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S^p stability of solutions of symmetric stochastic differential equations with discontinuous driving semimartingales (*)

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ABSTRACT. — We study the stability of a solution of a multidimensional symmetric (Stratonovich) stochastic differential equation

$$X_t = x + \int_0^t f(X_{s-}) \circ dZ_s$$

and of its "canonical extension" when the driving càdlàg semimartingale Z is perturbed in S^p.

Key words : Stability of solutions of SDE, discontinuous driving semimartingales, Stratonovich integrals, approximation of SDE.

RÉSUMÉ. — Nous étudions la stabilité de la solution d'équation différentielle stochastique symétrique (de Stratonovitch) multidimensionnelle du type $X_t = x + \int_0^t f(X_{s-}) \circ dZ_s$ et de son « extension canonique » lorsqu'on perturbe dans S^p la semimartingale càdlàg Z.

Mots clés : Stabilité de la solution d'EDS, semimartingales directrices discontinues, intégrale de Stratonovich, approximation d'EDS.

Classification A.M.S. : 60 H 10, 60 H 99.

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INTRODUCTION

Consider a multidimensional stochastic differential equation

$$X_i(t)=x_i+\sum_{j=1}^{r}\int_0^t f_{ij}(X(s-))\circ dZ_j(s),\qquad i=1,2,\ldots,d, \tag{1}$$

or in matrix notation,

$$X(t)=x+\int_0^t f(X(s-))\circ dZ(s), \tag{2}$$

where $Z=Z(t)=(Z_1(t), Z_2(t), \ldots, Z_r(t))$, $t\geqq 0$, is an r-dimensional càd-làg semimartingale, $f(x)=(f_{ij}(x), i=1, 2, \ldots, d, j=1, 2, \ldots, r)$, $x\in R^d$, is a sufficiently smooth matrix function : $f_{ij}\in C_b^3(R^d)$. The small circle $\circ$ denotes the symmetric (Stratonovich) integral, with the standard notation meaning the Itô integral. We are concerned with the behaviour of a solution of (1) when Z is perturbed in S^p. Ever since Wong and Zakai [24] showed the instability of one-dimensional Itô equations, this type of problem is usually considered for stochastic differential equations in the symmetric (Stratonovich) form. A very strong topology in the space of semimartingales is needed for the stability of solutions of Itô equations (Emery [5], Protter [21], Metivier-Pellaumail [16]; this topology is too strong to permit the approximation of Z by finite variation (FV) processes. On the other hand, one cannot expect to obtain the results of stability only under assumptions of a uniform type of convergence of driving semimartingales because such a stability is not valid even in the deterministic case. Therefore additional conditions on f or the perturbations of Z naturally appear. In the first case it is usually the condition of commutation of corresponding vector fields: $\sum_n\left(\frac{df_{ij}}{dx_n}f_{nk}-\frac{df_{ik}}{dx_n}f_{nj}\right)=0$ (Allain [1], Freedman-Willems [6], Sussmann [23], Doss [3], Krener [11], Marcus [13]). In the second case only certain classes of FV approximations are allowed (Nakao-Yamato [20], Protter [22], McShane [14], Marcus [13], Konecny [10]. Ikeda-Nakao-Yamato [8] considered a wide class (slightly extended in Ikeda-Watanabe [9]) of piecewise smooth approximations of multidimensional Brownian motion and the behaviour of solutions of corresponding equations. They obtained an equation for a limit of these "approximating" solutions which, in general, does not coincide with the initial one. The particular importance of this result is, in our opinion, that: the general situation is exhibited; at the same time a typical sub-class of so-called symmetric approximations is specified when the above mentioned limiting equation does coincide with the initial one. In this spirit in [12] we obtained a general result when a continuous driving semimartingale is approximated by continuous semimartingales. We extend here this result for discontinuous driving semimartingales. It appears that in order do require minimal assumptions on the jumps of the driving semimartingales it is convinient to replace (1) by another equation [coinciding with (1) in the continuous case]. In the spirit of McShane [14], [15] and Marcus [13] this equation can be considered as a sort of "canonical extension" of (1) or a corresponding Itô equation (note that in [13] the canonical extension is defined only for f satisfying a commutation restriction).

NOTATION

All processes considered in this paper are assumed to be càdlàg (right continuous with left limits) and adapted to a given filtration satisfying the "usual hypotheses" on a fixed probability space. For a fixed time interval $I=[0, T]$, $0<T<\infty$, or $I=[0, \infty[$ we denote by S^p, $1\leqq p\leqq\infty$, the Banach space of all processes X such that $||X||_{S^p}=||X^*||_{L^p}<\infty$, where $X^*=\sup_{t\in I}|X_t|$ (we shall use the same notations for multidimensional processes, the dimension being clear from the context). We also use freely the notation of H^p semimartingales, $1\leqq p\leqq\infty$ (*cf.* Emery [4], Dellacherie-Meyer [2], Meyer [19]), and the inequalities for norms of semimartingales: if $p\in[1,\infty[$, q, $r\in[1,\infty]$, $p^{-1}=q^{-1}+r^{-1}$, then

$$||X_-\cdot Y||_{H^p}\leqq||X||_{S^q}||Y||_{H^r}; \tag{3}$$

$$||X||_{S^p}\leqq c_p||X||_{H^p}. \tag{4}$$

For ease of notation we shall write $\langle X, Y\rangle^c$ instead of $\langle X^c, Y^c\rangle$. If X and Y are semimartingales, then symmetric (Stratonovich) integral $X\circ Y=\int_0^{\cdot} X_{s-}\circ dY_s$ is defined by

