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On a. s. convergence of classes of multivalued asymptotic martingales

by

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ABSTRACT. — Let E′ be the separable dual of a Banach space E, $\mathscr{K}$ the class of all non-empty convex, weak* compact subsets of E′. We prove weak* a. s. convergence for $\mathscr{K}$-valued amarts of class (B) and for pramarts such that a subsequence is L_1-bounded. If the limit random variable takes values in a separable subset of $\mathscr{K}$, then strong convergence obtains for pramarts. The martingale case of our results was obtained under the same assumptions by J. Neveu. Our proof uses Neveu's sub-martingale lemma in the subpramart form.

RÉSUMÉ. — Soit E′ le dual, supposé séparable, d'un espace de Banach E. Soit $\mathscr{K}$ la classe des parties convexes de E′ qui sont compactes pour la topologie $\sigma(E', E)$. Nous démontrons la convergence presque sûre pour cette topologie des amarts de classe (B) à valeur dans $\mathscr{K}$, et des pramarts tels qu'une sous-suite soit bornée. Si la variable aléatoire limite prend p. p. ses valeurs dans une partie séparable de $\mathscr{K}$, alors pour les pramarts on obtient la convergence forte p. p. Le cas de martingales a été obtenu par J. Neveu, et notre démonstration est basée sur une variante d'un lemme démontré par Neveu pour sous-martingales positives.

Let E be a Banach space. E-valued weak sequential amarts of class (B) converge weakly a. s. if E has the Radon-Nikodym property (RNP) and the dual E' is separable [Brunel and Sucheston, *1*]. E-valued L_1-bounded pramarts converge strongly a. s. if E itself is a separable dual [Frangos, *6*]. Neveu [*9*] has proved the a. s. convergence of L_1-bounded multivalued martingales which take values in the class of non-empty, convex, weak*-compact subsets of the separable dual E' of a Banach space E. The present work takes as a starting point the work of Neveu mentioned above. We give here complete generalizations to classes of asymptotic martingales, namely, amarts and pramarts. The theory of multivalued random variables has been studied by Castaing-Valadier [*2*], Neveu [*9*] and Hiai-Umegaki [*7*], among others.

First, we give some known definitions and results. In this paper we assume that E' (and hence E) is separable.

Let

$$\mathscr{K} = \{ K \subseteq E' : K \text{ is non-empty, convex and weak*-compact} \}.$$

Then $\mathscr{K}$ is precisely the class of all non-empty subsets of E' which are convex, strongly closed and strongly bounded. This is straightforward application of the Banach-Alaoglu theorem and the Uniform Boundedness principle. Let $\mathscr{H}$ be the class of all continuous sublinear functionals $\phi : E \to \mathbb{R}$. For a continuous sublinear map ϕ on E, define $\Delta(\phi) = \sup_{\|y\| \leq 1} |\phi(y)|$. A sublinear map $\phi : E \to \mathbb{R}$ is continuous iff $\Delta(\phi) < \infty$. For every $K \in \mathscr{K}$, define the map $K \to \phi(K, .)$ as follows:

$$\phi(K, y) = \sup_{x \in K} \langle y, x \rangle, \quad \text{for} \quad y \text{ in } E. \tag{1}$$

Also, for every ϕ in $\mathscr{H}$, define the map $\phi \to K_\phi$ as follows:

$$K_\phi = \{ x \in E' : \langle y, x \rangle \leq \phi(y) \quad \text{for all } y \text{ in } E \}. \tag{2}$$

The following lemma then is a consequence of the Hahn-Banach theorem.

LEMMA 1.1. — The maps $K \to \phi(K, .)$ and $\phi \to K_\phi$ as defined above are from $\mathscr{K}$ to $\mathscr{H}$ and from $\mathscr{H}$ to $\mathscr{K}$ respectively. Each of them is the inverse of the other and hence each is a one—one and onto map.

