ANNALES DE L'I. H. P., SECTION B

E. B. DYNKIN
P. J. FITZSIMMONS

Stochastic processes on random domains

Annales de l'I. H. P., section B, tome 23, nº S2 (1987), p. 379-396
<http://www.numdam.org/item?id=AIHPB_1987__23_S2_379_0>

Ann. Inst. Henri Poincaré,

Sup. au n° 2, 1987, Vol. 23, p. 379-396. *Probabilités et Statistiques*

Stochastic processes on random domains (*)

by

E. B. DYNKIN

Department of Mathematics, Cornell University,
White Hall, Ithaca, NY 14853, U.S.A.

and

P. J. FITZSIMMONS

Department of Mathematics, Cornell University,
White Hall, Ithaca, NY 14853, U.S.A.
Permanent adress:
Department of Mathematical Sciences,
The Universiity of Akron,
Akron, OH 44325, U.S.A.

ABSTRACT. — We study stochastic processes whose parameter sets are random. In particular, we characterize the finite dimensional distributions of a stochastic process whose parameter domain is a random open convex subset of $\mathbb{R}^d$. This result generalizes work of Kuznetsov, and others. It is based on a theorem of Choquet, and an "inverse limit" theorem that generalizes a result of K. Y. Hu. Applications are also made to Markov processes and to processes with continuous paths.

Key words : Markov processes, stochastic process, random convex domain, Choquet's theorem.

RÉSUMÉ. — Nous étudions les processus stochastiques dont l'ensemble des paramètres est aléatoire. En particulier, nous caractérisons les distributions fini-dimensionnelles d'un processus stochastique dont l'ensemble des paramètres est ouvert convexe aléatoire de $\mathbb{R}^d$. Ce résultat généralise le

(*) Partially supported by N.S.F. Grant DMS-8505020.

travail de Kuznetsov, et d'autres. Il repose sur un théorème de Choquet, et un théorème « limite inverse » qui généralise un résultat de K. Y. Hu. On donne aussi des applications aux processus de Markov et aux processus à trajectoires continues.

1. INTRODUCTION

1.1. Since the middle of the 1950's it has been conventional to consider Markov processes defined on a random time interval $[0, \zeta[$. For example, this is the natural domain for a process determined by "local characteristics" (a countable matrix in the case of a denumerable state space, a differential operator in the case of a diffusion, etc.).

Later, it was realized that there is an advantage in restoring symmetry by considering a random birth time α, dual to the random death time. In [D1] the class of Markov processes "alive" on a random interval $]\alpha, \beta[\subset \mathbb{R}$, having a given transition function, has been described. Kuznetsov [K1] (*see* also [K2]) extended this result, allowing the path-space measure to be σ-finite (and typically infinite). This is a very important generalization; for example, it makes it possible to associate a stationary Markov process to each excessive measure of a given transition function.

Actually, the arguments used in [D1] and [K1] are quite general and apply to the non-Markovian case as well. These arguments yield a general theorem for the existence of a stochastic process $(X_t : \alpha < t < \beta)$ with given "finite dimensional distributions"

$$m_{t_1, \ldots, t_n}(B_1, \ldots, B_n) = P(\{t_1, \ldots, t_n\} \subset]\alpha, \beta[, X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n).$$

Necessary and sufficient conditions for $m_{t_1, \ldots, t_n}(B_1, \ldots, B_n)$ to be the finite dimensional distributions of a process with random times of birth and death have been stated in [D2].

In this paper we establish analogous conditions for the existence of a stochastic process $(X_t : t \in \Delta)$ where Δ is a random open convex subset of $\mathbb{R}^d$. Our principle tools are a variation on a theorem of Choquet [C] concerning the distribution of random sets, and a generalization of the classical inverse (or projective) limit theorem. These methods actually apply to stochastic processes whose random domain Δ is a subset of an arbitrary parameter space T.

Our investigation was inspired by a recent paper of Hu [H] on stochastic processes with random parameter domains. One of the main results, the "inverse limit" Theorem 3.1, generalizes Theorem 1′ of [H].

1.2. Let $(\Omega, \mathscr{F}, P)$ be a measure space and for each $\omega \in \Omega$ let $\Delta(\omega)$ be a non-empty open convex subset of $\mathbb{R}^d$. We say that $\Delta(.)$ is a random open convex set (ROC) if $\{\omega : \Delta \supset \Lambda\} \in \mathscr{F}$ for every non-empty finite set $\Lambda \subset \mathbb{R}^d$.

Suppose that a measurable space $(E_t, \mathscr{E}_t)$ is given for each $t \in \mathbb{R}^d$, and that a point $X_t(\omega) \in E_t$ is specified for each $\omega \in \Omega$, $t \in \Delta(\omega)$. We say that $X_t(\omega)$ is a *tochastic process with random domain* Δ if, for each $t \in \mathbb{R}^d$, $B \in \mathscr{E}_t$,

$$\{\omega : t \in \Delta(\omega),\ X_t(\omega) \in B\} \in \mathscr{F},$$

and if each of the one-dimensional distributions

$$m_t(B) = P\{\omega : t \in \Delta(\omega),\ X_t(\omega) \in B\}$$

is a σ-finite measure.

For each non-empty set $G \subset \mathbb{R}^d$ we put

$$(E_G, \mathscr{E}_G) = \bigotimes_{t \in G} (E_t, \mathscr{E}_t). \tag{1.1}$$

The *finite dimensional distributions* of $X = (X_t : t \in \Delta)$ are defined by

$$m_\Lambda(B_\Lambda) = P(\Delta \supset \Lambda,\ X_\Lambda \in B_\Lambda), \qquad B_\Lambda \in \mathscr{E}_\Lambda. \tag{1.2}$$

Here $\Lambda = \{t_1, \ldots, t_n\}$ is a non-empty finite subset of $\mathbb{R}^d$ and $X_\Lambda = \{X_{t_1}, \ldots, X_{t_n}\}$. Obviously each m_Λ is a σ-finite measure on $(E_\Lambda, \mathscr{E}_\Lambda)$.

In the sequel $[\Lambda]$ denotes the closed convex hull of a set $\Lambda \subset \mathbb{R}^d$. Recall that a measurable space $(E, \mathscr{E})$, is a U-space if E is a topological space homeomorphic to a universally measurable subset of a compact metric space, and if $\mathscr{E}$ is the Borel σ-field on E.

THEOREM 1.1. — *Suppose that* $(E_t, \mathscr{E}_t)$ *is a* U*-space for each* $t \in \mathbb{R}^d$. *A collection* $\{m_\Lambda : \Lambda$ *a non-empty finite subset of* $\mathbb{R}^d\}$ *of* σ*-finite measures is the system of finite dimensional distributions of a stochastic process* $(X_t : t \in \Delta)$ *with* ROC *domain* Δ *if and only if*

$$\left.\begin{array}{c} m_\Lambda(.) = m_{\Lambda \cup \Gamma}(. \times E_\Gamma) \\ \textit{if} \quad \Lambda \cap \Gamma = \varnothing, \quad [\Lambda] = [\Lambda \cup \Gamma]; \end{array}\right\} \tag{1.3}$$

$$\left.\begin{array}{c} m_{\Lambda \cup \Gamma_n}(B_\Lambda \times E_{\Gamma_n}) \to m_\Lambda(B_\Lambda) \\ \textit{if} \quad \Lambda \cap \Gamma_n = \varnothing, \quad [\Lambda \cup \Gamma_n] \downarrow [\Lambda], \quad B_\Lambda \in \mathscr{E}_\Lambda; \end{array}\right\} \tag{1.4}$$

$$\left.\begin{array}{l}\sum_{\tilde{\Gamma}\subset\Gamma}(-1)^{|\tilde{\Gamma}|}m_{\Lambda\cup\tilde{\Gamma}}(B_\Lambda\times E_{\tilde{\Gamma}})\geqq 0\\ \qquad \textit{if}\quad \Lambda\cap\Gamma=\varnothing,\ B\in\mathscr{E}_\Lambda;\end{array}\right\}\tag{1.5}$$

where Λ, Γ, Γ_n *are arbitrary non-empty finite subsets of* $\mathbb{R}^d$, *and* $|\tilde{\Gamma}|$ *denotes the cardinality of* $\tilde{\Gamma}$.