$$X\circ Y=X_-\circ Y+\frac{1}{2}<X, Y>^c$$

(Meyer [18]; we prefer to write $X \circ Y$ rather than $X_- \circ Y$). We define here yet one more "symmetric" stochastic integral by

$$X \square Y = X_- \cdot Y + \frac{1}{2}[X, Y] = X \circ Y + \frac{1}{2} S(X, Y)$$

where

$$S(X, Y)_t = \sum_{s \leq t} \Delta X_s \Delta Y_s, \qquad S(X) = S(X, X).$$

This integral is "symmetric" in the following sense. It can be interpreted as the limit in probability of "Riemann type" sums $\sum_i \frac{1}{2}(X_{t_{i+1}} + X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$ and therefore does not depend on the direction of time. Nevertheless we use it only for the formulation of results in the more symmetric and short way.

FORMULATION OF RESULT

First we consider an equation different from (1) :

$$\begin{aligned} X_i(t) = x_i + \sum_{j=1}^{r} \int_0^t f_{ij}(X(s-)) \circ dZ_j(s) \\ + \frac{1}{2} \sum_{j,k=1}^{r} \int_0^t f_{ijk}(X(s-))\, dS(Z_j, Z_k)_s = x_i + \sum_{j=1}^{r} \int_0^t f_{ij}(X(s-))\, dZ_j(s) \\ + \frac{1}{2} \sum_{j,k=1}^{r} \int_0^t f_{ijk}(X(s-))\, d[Z_j, Z_k]_s, \quad (5) \end{aligned}$$

where

$$f_{ijk}(x) = \sum_{n=1}^{d} \frac{d f_{ij}(x)}{dx_n} f_{nk}(x), \qquad x = (x_1, x_2, \ldots, x_d) \in R^d.$$

Let $X^\delta = (X_1^\delta, X_2^\delta, \ldots, X_d^\delta)$, $\delta > 0$, be a solution of an analogous equation with Z replaced by a semimartingale $Z^\delta = (Z_1^\delta, Z_2^\delta, \ldots, Z_r^\delta)$. We are interested in the behaviour of X^δ as $Z^\delta \to Z^0 = Z$, $\delta \to 0$, in S^p. But first introduce some technical boundeness assumptions on the family $\{Z^\delta, \delta \geqq 0\}$.

Let $Z^\delta = Z^\delta(0) + M^\delta + A^\delta$, $\delta \geqq 0$, denote a decomposition of Z^δ into an r-dimensional local martingale M^δ and an FV process A^δ. Let $2 \leqq p < q \leqq \infty$, $\frac{1}{p} = \frac{1}{p'} + \frac{1}{q}$.

B1. $[M_i^\delta, M_i^\delta]$ are uniformly bounded in L^q, i. e.,

$$\sup_{\delta \geqq 0} \| [M_i^\delta, M_i^\delta]_T \|_{L^q} < \infty, \qquad i = 1, 2, \ldots, r;$$

B2. $S(Z_i^\delta, Z_i^\delta)$ are uniformly bounded in L^{2q};

B3. $\left\{ \left(\int_0^T | Z_j(s-) - Z_j^\delta(s-) | . | dA_k^\delta(s) | \right)^p, \delta > 0 \right\}$ is a uniformly integrable family of random variables, $j, k = 1, 2, \ldots, r$.

B1 and B2 imply a uniform boundedness of $< Z_i^\delta, Z_i^\delta >^c$ and $[Z_i^\delta, Z_i^\delta]$ in L^q and a uniform boundedness of M_i^δ in H^{2q} (but not of Z_i^δ!). B3 does not allow FV parts of Z_i^δ to increase too quickly (in comparison with rate of convergence of $Z - Z^\delta$ to 0).

Introduce the notation

$$Z_{jk}^\delta = [Z_k^\delta, Z_j] - [Z_j^\delta, Z_k] - 2(Z_j - Z_j^\delta) \square (Z_k - Z_k^\delta), \qquad j, k = 1, 2, \ldots, r.$$

THEOREM 1. — *Let the following conditions be satisfied:*

C1. $Z^\delta \to Z$ *in* $S^{p'}$, $\delta \to 0$;

C2. $Z_{jk}^\delta \to A_{jk}$ *in* $S^{p'}$, $\delta \to 0$,

where A_{jk}, $j, k = 1, 2, \ldots, r$, *are* FV *processes. Then* (*under hypotheses* B1, B2, B3) $X^\delta \to X$ *in* S^p, $\delta \to 0$, *where* X *is a solution of an equation*

$$X_i(t) = x_i + \sum_{j=1}^r \int_0^t f_{ij}(X(s-)) \circ dZ_j(s) + \frac{1}{2} \sum_{j,k=1}^r \int_0^t f_{ijk}(X(s-)) \, d(S(Z_j, Z_k) + A_{jk})_s, \quad (6)$$

$i = 1, 2, \ldots, d$.

COMMENTS AND REMARKS. — (1) In the formulation of theorem 1 Z_{jk}^δ can be replaced by their FV parts

$$A_{jk}^\delta = 2[Z_j, Z_k^\delta] - [Z_j, Z_k] - [Z_j^\delta, Z_k^\delta] - 2(Z_j - Z_j^\delta)_- \cdot (A_k - A_k^\delta)$$

but we wanted to state our results in terms of given driving semimartingales Z, Z^δ.

