]. $\langle M^f \rangle$ is easily established from (*d*) thanks to the Itô formula. □

THEOREM I.2. — *If on the boundary, either* $\gamma_1 - \rho b_1 > C > 0$, *or* $\gamma_1 + \rho > C > 0$ *and* $a_{11} > C > 0$, *then the martingale problem is equivalent to the following sub-martingale problem:*

$$S_t^f = f(X_t) - f(X_0) - \int_0^t L_s f(X_s) ds \qquad (I.4)$$

is a continuous $(P, \mathscr{F}_t^0)$-*submartingale for all* $f \in C_0^\infty$ *such that* $(\langle \gamma, \nabla \rangle + \Lambda - \rho L) f \geqq 0$ *on* $\partial \theta$.

Proof. — That the martingale problem implies the submartingale one is trivial. The other implication goes as follows:

Take $\Phi^\lambda (x) = \frac{1}{\lambda} (1 - \exp(-\lambda x^1))$. Then

$$\partial_1 \varphi^\lambda (x) = \exp(-\lambda x^1), \qquad \partial_{11}^2 \varphi^\lambda (x) = -\lambda \exp(-\lambda x^1),$$

and on the boundary,

$$(\langle \gamma, \nabla \rangle + \Lambda - \rho L) \Phi^\lambda = \gamma_1 + \rho \frac{\lambda}{2} a_{11} - \rho b_1 > C > 0$$

for a well-chosen λ. We can then use this Φ^λ instead of Φ in the classical proof of [3] and [8]. □

Remark 1. — The condition $\gamma_1 + \rho > C > 0$ and $a_{11} > C > 0$ is the most general condition considered in [3]. Generally in [3] and [8], the authors assume $\gamma_1 > C > 0$, $a_{11} > C > 0$.

Remark 2. — Naturally, in the case of a general Φ we should consider

$$\Phi^\lambda(x)=\frac{1}{\lambda}(1-\exp(-\lambda\Phi(x))).$$

An easy calculation yields $\Phi^\lambda(x)>C>0$ on $\partial\theta$, provided $\langle n, \gamma\rangle-\rho L\Phi>C>0$, or $\langle n, \gamma\rangle+\rho>C>0$ and $\langle n, an\rangle>C>0$.

THEOREM I.3. — *Assume that* $X_0\in\partial\theta$, *and that on a neighbourhood we either have* $a_{11}>0$ *and* $\frac{b_1}{a_{11}}<C$, *or* $b_1<0$. *Let* P *solve the martingale problem. Then* P$-$*a. s.*, $\forall t>0$, $K_t>0$.

Proof:

$$X_t^1=\int_0^t b_1(X_s)(ds-\rho(X_s)\,dK_s)+\int_0^t \gamma_1(X_s)\,dK_s+M_t,$$

with

$$\langle M\rangle_t=\int_0^t a_{11}(X_s)(ds-\rho(X_s)\,dK_s).$$

By the Girsanov Theorem [5], we have on $\mathscr{F}_1^0$ a probability $\bar{P}$ equivalent to P, under which $\bar{M}_t=M_t+\int_0^t b_1(X_s)\mathbf{1}_{b_1(X_s)>0}(ds-\rho(X_s)\,dK_s)$ is a martingale for $0\leqq t\leqq 1$.

Set $T=\inf\{t>0, K_t>0\}\wedge 1$. Then

$$\bar{E}(X_{t\wedge T}^1)=\bar{E}\left(\int_0^{t\wedge T} b_1(X_s)\mathbf{1}_{b_1(X_s)<0}\,ds\right)\leqq 0;$$

as $X_{t\wedge T}^1\geqq 0$, we get $X_{t\wedge T}^1=0$ and $\int_0^{t\wedge T}\mathbf{1}_{b_1(X_s)<0}\,ds=0$. Then we see that $\bar{M}_{t\wedge T}=0$, so $\bar{E}(\langle\bar{M}\rangle_T)=0$ and $\int_0^T \mathbf{1}_{a_{11}(X_s)>0}\,ds=0$. So $T=0$. All this is $\bar{P}$-a. s., and thus P-a. s. □

Remark. — We cannot have a better result. For an example, consider a Brownian motion B starting at zero, $\bar{B}_s=B_s$ if $B_s\neq 0$, $\bar{B}_s=1$ if $B_s=0$.

Then $B_t^2=2\int_0^t \bar{B}_s\,dB_s+t$, and $K=0$.

2. The sojourn condition and the time-change

We now give the fundamental results we will use when we take limits. We will first see that the usual sojourn condition $\mathbf{1}_{\partial\theta}(X_s)\,ds=\rho_s(X_s)\,dK_s$ is implied by the martingale problem as soon as $a_{11}>0$. If this inequality does not hold, we will use as a sojourn condition the measure inequality $\mathbf{1}_{\partial\theta}(X_s)\,ds \geqq \rho(X_s, s)\,dK_s$; $\partial\theta$ being closed, this inequality will behave properly when we take limits, and is trivial when $\rho=0$ as in [1], [2], [6]. We then generalize the time-change theory, provided $\gamma_1>C>0$. Note that in [3], [5], [8], the authors assume $a_{11}>C>0$, $\gamma_1>C>0$.

THEOREM I.4. — *Suppose that* P *solves the martingale problem. Then* P-*a. s.*, $\forall t \in \mathbb{R}+$,

$$\begin{aligned}\int_0^t \mathbf{1}_{\partial\theta}(X_s)\,dX_s^1 &= \frac{1}{2} L_t^0(X^1)\\ &= \int_0^t b_1(X_s, s)\,(\mathbf{1}_{\partial\theta}(X_s)\,ds-\rho(X_s, s)\,dK_s)\\ &\qquad + \int_0^t \gamma_1(X_s, s)\,dK_s \qquad (I.5)\end{aligned}$$

and these are all increasing processes.

Furthermore,

$$\int_0^t \mathbf{1}_{a_{11}(X_s, s)\neq 0}\,\mathbf{1}_{\partial\theta}(X_s)\,ds=\int_0^t \mathbf{1}_{a_{11}(X_s, s)\neq 0}\,\rho(X_s, s)\,dK_s \qquad (I.6)$$

and naturally if for $x \in \partial\theta$, $a_{11}(x, s) \neq 0$, *then*

$$\int_0^t \mathbf{1}_{\partial\theta}(X_s)\,ds=\int_0^t \rho(X_s, s)\,dK_s. \qquad (I.7)$$

Proof. — The first equality of (I.5) follows from the fact that $X_t^1 \geqq 0$ and thus $X_t^1=(X_t^1)^+$, and using the Tanaka formula [12]:

$$0=X_t^1-(X_t^1)^+=\int_0^t \mathbf{1}_{X_s^1=0}\,dX_s^1-\frac{1}{2} L_t^0(X^1).$$

We then use Theorem I.1.(*a*) with v the first vector of the canonical basis of $\mathbb{R}^d$, and integrate $\mathbf{1}_{\partial\theta}(X)$ with respect to M^1. $\int_0^t \mathbf{1}_{\partial\theta}(X_s)\,dM_s^1$ is then a continuous martingale which is also of bounded variation, and is therefore constantly equal to 0. We thus get the second equality in (I.5), and (I.6) by considering the increasing process of this martingale. □

Since $L^0(X^1)$ is increasing, we have:

$$\mathbf{1}_{b^1>0}\mathbf{1}_{\partial\theta}(X_s)\,ds \geqq \mathbf{1}_{b^1>0}(\rho-\gamma^1/b^1)(X_s)\,dK_s \tag{I.8}$$

$$\mathbf{1}_{b^1<0}\mathbf{1}_{\partial\theta}(X_s)\,ds \leqq \mathbf{1}_{b^1<0}(\rho-\gamma^1/b^1)(X_s)\,dK_s. \tag{I.8$'$}$$

THEOREM I.5. — *Let* P *solve the martingale problem, and set* $\theta_\varepsilon=\{x\in\theta,\ 0<x^1<\varepsilon\}$. *Then* P-*a. s.* $\frac{1}{2\varepsilon}\int_0^t a_{11}(X_s, s)\mathbf{1}_{\theta_\varepsilon}(X_s)\,ds$ *converges, uniformely on compact sets, to*

$$\int_0^t b_1(X_s, s)(\mathbf{1}_{\partial\theta}(X_s)\,ds-\rho(X_s, s)\,dK_s)+\int_0^t \gamma_1(X_s, s)\,dK_s \tag{I.9}$$

the convergence being also in all the L^p, $1\leqq p<+\infty$.