(1.6) *Remarks.* – (*a*) The necessity of conditions (1.3) and (1.4) is obvious. The necessity of (1.5) follows from the inequality

$$\begin{aligned}0&\leqq P(\Delta\supset\Lambda,\ \Delta\cap\Gamma=\varnothing,\ X_\Lambda\in B_\Lambda)\\ &=E(1_{B_\Lambda}(X_\Lambda)\prod_{t\in\Gamma}(1-1_{E_t}(X_t)).\end{aligned}$$

(*b*) If $d=1$, then Δ is a random open interval $]\alpha,\beta[$ and one needs to check (1.5) only for $|\Gamma|=1$ or 2. Indeed, assuming (1.3) holds, each term in (1.5) corresponding to a $\tilde{\Gamma}$ with $|\tilde{\Gamma}|\geqq 3$ has a companion term of equal magnitude and opposite sign; these terms cancel by pairs leaving only the terms with $|\tilde{\Gamma}|=0$, 1, or 2. As mentioned previously, when $d=1$, Theorem 1.1 can be proved by the arguments in [K1]. Another proof was given in [H].

1.3. Let $(X_t: t\in\Delta)$ be a stochastic process with ROC dòmain $\Delta\subset\mathbb{R}^d$, defined on a measure space $(\Omega,\mathscr{F},P)$. Fix a non-empty finite set $\Lambda\subset\mathbb{R}^d$, restrict P to $\Omega_\Lambda=\{\omega:\Delta(\omega)\supset\Lambda\}$, and let μ_Λ denote the image measure under the mapping $\omega\mapsto(\Delta(\omega), X_\Lambda(\omega))$ from Ω_Λ to $(C_\Lambda\times E_\Lambda)$. Here C_Λ is the class of open convex subsets of $\mathbb{R}^d$ which contain Λ.

The proof of Theorem 1.1 consists of two parts: starting from the system $\{m_\Lambda\}$ we *construct* a family $\{\mu_\Lambda\}$; then on an appropriate sample space Ω (endowed with mappings $\Delta(\omega)$, $X_t(\omega)$, $t\in\Delta(\omega)$) we construct a measure P such that the measures μ_Λ derive from $(X_t: t\in\Delta)$ as in the first paragraph of this subsection. The first step is based on Theorem 2.1 which characterizes the "finite dimensional distributions" of a ROC Δ. The second step is accomplished by noting that $\{\mu_\Lambda\}$ is a "projective system" of measures indexed by the partially ordered class of finite subsets of $\mathbb{R}^d$; an "inverse limit" theorem (Theorem 3.1) allows us to conclude that the μ_Λ's are the "projections" of a single measure P. In rough outline, this construction was suggested by Hu's treatment of Theorem 1.1 in the case $d=1$.

In section 2 we investigate ROCs and we construct the measures μ_Λ. In section 3 our inverse limit Theorem 3.1 is stated and used to finish the proof of Theorem 1.1. We also consider several variations of Theorem 1.1. The proof of Theorem 3.1 is given in section 4.

1.4. NOTATION. — $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, \ldots\}$. If $\mathscr{A}$ is a class of subsets of a set A, and if $B \subset A$, then $\mathscr{A} \cap B = \{C \cap B : C \in \mathscr{A}\}$ denotes the trace of $\mathscr{A}$ on B. If $(A, \mathscr{A})$ and $(B, \mathscr{B})$ are measurable spaces then $\mathscr{A} \times \mathscr{B}$ denotes Cartesian product while $\mathscr{A} \otimes \mathscr{B}$ is the σ-field generated by $\mathscr{A} \times \mathscr{B}$.

2. RANDOM OPEN CONVEX SETS

2.1. Let C denote the class of open convex subsets of $\mathbb{R}^d$, and let $\mathscr{C}$ denote the σ-field of subsets of C generated by the events $\{\Delta \in C : \Delta \supset K\}$ where K ranges over the class $\mathscr{K}$ of compact subsets of $\mathbb{R}^d$. It can be shown (*see* e. g. Matheron [Ma]) that $\mathscr{C}$ is the Borel σ-field corresponding to a separable metrizable topology on C. Let $\mathscr{I}$ denote the class of nonempty finite subsets of $\mathbb{R}^d$, and set $\mathscr{I}_0 = \mathscr{I} \cup \{\varnothing\}$. Recall that for $A \subset \mathbb{R}^d$, [A] denotes the closed convex hull of A.

THEOREM 2.2. — *Let* $M : \mathscr{I}_0 \to [0, 1]$ *satisfy the conditions:*

$$M(\varnothing) = 1; \tag{2.1}$$

$$M(\Lambda) = M(\Lambda \cup \{t\}), \qquad \text{if} \quad [\Lambda] = [\Lambda \cup \{t\}], \quad t \in \mathbb{R}^d \setminus \Lambda; \tag{2.2}$$

$$M(\Lambda_n) \to M(\Lambda), \qquad \text{if} \quad [\Lambda_n] \downarrow [\Lambda] \text{ as } n \uparrow \infty; \tag{2.3}$$

$$\sum_{\tilde{\Gamma} \subset \Gamma} (-1)^{|\tilde{\Gamma}|} M(\Lambda \cup \tilde{\Gamma}) \geqq 0, \qquad \text{if} \quad \Lambda \cap \Gamma = \varnothing. \tag{2.4}$$

(Here Λ, Λ_n, Γ *are arbitrary elements of* $\mathscr{I}$ *and* $|\tilde{\Gamma}|$ *denotes the cardinality of* $\tilde{\Gamma}$.*) Then there exists a unique probability measure* P *on* $(C, \mathscr{C})$ *such that*

$$P(\Delta : \Delta \supset \Lambda) = M(\Lambda), \qquad \Lambda \in \mathscr{I}_0. \tag{2.5}$$

Conversely, if P *is any probability measure on* $(C, \mathscr{C})$, *then* $M(\Lambda)$ *defined by* (2.5) *satisfies* (2.1) *through* (2.4).

Proof. — The necessity of (2.1)-(2.4) follows in the same way as that of (1.3)-(1.5). For the sufficiency first note that (2.2) [resp. (2.4)] implies the apparently stronger condition (2.6) [resp. (2.7)]:

$$M(\Lambda \cup \Gamma) = M(\Lambda), \qquad \text{if} \quad [\Lambda \cup \Gamma] = [\Lambda]; \tag{2.6}$$

$$\left.\begin{array}{c}\Sigma(-1)^k \mathrm{M}(\Lambda_0 \cup \Lambda_{i_1} \cup \ldots \cup \Lambda_{i_k}) \geqq 0, \\ \text{for disjoint } \Lambda_0, \Lambda_1, \ldots, \Lambda_n,\end{array}\right\} \tag{2.7}$$

where the sum in (2.7) extends over all subsets $\{i_1, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$, and $n \geqq 1$ is arbitrary. Indeed (2.2) implies (2.6) by an obvious induction. To see that (2.4) implies (2.7) first note that (2.7) reduces to (2.4) when $\Lambda_1, \ldots, \Lambda_n$ are singletons; in general the sum in (2.7) is, for a fixed Λ_0, a symmetric function of $\Lambda_1, \ldots, \Lambda_n$, and if F_n denotes this sum then

$$\begin{aligned}&\mathrm{F}_n(\Lambda_0; \Lambda_1, \ldots, \Lambda_{n-1}, \Lambda_n \cup \Lambda'_n) \\ &\qquad = \mathrm{F}_n(\Lambda_0; \Lambda_1, \ldots, \Lambda_{n-1}, \Lambda_n) + \mathrm{F}_n(\Lambda_0 \cup \Lambda_n; \Lambda_1, \ldots, \Lambda_{n-1}, \Lambda'_n)\end{aligned}$$

so (2.7) also follows by induction. Taking $n=1$ in (2.7) we see that M is a decreasing function.