(2) As will be seen from the proof, under the hypotheses of theorem 1 the processes A_{jk} are, in fact, continuous; moreover, $A_{jk} = -A_{kj}$ and, in

particular, $A_{jj}=0$. This can be easily seen from

$$Z^{\delta}_{jk}+Z^{\delta}_{kj}=-2\,(Z_j-Z^{\delta}_j)\,(Z_k-Z^{\delta}_k)\to 0 \quad \text{in } S^{p'/2}, \qquad \delta\to 0.$$

(3) We shall say that $\{Z^{\delta},\ \delta>0\}$ is a *symmetric* approximation of Z if C1 and C2 are satisfied with $A_{jk}=0$. Hence, one can say that the equation (5) is "stable under *symmetric* perturbations of Z".

(4) In the case of continuous Z and Z^{δ} the theorem also slightly generalizes theorem 1 in [12] where $q=\infty$ $(p'=p)$ is taken.

(5) It is clear that the local version of the theorem is true: if the hypotheses of theorem 1 are satisfied locally, then $X^{\delta}\to X$ in S^p locally.

(6) To compare with known results (*cf.* Protter [22], Marcus [13] and the articles referenced there) we should like to underline the following:

(*a*) A commutation restriction on f is not assumed (and this is the reason for the appearance, in general, of the processes A_{jk} in the limiting equation);

(*b*) the number of jumps of Z and Z^{δ} is permitted to be infinite; two components of Z and Z^{δ} are allowed to jump at the same time;

(*c*) continuous processes Z as well as their continuous parts can be approximated by processes with jumps.

(7) The condition C1 implies the uniform convergence (in probability) of $\Delta Z^{\delta}_j \Delta Z^{\delta}_k$ to $\Delta Z_j \Delta Z_k$ but not of $S(Z^{\delta}_j, Z^{\delta}_k)$ to $S(Z_j, Z_k)$. If the latter is assumed we have the same result for the equation (1) :

THEOREM 2. — *Assume the hypotheses of theorem* 1 *and, in addition, the hypothesis*

C3. $S(Z^{\delta}_j, Z^{\delta}_k)\to S(Z_j, Z_k)$ *in* $S^{p'}$, $\delta\to 0$.

Let X^{δ}, $\delta>0$, *be a solution of an equation* (2) *with* Z *replaced by* Z^{δ}. *Then* $X^{\delta}\to X$, *where* X *is a solution of an equation*

$$X_i(t)=x_i+\sum_{j=1}^{r}\int_0^t f_{ij}(X(s-))\circ dZ_j(s)+\frac{1}{2}\sum_{j,\,k=1}^{r}\int_0^t f_{ijk}(X(s))\,dA_{jk}(s),$$

$i=1, 2, \ldots, d.$

PROOF OF THEOREM 1

It suffices to prove the theorem for $I=[0, \infty[$. We shall use the notation $g(x; y)=g(x)-g(y)$ and

$$\Delta_2 g(X)_s=\Delta g(X)_s-\Sigma_k D_k g(X_{s-})\Delta X^k_s,\ D_k g=\frac{dg}{dx_k}$$

for functions $g: R^d \to R$ and processes $X=(X^1, X^2, \ldots, X^d)$ in R^d. We shall also use the convention of summation over repeating indices. By M^δ_i and A^δ_i, M^δ_{jk} and A^δ_{jk} etc. we shall always denote the martingale and FV parts of semimartingales Z^δ_i, Z^δ_{jk} etc. All constants $C, C_1, C_2, \ldots$ do not depend on $\delta>0$ and may be different in different expressions.

We shall need some auxilliary representations for $f_{ij}(X^\delta)$, $f_{ij}(X^\delta)\circ(Z_j-Z^\delta_j)$ etc. Using Itô's formula we have

$$\begin{aligned} f_{ij}(X^\delta) &= D_n f_{ij}(X^\delta)\circ X^\delta_n+\sum_{s\leq\cdot}\Delta_2 f_{ij}(X^\delta)_s \\ &= D_n f_{ij}(X^\delta)\circ\left(f_{nk}(X^\delta)\circ Z^\delta_k+\frac{1}{2}f_{nkm}(X^\delta)_-\cdot S(Z^\delta_k, Z^\delta_m)\right) \\ &\quad +\sum_{s\leq\cdot}\Delta_2 f_{ij}(X^\delta)_s=f_{ijk}(X^\delta)\circ Z^\delta_k \\ &\quad +\frac{1}{2}f_{ijkm}(X^\delta)_-\cdot S(Z^\delta_k, Z^\delta_m)+\sum_{s\leq\cdot}\Delta_2 f_{ij}(X^\delta)_s, \end{aligned} \tag{7}$$

where we denoted $f_{ijkm}=D_n f_{ij} f_{nkm}$. From (7) we also have

$$\langle f_{ij}(X^\delta), Z_j\rangle^c=f_{ijk}(X^\delta)_-\cdot\langle Z^\delta_k, Z_j\rangle^c, \qquad \delta\geqq 0; \tag{8}$$

$$\begin{aligned}\Delta f_{ij}(X^\delta)_s &= f_{ijk}(X^\delta_{s-})\Delta Z^\delta_k(s) \\ &\quad +\frac{1}{2}f_{ijkm}(X^\delta_{s-})\Delta Z^\delta_k(s)\Delta Z^\delta_m(s)+\Delta_2 f_{ij}(X^\delta)_s.\end{aligned} \tag{9}$$