For K_1, K_2 in $\mathscr{K}$, x_1 in K_1 and x_2 in K_2, we define the Hausdorff's metric Δ as follows:

$$\Delta(K_1, K_2) = \max \left[\sup_{K_1} \inf_{K_2} \| x_1 - x_2 \|, \sup_{K_2} \inf_{K_1} \| x_1 - x_2 \| \right]. \tag{3}$$

It can be checked that

$$\Delta(K_1, K_2) = \sup_{\|y\| \leq 1} | \phi(K_1, y) - \phi(K_2, y) |.$$

With this alternate definition for $\Delta(K_1, K_2)$, it is easy to check that $(\mathcal{K}, \Delta)$ is a complete metric space. In general $\mathcal{K}$ need not be separable. In view of lemma 1.1, let us introduce the following notation: For K in $\mathcal{K}$, $\Delta(K) = \Delta(K, \{0\})$.

Then, $\Delta(K) = \sup_{x \in K} \| x \| = \sup_{\|y\| \leq 1} | \phi(K, y) | = \Delta(\phi(K, .))$.

Let $(\Omega, \mathfrak{F}, P)$ be a probability space. A map X from $(\Omega, \mathfrak{F})$ to $\mathcal{K}$ is called a multivalued random variable (m. r. v.) if for every y in E, the map $\omega \to \phi(X(\omega), y)$ is a real valued random variable. Let D be a fixed countable dense subset of E. Since $\Delta(X) = \sup_{y \in D} \phi(X, y)$ it follows that if X is an m. r. v., $\Delta(X)$ is a real valued random variable. Moreover, due to the inequality $| \phi(X, y_1) - \phi(X, y_2) | \leqslant \Delta(X) \| y_1 - y_2 \|$, it follows that X is an m. r. v., if $\phi(X, y)$ is measurable for all y in D.

An m. r. v. X is called integrable if $\Delta(X)$ is. It follows that if X is an integrable m. r. v., then for each y in E, $\phi(X, y)$ is also integrable.

Suppose that X is an integrable m. r. v.. Then $\phi(y) = E[\phi(X, y)]$ is a continuous sublinear functional from E to $\mathbb{R}$. Hence by lemma 1.1, there is K in $\mathcal{K}$ such that $\phi(y) = \phi(K, y)$. We define the expectation of the m. r. v. X to be K, i. e., $E(X) = K$. Thus $E[\phi(X, y)] = \phi(EX, y)$ for y in E. The following lemma has been proved by Neveu ([9], p. 3).

LEMMA 1.2. — Let $y \to Z(., y)$ be a sublinear map from E to $L_1(\Omega, \mathfrak{F}, P)$; such that $E[\sup_{\|y\| \leq 1} | Z(., y) |] < \infty$. Then there exists an integrable m. r. v. X such that $\phi(X, y) = Z(y)$, a. e., for every y in E.

Let $\mathcal{G}$ be a sub-σ-field of $\mathfrak{F}$ and let X be an m. r. v.. Define $Z(y) = E[\phi(X, y) | \mathcal{G}]$. Then the map $y \to Z(y)$ satisfies all the hypotheses of lemma 1.2. Hence there is an integrable m. r. v. Y such that $\phi(Y, y) = Z(y)$, a. e. We define $E(X | \mathcal{G}) = Y$. In that case, we have that

$$E[\phi(X, y) | \mathcal{G}] = \phi(E(X | \mathcal{G}), y)$$

a. e., for y in E.