We construct the measure P on (C, $\mathscr{C}$) by applying Choquet's theorem concerning capacities which are "alternating of order ∞". To this end we define $\mathrm{N}: \mathscr{K} \to [0, 1]$ which satisfies the analogs of (2.3), (2.6), (2.7), and which agrees with M on $\mathscr{I}_0$. For $\mathrm{K} \in \mathscr{K}$ set

$$\mathrm{N}(\mathrm{K}) = \sup\{\mathrm{M}(\Lambda) : \Lambda \in \mathscr{I}_0, [\Lambda] \supset \mathrm{K}\}.$$

It is clear that

$$\mathrm{N}(\mathrm{K}_1) = \mathrm{N}(\mathrm{K}_2) \qquad \text{if} \quad [\mathrm{K}_1] = [\mathrm{K}_2].$$

Moreover, $\mathrm{N}(\Lambda) = \mathrm{M}(\Lambda)$ if $\Lambda \in \mathscr{I}_0$, because of (2.6). If $\mathrm{K}_n \downarrow \mathrm{K}$ then $[\mathrm{K}_n] \downarrow [\mathrm{K}]$, and if $\mathrm{U} \supset [\mathrm{K}]$, U open, then $\mathrm{U} \supset [\mathrm{K}_n]$ for all large n. These observations coupled with (2.3) and the monotonicity of M lead easily to

$$\mathrm{N}(\mathrm{K}_n) \uparrow \mathrm{N}(k) \qquad \text{if} \quad \mathrm{K}_n \downarrow \mathrm{K}, \mathrm{K}_n, \mathrm{K} \in \mathscr{K}. \tag{2.8}$$

Using (2.3) and (2.8) one now checks that

$$\left.\begin{array}{c}\Sigma(-1)^k \mathrm{N}(\mathrm{K}_0 \cup \mathrm{K}_{i_1} \cup \ldots \cup \mathrm{K}_{i_k}) \geqq 0, \\ \mathrm{K}_0, \mathrm{K}_1, \ldots, \mathrm{K}_n \in \mathscr{K},\end{array}\right\} \tag{2.9}$$

where the sum is over all subsets $\{i_1, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$, and $n \in \mathbb{N}$ is arbitrary.

The inequalities (2.9) and the continuity property (2.8) amount to the statement that $\mathrm{T} \equiv 1 - \mathrm{N}$ is a Choquet $\mathscr{K}$-capacity, alternating of order ∞. (*See* [C] or [Ma].) Let $\mathbb{F}$ denote the class of closed subsets of $\mathbb{R}^d$ endowed with the σ-field $\mathscr{B}(\mathbb{F})$ generated by the events $\{\mathrm{F} \in \mathbb{F} : \mathrm{F} \cap \mathrm{K} \neq \varnothing\}$, $\mathrm{K} \in \mathscr{K}$.

By Choquet's theorem ([C], [Ma, p. 30]) there is a unique probability measure P_0 on $(\mathbb{F}, \mathscr{B}(\mathbb{F}))$ such that

$$P_0(F \in \mathbb{F} : F \cap K \neq \varnothing) = T(K), \qquad K \in \mathscr{K}. \tag{2.10}$$

Let P_1 denote the image law of P_0 under the mapping $F \mapsto \mathbb{R}^d \setminus F$ from $(\mathbb{F}, \mathscr{B}(\mathbb{F}))$ to $(\mathbb{G}, \mathscr{B}(\mathbb{G}))$, where $\mathbb{G}$ is the class of open subsets of $\mathbb{R}^d$ and $\mathscr{B}(\mathbb{G})$ is generated by $\{G \in \mathbb{G} : G \supset K\}$, $K \in \mathscr{K}$. Clearly $P_1(G : G \supset K) = N(K)$. Let Λ, $\Gamma \in \mathscr{I}$ and let $\mathbb{G}_{\Lambda\Gamma}$ denote the set of those $G \in \mathbb{G}$ such that $G \supset \Lambda$, $G \not\supset \Lambda \cup \Gamma$. Because of (2.6) we have $P_1(\mathbb{G}_{\Lambda\Gamma}) = 0$ whenever $[\Lambda] = [\Lambda \cup \Gamma]$. It follows that P_1 is carried by C. Since $\mathscr{C} = \mathscr{B}(\mathbb{G}) \cap C$, we obtain the desired probability P by restricting P_1 to $(C, \mathscr{C})$. The uniqueness of P follow easily from the uniqueness in Choquet's theorem and our construction of N. □

2.2. Recall that $C_\Lambda = \{\Delta \in C : \Delta \supset \Lambda\}$, $\Lambda \in \mathscr{I}$. The measures μ_Λ on $C_\Lambda \times E_\Lambda$ described in subsection 1.3 satisfy the relation

$$\begin{aligned}\mu_\Lambda(C_{\Lambda \cup \Gamma} \times B) &= P(\Delta \supset \Lambda \cup \Gamma, X_\Lambda \in B)\\ &= m_{\Lambda \cup \Gamma}(B \times E_\Gamma), \qquad \text{if} \quad \Lambda \cap \Gamma = \varnothing, \quad B \in \mathscr{E}_\Lambda.\end{aligned} \tag{2.11}$$

Conversely, starting from a system $\{m_\Lambda\}$ satisfying (1.3)-(1.5), we can construct a system $\{\mu_\Lambda\}$ satisfying (2.11). To this end fix $\Lambda \in \mathscr{I}$ and $B \in \mathscr{E}_\Lambda$ such that $0 < m_\Lambda(B) < \infty$. The function

$$M(\Gamma) = m_{\Lambda \cup \Gamma_1}(B \times E_{\Gamma_1}) / m_\Lambda(B), \qquad \Gamma \in \mathscr{I}_0, \quad \Gamma_1 \equiv \Gamma \setminus \Lambda,$$

satisfies (2.1), and it follows from (1.3)-(1.5) that M satisfies (2.2)-(2.4) as well. By Theorem 2.1 there is a unique probability $Q_{\Lambda, B}$ on $(C, \mathscr{C})$ such that $Q_{\Lambda, B}(\Gamma) = M(\Gamma)$, $\Gamma \in \mathscr{I}_0$. Clearly $Q_{\Lambda, B}$ is concentrated on C_Λ. For each $D \in \mathscr{C}_\Lambda = \mathscr{C} \cap C_\Lambda$ and each $B \in \mathscr{E}_\Lambda$ with $m_\Lambda(B) < \infty$, we put

$$\begin{aligned}\mu_\Lambda(D \times B) &= Q_{\Lambda, B}(D)\, m_\Lambda(B), \qquad \text{if} \quad m_\Lambda(B) > 0;\\ &= 0, \qquad \text{if} \quad m_\Lambda(B) = 0.\end{aligned}$$

Evidently $\mu_\Lambda(\,.\times B)$ is a finite measure on $\mathscr{C}_\Lambda$. By the uniqueness part of Theorem 2.1, $\mu_\Lambda(D \times (\bigcup B_i)) = \sum_i \mu_\Lambda(D \times B_i)$ if the B_i are disjoint and $m_\Lambda(\bigcup B_i) < \infty$. Since m_Λ is σ-finite, for each $D \in \mathscr{C}_\Lambda$, $\mu_\Lambda(D \times .)$ can be uniquely extended to a σ-finite measure on $\mathscr{E}_\Lambda$. Since $(E_\Lambda, \mathscr{E}_\Lambda)$ is a U-space, it follows from a result of Morando [Mo] that there is a unique extension of μ_Λ to a measure on $\mathscr{C}_\Lambda \otimes \mathscr{E}_\Lambda$. The reader can check that (2.11) (without the middle term) holds for each μ_Λ, $\Lambda \in \mathscr{I}$.