Proof. — We use (I.5) and

$$L_t^0(X^1)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t \mathbf{1}_{0<X_s^1<\varepsilon}\,d\langle X^1\rangle_s,$$

plus the fact that a sequence of increasing continuous functions converging pointwise to a continuous function actually converges uniformely on compact sets.

Since

$$\frac{1}{\varepsilon}\int_0^t \mathbf{1}_{0<X_s^1<\varepsilon}\,d\langle X^1\rangle_s=\frac{1}{\varepsilon}\int_0^\varepsilon L_t^a\,da\leqq \sup_{0<a<\varepsilon} L_t^a,$$

the Tanaka equation, the boundedness of the coefficients and the explicit solution of the Skorokhod equation (Lemma III 4.2 in [5]) show that the convergence is bounded in all the L^p and thus takes place in all the L^p, $1\leqq p<+\infty$. □

THEOREM I.6. — *Suppose* P *solves the martingale problem*, $\gamma_1>C>0$, *and that* P-*a. s.*, $\mathbf{1}_{\partial\theta}(X_s)\,ds\geqq\rho(X_s, s)\,dK_s$.

Set $A_t=t-\int_0^t \rho(X_s, s)\,dK_s$. *Then* P-*a. s.* A *is strictly increasing*, $\lim_{t\to\infty} A_t=+\infty$, *and if* $\bar{A}$ *denotes its inverse,*

$$\bar{A}_t=t-\int_0^t \rho(X_{\bar{A}_s}, \bar{A}_s)\,dK_{\bar{A}_s}. \tag{I.10}$$

Proof. — A is increasing, denote by $\bar{A}$ its right-continuous inverse: $\bar{A}_t=\inf\ \{s\geqq 0,\ A_s>t\}$. Then: $\forall s\in[\bar{A}_{t-},\ \bar{A}_t[,\ A_s=t$ and so $\forall s\in[\bar{A}_{t-},\ \bar{A}_t[,\ X_s\in\partial\theta$.

Set $\Phi(x)=1-\exp(-x^1)$,

$$M_u=\Phi(X_u)-\Phi(X_0)-\int_0^u (b_1-a_{11}/2)(X_s,s)\times\exp(-X_s^1)\,dA_s-\int_0^u \gamma_1(X_s,s)\,dK_s \quad (I.11)$$

is a continuous $(\mathscr{F}_u^0)$-martingale, and thus a $(\mathscr{F}_u^+)$-martingale where $\mathscr{F}_u^+=\bigcap_{s\geqq u}\mathscr{F}_s^0$. $\bar{A}_{t-}$ and $\bar{A}_t$ are $(\mathscr{F}_u^+)$-stopping times.

$$\langle M\rangle_u=\int_0^u a_{11}(X_s,s)\exp(-2X_s^1)\,dA_s$$

and M is uniformely bounded in L^2 before $\bar{A}_t$. We may then use the optional sampling theorem to get $E(M_{\bar{A}_t}-M_{\bar{A}_{t-}})=0$. Since A does not increase between $\bar{A}_{t-}$ and $\bar{A}_t$, $\Phi(x)=0$ for $x\in\partial\theta$, and $\gamma_1>0$, we see that $K_{\bar{A}_{t-}}=K_{\bar{A}_t}$ and thus $\bar{A}_{t-}=\bar{A}_t$. $\bar{A}$ is then continuous and hence A is strictly increasing.

By the optional sampling theorem, $E(M_{\bar{A}_n})=0$, and since

$$E\left[\Phi(X_{\bar{A}_n})-\Phi(X_0)-\int_0^{\bar{A}_n}(b_1-a_{11}/2)\exp(-X_s^1)\,dA_s\right]$$

is bounded, we get $E\left(\int_0^{\bar{A}_n}\gamma_1\,dK_s\right)<+\infty$. It is clear that on $\{\bar{A}_n=+\infty\}$, $K_{\bar{A}_n}=K_\infty=+\infty$ and since $\gamma_1>C>0$, $\int_0^{\bar{A}_n}\gamma_1\,dK_s=+\infty$; so $\forall n\in\mathbb{N}$, $\bar{A}_n<+\infty$ P-a. s., and $A_\infty=+\infty$.

$$t=A_t+\int_0^t \rho(X_s,S)\,dK_s \Rightarrow \bar{A}_t=t+\int_0^t \rho(X_{\bar{A}_s},\bar{A}_s)\,dK_{\bar{A}_s}$$

by a simple time-change. □

Remark. — If $\int_0^t \mathbf{1}_{\partial\theta}(X_s)\,ds=\int_0^t \rho(X_s,s)\,dK_s$, then $A_t=\int_0^t \mathbf{1}_\theta(X_s)\,ds$.

Moreover by using Theorem I.4 we see that $\mathbf{1}_{\partial\theta}(X_s)\,ds\geqq\rho(X_s,s)\,dK_s$ as soon as $\mathbf{1}_{a^{11}=0}\mathbf{1}_{\partial\theta}(X_s)\geqq\mathbf{1}_{a^{11}=0}\,\rho(X_s,s)\,dK_s$.

If $\alpha=0$, Theorem I.1 (*d*) shows that it is enough that $\mathbf{1}_{a=0}\,\mathbf{1}_{\partial\theta}(X_s)\,ds \geqq \mathbf{1}_{a=0}\,\rho(X_s, s)\,dK_s$ (since the increasing process of a martingale is indeed increasing). Consider also (I.8), (I.8)′.

We can also note that as soon as $\gamma_1>0$, A is strictly increasing but could have a finite limit.

From now on, we only consider time-homogenous problems, that is, a, b, α, γ, ρ do not depend on time. We will discuss later how to get results for time-homogenous problems by adding time as a new coordinate.

We will consider the martingale problem with a boundary sojourn condition.

We will say that P is a solution to the martingale problem (L, Γ, ρ) with sojourn condition if for a certain K, P is a solution to the martingale problem (L, Γ, ρ) as in (I.3) and moreover P-a.s., we have

$$\mathbf{1}_{\partial\theta}(X_s)\,ds \geqq \rho(X_s)\,dK_s. \tag{I.12}$$

The previous remark is still of interest, as is the discussion at the beginning of 2. Naturally, $a_{11}\geqq 0$, but that is all we need. We also see that the boundedness of ρ is not important.

We see that when $\rho=0$ as in [1], [2], [6], (I.12) is automatically true and doesn't have to be stated.

Also, (I.12) can be restated by saying that

$$ds \geqq \rho(X_s)\,dK_s \qquad \text{(I.12)}'$$

or that $t-\int_0^t \rho(X_s)\,dK_s$ is an increasing process.

THEOREM I.7. — *Let* P *be a solution to the martingale problem* (L, Γ, ρ) *with sojourn condition, and* K *be the increasing process in* (I.3). *Let* ρ *be another positive measurable function on* $\partial\theta$, *and suppose either* $\bar\rho\geqq\rho$ *or* $\gamma_1>C>0$. *Set*

$$A_t = t + \int_0^t (\bar\rho-\rho)(X_s)\,dK_s \tag{I.13}$$

Then A *is a strictly increasing process which tends towards infinity. Denote by* $\bar A$ *its inverse,* $\bar X = X_{\bar A}$, $\bar K = K_{\bar A}$, $\bar P$ *the law on* Ω^0 *of* $\bar X$.

Then $\bar P$ *solves the martingale problem* (L, Γ, $\bar\rho$) *with sojourn condition, and* $\bar K$ *will be the corresponding increasing process.*

Moreover, $\bar A_t = t + \int_0^t (\rho-\bar\rho)(\bar X_s)\,d\bar K_s$.

Proof. — That $\bar{A}$ is strictly increasing to infinity is obvious using either $\bar{\rho} \geqq \rho$ or Theorem I.6 when $\gamma_1 > C > 0$. We get $\bar{A}$ in much the same way as in Theorem I.6 $\{\bar{A}_{t-} \leqq T\} = \{t \leqq A_T\} \in \mathscr{F}_T^0$ and thus $\bar{A}_{t-}$ is a $(\mathscr{F}_t^0)$-stopping time; A being strictly increasing, $\bar{A}_{t-} = \bar{A}_t$. That $\bar{K}$ increases only when $\bar{X}$ is on the boundary is trivial from the similar property of K and X. Also, $\bar{X}$ and $\bar{K}$ are continuous.