Using the stochastic integration by parts formula

$$XY = X \circ Y + Y \circ X + S(X, Y)$$

and (8) we have

$$\begin{aligned}
& f_{ij}(X^\delta) \circ (Z_j - Z_j^\delta) \\
&\quad = (Z_j - Z_j^\delta) f_{ij}(X^\delta) - (Z_j - Z_j^\delta) \circ f_{ij}(X^\delta) - S(Z_j - Z_j^\delta, f_{ij}(X^\delta)) \\
&\qquad = (Z_j - Z_j^\delta) f_{ij}(X^\delta) - f_{ijk}(X^\delta) \circ ((Z_j - Z_j^\delta) \circ Z_k^\delta) \\
&\quad - \sum_{s \leq \cdot} (Z_j - Z_j^\delta)_{s-} \left(\frac{1}{2} f_{ijkm}(X_{s-}^\delta) \Delta Z_k^\delta(s) \Delta Z_m^\delta(s) + \Delta_2 f_{ij}(X^\delta)_s \right) \\
&\qquad - \sum_{s \leq \cdot} \Delta (Z_j - Z_j^\delta)_s \Big[f_{ijk}(X_{s-}^\delta) \Delta Z_k^\delta(s) \\
&\qquad + \frac{1}{2} f_{ijkm}(X_{s-}^\delta) \Delta Z_k^\delta(s) \Delta Z_m^\delta(s) + \Delta_2 f_{ij}(X^\delta)_s \Big] \\
&\quad = (Z_j - Z_j^\delta) f_{ij}(X^\delta) - f_{ijk}(X^\delta)_- \cdot ((Z_j - Z_j^\delta) \circ Z_k^\delta + S(Z_j - Z_j^\delta, Z_k^\delta)) \\
&\quad - \frac{1}{2} (Z_j - Z_j^\delta)_- \cdot \langle f_{ijk}(X^\delta), Z_k^\delta \rangle^c - \sum_{s \leq \cdot} (Z_j - Z_j^\delta)_s \Big[\frac{1}{2} f_{ijkm}(X_{s-}^\delta) \\
&\qquad \Delta Z_k^\delta(s) \Delta Z_m^\delta(s) + \Delta_2 f_{ij}(X^\delta)_s \Big].
\end{aligned} \tag{10}$$

Consider

$$\begin{aligned}
X_i - X_i^\delta = f_{ij}(X; X^\delta) \circ Z_j + f_{ij}(X^\delta) \circ (Z_j - Z_j^\delta) \\
+ \frac{1}{2} (f_{ijk}(X)_- \cdot S(Z_j, Z_k) \\
- f_{ijk}(X^\delta)_- \cdot S(Z_j^\delta, Z_k^\delta)) + \frac{1}{2} f_{ijk}(X)_- \cdot A_{jk}.
\end{aligned} \tag{11}$$

Using (8) one can see that the first term in (11) is equal to

$$f_{ij}(X; X^\delta)_- \cdot Z_j + \frac{1}{2} f_{ijk}(X; X^\delta)_- \cdot \langle Z_k, Z_j \rangle^c + \frac{1}{2} f_{ijk}(X^\delta)_- \cdot \langle Z_k - Z_k^\delta, Z_j \rangle^c.$$

Therefore using (10) we obtain

$$X_i - X_i^\delta = I_1^\delta - \frac{1}{2} I_2^\delta - I_3^\delta + I_4^\delta + \frac{1}{2} I_5^\delta + \frac{1}{2} I_6^\delta + \frac{1}{2} I_7, \tag{12}$$

where

$$I_1^\delta=(Z_j-Z_j^\delta)f_{ij}(X^\delta),$$
$$I_2^\delta=(Z_j-Z_j^\delta)_-\cdot\langle f_{ijk}(X^\delta), Z_k^\delta\rangle^c,$$
$$I_3^\delta=\sum_{s\leqq\cdot}(Z_j-Z_j^\delta)_s\left[\frac{1}{2}f_{ijkm}(X_{s-}^\delta)\Delta Z_k^\delta(s)\Delta Z_m^\delta(s)+\Delta_2 f_{ij}(X^\delta)_s\right],$$
$$I_4^\delta=f_{ij}(X;X^\delta)_-\cdot Z_j,$$
$$I_5^\delta=f_{ijk}(X;X^\delta)_-\cdot[Z_j, Z_k],$$
$$I_6^\delta=f_{ijk}(X^\delta)_-\cdot\tilde{Z}_{jk}^\delta,$$
$$\tilde{Z}_{jk}=\langle Z_k-Z_k^\delta, Z_j\rangle^c-2(Z_j-Z_j^\delta)\circ Z_k^\delta$$
$$-2S(Z_j-Z_j^\delta, Z_k^\delta)+S(Z_j, Z_k)-S(Z_j^\delta, Z_k^\delta),$$
$$I_7=f_{ijk}(X)_-\cdot A_{jk}.$$

Let B be an increasing process that "controls" all Z^j, $j=1, 2, \ldots, r$, in the following sense:

$$E\left\{\sup_{t<T}\left(\int_0^t H_{s-}\,dZ_j(s)\right)^2\right\}\leqq E\left\{B_{T-}\int_0^{T-}H_{s-}^2\,dB_s\right\}$$

for any locally bounded process H and any stopping time T (*cf.* Metivier-Pellaumail [17], proposition 2).