3. A GENERAL INVERSE LIMIT THEOREM AND ITS APPLICATIONS

3.1. Our objective now is to construct a stochastic process $(X_t(\omega), P)$ with ROC domain $\Delta(\omega)$, such that

$$P(\Delta \supset \Lambda, \Delta \in D, X_\Lambda \in F) = \mu_\Lambda(D \times F) \tag{3.1}$$

for all $\Lambda \in \mathscr{I}$, $D \in \mathscr{C}_\Lambda$, $F \in \mathscr{E}_\Lambda$, where the measures μ_Λ are defined in subsection 2.2. The sample space Ω can be chosen in a canonical way: Ω is the collection of all functions ω defined on a nonempty open convex domain $\Delta(\omega) \subset \mathbb{R}^d$ and such that $\omega(t) \in E_t$ for each $t \in \Delta(\omega)$. As usual we set $X_t(\omega) = \omega(t)$ for $t \in \Delta(\omega)$. Our problem is reduced to constructing a measure P on Ω subject to condition (3.1).

To this end we invoke a general inverse limit theorem which deals with the following objects:

3.1.A. A partially ordered index set $(\mathscr{I}, \leqq)$ such that for each $\Lambda, \Gamma \in \mathscr{I}$ the least upper bound $\Lambda \vee \Gamma$ exists. [In our situation, $\leqq$ is inclusion and $\vee$ is the union of sets.]

3.1.B. A measurable space $(C, \mathscr{C})$, and a family $\{C_\Lambda : \Lambda \in \mathscr{I}\} \subset \mathscr{C}$ such that $C_\Lambda \cap C_\Gamma = C_{\Lambda \vee \Gamma}$ and $\bigcup \{C_\Lambda : \Lambda \in \mathscr{I}\} = C$. We put $\mathscr{C}_\Lambda = \mathscr{C} \cap C_\Lambda$. For $x \in C$ put $\mathscr{I}(x) = \{\Lambda \in \mathscr{I} : x \in C_\Lambda\}$. We say that $\mathscr{J} \subset \mathscr{I}$ *determines* $x \in C$ if for each $y \in C$ with $\mathscr{I}(y) \not\supset \mathscr{I}(x)$ there exists $\Lambda \in \mathscr{J}$ with $x \in C_\Lambda$, $y \notin C_\Lambda$. We assume that there is a countable set $\mathscr{I}' \subset \mathscr{I}$ such that for each $x \in C$, $\mathscr{I}'$ determines x. [In our situation C is the class of nonempty open convex subsets of $\mathbb{R}^d$, and $\mathscr{C}$ is the σ-field on C described in Theorem 2.1. Of course $C_\Lambda = \{\Delta \in C : \Delta \supset \Lambda\}$. We leave it to the reader to exhibit a set $\mathscr{I}'$ as above.]

3.1.C. A mapping Δ from a set Ω to C. For $\Lambda \in \mathscr{I}$ we set $\Omega_\Lambda = \{\omega \in \Omega : \Delta(\omega) \in C_\Lambda\}$.

3.1.D. Measurable spaces $(E_\Lambda, \mathscr{E}_\Lambda)$, $\Lambda \in \mathscr{I}$, which we assume to be U-spaces.

3.1.E. Mappings $X_\Lambda : \Omega_\Lambda \to E_\Lambda$, $\Lambda \in \mathscr{I}$.

3.1.F. Mappings $\pi_{\Lambda\Gamma} : E_\Lambda \to E_\Gamma$, $\Gamma \leqq \Lambda$. We assume that $\pi_{\Lambda\Gamma}$ is $\mathscr{E}_\Lambda/\mathscr{E}_\Gamma$-measurable and surjective, and that

$$\pi_{\Lambda\Sigma} = \pi_{\Gamma\Sigma} \circ \pi_{\Lambda\Gamma}, \qquad \Sigma \leqq \Gamma \leqq \Lambda; \tag{3.2}$$

$$X_\Sigma = \pi_{\Gamma\Sigma} \circ X_\Gamma \quad \text{on } \Omega_\Gamma,\ \Sigma \leqq \Gamma. \tag{3.3}$$

[In the context of Theorem 1.1, $\pi_{\Lambda\Gamma}$ is the natural restriction mapping from E_Λ to E_Γ.]

3.1.G. Measures μ_Λ on $\mathscr{C}_\Lambda \otimes \mathscr{E}_\Lambda$ such that

$$\mu_\Lambda(D \times \pi_{\Lambda\Gamma}^{-1} F) = \mu_\Gamma(D \times F), \qquad D \in \mathscr{C}_\Lambda,\quad F \in \mathscr{E}_\Gamma,\quad \Gamma \leqq \Lambda. \tag{3.4}$$

We assume that for each Λ there is a sequence $\{D_n \times F_n\} \subset \mathscr{C}_\Lambda \times \mathscr{E}_\Lambda$ with $\bigcup (D_n \times F_n) = C_\Lambda \times E_\Lambda$ and $\mu_\Lambda(D_n \times F_n) < \infty$ for all n. [In our situation (3.4) follows easily from (2.11).]

Finally, we need the following technical condition; *see* section 5 for a discussion of this condition. Recall that a sequence $\{x_n\}$ from a directed set $(D, \leqq)$ is *cofinal* if for each $y \in D$ there is an n such that $y \leqq x_n$.

3.1.H. For each $x \in C$, each sequence $\Lambda(1) \leqq \Lambda(2) \leqq \ldots$ from $\mathscr{I}(x)$ which is cofinal in some set $\mathscr{J}$ which determines x, each sequence $\{y_n : n \in \mathbb{N}\}$ such that $y_n \in E_{\Lambda(n)}$, $\pi_{\Lambda(n+1),\Lambda(n)} y_{n+1} = y_n$, $n \in \mathbb{N}$, there is a point $\omega \in \Omega$ such that $\Lambda(\omega) = x$, $X_{\Lambda(n)}(\omega) = y_n$, $n \in \mathbb{N}$. [This condition is trivially satisfied in the context of Theorem 1.1.]

Theorem 1.1 follows immediately from

THEOREM 3.1. — *Under hypotheses* 3.1.*A*-3.1.*H there exists a measure* P *on* Ω *such that for all* $\Lambda \in \mathscr{I}$, $D \in \mathscr{C}_\Lambda$, $F \in \mathscr{E}_\Lambda$,

$$P(\omega \in \Omega_\Lambda : \Delta(\omega) \in D,\ X_\Lambda(\omega) \in F) = \mu_\Lambda(D \times F). \tag{3.5}$$

The domain of P *is the* σ*-ring generated by the events on the left side of* (3.5), *and* P *is the unique measure on this domain satisfying* (3.5). *Moreover,* P *is* σ*-finite.*

If each C_Λ is a singleton, then each space $C_\Lambda \times E_\Lambda$ can be identified with E_Λ. In this case (3.4) means that $\pi_{\Lambda\Gamma}(\mu_\Lambda) = \mu_\Gamma$ for $\Gamma \leqq \Lambda$ and we have a generalization of the classical inverse limit theorem; *see* [DM, III-53].

Suppose that for each $x \in C$ there exists $y \in C$ such that $\mathscr{I}(y) \not\supseteq \mathscr{I}(x)$. (This is certainly the case in the context of Theorem 1.1). Then by 3.1.B we must have $\Omega = \bigcup \{C_\Lambda : \Lambda \in \mathscr{I}'\}$; since $\mathscr{I}'$ is countable it follows that Ω is an element of the σ-ring described in Theorem 3.1. Said σ-ring is thus a σ-field in this case.

3.2. Markov processes. Theorem 1.1 can be used to construct Markov processes with random times of birth and death but the paths of a process so constructed enjoy no regularity properties. If one desires a Markov process with, for example, right continuous paths, then a slightly different application of Theorem 3.1 must be made. We restrict our attention to the case of right continuous paths, but it will be obvious that quite similar considerations apply to the cases of continuous paths, cadlag paths, etc.

