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Exact convergence rates in Erdös-Rényi type theorems for renewal processes

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ABSTRACT. — Let $\{ N(t), t \geqslant 0 \}$ be the renewal process associated with a sequence $X_1, X_2, \ldots$ of non-degenerate non-negative i. i. d. random variables. Let $C > 0$ be a fixed constant, and consider $\Delta_T = \sup_{0 \leqslant t \leqslant T} \{ N(t + C \log T) - N(t) \}$ and $\delta_T = \inf_{0 \leqslant t \leqslant T} \{ N(t + C \log T) - N(t) \}$. In this paper, we prove that $\lim \left\{ \begin{array}{l} \sup \\ \inf_{T \to \infty} \end{array} \right. \{ \Delta_T - CA \log T \} / \log \log T = \pm h(C)$ a. s., where A and $h(C)$ are constants depending upon C and the distribution of X_1, together with a similar result for δ_T under the condition that $E(\exp(sX_1)) < \infty$ for some $s > 0$.

Key-words: Erdös-Rényi laws, laws of large numbers, renewal processes, law of the iterated logarithm, almost sure convergence.

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RÉSUMÉ. — Soit $\{ N(t), t \geq 0 \}$ le processus de renouvellement associé à une suite $X_1, X_2, \ldots$ de variables aléatoires non-dégénérées positives ou nulles. Soit $C > 0$ une constante fixée, et soit

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$\Delta_T = \sup_{0 \le t \le T} \{ N(t + C \log T) - N(t) \}$ et $\delta_T = \inf_{0 \le t \le T} \{ N(t + C \log T) - N(t) \}$. Dans cet article, nous établissons que

$$\lim \left\{ \begin{matrix} \sup \\ \inf \end{matrix} \right. (\Delta_T - CA \log T)/\log \log T = \pm h(C)$$

p. s., où A et $h(C)$ sont des constantes ne dépendant que de C et de la loi de X_1. Nous prouvons un résultat analogue pour δ_T sous la condition que $E(\exp(sX_1)) < \infty$ pour une valeur de $s > 0$.

Mots-clés : Lois d'Erdös-Rényi, Loi des grands nombres, processus de renouvellement, loi du logarithme itéré, convergence presque sûre.

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1. INTRODUCTION AND RESULTS

Let $X_1, X_2, \ldots$ be independent non-negative and identically distributed random variables such that:

(A) $$E(X_1) = \mu \in (0, \infty);$$

(B) $$P(X_1 = x) < 1 \qquad \text{for all } x.$$

Define the corresponding renewal process by

$$N(t) = \max \{ n \geqslant 0 : S_n \leqslant t \}, \qquad t \geqslant 0,$$

where $S_0 = 0, S_n = X_1 + \ldots + X_n$. Let $C > 0$ be fixed and let $K_T = C \log T$, $T > 1$.

In this paper, we will be concerned with the limiting behavior of the Erdös-Rényi type statistics Δ_T and δ_T as $T \to \infty$, where

$$\Delta_T = \sup_{0 \le t \le T} \{ N(t + K_T) - N(t) \},$$

and

$$\delta_T = \inf_{0 \le t \le T} \{ N(t + K_T) - N(t) \}.$$

Before stating our theorems, it will be convenient to give some preliminary results and notations. Let $\phi(s) = E(\exp(sX_1))$, and let $s_0 = \sup \{ s : \phi(s) < \infty \}$. Clearly $\phi(.)$ is increasing on $(-\infty, s_0)$ and such that $\phi(0) = 1$. Hence $s_0 \in [0, \infty)$. We shall make use of the following properties of the moment generating function $\phi(.)$ see e. g. Deheuvels and Devroye (1983), Deheuvels, Devroye and Lynch (1986).

Let $m(s) = \phi'(s)/\phi(s)$. Observe that $m(.)$ is increasing on $(-\infty, s_0)$ and that $m(0) = \mu$, so that $m(.)$ is continuous on $(-\infty, 0] \cup (-\infty, s_0)$. Set

$$0 \leq b = \text{ess inf } X_1 = \lim_{s \downarrow -\infty} m(s) < \mu \leq a = \lim_{s \uparrow s_0} m(s) \leq \text{ess sup } X_1 \leq \infty .$$

It is noteworthy that $a = \text{ess sup } X_1$ whenever $\text{ess sup } X_1 < \infty$ or when $s_0 = \infty$ and $\text{ess sup } X_1 = \infty$. In general we have $a \leq \text{ess sup } X_1$ in the other cases.