LEMMA 1. — *Let us denote*

$$\tau_K=\inf\{t\geqq 0:\max_i[Z_i, Z_i]_t\vee\max_{j,k}|A_{jk}|_t\vee B_t>K\},$$
$$\sigma_K^\delta=\inf\{t\geqq 0:|A_{jk}^\delta|_t>K\},$$
$$\tau_K^\delta=\tau_K\wedge\sigma_K^\delta$$

(A_{jk}^δ — FV *part of* Z_{jk}^δ, $|A|_t$ — *total variation process of the* FV *process* A). *Then*

$$\lim_{K\to\infty}\ \limsup_{\delta\downarrow 0}E(\sup_{t\geqq 0}|X(t)-X^\delta(t)|^p:\tau_K^\delta<\infty)=0. \tag{13}$$

Proof. — It is clear that

$$\|I_1^\delta\|_{S^p}\leqq C\|Z-Z^\delta\|_{S^p}\to 0,\qquad \delta\to 0. \tag{14}$$

Using Itô's formula for any $g\in C_b^2(R^d)$ we have

$$\langle g(X^\delta)\rangle^c=D_n g f_{nk}D_{n'}g f_{n'k'}(X^\delta)_-\cdot\langle Z_k, Z_k\rangle^c$$

and therefore

$$||\langle g(X^\delta)\rangle^c||_{L^q} \leqq C \sum_k ||\langle Z_k^\delta \rangle^c||_{L^q}. \tag{15}$$

Thus

$$||I_2^\delta||_{S^p} \leqq C ||Z - Z^\delta||_{S^{p'}} \times \sum_k (||\langle f_{ijk}(X^\delta)\rangle^c||_{L^q} + ||\langle Z_k^\delta \rangle^c||_{L^q}) \to 0, \qquad \delta \to 0. \tag{16}$$

Consider I_3^δ: since

$$|\Delta_2 f(X^\delta)_s| \leqq C \sum_i |\Delta X_i^\delta(s)|^2$$

$$= C \left| f_{ij}(X_{s-}^\delta) \Delta Z_j^\delta(s) + \frac{1}{2} f_{ijk}(X_{s-}^\delta) \Delta Z_j^\delta(s) \Delta Z_k^\delta(s) \right|^2$$

$$\leqq C_1 \sum_k (\Delta Z_k^\delta)^2(s)(1 + S(Z_k^\delta)_\infty)$$

and hence

$$I_3^{\delta *} \leqq C_2 (Z - Z^\delta)^* \sum_k S(Z_k^\delta)_\infty (1 + S(Z_k^\delta)_\infty),$$

using B2 we have

$$||I_3^\delta||_{S^p} \leqq C_3 ||Z - Z^\delta||_{S^{p'}} \times \sum_k (||S(Z_k^\delta)_\infty||_{L^{2q}}^{1/2} + ||S(Z_k)_\infty||_{L^q}) \to 0, \qquad \delta \to 0. \tag{17}$$

It is clear that $|I_7^*|^p$ is integrable. The families of r. v.'s $\{|I_k^{\delta *}|^p, \delta > 0\}$, $k = 4, 5, 6$, are uniformly integrable. Consider, for example, $k = 4$. If $q < \infty$ and hence $p' > p$ we have by (3), (4):

$$||I_4^\delta||_{S^{p'}} \leqq C_{p'} ||f(X; X^\delta)||_{S^\infty} \sum_j ||Z_j||_{H^{p'}} \leqq C.$$

If $q = \infty$ and hence $p' = p$ we have for any $\alpha > p$

$$(E \sup_{t \geqq 0} |f_{ij}(X; X^\delta)_- \cdot M_j(t)|^\alpha)^{1/\alpha} \leqq C_\alpha ||f_{ij}(X; X^\delta)||_{S^\alpha} ||M_j||_{H^\infty} \leqq C$$

and

$$E(\sup_{t\geq 0}|f_{ij}(X;X^{\delta})_{-}\cdot A_j(t)|^p : \sup_{t\geq 0}|f_{ij}(X;X^{\delta})_{-}\cdot A_j(t)|>\lambda)$$
$$\leqq (2C)^p E(|A_j|^p_{\infty} : |A_j|_{\infty}>\lambda/(2C)) \to 0, \qquad \lambda\to\infty$$

[here $C=\sup_{x,i,j}|f_{ij}(x)|$; $|A|^p_{\infty}$ is integrable because $M\in H^{\infty}\subset S^p$ and $M+A\in S^p$].

One can easily show by analogous argument the uniform integrability of $\{|I^{\delta *}_6|^p, \delta>0\}$, using the boundedness of $[M^{\delta}_k]$, $\langle Z^{\delta}_k\rangle^c$ and $S(Z^{\delta}_k)$ in L^q and the condition B3. The same is true for $\{|A^{\delta}_{jk}|^p_{\infty}, \delta>0\}$ and therefore $\sup_{\delta>0} P\{\tau^{\delta}_K<\infty\}\to 0, K\to\infty$. Summarizing the above we have (13).

LEMMA 2. — *Let f and Y be two increasing nonnegative processes such that $Ef(\infty)<\infty$ and $Y_{\infty}\leqq K$ for some constant K. If for every Markov moment T*

$$Ef(T-)\leqq E\int_0^{T-} f(s-)\,dY_s+\varepsilon$$

then $Ef(\infty)\leqq \varepsilon e^K$.

Proof. — In fact, this "Gronwall type" proposition is a special case of lemma 2 of Grigelionis-Mikulevičius [7]. The proof (slightly modified for our case) is reproduced here for the sake of completeness. Denote $\tau_t=\inf\{s\geqq 0: Y_s>t\}$, $t\geqq 0$. Then we have

$$Ef(\tau_t-)\leqq E\int_0^{\tau_t-} f(s-)\,dY_s+\varepsilon=E\int_0^{\infty} f(s-)\,1_{\{s<\tau_t\}}\,dY_s+\varepsilon$$
$$\leqq E\int_0^{\infty} f(\tau_u-)\,1_{\{\tau_u<\infty\}}\,1_{\{\tau_u<\tau_t\}}\,du+\varepsilon\leqq \int_0^t Ef(\tau_u-)\,du+\varepsilon.$$

Using the "classical" Gronwall lemma for $\Phi(t)=Ef(\tau_t-)$ we obtain inequality $Ef(\tau_t-)\leqq \varepsilon e^t$. The passing to the limit as $t\uparrow K$ then gives $Ef(\infty)\leqq \varepsilon e^K$.