For $f \in C_b^2$,

$$M_u^f = f(X_u) - f(X_0) - \int_0^u Lf(X_s)(ds - \rho(X_s)\, dK_s) - \int_0^u \Gamma f(X_s)\, ds$$

is a $(P, \mathscr{F}_u^0)$-martingale.

First, let's prove $E(\bar{K}_t) < +\infty$. If $\bar{\rho} \geqq \rho$, then $E(\bar{K}_t) \leqq E(K_t) < +\infty$. If $\gamma_1 > C > 0$, take $\Phi(x) = 1 - \exp(-x^1)$,

$$M_u^\Phi = \Phi(X_u) - \Phi(X_0) - \int_0^u \left(b_1 - \frac{1}{2} a_{11}\right)(X_s) \times \exp(-X_s^1)\,(ds - \rho(X_s)\, dK_s) - \int_0^u \gamma_1(X_s)\, dK_s$$

is a martingale, and

$$\langle M^\Phi \rangle_u = \int_0^u a_{11}(X_s) \exp(-2X_s^1)\,(ds - \rho(X_s)\, dK_s);$$

$$\langle M^\Phi \rangle (\bar{A}_t) = \int_0^t a_{11}(\bar{X}_s) \exp(-2\bar{X}_s^1)\,(ds - \bar{\rho}(\bar{X}_s)\, dK_s)$$

and thus M^Φ is uniformely bounded in L^2 before $\bar{A}_t$. So $E(\bar{M}_t^\Phi) = 0$ and by a boundedness argument, as $\gamma_1 > C > 0$, $E(\bar{K}_t) < +\infty$.

Now using Theorem I.1 we see that $\langle M^f \rangle$ is uniformely bounded in L^1 before $\bar{A}_t$. So M^f is uniformely bounded in L^2 before $\bar{A}_t$, and by the optional sampling theorem, $M^f(\bar{A})$ is a $(P, \mathscr{F}^0(\bar{A}_t))$-martingale.

$$d\bar{A}_s - \rho(X(\bar{A}_s))\, d\bar{K}_s = d\bar{A}_s - \rho(\bar{X}_s)\, d\bar{K}_s$$
$$= ds + (\rho - \bar{\rho})(\bar{X}_s)\, d\bar{K}_s - \rho(\bar{X}_s)\, d\bar{K}_s = ds - \bar{\rho}(\bar{X}_s)\, d\bar{K}_s.$$

So $\bar{P}$ solves the martingale problem $(L, \Gamma, \bar{\rho})$ with $\bar{K}$ as an increasing process.

$$\mathbf{1}_{\partial\theta}(\bar{X}_s)\, ds = \mathbf{1}_{\partial\theta}(\bar{X}_s)\,(d\bar{A}_s - (\rho - \bar{\rho})(X_s)\, d\bar{K}_s)$$

and since $\mathbf{1}_{\partial\theta}(X_s)\,ds \geqq \rho(X_s)\,dK_s$, we have the sojourn condition $\mathbf{1}_{\partial\theta}(\bar{X}_s)\,ds \geqq \bar{\rho}(\bar{X}_s)\,d\bar{K}_s$. □

Let us remark that we get X and K from $\bar{X}$ and $\bar{K}$ by taking $\bar{A}_t = t + \int_0^t (\rho - \bar{\rho})(\bar{X}_s)\,d\bar{K}_s$ and $X = \bar{X}(\bar{A}^{-1})$ $K = \bar{K}(\bar{A}^{-1})$. Hence:

THEOREM I.8. — *Suppose* $\gamma_1 > C > 0$. *We have a bijective mapping between the set of solutions of the martingale problem* (L, Γ, ρ) *with sojourn condition and the one for* $(L, \Gamma, \bar{\rho})$, *and between the corresponding increasing processes* K *and* $\bar{K}$.

Thus the existence, or the uniqueness, for these two martingale problems are equivalent notions.

Also, if P *is a solution for the martingale problem* (L, Γ, ρ) *with sojourn condition there is an unique* K *such that* (I.3) *holds.*

Proof. — The bijection is given in Theorem I.7 and in the remark that follows its proof. If $\rho = 0$, Theorem I.5 yields the uniqueness of K using $\gamma_1 > C > 0$, and the general case follows by use of the bijection. □

Remark. — We actually have a much better result of uniqueness of K. Indeed, using Theorem I.1 (*d*) and the fact that $\langle M^v \rangle$ is the quadratic variational process of X^v, we see that $\int_0^t \mathbf{1}_{\alpha - \rho a \neq 0}(X_s)\,dK_s$ is a function of X_s, $0 \leqq s \leqq t$. Using the time-change in Theorem I.7 for $\bar{\rho} = \rho + 1 \geqq \rho$, and the reverse time change, we see that $\int_0^t \mathbf{1}_{\alpha - (\rho+1) a \neq 0}(X_s)\,dK_s$ is a function of X_s, $0 \leqq s \leqq t$, and so are both $\int_0^t \mathbf{1}_{a \neq 0}(X_s)\,dK_s$ and $\int_0^t \mathbf{1}_{\alpha \neq 0}(X_s)\,dK_s$. In a similar way, so are $\int_0^t \mathbf{1}_{b \neq 0}\mathbf{1}_{a = 0}\mathbf{1}_{\alpha = 0}\,dK_s$ and $\int_0^t \mathbf{1}_{\gamma \neq 0}\mathbf{1}_{a = 0}\mathbf{1}_{\alpha = 0}\,dK_s$. In fact, dK_t is determined as a function depending only on a, α, b, γ, ρ of X_s, $0 \leqq s \leqq t$, as soon as $a(X_t) \neq 0$ or $\alpha(X_t) \neq 0$ or $b(X_t) \neq 0$ or $\gamma(X_t) \neq 0$, and is arbitrary whenever these four quantities vanish simultaneously on the boundary. A natural choice in this case would be $dK_t = 0$, wich is compatible with (I.12).

It is important to see all this is true even if we consider the martingale problem on $C(\mathbb{R}+, \bar{\theta}) \times C(\mathbb{R}+, \mathbb{R})$; it enables us to get back to $\Omega^0 = C(\mathbb{R}+, \bar{\theta})$.

Using this, the sojourn condition $\mathbf{1}_{\partial\theta}(X_s)\,ds \geqq \rho(X_s)\,dK_s$, the Radon-Nykodym Theorem, we first see that

$$\rho(X_s)\,dK_s = r_s\,ds \tag{I.14}$$

with $0 \leqq r_s \leqq \mathbf{1}_{\partial\theta}(X_s)$, and $r_s = \lim_{\eta \to 0} \frac{1}{\eta} \int_s^{s+\eta} \rho(X_u)\,dK_u$ and so for any $\eta > 0$, $r_s = r(X_u,\ s-\eta \leqq u \leqq s)$ with r depending only on a, α, b, γ, ρ.

Let's summarize all this: let (X, K) be the coordinate process on $C(\mathbb{R}+, \bar{\theta}) \times C(\mathbb{R}+, \mathbb{R})$.

THEOREM I.9. — *Let $\bar{P}$ be a solution on $C(\mathbb{R}+, \bar{\theta}) \times C(\mathbb{R}+, \mathbb{R})$ of the martingale problem. If for any x on the boundary, either $a(x) \neq 0$ or $\alpha(x) \neq 0$ or $b(x) \neq 0$ or $\gamma(x) \neq 0$, then K_t is an unambiguous function of X_s, $0 \leqq s \leqq t$. We may always take*

$$K'_t = K_t - \int_0^t \mathbf{1}_{a\,(X_s)=0}\,\mathbf{1}_{\alpha\,(X_s)=0}\,\mathbf{1}_{b\,(X_s)=0}\,\mathbf{1}_{\gamma\,(X_s)=0}\,dK_s,$$

and K' will be an adapted function of X. Then the law of X under $\bar{P}$ will solve the martingale problem on Ω^0, with K' as an increasing process.

If moreover the sojourn condition holds, then there exists a function r depending only on a, α, b, γ, ρ, such that

$$\rho(X_s)\,dK'_s = r(X_u,\ u \leqq s)\,ds \tag{I.15}$$

and r actually is a function of X_u, $s-\eta \leqq u \leqq s$, for any $\eta > 0$.