Define A_0 and B_0 by

$$A_0 = 1/b \quad \text{if} \quad b > 0 . \qquad A_0 = \infty \quad \text{if} \quad b = 0 ,$$
$$B_0 = 1/a \quad \text{if} \quad a < \infty , \qquad B_0 = 0 \quad \text{if} \quad a = \infty .$$

It is straightforward that $0 \leq B_0 < A_0 \leq \infty$ and that the equation $\theta m(s) = 1$ has a unique solution $\hat{s} = \hat{s}(\theta)$ for all $B_0 < \theta < A_0$. Furthermore $\hat{s}(.)$ is decreasing on (B_0, A_0) and such that $\hat{s}(1/\mu) = 0$ while

$$\lim_{\theta \downarrow B_0} \hat{s}(\theta) = s_0 \qquad \text{and} \qquad \lim_{\theta \uparrow A_0} \hat{s}(\theta) = -\infty .$$

Consider now $s - \theta \log \phi(s)$, and note that this function of $s \in (-\infty, s_0)$ has first derivative $1 - \theta m(s)$ and strictly negative second derivative. Thus, for $B_0 < \theta < A_0$, it has a unique maximum on $(-\infty, s_0)$ reached for $s = \hat{s}(\theta)$.

For $B_0 < \theta < A_0$, let

$$\Gamma(\theta) = \sup_s \{ s - \theta \log \phi(s) \} = \hat{s}(\theta) - \theta \log \phi(\hat{s}(\theta)) .$$

Because of the analycity of $\phi(.)$ on $(-\infty, s_0)$, $\hat{s}(\theta)$ is differentiable on (B_0, A_0). Furthermore

$$\Gamma'(\theta) = \hat{s}'(\theta) - \log \phi(\hat{s}(\theta)) - \theta m(\hat{s}(\theta)) \hat{s}'(\theta) = -\log \phi(\hat{s}(\theta)) .$$

It follows that $\Gamma(.)$ decreases on $(B_0, 1/\mu]$ and increases on $[1/\mu, A_0)$. Clearly $\Gamma(1/\mu) = 0$ so that $\Gamma(\theta) > 0$ for all $B_0 < \theta \neq 1/\mu < A_0$.

We investigate now the limiting behavior of $\Gamma(\theta)$ as $\theta \to A_0$ (resp. $\theta \to B_0$). Observe that $\Gamma(\theta) - \theta\Gamma'(\theta) = \hat{s}(\theta)$. This, jointly with $\Gamma(1/\mu) = 0$, implies that

$$\Gamma(\theta) = -\theta \int_{1/\mu}^{\theta} t^{-2} \hat{s}(t) dt = \theta \int_{\mu}^{1/\theta} \hat{s}(1/u) du = \theta \int_0^{\hat{s}(\theta)} t m'(t) dt .$$

It follows from the above equalities that

$$\lim_{\theta \downarrow B_0} \Gamma(\theta) = B_0 \int_0^{s_0} t m'(t) dt \quad \text{if} \quad B_0 > 0 , \qquad \lim_{\theta \downarrow B_0} \Gamma(\theta) = s_0 \quad \text{if} \quad B_0 = 0,$$

and

$$\lim_{\theta \uparrow A_0} \Gamma(\theta) = - A_0 \int_{-\infty}^{0} tm'(t)dt .$$

By Theorem 2 of Deheuvels, Devroye and Lynch (1986), we have

$$\begin{aligned}\int_0^{s_0} tm'(t)dt \\ &= -\log P(X_1 = a) \quad \text{if} \quad a = 1/B_0 = \text{ess sup } X_1 < \infty \quad \text{and} \quad P(X_1 = a) > 0 , \\ &= as_0 - \log \phi(s_0) \quad \text{if} \quad a = 1/B_0 < \text{ess sup } X_1 = \infty \quad \text{and} \quad s_0 < \infty , \\ &= \infty \quad \text{otherwise.}\end{aligned}$$