Let $(\mathfrak{F}_n)_{n=1}^{\infty}$ be a sequence of increasing sub-σ-fields of $\mathfrak{F}$. We shall assume that $\mathfrak{F}$ is generated by $\bigcup_{n=1}^{\infty} \mathfrak{F}_n$. Let T denote the class of simple stopping times. $(T, \leq)$ is a directed set filtering to the right, where $\leq$ is the usual order on T. A sequence of multivalued random variables (X_n) such that each X_n is $\mathfrak{F}_n$-measurable is called a multivalued process. For τ

in T, define $X_\tau = \sum_{i=1}^{n} X_i 1_{\{\tau = i\}}$, where $\tau \leq n$. A multivalued process $(X_n, \mathfrak{F}_n)$ is called (*i*) integrable, if for every $n \geq 1$, $E[\Delta(X_n)] < \infty$, (*ii*) L_1-bounded, if $\sup_{n \geq 1} E[\Delta(X_n)] < \infty$ and (*iii*) of class (B) if $\sup_T E[\Delta(X_\tau)] < \infty$. We now define various kinds of processes.

DEFINITION 2.1. — An integrable multivalued process $(X_n, \mathfrak{F}_n)$ is called

(*i*) a martingale if $E(X_{n+1} \mid \mathfrak{F}_n) = X_n$, a. e., for every $n \geq 1$.

(*ii*) an amart if the net $(EX_\tau)_{\tau \in T}$ converges in the Δ-metric, i. e., if there is K in $\mathscr{K}$ such that $\lim_T \Delta(EX_\tau, K) = 0$,

(*iii*) a w^*-amart if there is K in $\mathscr{K}$ such that for every y in E,

$$\lim_T \phi(EX_\tau, y) = \phi(K, y),$$

(*iv*) a pramart if $\operatorname*{slim}_{\substack{\sigma \leq \tau \\ \sigma, \tau \in T}} \Delta(X_\sigma, E(X_\tau \mid \mathfrak{F}_\sigma)) = 0$,

(*v*) a w^*-pramart if for every y in E, $\operatorname*{slim}_{\substack{\sigma \leq \tau \\ \sigma, \tau \in T}} | \phi(X_\sigma, y) - E(\phi(X_\tau, y) \mid \mathfrak{F}_\sigma) | = 0$.

Here « slim » is an abbreviation for stochastic limit (limit in probability). Clearly, an amart is a w^*-amart and a pramart is a w^*-pramart. A martingale is both an amart and a pramart. There is an example ([*8*], p. 123) to show that an amart need not be a pramart even in the single valued case. A pramart need not be an amart even in the real (single) valued case as shown in ([*8*], p. 108). Finally a w^*-amart is a w^*-pramart. We now proceed to prove some convergence theorems for w^*-amarts and pramarts.

THEOREM 2.2. — Suppose that (X_n) is a multivalued w^*-amart of class (B). Then there is an integrable m. r. v. X_∞ such that $\lim_{n \to \infty} \phi(X_n, y) = \phi(X_\infty, y)$, for every y in E, outside of a null set independent of y.

First, we state a lemma which is a direct application of the maximal inequality proved by Chacon and Sucheston ([*3*], p. 56).

LEMMA 2.3. — Let (X_n) be a multivalued process of class (B). Then for any fixed $a > 0$, $P(\{ \sup_n \Delta(X_n) \geq a \}) \leq \frac{1}{a} \sup_T E[\Delta(X_\tau)]$.

Proof. — Apply the maximal inequality ([*3*], p. 56) to the real valued process $(\Delta(X_n))$.

Now consider a multivalued process of class (B) and for a fixed $a > 0$ define a stopping time σ as follows:

$$\sigma = \min \{ n : \Delta(X_n) \geq a \} \quad \text{if there is } n \text{ such that } \Delta(X_n) \geq a,$$
$$= \infty \qquad \text{otherwise}.$$

$$P(\{ X_{n \wedge \sigma} \neq X_n \text{ for some } n \}) \leqslant P(\{ \sigma \neq \infty \}) \leqslant P(\{ \sup_n \Delta(X_n) \geq a \}) \leq \frac{1}{a} \sup_T E[\Delta(X_\tau)].$$