Remark. — To have examples, *see* I.4.

3. The stochastic differential equation, and the results on existence and uniqueness of solutions

We are going to see that under certain assumptions the martingale problem is equivalent to a stochastic differential equation. This will enable us to use the results on weak existence and uniqueness for the latter. We know that strong (trajectorial) uniqueness implies weak uniqueness; *see* [3], [5].

We also know that the martingale problem (L, Γ, ρ) has the same solutions as $(L, g\Gamma, g\rho)$, for any bounded g such that $g > C > 0$. So if

$\gamma_1 > C > 0$, we may take $\gamma_1 = 1$, or any such normalization. We can remark that if the coefficients are continuous, or bounded and Lipschitz, dividing by γ_1 doesn't change this property.

THEOREM I.10. — *Suppose that σ and τ are two matrices such that $a = \sigma\sigma^*$, $\alpha = \tau\tau^*$. Naturally, $\tau^* n = 0$. Then the following propositions are equivalent:*

(*a*) P *solves the martingale problem* (L, Γ, ρ) *with sojourn condition starting at* $x \in \bar{\theta}$, *with increasing process* K.

(*b*) *There exists a probability space* $(\bar{\Omega}, (\bar{\mathscr{F}}_t)_{t \geq 0}, \bar{P})$, *a d-dimensional Brownian motion* C, *a d-dimensional continuous martingale* N, *and two continuous processes* X *and* K, *such that* $K_0 = 0$, K *is increasing,* $dK_t = \mathbf{1}_{\partial\theta}(X_t)\, dK_t$, $X_0 = x$, $X_t \in \bar{\theta}$,

$$\langle N^i, N^j \rangle_t = \delta_{ij}\left(t - \int_0^t \rho(X_s)\, dK_s\right) \tag{I.16}$$

and

$$dX_t = \sigma(X_t)\, dN_t + b(X_t)(dt - \rho(X_s)\, dK_s) + \gamma(X_t)\, dK_t + \tau(X_t)\, dC_{K_t}, \tag{I.17}$$

and P *is the law of* X.

We may take $(\bar{\Omega}, (\bar{\mathscr{F}}_t), X)$ *to be an extension of* $(\Omega^0, (\mathscr{F}_t^0), X)$ *in the sense of* [5]. *As soon as* $\gamma_1 + \rho > C > 0$, *for* $t \in \mathbb{R}+$ *and* $\lambda \in \mathbb{R}$ *we have* $E(\exp \lambda K_t) < +\infty$.

Proof. — (*a*) $\Rightarrow$ (*b*): We use Theorem I.1 (*d*) and the representation theorem for martingales: Theorem II.7.1′ in [5]. We reason as in [3], [5].

(*b*) $\Rightarrow$ (*a*): The Itô formula gives us the martingale problem on $C(\mathbb{R}+, \bar{\theta}) \times C(\mathbb{R}+, \mathbb{R})$, and Theorem I.9 gets us back to Ω^0. The sojourn condition holds because $\langle N^1 \rangle$ is an increasing process.

Take the projection of the stochastic differential equation on the first axis:

$$X_t^1 = X_0^1 + \int_0^t b_1(X_s)\, d\langle N^1 \rangle_s + \left(\int_0^t \sigma(X_n)\, dN_s\right)^1 + \int_0^t \gamma_1(X_s)\, dK_s.$$

We use $d\langle N^i \rangle_s \leq ds$, the classical estimates on Brownian motion, the boundedness of the coefficients, and the explicit solution to the Skhorokhod equation (Lemma III.4.2. in [5]) to get $E\left(\exp \lambda \int_0^t \gamma_1(X_s)\, dK_s\right) < +\infty$. Since $\int_0^t \rho(X_s)\, dK_s \leq t$, when $\gamma_1 + \rho > C > 0$ we get $E(\exp \lambda K_t) < +\infty$. □

Remark 1. — Thanks to the martingale representation theorem, we see that (perhaps after enlarging the probability space) there is a brownian motion B independent from C such that

$$N_t = \int_0^t \sqrt{1 - r(X_u, u \leq s)}\, dB_s \tag{I.18}$$

Naturally, if $\rho = 0$ then $r = 0$ and $N = B$. We may give a definition of a strong solution for the Stochastic differential equation, by giving first $\bar{\Omega}$, $\bar{\mathscr{F}}_t$, $\bar{P}$, B, C and by looking for an adapted continuous $(\bar{X}, \bar{K})$ such that (I.17) holds with N given as in (I.18).

If moreover $\tau = 0$, $\rho = 0$, we get the (strong) equations of [1], [2], [6].

Remark 2. — Proposition IV.6.2 in [5] tells us that as soon as a is positive definite and $a \in C_b^2$, we may choose a Lipschitz continuous σ such that $\sigma\sigma^* = a$. Same for α.

Remark 3. — The condition $\alpha n = 0$ translates into $\tau^* n = 0$; the condition $n^* an > 0$ into $\sigma^* n \neq 0$.

THEOREM I.11. — *Suppose that* $\alpha = 0$, *that* σ, b, *are uniformely Lipschitz and bounded,* $\gamma \in C_b^2$ *or* $\gamma = n$, $\gamma_1 > C > 0$. *Then for all positive* ρ, *all* $x \in \bar{\theta}$, *there is an unique solution* P *to the martingale problem* (L, Γ, ρ) *starting at* X, *with sojourn condition.*

If $\rho = 0$, *then there is an unique strong solution to the corresponding stochastic differential equation.*

Proof. — When $\rho = 0$, this is the result on stochastic differential equations of [1], [2], [6], where strong trajectorial results are given. The general case follows by Theorem 1.7, that is, by time-change. *See* Theorem IV.7.2 in [5] for a direct calculation on the stochastic differential equation. □

Remark 1. — By theorem I.8 there is an unique K satisfying (I.3) and (I.12).

Remark 2. — We could give results under assumptions of uniform strict ellipticity and of continuity on a. We can also diminish the assumptions on the drifts by the Girsanov Theorem. We also only need to find a bounded $g > C > 0$ such that (L, $g\Gamma$, $g\rho$) verifies the assumptions. For details, *see* [3], [5], [8].

Remark 3. — We don't suppose $a_{11} > 0$, and so can only use the strong results that authors get when $\gamma \in C_b^2$ or $\gamma = n$, $\alpha = 0$, $\rho = 0$. These are the only equations giving uniqueness results if we don't take $a_{11} > C > 0$.

We now will start to use the fact that our formulation of the martingale problem enables us to take limits. First, a proposition to get continuity results we will need for such proofs.

PROPOSITION I.12. — *Let $(v_n)_{n\geqq 0}$ be a sequence of real functions, with uniformely bounded variation on each compact set. Suppose the sequence converges pointwise to zero.*

Then for any $T\in\mathbb{R}$ and any $f\in C([0, T], \mathbb{R})$, $\int_0^T f(t)\,dv_n(t)$ converges to zero.

Proof. — Suppose the variations of the v_n are bounded by M on [0, T]. For $\varepsilon>0$, there is an η such that if

$$0\leqq s,\ t\leqq T,\ |s-t|<\eta \Rightarrow |f(s)-f(t)|<\frac{\varepsilon}{M}.$$

Let σ be a subdivision (s_i) of [0, T] such that $\operatorname{Max}(s_{i+1}-s_i)\leqq\eta$, and τ be a refinement of σ. Set $\tau(i)=\{t\in\tau,\ s_i\leqq t<s_{i+1}\}$. Then:

$$\begin{aligned}|\sum_{\tau} f(t_j)(v_n(t_{j+1})-v_n(t_j))| &= |\sum_i \sum_{\tau(i)} f(t_j)(v_n(t_{j+1})-v_n(t_j))| \\ &\leqq |\sum_i \sum_{\tau(i)} f(s_i)(v_n(t_{j+1})-v_n(t_j))| \\ &\quad + |\sum_i \sum_{\tau(i)} (f(t_j)-f(s_i))(v_n(t_{j+1})-v_n(t_j))|\end{aligned}$$

The first of these two terms is equal to $|\sum_{\sigma} f(s_i)(v_n(s_{i+1})-v_n(s_i))|$ and thus is less than ε as soon as n is larger than a $n(\sigma)$; the second term is less than $\frac{\varepsilon}{M}\times M=\varepsilon$.