Note also that $B_0 > 0$ and $s_0 = \infty$ is equivalent to ess sup $X_1 < \infty$, so that finally

$$\begin{aligned}\lim_{\theta \downarrow B_0} \Gamma(\theta) &= -\frac{1}{a} \log P(X_1 = a) \quad \text{if} \quad a = \text{ess sup } X_1 < \infty \quad \text{and} \quad P(X_1 = a) > 0 , \\ &= s_0 - \frac{1}{a} \log \phi(s_0) \quad \text{if} \quad a < \text{ess sup } X_1 = \infty \quad \text{and} \quad s_0 < \infty , \\ &= s_0 \quad \text{otherwise.}\end{aligned}$$

Likewise

$$\begin{aligned}\lim_{\theta \uparrow A_0} \Gamma(\theta) &= -\frac{1}{b} \log P(X_1 = b) \quad \text{if} \quad b > 0 \quad \text{and} \quad P(X_1 = b) > 0 , \\ &= \infty \quad \text{otherwise.}\end{aligned}$$

Define now $c_1 = \lim_{\theta \uparrow A_0} (1/\Gamma(\theta))$ and $c_0 = \lim_{\theta \downarrow B_0} (1/\Gamma(\theta))$, corresponding to

$$\begin{cases} c_1 = -b/\log P(X_1 = b) & \text{if} \quad b = \text{ess inf } X_1 > 0 \quad \text{and} \quad P(X_1 = b) > 0 , \\ c_1 = 0 & \text{otherwise,} \end{cases}$$

and

$$\begin{cases} c_0 = -a/\log P(X_1 = a) & \text{if} \quad a = \text{ess sup } X_1 < \infty \quad \text{and} \quad P(X_1 = a) > 0, \\ c_0 = a/\{as_0 - \log \phi(s_0)\} & \text{if} \quad a < \text{ess sup } X_1 = \infty \quad \text{and} \quad 0 < s_0 < \infty , \\ c_0 = \infty & \text{if} \quad s_0 = 0 , \\ c_0 = 1/s_0 & \text{otherwise (with the notation } 1/\infty = 0 \text{ if } s_0 = \infty). \end{cases}$$

We may now state our main results in the following theorems.

THEOREM 1. — For any $c_1 < C < \infty$, let $A \in \left(\frac{1}{\mu}, A_0\right)$ and $s^{**} < 0$ be the unique solutions of the equations

$$\frac{1}{C} = \sup_{s<0} \{ s - A \log \phi(s) \} = s^{**} - A \log \phi(s^{**}).$$

Then

$$\limsup_{T\to\infty} \frac{\Delta_T - CA \log T}{\log\log T} = \frac{-CA}{2(Cs^{**} - 1)}, \tag{1}$$

and

$$\liminf_{T\to\infty} \frac{\Delta_T - CA \log T}{\log\log T} = \frac{CA}{2(Cs^{**} - 1)} \quad \text{a. s.} \tag{2}$$

THEOREM 2. — For any $c_0 < C < \infty$, let $B \in \left(B_0, \frac{1}{\mu}\right)$ and $s^* > 0$ be the unique solutions of the equations

$$\frac{1}{C} = \sup_{s>0} \{ s - B \log \phi(s) \} = s^* - A \log \phi(s^*).$$

Then

$$\limsup_{T\to\infty} \frac{\delta_T - CB \log T}{\log\log T} = \frac{CB}{2(Cs^* - 1)}, \tag{3}$$

and

$$\liminf_{T\to\infty} \frac{\delta_T - CB \log T}{\log\log T} = \frac{-CB}{2(Cs^* - 1)} \quad \text{a. s.} \tag{4}$$

REMARK 1. — Note that the assumption $c_0 < \infty$ in Theorem 2 requires that $s_0 > 0$.

Retka (1982) and Steinebach (1982) proved that, under the assumption of Theorem 1 we have

$$\Delta_T - CA \log T = o((\log T)^{1/2}) \quad \text{a. s.}$$

Recently Steinebach (1986) proved that, under the same assumptions

$$\Delta_T - CA \log T = O(\log\log T) \quad \text{a. s.} \tag{5}$$

The following extension of Theorem 5 in Deheuvels, Devroye and Lynch (1986), given by Bacro (1985), enables us to also provide the best constants in (5) as well as in an analogue assertion with Δ_T replaced by δ_T:

Using the notations introduced above, let $K'_N = [c \log N + \lambda \log\log N]$, $N = 1, 2, \ldots$, and

$$D_N = \max_{0 \leq n \leq N} \{ S_{n+K'_N} - S_n \},$$

$$d_N = \min_{0 \leq n \leq N} \{ S_{n+K'_N} - S_n \}.$$

Let c_0' and c_1' be defined by (using the notations $1/\infty = 0$, $1/0 = \infty$)

$$c_0' = 1 \Big/ \int_0^{s_0} tm'(t)dt\,,$$

and

$$c_1' = 1 \Big/ \int_0^{-\infty} tm'(t)dt\,.$$

THEOREM A (Bacro, 1985). — For $c > c_0'$ let $\alpha \in (\mu, a)$ and $t^* > 0$ be the unique solutions of the equations

$$\frac{1}{c} = \sup_{t>0} \{ t\alpha - \log \varphi(t) \} = t^*\alpha - \log \varphi(t^*)\,.$$

Then

$$\text{(6)} \qquad \limsup_{N\to\infty} \frac{D_N - \alpha[c \log N]}{\log\log N} = \alpha\lambda - \frac{1}{t^*}\left\{\frac{\lambda}{c} - \frac{1}{2}\right\} \quad \text{a. s.},$$

$$\text{(7)} \qquad \liminf_{N\to\infty} \frac{D_N - \alpha[c \log N]}{\log\log N} = \alpha\lambda - \frac{1}{t^*}\left\{\frac{\lambda}{c} + \frac{1}{2}\right\} \quad \text{a. s.}$$

By considering $\{ - S_n \}_{n=0,1,\ldots}$ instead of $\{ S_n \}_{n=0,1,\ldots}$ an immediate corollary of Theorem A is as follows:

THEOREM B. — For $c > c_1'$ let $\beta \in (b, \mu)$ and $t^{**} < 0$ be the unique solutions of the equations

$$\frac{1}{c} = \sup_{t<0} \{ t\beta - \log \varphi(t) \} = t^{**}\beta - \log \varphi(t^{**})\,.$$

Then

$$\text{(8)} \qquad \limsup_{N\to\infty} \frac{d_N - \beta[c \log N]}{\log\log N} = \beta\lambda - \frac{1}{t^{**}}\left\{\frac{\lambda}{c} + \frac{1}{2}\right\} \quad \text{a. s.},$$

$$\text{(9)} \qquad \liminf_{N\to\infty} \frac{d_N - \beta[c \log N]}{\log\log N} = \beta\lambda - \frac{1}{t^{**}}\left\{\frac{\lambda}{c} - \frac{1}{2}\right\} \quad \text{a. s.}$$

REMARK 2. — The random variables $X_1, X_2, \ldots$ in Theorems A and B need not be restricted to possess non-negative values only (see Bacro (1985)). Moreover, it is obvious from the proofs that the results are the same if $\left\{\begin{matrix}\max\\ \min\end{matrix}\right\}_{0 \le n \le N}$ in the definition of $\left\{\begin{matrix}D_N\\ d_N\end{matrix}\right\}$ is replaced by $\left\{\begin{matrix}\max\\ \min\end{matrix}\right\}_{0 \le n \le N - K_N'}$.

As will be seen in the sequel, the assumption that $X_1, X_2, \ldots$ are non-negative can be relaxed to $\phi(s) < \infty$ for some $s < 0$. We will not state the corresponding results for sake of brevity.

Before we give the proofs of Theorems 1 and 2, let us, for example, consider the case of the standard Poisson process, which possesses independent and stationary increments and hence can directly be treated by using Theorems A and B.

2. ERDÖS-RÉNYI TYPE INCREMENTS OF POISSON PROCESSES

Let $\{ N(t) \}_{t \geqslant 0}$ be a standard Poisson process, i. e. a renewal process associated with an i. i. d. sequence $X_1, X_2, \ldots$, where $P(X_1 \geqslant x) = e^{-x}$, $x > 0$. Since $\varphi(s) = 1/(1 - s)$, $s < 1$, and $\mu = 1$, we have the following corollaries of Theorems 1 and 2:

COROLLARY 1. — For any $C > 0$, let $1 < A < \infty$ be the solution of

$$\frac{1}{C} = 1 - A + A \log A .$$

Then

$$\limsup_{T \to \infty} \frac{\Delta_T - CA \log T}{\log \log T} = \frac{1}{2 \log A} \quad \text{a. s.,} \tag{10}$$

$$\liminf_{T \to \infty} \frac{\Delta_T - CA \log T}{\log \log N} = - \frac{1}{2 \log A} \quad \text{a. s.} \tag{11}$$