Let E be a topological space homeomorphic to a Borel subset of a compact metric space, with Borel sets $\mathscr{E}$; let $\partial \notin E$ be a cemetary point, and let Ω denote the space of paths $\omega : \mathbb{R} \to E \cup \{\partial\}$ which are E-valued and right continuous on some nonempty open interval $\Delta(\omega) =]\alpha(\omega), \beta(\omega)[$ and which take the value ∂ outside of $\Delta(\omega)$. Let $Y_t(\omega) = \omega(t)$, $t \in \mathbb{R}$, $\omega \in \Omega$, and let $\mathscr{F} = \sigma\{Y_t : t \in \mathbb{R}\}$. Then $(\Omega, \mathscr{F})$ is a U-space (*see* [DM, III]). We shall consider measures P on $(\Omega, \mathscr{F})$ under which the process $(Y_t : t \in \mathbb{R})$ is a Markov process.

Let $(P_t^s : s < t)$ be a nonhomogeneous transition function: each P_t^s is a subMarkov kernel on $(E, \mathscr{E})$, and $P_t^s P_u^t = P_u^s$ for $s < t < u$ $(s, t, u \in \mathbb{R})$. We say that a measure P on $(\Omega, \mathscr{F})$ is *Markovian with transition function* P_t^s *if*

$$\nu_t \equiv P(Y_t \in .) \text{ is a } \sigma\text{-finite measure on } (E, \mathscr{E}), t \in \mathbb{R}, \tag{3.6}$$

and

$$P(\alpha < t_1, Y_{t_1} \in dx_1, \ldots, Y_{t_n} \in dx_n, t_n < \beta)$$
$$= \nu_{t_1}(dx_1) P_{t_2}^{t_1}(x_1, dx_2) \ldots P_{t_n}^{t_{n-1}}(x_{n-1}, dx_n) \tag{3.7}$$

for $t_1 < t_2 < \ldots < t_n$. Note that

$$\nu_s P_t^s \uparrow \nu_t \quad \text{as} \quad s \uparrow t, \quad t \in \mathbb{R}. \tag{3.8}$$

A family $\{\nu_t : t \in \mathbb{R}\}$ of σ-finite measures which satisfies (3.8) is called an *entrance rule for* P_t^s. Conversely, each entrance rule corresponds to some Markovian measure P, at least under the following regularity hypothesis:

3.2.A. For each $s \in \mathbb{R}$ and $x \in E$ there exists a probability measure $P_{s,x}$ on $\mathscr{F} \cap \{\alpha = s < \beta\}$ such that for $s < t_1 < t_2 < \ldots < t_n$

$$P_{s,x}(Y_{t_1} \in dx_1, \ldots, Y_{t_n} \in dx_n, t_n < \beta)$$
$$= P_{t_1}^s(x, dx_1) \ldots P_{t_n}^{t_{n-1}}(x_{n-1}, dx_n). \tag{3.9}$$

THEOREM 3.2. — *Let* $\{\nu_t\}$ *be an entrance rule for* P_t^s *and assume that* 3.2.*A holds. Then there is a unique measure* P *on* $(\Omega, \mathscr{F})$ *such that* (3.7) *holds. Moreover,* P *is necessarily* σ*-finite.*

Proof. — We will apply Theorem 3.1 to the present situation, following the notational scheme laid down in 3.1.A-3.1.H. We take the index set $\mathscr{I}$ to be the class of compact intervals ordered by inclusion; if $\Lambda = [a, b]$, $\Gamma = [c, d]$, then $\Lambda \vee \Gamma = [\min(a, c), \max(b, d)]$. For $\Lambda \in \mathscr{I}$ we take

$\Omega_\Lambda = \{\omega \in \Omega : \Delta(\omega) \supset \Lambda\}$ and we identify C_Λ, the class of open subintervals of $\mathbb{R}$ containing $\Lambda = [a, b]$, with the quadrant $H_\Lambda = \{(c, d) : -\infty \leqq c < a < b < d \leqq +\infty\}$. We take $\mathscr{C}_\Lambda$ to be the natural Borel field on H_Λ. Let E_Λ denote the space of right continuous paths from Λ to E, and let $\mathscr{E}_\Lambda$ be the σ-field on E_Λ generated by the coordinate mappings. We take $X_\Lambda : \Omega_\Lambda \to E_\Lambda$, $\pi_{\Lambda, \Gamma} : E_\Lambda \to E_\Gamma$ to be the natural restriction mappings. The reader can easily check that with these choices 3.1.A-3.1.F are satisfied. (Take $\mathscr{I}'$ in 3.1.B to be the class of those intervals in $\mathscr{I}$ with rational endpoints.) If x, $\Lambda(n)$, y_n are as in 3.1.H, then clearly there is a *unique* path $\omega \in \Omega$ such that $\Delta(\omega) = x$ and $\omega|_{\Lambda(n)} = y_n$ for all $n \in \mathbb{N}$. It remains to construct the measures μ_Λ subject to 3.1.G.

For an interval $I \subset \mathbb{R}$ we let $\mathscr{F}(I) = \sigma\{Y_t : t \in I\}$. Fix $\Lambda = [a, b]$ and $B \in \mathscr{E}$ with $\nu_a(B) < \infty$. Making the obvious identifications, for any $s < a$, $b \leqq t$ we can regard $P_{s, x}$ restricted to $\mathscr{F}(\Lambda) \cap \{t < \beta\}$ as a measure on $(E_\Lambda, \mathscr{E}_\Lambda)$. Thus, for $s \leqq a$, $b \leqq t$, $F \in \mathscr{E}_\Lambda$ we may define

$$M(s, t, F) = \int_E \nu_s P^s_u(dx) P_{u, x}(F, t < \beta, Y_a \in B), \quad s < a \tag{3.10}$$

$$M(a, t, F) = \uparrow \lim_{s \uparrow a} M(s, t, F), \tag{3.11}$$

where $u \in]s, a[$ is arbitrary in (3.10). The right side of (3.10) does not depend on u because of (3.9). Clearly $t \to M(s, t, F)$ is decreasing and right continuous on $[b, +\infty[$. Also, $\nu_s P^s_u = \nu_s P^s_v P^v_u \uparrow \nu_v P^v_u$ as $s \uparrow v < u < a$. It follows that $s \to M(s, t, F)$ is increasing and left continuous on $]-\infty, a]$. Finally, $M(s, t, F) \leqq \nu_a(B) < \infty$ and so from (3.10), (3.11) we can deduce

$$M(a, b, F) - M(s, b, F) - M(a, t, F) + M(s, t, F) \geqq 0.$$

Recalling our identification of C_Λ with H_Λ, and taking note of Remark (1.6-*b*), we can argue as in subsection 2.2 to produce a measure μ_Λ on $\mathscr{C}_\Lambda \otimes \mathscr{E}_\Lambda$ such that

$$\mu_\Lambda(C_\Gamma \times (F \cap \{Y_a \in B\}) = M(s, t, F), \qquad F \in \mathscr{E}_\Lambda, \tag{3.12}$$

where $\Gamma = [s, t] \supset \Lambda = [a, b]$. Since ν_a is a σ-finite, it is clear from (3.12) that μ_Λ satisfies the σ-finiteness hypothesis in 3.1.G. The relation (3.4) follows from (3.9)-(3.12). Clearly $\mathscr{F}$ coincides with the σ-ring described in Theorem 3.1 (*see* the remark following Theorem 3.1). By Theorem 3.1 there exists a unique measure P (necessarily σ-finite) on $(\Omega, \mathscr{F})$ such that

$$P(\Delta \supset \Gamma,\ X_\Lambda \in F) = \mu_\Lambda(C_\Gamma \times F), \qquad \Lambda \leqq \Gamma, \quad F \in \mathscr{E}_\Lambda. \tag{3.13}$$