Note that if k^n is a sequence of increasing functions converging pointwise to k, then $\int_0^T f(t)\,dk^n(t)$ converges to $\int_0^T f(t)\,dk(t)$ because $|k^n-k|_T\leqq k_T^n-k_0^n+k_T-k_0\leqq 3(k_T-k_0)$ as soon as n is large enough.

If we notice that v_n need only converge to zero outside a countable set, the proposition appears as a generalization of the classical result relating weak ("narrow") convergence on $\mathbb{R}$ to the pointwise convergence of distribution functions wherever the limit distribution is continuous.

We now get our first result using a limiting procedure.

THEOREM I.13. — *Assume that* σ, b, τ, γ *are continuous and bounded,* $\gamma_1 > C > 0$, *and* ρ *is a positive measurable function. Let* $x \in \bar{\theta}$. *Then there exists at least one solution to the martingale problem* (L, Γ, ρ) *with sojourn condition, starting at* x.

If moreover σ, b, τ, γ *are Lipschitz, and* $a_{11} > C > 0$, *then the solution is unique.*

Proof. — The last result is Theorem IV.7.2 in [5]. *See* also [3]. For $\varepsilon > 0$, put $L^\varepsilon = L + \varepsilon \Delta$. We first modify the proof of Theorem IV.7.2 in [5], by replacing the existence of solutions for SDE with Lipschitz coefficients by the existence for SDE with continuous bounded coefficients (Theorem IV.2.2 in [5]). Since $a_{11} + \varepsilon \geqq \varepsilon$, the proof can be carried through and we see there is at least one solution P^ε for the martingale problem $(L^\varepsilon, \Gamma, \rho)$.

As we plan future use of Theorem I.7, we take $\rho = 0$ at first, and for simplicity we take $\gamma_1 = 1$ $\left(\text{and multiply the rest of } \Gamma \text{ by } \dfrac{1}{\gamma_1}\right)$.

We will first prove tightness for $\{P^\varepsilon, 0 < \varepsilon < 1\}$, and then prove than any accumulation point for $\varepsilon \to 0$ solves the martingale problem (L, Γ, ρ). That will prove the theorem by using Theorem I.7.

Under P^ε, we have

$$X_t^1 = X_0^1 + \left(\int_0^t \sigma^\varepsilon(X_s)\, dB_s\right)^1 + \int_0^t b_1(X_s)\, ds + dK_t \tag{I.19}$$

with $dK_t = \mathbf{1}_{X_t^1 = 0}\, dK_t \geqq 0$ and $K_0 = 0$ and X and K continuous. The explicit solution to the Shorokhod problem yields:

$$K_t = \sup_{0 \leqq s \leqq t} \left\{ \left[X_0^1 + \left[\int_0^s \sigma^\varepsilon(X_u)\, dB_u\right]^1 + \int_0^s b_1(X_u)\, du \right]^- \right\}. \tag{I.20}$$

The coefficients are uniformely bounded for $\varepsilon < 1$, and (I.20) plus the classical results on Brownian motion yield that for any $T \in \mathbb{R}+$, there is C_T such that for $\varepsilon < 1$ and $0 \leqq s \leqq t \leqq T$ we have

$$E^\varepsilon(|K_t - K_s|^4) \leqq C_T |t-s|^2 \tag{I.21}$$

$$E^\varepsilon(|X_t - X_s|^8) \leqq C_T |t-s|^2 \tag{I.22}$$

(the eighth power coming from C_K).

Theorem I.4.3 in [5] then gives the tightness for the laws of (X, K) under the P^ε, $0 < \varepsilon < 1$.

Take $\varepsilon_n \to 0$ such that the laws of the couple converges to $\bar{P}^\infty$ on $C(\mathbb{R}+, \mathbb{R}^d) \times C(\mathbb{R}+, \mathbb{R})$, and L^n, P^n corresponding to ε_n, $0 \leqq s_1 \leqq \ldots \leqq s_p \leqq s \leqq t$, $g_1, \ldots, g_p$ continuous bounded, and set

$$M_t^{f,n} = f(X_t) - f(X_0) - \int_0^t L^n f(X_s)\, ds - \int_0^t \Gamma f(X_s)\, dK_s \qquad (I.23)$$

then we have

$$E^n[(M_t^{f,n} - M_s^{f,n})\, g_1(X_{s_1}, K_{s_1}) \ldots g_p(X_{s_p}, K_{s_p})] = 0. \quad (I.24)$$

Since the coefficients are continuous, we see using Proposition I.12 and what follows that $(X, K) \to \int_s^t \Gamma f(X_u)\, dK_u$ is continuous. Using (I.21) to get the needed uniform integrability, we obtain at the limit

$$\bar{E}^\infty[(M_t - M_s^f)\, g_1(X_{s_1}, K_{s_1}) \ldots g_p(X_{s_p}, K_{s_p})] = 0. \qquad (I.25)$$

where naturally M^f is like $M^{f,n}$ with L instead of L^n. So M^f is a $(\bar{P}^\infty, \bar{\mathscr{F}}_t^0)$-martingale, where $(\bar{\mathscr{F}}_t^0)$ is the natural filtration on $C(\mathbb{R}+, \mathbb{R}^d) \times C(\mathbb{R}+, \mathbb{R})$. Now if g is continuous positive with compact support in θ,

$$\bar{E}^\infty\left(\int_0^t g(X_s)\, dK_s\right) \leqq \overline{\lim}\, E^n\left(\int_0^t g(X_s)\, dK_s\right) = 0 \qquad (I.26)$$

and K increases only when X is on $\partial\theta$. Since $\bar{\theta}$ is closed, $\bar{P}^\infty(X_t \in \bar{\theta}) \geqq \overline{\lim}\, P^n(X_t \in \bar{\theta}) = 1$. Similarily, K is increasing, and $K_0 = 0$; $\bar{E}^\infty(K_t) < +\infty$ comes from (I.21). Using Theorem I.9, or directly (I.20), we see that K is actually an adapted process of X, and thus if P^∞ is the projection of $\bar{P}^\infty$ on Ω^0, M^f is a $(P^\infty, \mathscr{F}_t^0)$-martingale. So P^∞ is a solution to the martingale problem $(L, \Gamma, 0)$. Time change gives the general case. □

Up to now, we require $\gamma_1 > C > 0$ as most authors do. In [5], Chapter IV, at the end of Section 7, the authors argue that this assumption implies that there is reflection at the boundary and is too restrictive; instead we should suppose $\gamma_1 + \rho > C > 0$, which would allow sojourn without reflection. Then they construct a process under that assumption (and $a_{11} > C > 0$) thanks to the Poisson point process of excursions and a jump-type stochastic differential equation on the boundary. Naturally, they don't get uniqueness this way, and their construction is quite intricate. *See* [5], IV (7.13) and following, and the references given there.

We will now give a generalization of Theorem I.13 when $\gamma_1+\rho>C>0$ by a limiting method. This englobes the result in [5]. Naturally, $\gamma_1\geqq 0$ and $\rho\geqq 0$. A similar idea appears in [3], Proposition 20, when $\rho>C>0$, $a_{11}>C>0$.

THEOREM I.14. — *Assume that* σ, b, τ, γ *are continuous and bounded,* ρ *is a positive measurable function, and* $\gamma_1+\rho>C>0$. *Let* $x\in\bar{\theta}$.

Then there exists at least one solution to the martingale problem (L, Γ, ρ) *with sojourn condition, starting at* x.