COROLLARY 2. — For any $C > 1$, let $0 < B < 1$ be the solution of

$$\frac{1}{C} = 1 - B + B \log B .$$

Then

$$\limsup_{T \to \infty} \frac{\delta_T - CB \log T}{\log \log T} = - \frac{1}{2 \log B} \quad \text{a. s.,} \tag{12}$$

$$\liminf_{T \to \infty} \frac{\delta_T - CB \log T}{\log \log T} = \frac{1}{2 \log B} \quad \text{a. s.} \tag{13}$$

As indicated above we can give a direct proof of Corollary 1 by making use of Theorem A. Similarly, Corollary 2 can directly be deduced from Theorem B.

Proof of Corollary 1. — Since we have

$$\Delta_T \leqslant \max_{0 \leqslant n \leqslant N+1} \{ N(n + K'_{N+1} + 2) - N(n) \},$$
$$\Delta_T \geqslant \max_{0 \leqslant n \leqslant N} \{ N(n + K'_N) - N(n) \},$$

where $n = [t]$, $N = [T]$, $K'_N = [C \log N]$, and $\{ N(t) \}_{t \geqslant 0}$ has independent and stationary increments, relations (10) and (11) follow from assertions (6) and (7) of Theorem A (with $\lambda = 0$), provided the constants can be determined in a proper way. But observing that

$$\varphi(t) = E \exp(tN(1)) = \exp(e^t - 1), \qquad t \in \mathbb{R},$$

we have $m(t) = e^t$, $\mu = 1$, $a = \infty$, $c'_0 = 0$ (i. e. $c \in (0, \infty)$), $\alpha \in (1, \infty)$, $t^* = \log \alpha$, and

$$\frac{1}{c} = 1 - \alpha + \alpha \log \alpha$$

for this special case of Theorem A. Replacing $\alpha = A$ and $c = C$ completes the proof of Corollary 1.

In order to deduce Corollary 2 from Theorem B note that $b = 0$, $P(N(1) = b) = 1/e$. Hence $c'_1 = 1$, $c \in (1, \infty)$, $\beta \in (0, 1)$, $t^{**} = \log \beta$ and

$$\frac{1}{c} = 1 - \beta + \beta \log \beta .$$

It is interesting to compare these direct results of Corollaries 1 and 2 with the results of Theorems 1 and 2 in the special case of exponentially E(1)—distributed random variables. Since

$$\varphi(s) = E \exp(sX_1) = 1/(1 - s), \qquad s < s_0 = 1,$$

we obtain

$$m(s) = 1/(1-s), \quad \mu = 1, \quad a = \infty, \quad c_1 = 0, \quad C \in (0, \infty), \quad A \in (1, \infty),$$

and $$\frac{1}{C} = \sup_{s<0} \{ s + A \log(1 - s) \} = s^{**} + A \log(1 - s^{**}).$$

The latter equations yield $s^{**} = 1 - A < 0$ and $1/C = 1 - A + A \log A$, which implies

$$\frac{-CA}{2\{Cs^{**} - 1\}} = \frac{-A}{2\{s^{**} - (1/C)\}} = \frac{1}{2 \log A}.$$

Similarly, in Theorem 2, $s_0 = 1$, $c_0 = 1$, $B \in (0, 1)$, $s^* = 1 - B > 0$, $1/C = 1 - B + B \log B$, and

$$\frac{CB}{2\{Cs^* - 1\}} = -\frac{1}{2\log B}.$$

Hence the direct approaches to Corollaries 1 and 2 are in full agreement with Theorems 1 and 2.