Therefore, for a large enough $a > 0$, the processes $(X_{n \wedge \sigma})$ and (X_n) coincide except on a set of arbitrarily small probability. Let $\gamma = \sup_n \Delta(X_{n \wedge \sigma})$. On $\{ \sigma < \infty \}$, $\Delta(X_{n \wedge \sigma}) \leq \Delta(X_\sigma)$ and hence $Y \leq \Delta(X_\sigma)$. On $\{ \sigma = \infty \}$, $\Delta(X_n) \leq a$ for every n and thus $Y \leq a$. Moreover, on $\{ \sigma < \infty \}$, $\lim_{n \to \infty} \Delta(X_{n \wedge \sigma}) = \Delta(X_\sigma)$. Therefore,

$$EY = \int_{\{\sigma = \infty\}} Y + \int_{\{\sigma < \infty\}} \leq aP(\{ \sigma = \infty \}) + \liminf_{n \to \infty} \int \Delta(X_{n \wedge \sigma}) < \infty .$$

Hence $(X_{n \wedge \sigma})$ is a process such that $\sup_n \Delta(X_{n \wedge \sigma}) \in L_1$. Thus we shall assume that (X_n) itself is a process such that $\sup_n \Delta(X_n) \in L_1$.

Let $(X_n, \mathfrak{F}_n)$ be a multivalued w^*-amart of class (B). For every fixed y in E, the process $(\phi(X_n, y), \mathfrak{F}_n)$ is an L_1-bounded real-valued amart. By the real valued amart convergence theorem [*4*], there is $Z(y, .)$ such that

$$\lim \phi(X_n, y) = Z(y) \qquad \text{a. e.}$$

Let $\{ y_i \}$ be a countable dense subset of E. Since $\sup_n \Delta(X_n)$ is in L_1, there is a set Ω_1, such that $P(\Omega_1) = 1$ and for every $\omega \in \Omega_1$, $\sup_n \Delta(X_n(\omega)) < \infty$ and $\lim_{n \to \infty} \phi(X_n(\omega), y_i) = Z(y_i, \omega)$.

Thus,

$$\lim_{n \to \infty} \phi(X_n(\omega), y) = Z(y, \omega), \quad \text{for all } y \text{ in E and } \omega \text{ in } \Omega_1 .$$

Since $\phi(X_n(\omega), .)$ is a sublinear map on E for every n, so is $Z(., \omega)$ and moreover.

$$| Z(y, \omega) | \leq \liminf_{n \to \infty} \| y \| . \Delta(X_n(\omega)) .$$

Hence

$$E[\sup_{\|y\| \leq 1} | Z(y) |] \leq \liminf_{n \to \infty} E[\Delta(X_n)] < \infty .$$

Thus, by lemma 1.2, there is an m. r. v. X_∞ such that

$$\lim_{n\to\infty} \phi(X_n(\omega), y) = Z(y, \omega) = \phi(X_\infty, y) \quad \text{for every } y \text{ in E and every } \omega \in \Omega_2$$

where $P(\Omega_2) = 1$.

Moreover,

$$E[\Delta(X_\infty)] = E[\sup_{\|y\|\leq 1} |\phi(X_\infty, y)|] < \infty .$$

This proves the theorem. Our next theorem is about the strong convergence of multivalued pramarts. Real valued pramarts were introduced by Millet and Sucheston in [*8*].

THEOREM 2.4. — (*i*) Let (X_n) be a multivalued w^*-pramart such that $\liminf_{n\to\infty} E[\Delta(X_n)] < \infty$. Then there exists an integrable m. r. v. X_∞ such that we have, for every y in E,

$$\lim_{n\to\infty} \phi(X_n, y) = \phi(X_\infty, y) \qquad \text{a. e.}$$

(The exceptional null set depends on y even in the single-valued case).