We leave it to the reader to verify (3.7), using (3.9)-(3.13). □

3.3. Continuous processes on random open domains. Now we are interested in the case where $\Delta(\omega)$ is a nonempty open subset of a locally compact, second countable Hausdorff space T, and $t \to X_t(\omega)$ is a continuous mapping from $\Delta(\omega)$ to $\mathbb{R}$. We let $\mathscr{I}$ denote the class of nonempty compact subsets of T (ordered by inclusion), and for $\Lambda \in \mathscr{I}$ we let E_Λ denote the space of continuous paths from Λ to $\mathbb{R}$. The Borel sets in E_Λ are denoted $\mathscr{E}_\Lambda$. If $\Delta(\omega) \supset \Lambda$, then $X_\Lambda(\omega)$ denotes the restriction of ω to Λ. We put

$$m_\Lambda(B) = P(\Delta \supset \Lambda,\ X_\Lambda \in B), \qquad B \in \mathscr{E}_\Lambda, \quad \Lambda \in \mathscr{I} \tag{3.15}$$

where P is a measure whose domain contains the events on the right side of (3.15). For $\Gamma \leqq \Lambda$ (both in $\mathscr{I}$) we let $\pi_{\Lambda\Gamma}$ denote the natural restriction mapping from E_Λ to E_Γ. It is easy to see that

$$m_{\Lambda_n}(\pi_{\Lambda_n, \Lambda}^{-1} B) \to m_\Lambda(B) \qquad \text{if} \quad \Lambda_n \downarrow \Lambda, \quad B \in \mathscr{E}_\Lambda, \tag{3.16}$$

and that for $n \in \mathbb{N}$, $\Lambda_0, \Lambda_1, \ldots, \Lambda_n \in \mathscr{I}$,

$$\Sigma(-1)^k m_{\Lambda_0 \vee \Lambda_{i_1} \vee \ldots \vee \Lambda_{i_k}}(\pi_{\Lambda_0 \vee \Lambda_{i_1} \ldots \vee \Lambda_{i_k}, \Lambda_0}^{-1}(B)) \geqq 0, \quad B \in \mathscr{E}_{\Lambda_0}, \tag{3.17}$$

where the sum extends over all subsets $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$.

THEOREM 3.3. — *Let $\{m_\Lambda : \Lambda \in \mathscr{I}\}$ be a family of σ-finite measures $[m_\Lambda$ a measure on $(E_\Lambda, \mathscr{E}_\Lambda)]$ satisfying* (3.16) *and* (3.17). *Then there exists a continuous $\mathbb{R}$-valued stochastic process $(X_t(\omega), P)$ with a random open domain $\Delta(\omega)$, such that* (3.15) *holds.*

Remark. — If $T = \mathbb{R}^d$, then $\Delta(\omega)$ is convex (a. s. P) if and only if

$$m_{\Lambda \vee \Gamma}(\pi_{\Lambda \vee \Gamma, \Lambda}^{-1}(B)) = m_\Lambda(B), \qquad B \in \mathscr{E}_\Lambda, \tag{3.18}$$

for every pair $\Lambda, \Gamma \in \mathscr{I}$ such that $[\Lambda \cup \Gamma] = [\Lambda]$.

Theorem 3.3 follows from Choquet's theorem and Theorem 3.1 in much the same way as Theorem 1.1. We leave to the reader the details of the proof, contenting ourselves with a few remarks. Let C denote the class of nonempty open subsets of T, and let $\mathscr{C}$ be the σ-field on C generated by the events

$$C_\Lambda \equiv \{x \in C : x \supset \Lambda\}, \qquad \Lambda \in \mathscr{I}.$$

If $\mathscr{B}$ is a countable base for the topology of T consisting of relatively compact open sets, then 3.1.B holds with $\mathscr{I}'=\{\bar{U} : U\in\mathscr{B}\}$. The fact that each $\pi_{\Lambda\Gamma}$ is surjective (3.1.F) follows from Tietze's extension theorem. Hypothesis 3.1.H is automatically satisfied (*see* subsection 5.4).

Remark. – In Theorem 3.3 the state space $\mathbb{R}$ can be replaced by a topological space E with the property that any continuous mapping, of a compact subset of a compact metric space K, into E, admits a continuous extension to all of K.

4. PROOF OF THE GENERAL INVERSE LIMIT THEOREM

4.1. We begin with the following

LEMMA 4.1. – *For each* $\Lambda\in\mathscr{I}$, *the mapping* $\omega\rightarrow(\Delta(\omega), X_\Lambda(\omega))$, *from* Ω_Λ *to* $C_\Lambda\times E_\Lambda$, *is surjective.*

Proof. – Fix $\Lambda\in\mathscr{I}$, $x\in C_\Lambda$, $y_\Lambda\in E_\Lambda$. Let $\mathscr{I}'$ be as in 3.1.B and let $\mathscr{J}$ denote the $\vee$-stabilization of $\{\Lambda\vee\Gamma : \Gamma\in\mathscr{I}'\cap\mathscr{I}(x)\}\cup\{\Lambda\}$. Since $\mathscr{I}'$ determines x, so does $\mathscr{J}$. Clearly there exists an increasing sequence $\{\Lambda(n): n\in\mathbb{N}\}\subset\mathscr{J}$ which is cofinal in $\mathscr{J}$. By 3.1.F, each $\pi_{\Gamma,\Gamma'}$ is surjective and so we can choose inductively a sequence $\{y_n\}$ with $y_n\in E_{\Lambda(n)}$, $\pi_{\Lambda(n+1),\Lambda(n)}y_{n+1}=y_n$, and $\pi_{\Lambda(1),\Lambda}y_1=y_\Lambda$. By 3.1.H there exists an $\omega\in\Omega$ with $\Delta(\omega)=x$, $X_{\Lambda(n)}(\omega)=y_n$, $n\in\mathbb{N}$. Clearly $(\Delta(\omega), X_\Lambda(\omega))=(x, y_\Lambda)$ as required. □

Now note that the sets

$$\{\omega : \Delta(\omega)\in D, X_\Lambda(\omega)\in F\},\qquad \Lambda\in\mathscr{I},\quad D\in\mathscr{C}_\Lambda,\quad F\in\mathscr{E}_\Lambda,$$

form a semiring $\mathscr{A}$ in Ω. The first step in the proof is to define a function $\mu:\mathscr{A}\rightarrow[0, +\infty]$ such that

$$\mu(A)=\mu_\Lambda(D\times F)\qquad\text{if}\quad A=\{\Delta\in D, X_\Lambda\in F\}\tag{4.1}$$

where $D\in\mathscr{C}_\Lambda$, $F\in\mathscr{E}_\Lambda$. To justify (4.1) we need to verify that if A has a second representation $A=\{\Delta\in D', X_{\Lambda'}\in F'\}$, where $D'\in\mathscr{C}_{\Lambda'}$, $F'\in\mathscr{E}_{\Lambda'}$, then

$$\mu_{\Lambda'}(D'\times F')=\mu_\Lambda(D\times F).\tag{4.2}$$

If $A=\varnothing$, then (4.2) follows trivially from Lemma 4.1. If $A \neq \varnothing$, then $A \subset \Omega_\Lambda \cap \Omega_{\Lambda'} = \Omega_\Gamma$, where $\Gamma = \Lambda \vee \Lambda'$. By (3.3)

$$A=\{\Delta \in D,\ X_\Gamma \in \pi_{\Gamma,\Lambda}^{-1} F\} = \{\Delta \in D',\ X_\Gamma \in \pi_{\Gamma,\Lambda'}^{-1} F'\},$$

and so $D=D'$, $\pi_{\Gamma,\Lambda}^{-1} F = \pi_{\Gamma,\Lambda'}^{-1} F'$ by Lemma 4.1. Relation (4.2) now follows from (3.4).

Thus μ is well-defined on $\mathscr{A}$ by (4.1). It is easy to check that μ is additive on $\mathscr{A}$. Moreover, by 3.1.G, each $A \in \mathscr{A}$ is contained in a countable union of sets from $\mathscr{A}$ with finite μ measure (i. e., μ is σ-finite on $\mathscr{A}$).

4.2. By a classical theorem of measure theory (*see* 3.13, Theorem A of [Ha]), the proof will be completed once we show that μ is σ-additive on $\mathscr{A}$. Our proof of this point follows in outline an argument of Hu [H].