3. PROOFS OF THEOREMS 1 AND 2

The proofs are mainly based upon Bacro's (1985) Theorems A and B and a duality argument comparing the increments of a renewal process $\{N(t)\}_{t\geqslant 0}$ with suitable increments of the corresponding partial sum sequence $\{S_n\}_{n=0,1,\dots}$. Consider Δ_T, δ_T as introduced before and let

$$\begin{aligned} A_T(\lambda) &= \{\Delta_T \geqslant CA\log T + \lambda \log\log T\} \\ A'_N(\lambda, d) &= \{d_N(\lambda) \leqslant C\log N + d\}, \end{aligned}$$

where, for $N = 1, 2, \dots$,

$$\begin{aligned} d_N(\lambda) &= \min_{0\leqslant n\leqslant N-K'_N(\lambda)} \{S_{n+K'_N(\lambda)} - S_n\}, \\ K'_N(\lambda) &= [CA\log N + \lambda\log\log N], \end{aligned}$$

and d is a suitable constant chosen below. Then

LEMMA 1. — For all $\lambda' < \lambda < \lambda''$, there exist d' and d'' such that

a) $P(A_T(\lambda) \text{ i. o. (in T)}) \leqslant P(A'_N(\lambda', d') \text{ i. o. (in N)})$,
b) $P(A_T(\lambda) \text{ i. o. (in T)}) \geqslant P(A'_N(\lambda'', d'') \text{ i. o. (in N)})$.

Proof. — Let $[u]$, $]u[$ denote the lower and upper integer part of u, i. e. $[u] \leqslant u < [u] + 1$, $]u[- 1 < u \leqslant]u[$.

a) Observe that Δ_T is integer-valued and it attaines its maximum at a random point t such that $t + C\log T = S_n$, where $S_n \leqslant T$ is a renewal point, i. e. $n \leqslant N(T)$. Now

$$\Delta_T = N(S_n) - N(S_n - C\log T) \geqslant\ ]CA\log T + \lambda\log\log T[$$

implies

$$S_n - S_{n-]CA\log T+\lambda\log\log T[} \leqslant C\log T.$$

Setting $N = [\delta' T]$, where $\delta' > 1/\mu$, we know from the law of large numbers for the renewal process that

$$N(T) \leqslant N, \qquad \text{for} \quad T \geqslant T_0 \text{ chosen large enough},$$

and also

$$\log N = \log T + \log \delta' + o(1),$$
$$\log \log N = \log \log T + o(1).$$

Hence, for $\lambda' < \lambda$ and T sufficiently large,

$$S_n - S_{n-K'_N(\lambda')} \leqslant C \log N + d$$

where $n \leqslant N$, $d = |2C \log \delta'|$, $K'_N(\lambda')$ as introduced before. This proves part *a*) of our lemma.

b) Suppose that

$$d_N(\lambda'') \leqslant C \log N + d,$$

where $\lambda'' > \lambda$, i. e. for some $n \leqslant N$,

$$S_n - S_{n-K'_N(\lambda'')} \leqslant C \log N + d.$$

Choose $T = \delta'' N$, where $\delta'' > \mu$. Then, by the SLLN again,

$$N(T) \geqslant N, \qquad \text{for} \quad N \geqslant N_0 \text{ sufficiently large}, \quad \text{and}$$
$$\log T = \log N + \log \delta''$$
$$\log \log T = \log \log N + o(1).$$

Since $\lambda'' > \lambda$, for N sufficiently large,

$$K'_N(\lambda'') \geqslant \,]CA \log T + \lambda \log \log T[\,, \qquad \text{and}$$
$$N(T) \geqslant N \geqslant n, \quad \text{i. e.} \quad S_n \leqslant T.$$

Choosing $d = C \log \delta''$, we have

$$N(S_n) - N(S_n - C \log T) \geqslant \,]CA \log T + \lambda \log \log T[\,, \quad \text{i. e.}$$
$$\Delta_T \geqslant CA \log T + \lambda \log \log T,$$

which completes the proof.

Considering the events

$$B_T(\lambda) = \{ \Delta_T \leqslant CA \log T + \lambda \log \log T \},$$
$$B'_N(\lambda, d) = \{ d_N(\lambda) \geqslant C \log N + d \},$$

we also have

LEMMA 2. — For all $\lambda' < \lambda < \lambda''$, there exist d' and d'' such that

a) $P(B_T(\lambda) \text{ i. o. (in T)}) \leqslant P(B'_N(\lambda'', d'') \text{ i. o. (in N)})$,
b) $P(B_T(\lambda) \text{ i. o. (in T)}) \geqslant P(B'_N(\lambda', d') \text{ i. o. (in N)})$.