Let us return to the proof of the theorem. Consider

$$\|X-X^{\delta}\|^p_{S^p}\leqq E(\sup_{t\geq 0}|X(t)-X^{\delta}(t)|^p : \tau^{\delta}_K<\infty)+E(\sup_{t<\tau^{\delta}_K}|X(t)-X^{\delta}(t)|^p). \quad (18)$$

By lemma 1 it is sufficient to prove that the second term on the right side of (18) tends to 0 as $\delta\to 0$ for every $K<\infty$. In the proof of lemma 1 we have seen that $I_1^{\delta *}$, $I_2^{\delta *}$, $I_3^{\delta *}$ converge to 0 in L^p, I_7 is in L^p and $\{ | I_k^{\delta *} |^p, \delta > 0, k = 4, 5, 6 \}$ is uniformly integrable. Therefore finally it is sufficient to prove that

$$\lim_{\delta \to 0} E(\sup_{t < \tau_K^\delta} | X(t) - X^\delta(t) |^2) = 0$$

for every $K < \infty$. Note that

$$[Z_i]_t \leqq K, \qquad | A_{jk} |_t \leqq K, \qquad | B_t | \leqq K, \qquad | A_{jk}^\delta |_t \leqq K \quad \text{on } \{ t < \tau_K^\delta \}.$$

For typographical reasons we shall often write τ instead of τ_K^δ. In order to use lemma 2 we shall try to get an estimate for $f^\delta(t) = \sup_{s < t \wedge \tau} | X(s) - X^\delta(s) |^2$ (as an exception to our convention, f^δ is left continuous with right limits). Let T be an arbitrary Markov moment. Consider I_4^δ:

$$E | I_4^{\delta *}((T \wedge \tau) -) |^2 = E \left\{ \sup_{t < T \wedge \tau} \left| \int_0^t f_{ij}^\delta (X; X^\delta)_{s-} \, dZ_j(s) \right|^2 \right\}$$

$$\leqq E \left\{ B_{(T \wedge \tau)-} \sum_j \int_0^{(T \wedge \tau)-} f_{ij}^2 (X; X^\delta)_{s-} \, dB_s \right\} \leqq C \int_0^{T-} f^\delta(s) \, dB_{s \wedge \tau -} \qquad (19)$$

[we use the notation $X^*(t) = \sup_{s \leqq t} | X_s |$, $X^* = X_\infty^*$]. Analogously,

$$E | I_5^{\delta *}((T \wedge \tau) -) |^2 \leqq C \sum_j E \int_0^{T-} f^\delta(s) \, d[Z_j]_{s \wedge \tau -}. \qquad (20)$$

It is easy to check that $\tilde{Z}_{jk} = -Z_{jk}^\delta - 2(Z_j - Z_j^\delta)_- \cdot Z_k$. Therefore

$$I_6^\delta + I_7 = -2 f_{ijk}(X^\delta)_- (Z_j - Z_j^\delta)_- \cdot Z_k$$
$$+ f_{ijk}(X; X^\delta)_- \cdot M_{jk}^\delta + f_{ijk}(X; X^\delta)_- . A_{jk}^\delta$$
$$+ f_{ijk}(X)_- \cdot (A_{jk} - Z_{jk}^\delta) =: -2 I_9^\delta + I_{10}^\delta + I_{11}^\delta + I_{12}^\delta.$$

For I_9^δ we have

$$\| I_9^\delta \|_{S^2} \leqq C \| Z - Z^\delta \|_{S^{p'}} \sum_k \| Z_k \|_{H^q} \to 0, \qquad \delta \to 0. \qquad (21)$$

Analogously, by hypothesis B1,

$$\| I_{10}^{\delta} \|_{S^2}^{p} \leqq \| I_{10}^{\delta} \|_{S^p}^{p} = E\left\{ \sup_{t \geqq 0} \left| \int_0^t f_{ijk}(X; X^{\delta})_{s-} (Z_j - Z_j^{\delta})_{s-} \, dM_k^{\delta}(s) \right|^p \right\}$$
$$\leqq C \| Z - Z^{\delta} \|_{S^{p'}}^{p} \| M_k^{\delta} \|_{H^q}^{p} \to 0, \qquad \delta \to 0. \quad (22)$$

The estimate for I_{11}^{δ} is obtained as in (19):

$$E | I_{11}^{\delta *}((T \wedge \tau)-) |^2 \leqq C \sum_{j,k} \int_0^{T-} f^{\delta}(s) | dA_{jk}^{\delta}(s \wedge \tau -) |. \qquad (23)$$

Now it only remains to consider

$$I_{12}^{\delta} = f_{ijk}(X)(A_{jk} - Z_{jk}^{\delta}) - (A_{jk} - Z_{jk}^{\delta})_- \cdot f_{ijk}(X) + \frac{1}{2} \langle Z_{jk}^{\delta}, f_{ijk}(X) \rangle^c$$
$$+ S(Z_{jk}^{\delta} - A_{jk}, f_{ijk}(X)) =: I_{121}^{\delta} - I_{122}^{\delta} + \frac{1}{2} I_{123}^{\delta} + I_{124}^{\delta}.$$

