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G. LETAC
Q. I. RAHMAN

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A factorisation of the Askey's characteristic function $(1 - \| t \|_{2n+1})_+^{n+1}$

by

G. LETAC (*) and Q. I. RAHMAN (**)

ABSTRACT. — g_r is the indicator of the ball centered on 0 with radius r in the Euclidean space E with odd dimension $2n + 1$. Richard Askey has shown that $\varphi_n(t) = (1 - \| t \|)^{n+1} g_1(t)$ is a positive definite function on E. We give here a new proof of this fact by showing that $\varphi_n/g_{1/2} * g_{1/2}$ is positive definite. The technique of the proof is to locate the zeros of the polynomial $\int_z^1 (1 - t^2)^n dt$.

RÉSUMÉ. — Soit g_r l'indicatrice de la boule de centre 0 et rayon r de l'espace euclidien E de dimension impaire $2n + 1$. Richard Askey a montré que dans E, $\varphi_n(t) = (1 - \| t \|)^{n+1} g_1(t)$ est une fonction définie positive. La note en donne une nouvelle démonstration, en montrant que $\varphi_n/g_{1/2} * g_{1/2}$ est définie positive. Ce résultat est atteint en localisant les zéros du polynôme $\int_z^1 (1 - t^2)^n dt$.

(*) Université Paul Sabatier, Toulouse, France. This author gratefully aknowledges the support of the University of Montreal during the preparation of this paper.
(**) Université de Montréal, Montréal (P. Q.) Canada.

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Let n a non negative integer. As usual, the norm and the scalar product in the Euclidean space $\mathbb{R}^{2n+1}$ are denoted by $\| t \| = \| t \|_{2n+1}$ and $\langle t, x \rangle$. R. Askey in [*1*] proves the following result:

THEOREM 1. — *Let $g: [0, +\infty) \to \mathbb{R}$ continuous, such that*
1) $g(0) = 1$,
2) $(-1)^n g^{(n)}(r)$ *exists and is convex in* $(0, +\infty)$,
3) $\lim_{r \to +\infty} g(r) = \lim_{r \to +\infty} g^{(n)}(r) = 0$.

Then $t \mapsto g(\| t \|)$ is a characteristic function on $\mathbb{R}^{2n+1}$, i. e. is the Fourier transform of a probability distribution on $\mathbb{R}^{2n+1}$.

This is a non-trivial generalization of the well-known Polya's theorem (see e. g. Feller [*2*], p. 482) corresponding to $n = 0$. The proof splits in two parts. Define φ_n on $\mathbb{R}^{2n+1}$ by:

$$\varphi_n(t) = (1 - \| t \|)_+^{n+1}, \tag{1}$$

where $a_+ = \max(0, a)$. R. Askey proves first that $g: [0, +\infty) \to \mathbb{R}$ fulfills the hypothesis of Th. 1 if and only if there exists a probability measure $\nu(dr)$ on $(0, +\infty)$ such that:

$$g(\| t \|) = \int_0^\infty \varphi_n(t/r)\, \nu(dr) \tag{2}$$

Therefore, to achieve the proof, we have to prove that φ_n is a characteristic function over $\mathbb{R}^{2n+1}$. Since φ_n is continuous, integrable and invariant by rotation there must exist a continuous function f_n on $[0, +\infty)$ such that:

$$\varphi(t) = \int_{\mathbb{R}^{2n+1}} \exp(i < t, x >) f_n(\| x \|) dx, \tag{3}$$

where $\int_{\mathbb{R}^{2n+1}} f_n(\| x \|) dx < \infty$. The positivity of f_n is now elegantly obtained in [*1*] by means of the formula:

$$\int_0^\infty e^{-sr} r^{3n+2} f_n(r) dr = C_n \left[\int_0^\infty e^{-sr} (1 - \cos r) dr \right]^{n+1} \tag{4}$$

true for $s > 0$, where C_n is $2^{2n+1} \pi^n n! (n+1)!$. (This proof is also reproduced in Letac [*3*] Problems II 6 and III 5, with slight modifications avoiding the use of Bessel functions).