Outline of proof. — If $B_T(\lambda)$ occurs, then

$$N(S_n) - N(S_n - C \log T) \leqslant [CA \log T + \lambda \log \log T]$$

for all renewal points $S_n \leqslant N(T)$. Choosing $N = [\tilde{\delta} T]$, $\tilde{\delta} < 1/\mu$, by $\lambda'' > \lambda$ and along the lines of proof for Lemma 1, this also implies

$$d_N(\lambda'') \geqslant C \log N + d$$

with some suitable constant d and large N. Similar arguments apply for part *b*).

Proof of Theorem 1. — We apply Theorem B with $c = CA$, $\beta = 1/A$ and $\lambda_{\pm}$ such that

$$b_{\pm}(\lambda) = \frac{\lambda}{A} - \frac{1}{s^{**}}\left\{\frac{\lambda}{CA} \pm \frac{1}{2}\right\} = 0, \qquad \text{i. e.}$$

$$\lambda_+ = \frac{CA}{2(Cs^{**} - 1)}, \qquad \lambda_- = \frac{-CA}{2(Cs^{**} - 1)}.$$

Note that $b_{\pm}(\lambda)$ is strictly increasing in λ. Let $\lambda_0 < \lambda_+ < \lambda_1$. Then, by assertion (8) of Theorem B, for any d,

$$P(B'_N(\lambda_0, d) \text{ i. o.}) = 0,$$
$$P(B'_N(\lambda_1, d) \text{ i. o.}) = 1,$$

which, by Lemma 2, also implies

$$P(B_T(\lambda) \text{ i. o.}) = 0, \qquad \lambda < \lambda_0,$$
$$P(B_T(\lambda) \text{ i. o.}) = 1, \qquad \lambda > \lambda_1.$$

Hence follows,

$$\lambda_0 \leqslant \liminf_{T\to\infty} \frac{\Delta_T - CA \log T}{\log \log T} \leqslant \lambda_1,$$

which proves (2) by letting $\lambda_{0,1}$ tend to λ_+.

Assertion (1) can be proved in a similar way, making use of relation (9) in Theorem B and the duality inequalities given in Lemma 1.

The proof of Theorem 2 can be derived from Theorem A and the following analogues of Lemmas 1 and 2:

Let

$$a_T(\lambda) = \{ \delta_T \geqslant CB \log T + \lambda \log \log T \},$$
$$b_T(\lambda) = \{ \delta_T \leqslant CB \log T + \lambda \log \log T \},$$

and

$$a'_N(\lambda, d) = \{ D_N(\lambda) \leqslant C \log N + d \},$$
$$b'_N(\lambda, d) = \{ D_N(\lambda) \geqslant C \log N + d \},$$

where

$$D_N(\lambda) = \max_{0 \leqslant n \leqslant N - K''_N(\lambda)} \{ S_{n+K''_N(\lambda)} - S_n \},$$
$$K''_N(\lambda) = [CB \log N + \lambda \log \log N],$$

and d is a suitable constant. Then we have:

LEMMA 3. — For all $\lambda' < \lambda < \lambda''$, there exist d' and d'' such that

a) $P(a_T(\lambda)$ i. o. (in T)) $\leqslant P(a'_N(\lambda', d')$ i. o. (in N)),
b) $P(a_T(\lambda)$ i. o. (in T)) $\geqslant P(a'_N(\lambda'', d'')$ i. o. (in N)).

LEMMA 4. — For all $\lambda' < \lambda < \lambda''$, there exist d' and d'' such that

a) $P(b_T(\lambda)$ i. o. (in T)) $\leqslant P(b'_N(\lambda'', d'')$ i. o. (in N)),
b) $P(b_T(\lambda)$ i. o. (in T)) $\geqslant P(b'_N(\lambda', d')$ i. o. (in N)).

The proofs are analogous to those of Lemmas 1 and 2 and can be omitted.

Remark 3. — It can be verified by an extension of the preceding arguments that the results of Theorems 1 and 2 subsist in the definitions of Δ_T and δ_T we choose K_T as a nondecreasing function of $T > 0$ such that $K_T - C \log T = o(\log \log T)$ as $T \to \infty$.

Remark 4. — The duality argument we have used for the Erdös-Rényi increments Δ_T and δ_T enables for a general K_T to obtain similar results as above by relating the limiting behavior of increments of partial sums to the limiting behavior of the increments of the corresponding renewal process. An example of such techniques is given in Deheuvels (1985) for the Bernoulli process ($P(X_i = 1) = 1 - P(X_i = 0) = p \in (0, 1)$).

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