As before,

$$\| I_{121}^{\delta} \|_{S^2} \leqq C \| A_{jk} - Z_{jk}^{\delta} \|_{S^p} \to 0, \qquad \delta \to 0, \qquad (24)$$

and

$$\| I_{122}^{\delta} \|_{S^2} \leqq C \| A_{jk} - Z_{jk}^{\delta} \|_{S^{p'}} \| f_{ijk}(X) \|_{H^q} \to 0, \qquad \delta \to 0. \qquad (25)$$

For $I_{123}^{\delta} = (Z_j - Z_j^{\delta})_- \cdot \langle M_k^{\delta}, f_{ijk}(X) \rangle^c$ we have

$$\| I_{123}^{\delta} \|_{S^2} \leqq C \| Z - Z^{\delta} \|_{S^{p'}}$$
$$\times \sum_{j,k} (\| \langle M_k^{\delta} \rangle_{\infty}^c \|_{L^q} + \| \langle M_{ijk} \rangle_{\infty} \|_{L^q}) \to 0, \qquad \delta \to 0, \quad (26)$$

where by M_{ijk} we denote the continuous martingale part of $f_{ijk}(X)$.

Now consider I_{124}^{δ}. First we shall note that, in fact, the processes A_{jk} are continuous. This is seen from the equality

$$\Delta Z_{jk}^{\delta}(t) = \Delta(Z_k^{\delta} - Z_k)(t) \Delta Z_j(t) + \Delta(Z_j - Z_j^{\delta})(t) \Delta Z_k^{\delta}(t)$$
$$- 2(Z_j - Z_j^{\delta})(t) \Delta(Z_k - Z_k^{\delta})(t)$$

which yields

$$\sup_t | \Delta Z_{jk}^{\delta}(t) | \leqq 4(Z_k^{\delta} - Z_k)^* Z_j^*$$
$$+ 4(Z_j - Z_j^{\delta})^* Z_k^{\delta *} + 4(Z_j - Z_j^{\delta})^* (Z_k - Z_k^{\delta})^* \to 0, \qquad \delta \to 0,$$

in probability and hence $\sup_t |\Delta A_{jk}(t)| = 0$. Therefore

$$
\begin{aligned}
I^{\delta *}_{124}(\tau -) = S(Z^{\delta}_{jk}, f_{ijk}(X))^{*}_{\tau -} \\
&\leqq \sum_{t<\tau} |\Delta(Z^{\delta}_k - Z_k)(t)\,\Delta Z_j(t)\,\Delta f_{ijk}(X)_t| \\
&\quad + |\Delta(Z_j - Z^{\delta}_j)(t)\,\Delta Z^{\delta}_k(t)\,\Delta f_{ijk}(X)_t| \\
&\quad + 2|(Z_j - Z^{\delta}_j)(t)|\,(|\Delta Z_k(t)| + |\Delta Z^{\delta}_k(t)|)\,|\Delta f_{ijk}(X)_t| \\
&\leqq C(Z^{\delta} - Z)^{*} \sum_{t<\tau} (|\Delta Z_j(t)| + |\Delta Z_k(t)| \\
&\quad + |\Delta Z^{\delta}_k(t)|)\,|\Delta f_{ijk}(X)_t| \\
&\leqq C(Z^{\delta} - Z)^{*}\,(S(Z_j)^{1/2}_{\infty} + S(Z_k)^{1/2}_{\infty} \\
&\qquad + S(Z^{\delta}_k)^{1/2}_{\infty})\,S(f_{ijk}(X))^{1/2}_{\tau -}
\end{aligned}
$$

and thus

$$
\begin{aligned}
\| I^{\delta *}_{124}(\tau -) \|_{L^2} \leqq \| I^{\delta *}_{124}(\tau -) \|_{L^p} \\
&\leqq C_1 \| Z^{\delta} - Z \|_{S^{p'}} \times (\| S(Z_j)_{\infty} \|^{1/2}_{L^q} + \| S(Z_k)_{\infty} \|^{1/2}_{L^q} \\
&\quad + \| S(Z^{\delta}_k)_{\infty} \|^{1/2}_{L^q})\, \| S(f_{ijk}(X))_{\tau -} \|^{1/2}_{L^q} \to 0, \qquad \delta \to 0, \quad (27)
\end{aligned}
$$

where we have used B2 and the boundedness of $S(f_{ijk}(X))_t$ on $\{t<\tau\}$. The latter can be easily seen from the expression for $\Delta f_{ijk}(X)$, analogous to (9), and the inequality $S(Z_k)_t \leqq K$ on $\{t<\tau\}$.

Summarizing the estimates (14), (16), (17), (19)-(27), we obtain for f^{δ} a desired final estimate

$$
\begin{aligned}
E f^{\delta}(T-) = E f^{\delta}(T) = E \sup_{t<T \wedge \tau} |X(t) - X^{\delta}(t)|^2 \\
&\leqq E \int_0^{T-} f^{\delta}(s-)\, d\tilde{A}^{\delta}(s) + \varepsilon(\delta)
\end{aligned}
$$

with a certain increasing process $\tilde{A}^{\delta}$ such that $\tilde{A}^{\delta}_{\infty} \leqq \tilde{K}$ for some constant $\tilde{K}$ (not depending on $\delta > 0$) and $\varepsilon(\delta)$ tending to 0 as $\delta \to 0$. Finally, by lemma 2 we have

$$
E(\sup_{t<\tau^{\delta}_K} |X(t) - X^{\delta}(t)|^2) = E f^{\delta}(\infty) \leqq \varepsilon(\delta)\, e^{\tilde{K}} \to 0, \qquad \delta \to 0,
$$

and the proof of theorem 1 is complete.