The aim of this note is to offer an other proof of the positivity of f_n. To get the idea of it, let us come back to the case $n = 0$. Formula (4) gives

$f_0(r) = \frac{4}{r^2}(1 - \cos r)$ for $r > 0$. However, an other way to check the positive-definiteness of φ_0 is to consider $h_0 : \mathbb{R} \to \mathbb{R}$ defined by:

$$\begin{aligned} h_0(t) &= 1 \quad \text{if} \quad |t| < 1/2 \\ &= 0 \quad \text{if} \quad |t| \geq 1/2, \end{aligned}$$

and to observe that

$$\varphi_0 = h_0 * h_0, \tag{5}$$

where $*$ indicates convolution in $\mathbb{R}$. We deduce from (5) that φ_0 is positive-definite, as Prop. 2 below shows.

To extend this idea to higher dimensions, we introduce the indicator $h_n : \mathbb{R}^{2n+1} \to \mathbb{R}$ of the ball in $\mathbb{R}^{2n+1}$ centered in 0 with radius 1/2:

$$\begin{aligned} h_n(t) &= 1 \quad \text{if} \quad \|t\| < 1/2 \\ &= 0 \quad \text{if} \quad \|t\| \geq 1/2 \end{aligned}$$

and we make the following remark:

PROPOSITION 2. — *$t \mapsto (h_n * h_n)(t)$ is a continuous positive-definite function on $\mathbb{R}^{2n+1}$.*

Proof. — Since h_n is in $L^2(\mathbb{R}^d)$ and $h_n(t) = \overline{h_n(-t)}$, this fact is well known (see e. g. Rudin [5], Chap. 1, 1.4.2.). □

For $n > 0$, $h_n * h_n$ is no longer equal to φ_n, but it divides it, as our main result shows:

THEOREM 3. — *Consider the polynomial Q_n of n^{th} degree :*

$$Q_n(z) = (1 - z)^{-n-1} \int_z^1 (1 - u^2)^n du, \text{ and } B_n = 2^{-n-1}\pi^n/n!$$

Then for all t in $\mathbb{R}^{2n+1}$:

$$\varphi_n(t) = (h_n * h_n)(t) \frac{B_n}{Q_n(\|t\|)}, \tag{6}$$

and $t \mapsto 1/Q_n(\|t\|)$ is positive-definite on $\mathbb{R}^{2n+1}$.

As a corollary of this Th. 3, we get the second half of Askey's result, i. e. the positive-definiteness of φ_n, now written by (6) as the product of two positive definite functions. One minor advantage of this approach is that it gives a decomposition of the density in $\mathbb{R}^{2n+1} : x \mapsto f_n(\|x\|)$, as a convolution of two positive functions (Recall that (4) shows that the function on $(0, +\infty) : r \mapsto r^{3n+2} f_n(r)$ is the $(n+1)$th power of convolution of the function on $(0, +\infty) : r \mapsto C_n^{1/n+1}(1 - \cos r)$).

Although a probabilistic proof of Th. 3 would be desirable, we supply an analytic one. Let us begin with the easy part. n is now > 0.

Proof of (6). — There exists a function $V : [0, +\infty) \mapsto \mathbb{R}$ such that $V(\| t \|) = (h_n * h_n)(t)$. Clearly, V is the volume of the convex body obtain as the intersection of two balls in $\mathbb{R}^{2n+1}$ with radius 1/2 and centers at distance r.

Trivially for $r > 1$, $V(r) = 0$ and (6) is true for $\| t \| > 1$. Let us suppose now r *in* $[0, 1]$. In this case:

$$V(r) = 2\int_{B_r}\left(\left(\frac{1}{4} - (x_1^2 + x_2^2 + \ldots + x_{2n}^2)\right)^{1/2} - \frac{r}{2}\right)dx_1 \ldots dx_{2n},$$

where B_r is the ball in $\mathbb{R}^{2n}$ with center 0 and radius $\left(\frac{1}{4} - \frac{r^2}{4}\right)^{1/2}$, i. e $V(r)$ is twice the volume of a certain spherical cap.

Since the image in $(0, +\infty)$ of the Lebesgue measure in R^{2n} by the map $(x_1, \ldots, x_{2n}) \mapsto (x_1^2 + \ldots + x_{2n}^2)^{1/2} = \rho$ is $A_n \rho^{2n-1} d\rho$, where $A_n = 2\pi^n/(n-1)!$, then:

$$V(r) = 2A_n \int_0^{\left(\frac{1}{4} - \frac{r^2}{4}\right)^{1/2}} \left(\left(\frac{1}{4} - \rho^2\right)^{1/2} - \frac{r}{2}\right)\rho^{2n-1} d\rho.$$