Proof. — For $\varepsilon>0$, Γ^ε is like Γ with γ_1 replaced by $\gamma_1+\varepsilon$. We use Theorem I.13 to get existence for the martingale problem $(L, \Gamma^\varepsilon, \rho)$. We will follow the proof of Theorem I.13, except we don't take $\gamma_1=1$ and $\rho=0$. For $0\leqq s\leqq t\leqq T$, we will get instead of (I.21), using ds-$\rho(X_s)\,dK_s\leqq ds$

$$E^\varepsilon\left(\left|\int_s^t \gamma_1(X_s)\,dK_s\right|^4\right)\leqq C_T\,|t-s|^2 \tag{I.27}$$

and since $\displaystyle\int_s^t \rho(X_s)\,dK_s\leqq\int_s^t \mathbf{1}_{\partial\theta}(X_s)\,ds\leqq t-s$,

$$E^\varepsilon\left(\left|\int_s^t \rho(X_s)\,dK_s\right|^4\right)\leqq T^2\,|t-s|^2 \tag{I.28}$$

and using $\gamma_1+\rho>C>0$ and an easy convexity argument,

$$E^\varepsilon(|K_t-K_s|^4)\leqq C'_T\,|t-s|^2 \tag{I.29}$$

Now if ρ is continuous, we can finish our proof as for Theorem I.13, and since $\partial\theta$ is closed, the inequality $\mathbf{1}_{\partial\theta}(X_s)\,ds\geqq\rho(X_s)\,dK_s$ carries to the limit. We need the continuity of ρ for this, as well as to take limits in the martingale problem. Now if ρ is not continuous, take $\bar{\rho}=(c-\gamma_1)^+$. Then $\bar{\rho}$ is continuous, and $\bar{\rho}+\gamma_1\geqq C>0$, so $(L, \Gamma, \bar{\rho})$ has a solution. Now since $\rho\geqq\bar{\rho}$, we can use time-change as in Theorem I.7 (the notations are in reverse but coherent) to get existence for (L, Γ, ρ). We use Theorem I.9 to get back to Ω^0. □

Remark. — Assume $a_{11}>0$, and ρ is continuous. Then the law of $\mathbf{1}_{\partial\theta}(X_s)\,ds$ under the converging subsequence of P_n converges weakly to its law under P^∞. To see this, just use Theorem I.4 and the continuity of $(X, K)\to\displaystyle\int_0^t \rho(X_s)\,dK_s$.

4. Examples

We shall now deal with some examples of processes under our new reflection conditions. We will show the greater generality attained with respect to the usual ones. We use stochastic differential equations, which are more telling than martingale problems.

(*a*) Take the case $a_{11}=0$. This is close to a deterministic case, and could not be studied in the usual framework. Now by Theorem I.5, we have $b_1(X_s)\mathbf{1}_{\partial\theta}(X_s)\,ds=(\rho b_1-\gamma_1)(X_s)\,dK_s$. On the set of s such that $(\rho b_1-\gamma_1)(X_s)=0$, necessarily $b_1(X_s)\mathbf{1}_{\partial\theta}(X_s)\,ds=0$. We can remark that if γ_1 is allowed to vanish, and if $b_1(X_s)=0$, the process may well stay on the boundary and K be to a large extent arbitrary, even with the sojourn condition (I.12).

Suppose $\gamma_1>0$. Then on the set $(\rho b_1-\gamma_1)(X_s)=0$, necessarily $\rho\neq 0$, $b_1>0$, so $\mathbf{1}_{\partial\theta}(X_s)\,ds=0$ and (I.12) implies $dK_s=0$.

So $dK_s=\dfrac{b_1}{\rho b_1-\gamma_1}\mathbf{1}_{\partial\theta}(X_s)\,ds$, and (I.12) is equivalent to $\mathbf{1}_{b_1>0}\mathbf{1}_{\partial\theta}(X_s)\,ds=0$ which insures K is increasing.

We now give an example showing that we have to ask for the sojourn condition: take $d=1$, $b_1>0$, $\rho b_1-\gamma_1>0$, $X_0=0$. Clearly if we take $X_t=0$ and dK as above, we get a solution satisfying to the martingale problem, but it will not satisfy (I.12) since $\mathbf{1}_{\partial\theta}(X_s)\,ds=\left(\rho-\dfrac{\gamma_1}{b_1}\right)(X_s)\,dK_s$. The true solution will be obtained by taking $K=0$ and X a solution of $dX_t=b_1(X_t)\,dt$, starting at 0; b_1 being positive, $X_t\geqq 0$.

Also the usual sojourn condition $\mathbf{1}_{\partial\theta}(X_s)\,ds=\rho(X_s)\,dK_s$ is too restrictive: take $d=1$, $b_1\leqq 0$, $X_0=0$, clearly the only solution is $X_t=0$, so

$$\mathbf{1}_{\partial\theta}(X_s)\,ds=ds>\rho(X_s)\,dK_s=\frac{\rho b_1}{\rho b_1-\gamma_1}(X_s)\,ds.$$

(*b*) Let us recall some results on the square of the Bessel diffusion with index $\lambda\in\mathbb{R}+$. It's the solution to the stochastic differential equation

$$dX_t=2\sqrt{|X_t|}\,dB_t+\lambda\,dt. \tag{I.30}$$

If $X_0\geqq 0$ this equation has an unique solution, which is positive. If $\lambda<2$ the solution vanishes infinetely often at infinity, if $\lambda\geqq 2$ zero is a polar value, if $\lambda>0$ $\mathbf{1}(X_s=0)\,ds=0$, and if $\lambda=0$ X is absorbed at zero. *See*

IV . 8 . 2 and IV . 8 . 3 in [5] for more details, Let's now consider the equation

$$dX_t = 2\sqrt{|X_t|}\, dB_t + \lambda (dt - \rho\, dK_t) + dK_t \tag{I.31}$$

(we may take a Brownian motion in stead of N in Theorem I. 10, since $a_{11}(0) = 0$; we only have now to check that the sojourn condition (I. 12) holds, as it is no longer implied by the increasing nature of $\langle N \rangle$). An obvious solution is obtained by putting $K = 0$.

If $X_0 = 0$, let's take $\lambda t + (1 - \lambda\rho) K_t = 0$ and $X_t = 0$. If K should be increasing, we must have $\lambda\rho > 1$, but the sojourn condition will not be fulfilled.

Also if $\lambda\rho = 1$, the solution of (I. 30) will solve (I. 31) for any K, but only $K = 0$ will fulfill the sojourn condition, since $\mathbf{1}(X_s = 0)\, ds = 0$. By time-change, we see that for any ρ, the only solution to (1. 31) fulfilling the sojourn condition (1. 12) will be the solution for (1. 30), with $K = 0$.

We could generalize all this in some extent to the squares of norms of processes, or to "affine" equations as in IV . 8 . 1.

(*c*) Let us adapt the Langevin equation. *See* IV . 4 . 4 in [5]. We shall consider the system of equations

$$\begin{gathered} dX_t = V_t\, dt + dK_t \\ dV_t = \sigma(X_t, V_t)\, dB_t + b(X_t, V_t)\, dt + \gamma(V_t)\, dK_t + \tau(V_t)\, dC_{K_t} \end{gathered} \tag{I.32}$$

with $X_t \geqq 0$, $V_t \in \mathbb{R}$, which represents a reflecting particle with random acceleration. Note that here again $a_{11} = 0$. We saw in (*a*) that necessarily $dK_t = -V_t \mathbf{1}(X_t = 0)\, dt$, and we now get an unique solution as soon as σ, *b*, γ are Lipschitz and bounded and $\tau = 0$.

This particle is absorbed and freed depending on the sign of V. A sojourn coefficient ρ would accentuate this phenomenon.

All this gives examples for Theorem I . 9.

5. The non-homogenous martingale problem

We now suppose L, Γ, ρ depend on $x \in \bar{\theta}$, $s \in \mathbb{R}^+$. We can get results for this time-dependent martingale problem by the usual trick of adding time as a new coordinate. By using Theorem I . 1 (*d*) and the Itô formula, we see that for $f \in C_b^2(\bar{\theta} \times \mathbb{R})$,

$$f(X_t, t) - f(X_0, 0) - \int_0^t \left(\frac{\partial}{\partial s} + L\right) f(X_s, s)\, ds$$

$$-\int_0^t \left[\left(\langle \gamma, \nabla \rangle + \rho \frac{\partial}{\partial s}\right) + \Lambda - \rho\left(\frac{\partial}{\partial s} + \mathrm{L}\right)\right] f(\mathrm{X}_s, s)\, dK_s \quad (\mathrm{I}.33)$$

is a martingale. This is the martingale problem $(\bar{\mathrm{L}}, \bar{\Gamma}, \rho)$ on $C(\mathbb{R}+, \bar{\theta} \times \mathbb{R})$, where $\bar{\mathrm{L}} = \frac{\partial}{\partial s} + \mathrm{L}$, $\bar{\gamma} = (\gamma, \rho)$, the rest not being changed. Note that $\partial(\theta \times \mathbb{R}) = \partial\theta \times \mathbb{R}$, $\bar{n} = (n, 0)$, $\langle \bar{\gamma}, \bar{n} \rangle = \langle \gamma, n \rangle$, $\langle \bar{n}, \overline{an} \rangle = \langle n, an \rangle$, etc., and all our results may be used.