Remark. – The proof of theorem 2 is almost the same. The only difference is that the term I_5^δ becomes equal to $f_{ijk}(X; X^\delta) \cdot \langle Z_j, Z_k \rangle^c$ and the processes Z_{jk}^δ have to be replaced by the processes $\hat{Z}_{jk}^\delta = Z_{jk}^\delta + S(Z_j^\delta, Z_k^\delta) - S(Z_j, Z_k)$ which also converge to A_{jk} under the additional hypothesis C3.

EXAMPLE

In [12] several examples of approximations of continuous driving semimartingales by continuous semimartingales are presented. Protter [22] considered the approximation of a discontinuous driving semimartingale by FV processes, but the continuous and discontinuous parts were approximated separately. Theorem 1 does not exclude the possibility of an approximation of continuous driving semimartingale by the processes with jumps:

PROPOSITION. – *Let* $Z = \{ Z_t, t \in [0, T] \}$ $(T < \infty)$ *be a continuous r-dimensional semimartingale such that* $Z_i \in H^{8p}$, $i = 1, 2, \ldots, r$, *for some* $p \in [2, \infty[$. *Denote*

$$Z^\delta(t) = Z(k\delta) \qquad \textit{for} \quad t \in [k\delta, (k+1)\delta[, k = 0, 1, 2, \ldots$$

Then $\{ Z^\delta, \delta > 0 \}$ *is a symmetric approximation of* Z. *More precisely, the conditions* B1, B2, B3, C1, C2 *are satisfied with* $A_{jk} = 0$ *and* $p' = q = 2p$.

Proof. – The conditions B2 and B1 follow from the inequality

$$\| S(Z_j^\delta)_T \|_{L^{2q}} \leqq C \| Z \|_{H^{2q}}^2$$

([2], p. 320). For B3 let us note that the martingale part of Z_k^δ is 0 and therefore

$$E\left(\int_0^T |(Z_j - Z_j^\delta)(s-)| \, |dA_k^\delta(s)| \right)^{2p} = E(\sum_t |\Delta Z_j^\delta(t)| \, |\Delta Z_k^\delta(t)|)^{2p} \leqq \| Z_j \|_{H^{4p}}^{2p} \| Z_k \|_{H^{4p}}^{2p}.$$

The condition C1 is evidently satisfied. It remains to consider C2. Since $[Z_j^\delta, Z_k] = 0$, we have

$$Z_{jk}^\delta = -(Z_j - Z_j^\delta) \,\square\, (Z_k - Z_k^\delta) = -(Z_j - Z_j^\delta)_- \cdot Z_k - \frac{1}{2} \langle Z_j, Z_k \rangle + \frac{1}{2} S(Z_j^\delta, Z_k^\delta),$$

and since

$$\|(Z_j - Z_j^\delta)_- \cdot Z_k\|_{S^{2p}} \leq C_p \|Z_j - Z_j^\delta\|_{S^{4p}} \|Z_k\|_{H^{4p}} \to 0, \qquad \delta \to 0,$$

it suffices to check that $S(Z_j^\delta, Z_k^\delta) \to \langle Z_j, Z_k \rangle$ in S^{2p}. It is known that $\sup_{t \leq T} |S(Z_j^\delta, Z_k^\delta)_t - \langle Z_j, Z_k \rangle_t| \to 0, \delta \to 0$, in probability (*cf.* [2], [18]).

The estimate

$$E|S(Z_j^\delta, Z_k^\delta)_T|^{4p} \leq \|S(Z_j^\delta)_\infty\|_{L^{4p}}^{2p} \|S(Z_k^\delta)_\infty\|_{L^{4p}}^{2p} \leq C_p \|Z_j\|_{H^{8p}}^{4p} \|Z_k\|_{H^{8p}}^{4p}$$

([2], p. 320) then assures the convergence in S^{2p}.

Example. — Using the example of a symmetric approximation given in the proposition one can easily construct a simple example of a non-symmetric approximation ($A_{jk} \neq 0$). Let Z and $\tilde{Z}$ be two continuous semi-martingales satisfying the conditions of the proposition. Define an approximation of Z by

$$Z^\delta = \tilde{Z} + (Z - \tilde{Z})^\delta, \qquad \delta > 0,$$

where $(Z - \tilde{Z})^\delta$ is the symmetric approximation of the proposition applied to $Z - \tilde{Z}$. Then from the proposition we see that $\{Z^\delta, \delta > 0\}$ satisfies B1, B2, B3 and C1 with $p' = q = 2p$. We also have

$$Z_{jk}^\delta = \langle \tilde{Z}_k, Z_j \rangle - \langle \tilde{Z}_j, Z_k \rangle + (Z - \tilde{Z})_{jk}^\delta$$

$$\to \langle \tilde{Z}_k, Z_j \rangle - \langle \tilde{Z}_j, Z_k \rangle \quad \text{in } S^{p'}, \qquad \delta \to 0.$$

Thus C2 is satisfied with

$$A_{jk} = \langle \tilde{Z}_k, Z_j \rangle - \langle \tilde{Z}_j, Z_k \rangle,$$

which, in general, is not 0. The constructed approximation is symmetric only if the coordinates of martingale parts of Z and $\tilde{Z}$ "commute" in the following sense:

$$\langle \tilde{Z}_k, Z_j \rangle = \langle \tilde{Z}_j, Z_k \rangle \quad \text{for all } k, j.$$

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