Taking now

$$u = 2\left(\frac{1}{4} - \rho^2\right)^{1/2}, \text{ we get}$$

$$V(r) = A_n 2^{-n-1} \int_r^1 (u - r)(1 - u^2)^{n-1} u\, du.$$

Using integration by parts:

$$V(r) = \frac{A_n 2^{-n-2}}{n} \int_r^1 (1 - u^2)^n du,$$

and (6) is proved for $\| t \| \leq 1$. □

To complete the proof of Th. 3, we need two results.
The first one is likely to be known:

PROPOSITION 4. — *Let* P *be a polynomial with real coefficients without zeros in the closed right halfplane* $\{ z ; \operatorname{Re} z \geq 0 \}$. *Then for all integers* $d > 0$, *the function on the Euclidean space* $\mathbb{R}^d : t \mapsto P(0)/P(\| t \|)$ *is positive-definite.*

Proof. — If p and q are ≥ 0, $\exp(-p \| t \|)$ and $\exp\left(\frac{q}{2} \| t \|^2\right)$ are posi-

tive definite in $\mathbb{R}^d$. Therefore, if $s \geq 0$, $\exp\left(-s\left(p\|t\| + \frac{q}{2}\|t\|^2\right)\right)$ is positive definite in $\mathbb{R}^d$, as well as $1/\mathrm{P}(\|t\|) = \int_0^\infty \exp(-s\,\mathrm{P}(\|t\|))ds$,
if

$$\mathrm{P}(r) = 1 + pr + \frac{q}{2}r^2. \tag{7}$$

Since, for general P, P(0)/P is a product of functions of type (7), the result follows. □

PROPOSITION 5. — *The zeros of* $\mathrm{Q}_n(z) = (1-z)^{-n-1}\int_z^1 (1-t^2)^n dt$, *for* $n > 0$, *lie in the disk* :

$$\mathrm{D} = \left\{ z \,; \left| z + \frac{1}{2}\left(n + 1 + \frac{1}{n+1}\right)\right| < \frac{1}{2}\left(n + 1 - \frac{1}{n+1}\right)\right\},$$

which is contained in the half plane $\left\{ z \,; \mathrm{Re}\, z < -\frac{1}{n+1}\right\}$.

Clearly, Prop. 4 and 5 complete the proof of Th. 3.

Proof of Prop. 5. — For fixed n, and for an integer j in $\{0, \ldots, n\}$, we consider the polynomial

$$\mathrm{P}_j(z) = \int_z^1 (1-t)^{n+j}(1+t)^{n-j}\,dt.$$

An integration by parts gives, for $j < n$:

$$\mathrm{P}_j(z) = \frac{1}{n+j+1}(1-z)^{n+j+1}(1+z)^{n-j} + \frac{n-j}{n+j+1}\mathrm{P}_{j+1}(z) \tag{8}$$

and

$$\mathrm{P}_n(z) = \frac{1}{2n+1}(1-z)^{2n+1}. \tag{9}$$

Defining $c_j = \frac{1}{n+1} \cdot \frac{n}{n+2} \cdot \frac{n-1}{n+3} \cdot \ldots \cdot \frac{n-j+1}{n+j+1}$, one gets from (8) and (9) that:

$$\mathrm{P}_0(z) = \sum_{j=0}^{n} c_j(1-z)^{n+1+j}(1+z)^{n-j},$$

and

$$Q_n(z) = (1 + z)^n \sum_{j=0}^{n} c_j \left(\frac{1 - z}{1 + z}\right)^j . \tag{10}$$

Introducing now $a_j = (n + 1)\left(\frac{n + 2}{n}\right)^j c_j$, and the polynomial $H(w) = \sum_{j=0}^{n} a_j w^j$, we get from (10):

$$H\left(\frac{n}{n + 2} \cdot \frac{1 - z}{1 + z}\right) = (n + 1)\frac{Q_n(z)}{(1 + z)^n} .$$

The basic remark is now $a_0 = 1$ and $0 < \frac{a_{j+1}}{a_j} = \frac{n - j}{n + j + 2} \frac{n + 2}{n} < 1$. From the Eneström-Kakeya Theorem (see e. g. [4]), the zeros of H lie in $\{ w ; | w | > 1 \} = D_1$; the image of $D_1 \cup \infty$ by the reciprocal map of $z \mapsto w = \frac{n}{n + 2} \frac{1 - z}{1 + z}$ is D, and the proof is achieved. □

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