(*ii*) If moreover, (X_n) is an L_1-bounded multivalued pramart such that X_∞ takes values a. s. in a separable subset of $\mathscr{K}$, then we have that

$$\lim_{n\to\infty} \Delta(X_n, X_\infty) = 0 \qquad \text{a. s.} .$$

Proof. — If (X_n) is a w^*-pramart, then for every fixed y in E, $(\phi(X_n, y), \mathfrak{F}_n)$ is a real valued pramart. Since

$$\liminf_{n\to\infty} E[\phi(X_n, y)^+] + \liminf_{n\to\infty} E[\phi(X_n, y)^-] \leq 2 . \|y\| . \liminf_{n\to\infty} E[\Delta(X_n)] < \infty,$$

by the real-valued pramart convergence theorem of Millet and Sucheston ([*8*], p. 98) we have the existence of a real valued random variable $Z(y)$ such that for every y in E,

$$\lim_{n\to\infty} \phi(X_n, y) = Z(y) \qquad \text{a. s.} .$$

Now, $|Z(y)| \leq \liminf_{n\to\infty} \|y\| . \Delta(X_n)$. Hence, by Fatou's lemma, we have $E[\sup_{\|y\|<1} |Z(y)|] \leq \liminf_{n\to\infty} E[\Delta(X_n)] < \infty$. From lemma 1.2, it follows that there is an integrable m. r. v. X_∞ such that

$$\lim_{n\to\infty} \phi(X_n, y) = \phi(X_\infty, y) \qquad \text{a. s.} .$$

This proves (*i*). To prove (*ii*), we shall use a lemma by Egghe [*5*].

LEMMA 2.5. — *Let* I *be a counatble set. Suppose that for each* $i \in \mathrm{I}$, *there is a process* $(\mathrm{U}_n^i, \mathfrak{F}_n)_{n=1}^{\infty}$ *satisfying the following conditions:*

(*i*) $\mathrm{U}_n^i \geq 0$ *for each* i *in* I *and* $n \geq 1$;

(*ii*) $\operatorname{slim}_{\substack{\sigma \leq \tau \\ \sigma,\tau \in \mathrm{T}}} \sup \left[\sup_{\mathrm{I}} (\mathrm{U}_\sigma^i - \mathrm{E}(\mathrm{U}_\tau^i \mid \mathfrak{F}_\sigma))\right] \leq 0$;

(*iii*) $\sup_n \mathrm{E}\left[\sup_{\mathrm{I}} (\mathrm{U}_n^i)\right] < \infty$.

Then, for every i *in* I, *there is* U^i *such that* $\lim_n \mathrm{U}_n^i = \mathrm{U}^i$, *a. s. and moreover*

$$\sup_{\mathrm{I}} \mathrm{U}_n^i \to \sup_{\mathrm{I}} \mathrm{U}_\infty^i \qquad a.\,s. \quad as \quad n \to \infty .$$

Suppose now that (X_n) is an L_1-bounded multivalued pramart. Since E is separable, we can choose a countable set $\{ y_i : i \in \mathrm{I} \}$ dense in the unit ball of E. For Q in $\mathscr{K}$, define $\mathrm{U}_n^i = | \phi(\mathrm{X}_n, y_i) - \phi(\mathrm{Q}, y_i) |$. Clearly U_n^i is $\mathfrak{F}_n$-measurable and non-negative.

$$\begin{aligned} \mathrm{U}_\sigma^i - \mathrm{E}(\mathrm{U}_\tau^i \mid \mathfrak{F}_\sigma) &= | \phi(\mathrm{X}_\sigma, y_i) - \phi(\mathrm{Q}, y_i) | - \mathrm{E}(| \phi(\mathrm{X}_\tau, y_i) - \phi(\mathrm{Q}, y_i) | \mid \mathfrak{F}_\sigma) \\ &\leq | \phi(\mathrm{X}_\sigma, y_i) - \phi(\mathrm{Q}, y_i) | - | \mathrm{E}(\phi(\mathrm{X}_\tau, y_i) \mid \mathfrak{F}_\sigma) - \phi(\mathrm{Q}, y_i) | \\ &\leq | \phi(\mathrm{X}_\sigma, y_i) - \mathrm{E}(\phi(\mathrm{X}_\tau, y_i) \mid \mathfrak{F}_\sigma) | \\ &= | \phi(\mathrm{X}_\sigma, y_i) - \phi(\mathrm{E}(\mathrm{X}_\tau \mid \mathfrak{F}_\sigma), y_i) | . \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{\mathrm{I}} \left[\mathrm{U}_\sigma^i - \mathrm{E}(\mathrm{U}_\tau^i \mid \mathfrak{F}_\sigma)\right] &\leq \sup_{\mathrm{I}} | \phi(\mathrm{X}_\sigma, y_i) - \phi(\mathrm{E}(\mathrm{X}_\tau \mid \mathfrak{F}_\sigma), y_i) | \\ &= \Delta(\mathrm{X}_\sigma, \mathrm{E}(\mathrm{X}_\tau \mid \mathfrak{F}_\sigma)) . \end{aligned}$$