We fix a sequence $\{A_i : i \in \mathbb{N}\}$ of nonempty disjoint sets from $\mathscr{A}$ with union $A_0 \in \mathscr{A}$. Then for each $i \geqq 0$ there is an index $\Lambda(i) \in \mathscr{I}$ and sets $(D_i \times F_i) \in \mathscr{C}_{\Lambda(i)} \times \mathscr{E}_{\Lambda(i)}$ such that $A_i = \{\Delta \in D_i,\ X_{\Lambda(i)} \in F_i\}$. Now $A_i \subset A_0 \subset \Omega_{\Lambda(0)}$ and so $A_i \subset \Omega_{\Lambda(i) \vee \Lambda(0)}$ if $i \geqq 1$. Replacing $\Lambda(i)$ by $\Lambda(i) \vee \Lambda(0)$, and F_i by $\pi_{\Lambda(i) \vee \Lambda(0),\Lambda(i)}^{-1}(F_i)$ if necessary, we assume without loss of generality that $\Lambda(i) \geqq \Lambda(0)$ for $i \geqq 1$. If $x \in D_0$ then we can choose $y \in F_0$ (else $A_0 = \varnothing$); by Lemma 4.1 there exists an ω with $\Delta(\omega)=x$, $X_{\Lambda(0)}(\omega)=y$. Thus $\omega \in A_0 = \bigcup_{i=1}^{\infty} A_i$, so $x=\Delta(\omega) \in D_i$ for some $i \geqq 1$. In other words, $D_0 \subset \bigcup_{i=1}^{\infty} D_i$. Similary, $D_0 \supset \bigcup_{i=1}^{\infty} D_i$.

We need two more lemmas, which we prove in the next subsection.

LEMMA 4.2. — *There exists a probability measure ν on $(C_{\Lambda(0)}, \mathscr{C}_{\Lambda(0)})$, a family $\rho_\Lambda(x, dy)$, $\Lambda \geqq \Lambda(0)$ of Markov kernels from $(C_\Lambda, \mathscr{C}_\Lambda)$ to $(E_\Lambda, \mathscr{E}_\Lambda)$, and strictly positive functions $f_\Lambda \in \mathscr{C}_\Lambda \otimes \mathscr{E}_\Lambda$ such that*

$$\mu_\Lambda(D \times F) = \int_D \nu(dx) \int_F [f_\Lambda(x, y)]^{-1} \rho_\Lambda(x, dy), \tag{4.3}$$

where $\Lambda \geqq \Lambda(0)$, $D \in \mathscr{C}_\Lambda$, $F \in \mathscr{E}_\Lambda$.

LEMMA 4.3. — *There is a ν-null set $B \in \mathscr{C}_{\Lambda(0)}$ such that for $x \in D_0 \setminus B$,*

$$\int_{F_0} [f_{\Lambda(0)}(x, y)]^{-1} \rho_{\Lambda(0)}(x, dy) = \sum_{i \in I(x)} \int_{F_i} [f_{\Lambda(i)}(x, y)]^{-1} \rho_{\Lambda(i)}(x, dy), \tag{4.4}$$

where $I(x) = \{i \geqq 1 : x \in D_i\}$.

Let us now integrate both sides of (4.4) against $1_{D_0}(x)\nu(dx)$. On the left side, because of (4.3), we obtain $\mu_{\Lambda(0)}(D_0 \times F_0) = \mu(A_0)$. On the right side we obtain

$$\int_{D_0} \nu(dx) \sum_{i \in I(x)} \int_{F_i} [f_{\Lambda(i)}(x, y)]^{-1} \rho_{\Lambda(i)}(x, dy)$$

$$= \sum_{i \geqq 1} \int_{D_i} \nu(dx) \int_{F_i} [f_{\Lambda(i)}(x, y)]^{-1} \rho_{\Lambda(i)}(x, dy)$$

$$= \sum_{i \geqq 1} \mu_{\Lambda(i)}(D_i \times F_i) = \sum_{i \geqq 1} \mu(A_i).$$

Thus $\mu(A_0) = \sum_{i \geqq 1} \mu(A_i)$ and so μ is σ-additive on $\mathscr{A}$. The proof of Theorem 3.1 is complete.

4.3. It remains to prove Lemmas 4.2 and 4.3.

Proof of Lemma 4.2. — By 3.1.G, $\mu_{\Lambda(0)}$ is σ-finite and so we may choose a strictly positive function $f = f_{\Lambda(0)} \in \mathscr{C}_{\Lambda(0)} \otimes \mathscr{E}_{\Lambda(0)}$ such that $\mu_{\Lambda(0)}(f) = 1$. Define a probability measure ν on $(C_{\Lambda(0)}, \mathscr{C}_{\Lambda(0)})$ by setting $\nu(D) = \mu_{\Lambda(0)}(f\, 1_{D \times E_{\Lambda(0)}})$. For $\Lambda \geqq \Lambda(0)$ define $f_\Lambda \in \mathscr{C}_\Lambda \otimes \mathscr{E}_\Lambda$ by $f_\Lambda(x, y) = f_{\Lambda(0)}(x, \pi_{\Lambda, \Lambda(0)}\, y)$, and note that $f_\Lambda > 0$ on $C_\Lambda \times E_\Lambda$. By (3.4), $\mu_\Lambda(f_\Lambda) \leqq \mu_{\Lambda(0)}(f_{\Lambda(0)}) = 1$. The measure $\bar{\mu}_\Lambda \equiv f_\Lambda \cdot \mu_\Lambda$ is thus a subprobability measure on $\mathscr{C}_\Lambda \otimes \mathscr{E}_\Lambda$. It follows from (3.4) that for fixed $\Lambda \geqq \Lambda(0)$, $F \in \mathscr{E}_\Lambda$, the measure $\bar{\mu}_\Lambda(\,.\, \times F)$ is dominated by ν on $\mathscr{C}_\Lambda$. Since $(E_\Lambda, \mathscr{E}_\Lambda)$ is a U-space, standard techniques (*see* [DM], [G]) show that there are Markov kernels $\rho_\Lambda(x, dy)$ satisfying (4.3). □

Proof of Lemma 4.3. — First note that since $\mathscr{E}_\Lambda$ is countably generated, it follows from (3.4) that if $\Lambda(0) \leqq \Gamma \leqq \Lambda$ then there is a ν-null set $B_{\Lambda\Gamma} \in \mathscr{C}_\Lambda$ such that for $x \in C_\Lambda \setminus B_{\Lambda\Gamma}$,

$$\pi_{\Lambda\Gamma}(\rho_\Lambda(x, .)) = \rho_\Gamma(x, .). \tag{4.5}$$

Let $\mathscr{I}'$ be as in 3.1.B and let $\mathscr{I}$ denote the $\vee$-stabilization of $\{\Gamma \vee \Lambda(i) : \Gamma \in \mathscr{I}', i \geqq 0\} \cup \{\Lambda(i) : i \geqq 0\}$; clearly $\mathscr{I}$ is countable. Let $B = \bigcup \{D_0 \cap B_{\Lambda\Gamma} : \Lambda, \Gamma \in \mathscr{I}, \Gamma \leqq \Lambda\}$ so that $B \in \mathscr{C}_{\Lambda(0)}$, $B \subset D_0$, $\nu(B) = 0$.