In [8], Stroock and Varadhan could not use this idea because even if L is uniformely elliptic, $\bar{\mathrm{L}}$ is not.

In the stochastic differential equation point of view, we just added

$$dt = (dt - \rho_t(\mathrm{X}_t)\, d\mathrm{K}_t) + \rho_t(\mathrm{X}_t)\, d\mathrm{K}_t. \quad (\mathrm{I}.34)$$

II. INTERACTING PARTICLES REFLECTING IN A DOMAIN WITH STICKY BOUNDARY

1. The framework

We consider a system of interacting particles reflecting in a domain θ of $\mathbb{R}^d$, given by $\Phi \in C_b^2$ as in Part I, with a sticky boundary and a normal internal vector field n. There is interaction in the sojourn term ρ. We will give here a construction for such a system, using our previous results to obtain it as a limit in law of approximating processes.

The following step would be to investigate the behaviour of the system when the number of particles tends to infinity; that is, to study the associated non-linear equation (in the case of mean-field interaction), the propagation of chaos, Gaussian field fluctuations, and large deviation theory. This too will require our results on the martingale problem. The first results in this direction are in [4]. *See* [9] for the case $\rho = 0$, and [10], [11] when $\theta = \mathbb{R}^d$.

We describe our system of interacting particles by the following stochastic differential equations, as in Theorem I.10: for N particles X^i, set $\bar{\mathrm{X}} = (\mathrm{X}^1, \ldots, \mathrm{X}^\mathrm{N}) \in \bar{\theta}^\mathrm{N}$, and consider

$$\left.\begin{aligned} d\mathrm{X}_t^i &= \sigma^i(\bar{\mathrm{X}}_t)\, d\mathrm{N}_t^i + b^i(\bar{\mathrm{X}}_t)\, d\langle \mathrm{N}^{i1} \rangle_t + n(\mathrm{X}_t^i)\, d\mathrm{K}_t^i \\ d\mathrm{K}_t^i &= \mathbf{1}_{\partial\theta}(\mathrm{X}_t^i)\, d\mathrm{K}_t^i, \qquad d\langle \mathrm{N}^{ij} \rangle_t = dt - \rho^i(\bar{\mathrm{X}}_t)\, d\mathrm{K}_t^i. \end{aligned}\right\} \quad (\mathrm{II}.1)$$

The N^i are N orthogonal martingales, with orthogonal components N^{ij}. $X_t^i \in \bar{\theta}$, $\bar{X}_0$ has law U_N and is independent of the N^i, and the K^i are increasing processes with $K_0^i = 0$. σ^i is a $d \times d$ matrix field, b^i a d-dimensional vector field, and instead of $n(X_t^i)$ we could take $\gamma^i(\bar{X}_t)$ provided $\langle \gamma^i(\bar{X}_t), n(X_t^i) \rangle > C > 0$.

Naturally if we have the following non-degeneracy condition:

$$\left.\begin{array}{l} \forall x \in \partial\theta, \quad \forall \bar{x} \in \bar{\theta}^N \text{ such that } (\bar{x})^i = x, \\ \qquad \sigma^{i*}(\bar{x})\, n(x) \neq 0 \end{array}\right\} \tag{II.2}$$

then thanks to Theorem I.4, we can replace in (II.1) N_t^i by $\mathbf{1}_\theta(X_t^i)\, dB_t^i$ and the condition on $d\langle N^{ij}\rangle$ by $\mathbf{1}_{\partial\theta}(X_t^i)\, dt = \rho^i(\bar{X}_t)\, dK_t^i$.

The interesting interaction for asymptotics is the mean-field interaction, where $\sigma^i(\bar{X}_t) = \frac{1}{N} \sum_j \sigma(X_t^i, X_t^j)$, etc. In [9], A. S. Sznitman studied this problem when $\rho = 0$ (and so $N^i = B^i$ is a Brownian motion). He shows strong existence and uniqueness for the system and for the corresponding non-linear limit equation, under the assumption that σ and b are Lipschitz and bounded. This is done by a fixed point method which would apply to uniformely bounded and Lipschitz predictable coefficients. He then proves the propagation of chaos, and a Gaussian fluctuation result when $\sigma = \mathrm{Id}$.

From now on, we assume the σ^i and b^i are Lipschitz and bounded.

The classical technique of time-change, to get the existence and uniqueness of solutions for general ρ when you have it for $\rho = 0$, as in Theorems I.7, I.8, I.11, will not work here. *See* also Theorem IV.7.2 in [5] for a direct calculation on stochastic differential equations. The reason why is that if we use time-change on each particle separately, their time-scales will not coincide any more and we lose the temporal coherence of the system. If we try to use a global time-change, it will perturb far too much the equation of a particle inside the domain as soon as another particle is on the boundary. The reason the time-change worked for one equation is that the time-change acted only when the one particle was on the boundary, and thus perturbed the equation only in its boundary terms.

Nor can we consider the system globally as one process, for it would evolve in the unsmooth domain $\bar{\theta}^N$. We should also rewrite the equations, remplacing the $\mathbf{1}_\theta(X^i)$ by $\mathbf{1}_{\theta^N}(\bar{X})$ and giving a boundary condition with $\mathbf{1}_{\partial(\bar{\theta}^N)}(\bar{X}) = \mathbf{1}_{\bar{\theta}^N \setminus \theta^N}(\bar{X})$ instead of $\mathbf{1}_{\partial\theta}(X^i)$. This would lead to a complicated equation, with martingale terms diffusing on the boundary, and it would be difficult to state a boundary sojourn condition.

It is easy to see that the "corners" in $\bar{\theta}^N$ will be visited by the global process $\bar{X}$. Here is a very simple example where the global process spends positive time in the corners:

Example. – Suppose $\sigma^i(\bar{x})=\sigma(x^i)$, $b^i(\bar{x})=b(x^i)$, $\rho^i(\bar{x})=\rho(x^i)$, so that the particles are actually not interacting. Suppose $X_0^1, \ldots, X_0^N$ independent and identically distributed, then (using uniqueness) the particles will stay independent and identically distributed during time. The law of large numbers shows that $\frac{1}{N}\sum_{i=1}^{N}\int_0^t \rho(X_s^i)\,dK_s^i$ converges when N goes to infinity to $E\left[\int_0^t \rho(X_s^1)\,dK_s^1\right]$, a. s, and this expectation is strictly positive for $t>0$ as soon as we assume $\partial\theta\neq\varnothing$, $\rho>0$, and $\sigma\sigma^*$ uniformely elliptic. Then $\sum_{i=1}^{N}\int_0^t \rho(X_s^i)\,dK_s^i$ goes to infinity, and so does $\sum_{i=1}^{N}\int_0^t \mathbf{1}_{\partial\theta}(X_s^i)\,ds$, and it is enough to chose N such that this quantity excedes t to know that at least two particles spend time together on the boundary before t. Using the independence and the equirepartition, we see that it is enough that $N=2$, and that for any $N\geqq 2$ the N particles spend time together on the boundary for $t>0$.

In view of the propagation of chaos results we get, this is almost a "generic" result.

It is interesting to examine the problems we run into if we try to define a global boundary condition. If the ρ^i are strictly positive, $dK_t^i=\frac{\mathbf{1}_{\partial\theta}(X_t^i)}{\rho^i(\bar{X}_t)}dt$, and if $\bar{K}=(K^1, \ldots, K^N)$, $|\bar{K}|$ is the total variation of $\bar{K}$, then

$$d|\bar{K}|_t=\left[\sum_{i=1}^{N}\left(\frac{\mathbf{1}_{\partial\theta}(X_t^i)}{\rho^i(\bar{X}_t)}\right)^2\right]^{1/2}dt$$

and necessarily

$$\mathbf{1}(\bar{X}_t\in\partial\theta^N)\,dt=\left[\sum_{i=1}^{N}\left(\frac{\mathbf{1}_{\partial\theta}(X_t^i)}{\rho^i(\bar{X}_t)}\right)^2\right]^{-1/2}d|\bar{K}|_t.$$

But since many particles may be on the boundary at the same time, we have lost some information and it is difficult (if not impossible) to get the former N boundary conditions from the latter.