Hence,
$$\operatorname{slim}_{\substack{\sigma \leq \tau \\ \sigma,\tau \in \mathrm{T}}} \sup \left[\sup_{\mathrm{I}} (\mathrm{U}_\sigma^i - \mathrm{E}(\mathrm{U}_\tau^i \mid \mathfrak{F}_\sigma))\right] \leq 0 ,$$

since (X_n) is a pramart.

Also,

$$\sup_n \mathrm{E}\left[\sup_{\mathrm{I}} (\mathrm{U}_n^i)\right] = \sup_n \mathrm{E}\left[\Delta(\mathrm{X}_n, \mathrm{Q})\right] < \infty, \quad \text{since } (\mathrm{X}_n) \text{ is } \mathrm{L}_1\text{-bounded} .$$

Hence, by this lemma,

$$\sup_{\mathrm{I}} | \phi(\mathrm{X}_n, y_i) - \phi(\mathrm{Q}, y_i) | \to \sup_{\mathrm{I}} | \phi(\mathrm{X}_\infty, y_i) - \phi(\mathrm{Q}, y_i) | \quad \text{a. s.} \quad \text{as } n \to \infty .$$

That is,
$$\lim_{n \to \infty} \Delta(\mathrm{X}_n, \mathrm{Q}) = \Delta(\mathrm{X}_\infty, \mathrm{Q}) , \qquad \text{a. s.}$$

Suppose now that X_∞ takes its values a. s. in a separable subset $\mathscr{K}_0$ of $\mathscr{K}$.

Then there is a countable dense set $\{ Q_i : i \geq 1 \}$ and a set Ω_0 such that $P(\Omega_0) = 1$ and for every $\omega \in \Omega_0$ and every $i \geq 1$, we have that

$$\lim_{n \to \infty} \Delta(X_n(\omega), Q_i) = \Delta(X_\infty, Q_i).$$

Since $\{ Q_i : i \geq 1 \}$ is dense in $\mathscr{K}_0$, it follows that

$$\lim_{n \to \infty} \Delta(X_n(\omega), Q) = \Delta(X_\infty(\omega), Q)$$

for every $\omega \in \Omega_0$ and every Q in $\mathscr{K}_0$.

Taking $Q = X_\infty(\omega)$, we have that

$$\lim_{n \to \infty} \Delta(X_n, X_\infty) = 0 \qquad \text{a. s.}$$

This completes the proof of the theorem.

As a consequence of this theorem we obtain a theorem proved by Neveu [*9*].

THEOREM 2.6. — Let $(X_n, \mathfrak{F}_n)$ be an L_1-bounded martingale. Then there is an m. r. v. X_∞ such that $E[\Delta(X_\infty)] < \infty$ and $\lim_{n \to \infty} \phi(X_n, y) = \phi(X_\infty, y)$. a. s., for every $y \in E$. Moreover, if X_∞ takes its values a. s. in a separable subset of $\mathscr{K}$, then we have

$$\lim_{n \to \infty} \Delta(X_n, X_\infty) = 0 \qquad \text{a. s.}$$

Proof. — This follows from theorem 2.3 and the fact that a multivalued martingale is a multivalued pramart.

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