Fix $x \in D_0 \setminus B$ and put $\mathscr{I}(x) = \{\Lambda \in \mathscr{I} : x \in C_\Lambda\}$. Then $\mathscr{I}(x)$ determines x, and (4.5) holds whenever $\Gamma \leqq \Lambda$ both lie in $\mathscr{I}(x)$. Since $\mathscr{I}(x)$ has a cofinal sequence, the classical inverse limit theorem ([DM, III-53]) applies to the inverse system $\{\rho_\Lambda(x, .), \Lambda \in \mathscr{I}(x)\}$ under the mappings $\pi_{\Lambda\Gamma}$. More precisely, let E consist of those elements $(y_\Lambda : \Lambda \in \mathscr{I}(x))$ of the product space $\times\{E_\Lambda: \Lambda\in\mathscr{J}(x)\}$ which satisfy $\pi_{\Lambda\Gamma} y_\Lambda = y_\Gamma$ for all Λ, $\Gamma\in\mathscr{J}(x)$ with $\Gamma\leqq\Lambda$. Let $q_\Lambda: E\to E_\Lambda$ denote the coordinate projection and put $\mathscr{F}=\sigma\{q_\Lambda: \Lambda\in\mathscr{J}(x)\}$. Then there is a unique probability measure ρ_x on $(E, \mathscr{F})$ such that

$$\rho_x(q_\Lambda^{-1} F)=\rho_\Lambda(x, F), \qquad \Lambda\in\mathscr{J}(x), \quad F\in\mathscr{E}_\Lambda. \tag{4.6}$$

Recall that $A_0=\{\Delta\in D_0, X_{\Lambda(0)}\in F_0\}$ is the disjoint union of

$$A_i=\{\Delta\in D_i, X_{\Lambda(i)}\in F_i\}, \qquad i\geqq 1.$$

For $i\in I(x)=\{i\geqq 1: x\in D_i\}$ put $B_i=q_{\Lambda(i)}^{-1} F_i\in\mathscr{F}$. Then $B_0=\bigcup\{B_i: i\in I(x)\}$ and this is a disjoint union. Indeed, if $y=(y_\Lambda)\in E$ is a point of $B_i\cap B_j$ ($i\neq j$; $i, j\in I(x)$), then by 3.1.H there exists $\omega\in\Omega$ with $\Delta(\omega)=x$, $X_\Lambda(\omega)=y_\Lambda$, $\Lambda\in\mathscr{J}(x)$. But then $\omega\in A_i\cap A_j$ which contradicts $A_i\cap A_j=\varnothing$. By a similar argument we have $B_0=\bigcup\{B_i: i\in I(x)\}$. Finally, note that there is a uniquely determined $\mathscr{F}$-measurable function $g: E\to]0, +\infty[$ such that $g(y)=f_\Lambda(x, y_\Lambda)$ for any $y=(y_\Lambda)\in E$, $\Lambda\in\mathscr{J}(x)$. Since ρ_x is σ-additive we have

$$\rho_x(g^{-1} 1_{B_0})=\sum_{i\in I(x)} \rho_x(g^{-1} 1_{B_i}). \tag{4.7}$$

Now (4.4) follows immediately from (4.7) and (4.6). $\square$

5. CONCLUDING REMARKS

5.1. Theorem 2.1 is a refinement of a theorem of Choquet [C] which characterizes the distributions of random closed subsets of locally compact Hausdorff spaces. Choquet's theorem forms the foundation of Matheron's theory of random sets. (*See* [Ma].) In particular Theorem 4-2-1 of [Ma] gives necessary and sufficient conditions for a random closed subset of $\mathbb{R}^d$ to be convex. A more general theory of random sets has been developed by Kendall [Ke].

5.2. Theorem 1.1 implies a theorem of Kuznetsov [K1], mentioned already in section 1.

Theorem 3.2 (under an additional hypothesis on P_t^s) has been proved by quite different methods by Getoor and Glover [GG, (3.5)].

5.3. Under certain conditions the hypothesis 3.1.H of Theorem 3.1 can be dropped. For instance, if the elements of the index set $\mathscr{I}$ are subsets of some set T (the order $\leqq$ on $\mathscr{I}$ being inclusion), if each $(E_\Lambda, \mathscr{E}_\Lambda)$ is the product of spaces $(E_t, \mathscr{E}_t)$, $t \in \Lambda$, and if $\pi_{\Lambda\Gamma}$ is the natural projection, then 3.1.H is automatically satisfied. In this case the assumption in 3.1.B concerning $\mathscr{I}'$ can also be dropped, and Theorem 3.1 is essentially Theorem 1′ of [H].

5.4. In most cases of interest it is possible, starting from 3.1.A, 3.1.B, 3.1.D, 3.1.F [but not (3.3)], and 3.1.G, to construct Ω, Δ, X_Λ so that 3.1.C, 3.1.H, and (3.3) hold. For instance, suppose that we strengthen the second part of 3.1.B by assuming the existence of a countable set $\mathscr{I}' \subset \mathscr{I}$ such that $\mathscr{I}' \cap \mathscr{I}(x)$ is cofinal in $\mathscr{I}(x)$ for each $x \in C$. For fixed $x \in C$ let $E(x)$ denote the set of elements $(y_\Lambda : \Lambda \in \mathscr{I}(x))$ of the product space $\times \{E_\Lambda : \Lambda \in \mathscr{I}(x)\}$ such that $\pi_{\Lambda\Gamma} y_\Lambda = y_\Gamma$ for all $\Gamma, \Lambda \in \mathscr{I}(x)$ with $\Gamma \leqq \Lambda$. Now put

$$\Omega = \{(x, y) : x \in C,\ y \in E(x)\},$$

$$\Delta(\omega) = \Delta(x, y) = x,$$

$$X_\Lambda(\omega) = X_\Lambda(x, y) = y_\Lambda \quad \text{if} \quad (x, y) = (x, (y_\Lambda)) \in \Omega_\Lambda.$$

Clearly 3.1.C and (3.3) are satisfied, and 3.1.H follows easily from our assumption regarding $\mathscr{I}'$.

A class $\mathscr{I}'$ as above exists, for example, if C is the class of nonempty open subsets of a locally compact, second countable Hausdorff space T, if $\mathscr{I}$ is the class if nonempty compact subsets of T (ordered by inclusion), and if

$$C_\Lambda = \{x \in C : x \supset \Lambda\}, \qquad \Lambda \in \mathscr{I}.$$

REFERENCES

[C] G. Choquet, Theory of Capacities, *Ann. Inst. Fourier*, Vol. **5**, 1955, pp. 131-295.

[DM] C. Dellacherie and P. A. Meyer, *Probabilities and Potential*, Vol. **I**, North-Holland, Amsterdam-New York-Oxford, 1978.

[D1] E. B. Dynkin, Integral Representation of Excessive Measures and Excessive Functions, *Russian Math. Surveys*, Vol. **27**, 1972, pp. 43-84.

[D2] E. B. Dynkin, On Duality for Markov Processes, *Stochastic Analysis*, A. Friedman and M. Pinsky Eds., pp. 63-77, Academic Press, New York, 1978.

[G] R. K. GETOOR, On the Construction of Kernels, *Lecture Notes in Math.*, No. 465, pp. 443-463, Springer-Verlag, New York-Heidelberg-Berlin, 1975.

[GG] R. K. GETOOR and J. GLOVER, Constructing Markov Processes with Random Times of Birth and Death, *Seminar on Stochastic Processes*, 1986 (to appear).

[Ha] P. HALMOS, *Measure Theory*, Van Nostrand, 1950.

[H] K. Y. HU, A Generalization of Kolmogorov's Extension Theorem and an Application to the Construction of Stochastic Processes with Random Time Domains, *Annals of Probability* (to appear).

[Ke] D. G. KENDALL, Foundations of a Theory of Random Sets, *Stochastic Geometry*, E. F. HARDING and D. G. KENDALL Eds., pp. 322-376, Wiley, London-New York-Sydney-Toronto, 1974.

[K1] S. E. KUZNETSOV, Construction of Markov Processes with Random Times of Birth and Death, *Th. Prob. Appl.*, Vol. **18**, 1974, pp. 571-575.

[K2] S. E. KUZNETSOV, Nonhomogeneous Markov Processes, *Journal of Soviet Math.*, Vol. **25**, 1984, pp. 1380-1498.

[Ma] G. MATHERON, *Random Sets and Integral Geometry*, Wiley, New York-London-Sydney-Toronto, 1975.

[Mo] P. MORANDO, Mesures aléatoires, *Lecture Notes in Math.*, No. 88, Springer-Verlag, Berlin, 1969.

(*Manuscrit reçu le* 5 *janvier* 1987.)