2. The construction of the system of interacting particles

We are now going to give a construction of the system of particles as a limit in law. This naturally does not give uniqueness, even in the case of one particle. We naturally have to use the results in Part I, especially Theorem I.4 and Proposition I.12.

From now on we assume ρ^i is continuous and bounded on $\{\bar{x}\in\bar{\theta}^N, \bar{x}^i\in\partial\theta\}$. We will suppose σ^i and b^i uniformely lipschitz and bounded. We get then our main result.

THEOREM II.2. — *Under these assumptions, for any initial law* U_N *there is a weak solution to* (II.1) *such that the law of* $\bar{X}_0$ *is* U_N.

If moreover U_N *is symmetric, and* σ^i, b^i, ρ^i *are such that for any* $\bar{x}=(x^i)\in\bar{\theta}^N$, *and any permutation* τ *of* $\{1, \ldots, N\}$, *we have* $\sigma^i(x^{\tau(1)}, \ldots, x^{\tau(N)})=\sigma^{\tau(i)}(x^1, \ldots, x^N)$, *etc. (as in the mean-field interaction), then there is a weak symmetric solution.*

If $\rho=0$, *then there is strong existence and uniqueness.*

Proof. — The result for $\rho=0$ is the result of Sznitman [9]. We give ourselves a probability space Ω with a right-continuous and complete filtration (for convenience) and N independent d-dimensional Brownian motions $B^1, \ldots, B^N$. We are going to get a solution of (II.1) as a limit in law of processes indexed by $\varepsilon>0$, defined as follows.

For $\varepsilon>0$, we are going to define recursively an increasing sequence of stopping times T_n, a sequence of subsets J_n of $\{1, \ldots, N\}$, and the processes X^i, K^i, N^i between T_n and T_{n+1}. To the particle X^i we associate the clock

$$C_t^i=\int_0^t \rho^i(\bar{X}_s)\,dK_s^i. \tag{II.3}$$

Set $T_0=0$, $J_0=\varnothing$, and suppose we have defined $T_0, \ldots, T_n, J_0, \ldots, J_n$, and X^i, K^i, N^i up to T_n. Then starting at T_n, we freeze X^j, K^j, N^j for $j\in J_n$, and let the other particles X^i for $i\notin J_n$ go following an equation (II.1) with $\rho=0$, starting at $X_{T_n}^i$, with $N_{T_n+s}^i-N_{T_n}^i=B_{T_n+s}^i-B_{T_n}^i$. All we need is the results in Sznitman [9], with random coefficients because of the frozen particles, but the strong trajectorial proof in [9] carries easily through.

We then set

$$\varepsilon_t^i=\langle N^{i1}\rangle_t-(t-C_t^i) \tag{II.4}$$

and

$$T_{n+1}=\inf\{t\geqq T_n,\ \exists i\notin J_n,\quad \varepsilon_t^i\geqq\varepsilon, \text{ or } \exists j\in J_n,\ \varepsilon_t^j\leqq 0\} \qquad \text{(II.5)}$$
$$J_{n+1}=\{i\notin J_n,\ \varepsilon_t^i=\varepsilon\}\cup\{j\in J_n,\ \varepsilon_t^j\neq 0\}.$$

Note that $0\leqq\varepsilon_t^i\leqq\varepsilon$, ε^i is a process of bounded variation with $|\varepsilon^i|_t\leqq 2t+\varepsilon$, and that for $j\in J_n$, $X^j_{T_n}\in\partial\theta$. Whenever a particle X^i has not spent enough time on the boundary, that is whenever ε_t^i reaches ε, we stop it and wait during ε before freeing it again (and ε_t^i will have fallen to 0). It is easy to see that $T_{n+2N}>T_n+\varepsilon$, so $\lim T_n=+\infty$, and we have defined a process on $\mathbb{R}+$.

Note that if (II.2) holds, then $\varepsilon_t^i=C_t^i-\int_0^t \mathbf{1}_{\partial\theta}(X_s^i)\,ds$.

Our $\bar{X}$ will satisfy (II.1), with the exception that

$$d\langle N^{ij}\rangle_t=dt-\rho^i(\bar{X}_t)\,dK_t^i+d\varepsilon_t^i.$$

Let us set for $\bar{x}=(x^1,\ \ldots,\ x^N)\in\bar{\theta}^N$, $f\in C_b^2(\bar{\theta})$

$$L^i f(\bar{x})=\frac{1}{2}\sum_{j,\,k}(\sigma^i\sigma^{i*})_{jk}(\bar{x})\,\partial_{jk}^2 f(x^i)+\sum_j b_j^i(\bar{x})\,\partial_j f(x^i) \qquad \text{(II.6)}$$

then by the Itô formula the law of $\bar{X}$ satisfies the N martingale problems

$$M_t^{f,\,i}=f(X_t^i)-f(X_0^i)-\int_0^t L^i f(\bar{X}_s)\times(ds-\rho^i(\bar{X}_s)\,dK_s^i+d\varepsilon_s^i)-\int_0^t\langle n,\nabla\rangle f(X_s^i)\,dK_s^i \qquad \text{(II.7)}$$

is a martingale, with

$$\langle M^{f,\,i},\ M^{f,\,j}\rangle=0 \quad \text{if } i\neq j, \qquad \text{and} \qquad ds-\rho^i(\bar{X}_s)\,dK_s^i+d\varepsilon_s^i\geqq 0.$$

We can reason exactly as for Theorem I.13 to get the tightness of the laws of $(X^i,\ K^i)$ for $0<\varepsilon<1$, and using Proposition I.12, the continuity of all the coefficients, $0\leqq\varepsilon_t^i\leqq\varepsilon$, $|\varepsilon_t^i|\leqq 2t+\varepsilon$, we see that a limit point for $\varepsilon\to 0$ satisfies (II.7) with $\varepsilon^i=0$, and in particular $ds\geqq\rho^i(\bar{X}_s)\,dK_s^i$. Then this law is a weak solution to (II.1).

If the symmetry condition holds, we may either symmetrise our limit law or note that by construction, our processes indexed by ε are symmetric, and so is any limit law. □

REFERENCES

[1] R. F. ANDERSON and S. OREY, Small Random Perturbations of Dynamical Systems with Reflecting Boundary, *Nagoya Math. J.*, Vol. **60**, 1976, pp. 189-216.
[2] A. BENSOUSSAN and J.-L. LIONS, *Contrôle impulsionnel et inéquations quasi-variationnelles*, Dunod, Paris, 1982.
[3] N. EL KAROUI, Processus de diffusion associé à un opérateur élliptique dégénéré et à une condition frontière, *Thèse d'État*, Paris-VI, 1971.
[4] C. GRAHAM, Systèmes de particules en interaction dans un domaine à paroi collante et problèmes de martingales avec réflexion, *Thèse de Docteur de 3e cycle*, Paris-VI, 1985.
[5] N. IKEDA and S. WATANABE, *Stochastic Differential Equations and Diffusion Processes*, North Holland, Amsterdam, 1981.
[6] P.-L. LIONS and A.-S. SZNITMAN, Stochastic Differential Equations with Reflecting Boundary Conditions, *Comm. Pure Appl. Math.*, Vol. **37**, 1984, pp. 511-537.
[7] P.-A. MEYER, Un cours sur l'intégrale stochastique, *Sém. de Prob. de Strasbourg-X.*
[8] D. W. STROOCK and S. R. S. VARADHAN, Diffusion Processes with Boundary Conditions, *Comm. Pure Appl. Math.*, Vol. **24**, 1971, pp. 147-225.
[9] A.-S. SZNITMAN, Non-Linear Reflecting Diffusion Processes, and the Propagation of Chaos and Fluctuations Associed, *J.F.A.*, Vol. **56**, 1984, pp. 311-336.
[10] H. TANAKA, Limit Theorems for Certain Diffusion Processes with Interaction, *Taniguchi Symp. S.A.*, Katata, 1982, pp. 469-488.
[11] T. SHIGA and H. TANAKA, *Central Limit Theorem for a System of Markovian Particles with Mean Field Interaction*, Preprint.
[12] Temps locaux, *Astérisque*, Vol. **52/53**, 1978, S.M.F.

(*Manuscrit reçu le* 19 *juin* 1986.)